J Geom Anal (2016) 26:837–857
DOI 10.1007/s12220-015-9573-y

Sharp Moser–Trudinger Inequalities on Hyperbolic Spaces with Exact Growth Condition

Guozhen Lu · Hanli Tang

Received: 28 May 2014 / Published online: 3 February 2015

Abstract Let $\Phi_n(x) = e^x - \sum_{j=0}^{n-2} \frac{x^j}{j!}$ and $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ be the sharp constant in Moser's inequality (where ω_{n-1} is the area of the surface of the unit n-ball in $\mathbb{R}^n$), and dV be the volume element on the n-dimensional hyperbolic space $(\mathbb{H}^n, g)$ $(n \geq 2)$. In this paper, we establish the following sharp Moser–Trudinger type inequalities with the exact growth condition on $\mathbb{H}^n$:

For any $u \in W^{1,n}(\mathbb{H}^n)$ satisfying $\|\nabla_g u\|_n \leq 1$, there exists a constant $C(n) > 0$ such that

$$\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n |u|^{\frac{n}{n-1}})}{(1+|u|)^{\frac{n}{n-1}}} dV \leq C(n) \|u\|_{L^n}^n.$$

The power $\frac{n}{n-1}$ and the constant α_n are optimal in the following senses:

(i) If the power $\frac{n}{n-1}$ in the denominator is replaced by any $p < \frac{n}{n-1}$, then there exists a sequence of functions $\{u_k\}$ such that $\|\nabla_g u_k\|_n \leq 1$, but

$$\frac{1}{\|u_k\|_{L^n}^n} \int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n (|u_k|)^{\frac{n}{n-1}})}{(1+|u_k|)^p} dV \to \infty.$$

(ii) If $\alpha > \alpha_n$, then there exists a sequence of function $\{u_k\}$ such that $\|\nabla_g u_k\|_n \leq 1$, but

G. Lu · H. Tang (✉)
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
e-mail: hltang@bnu.edu.cn

G. Lu
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
e-mail: gzlu@wayne.edu

$$\frac{1}{\|u_k\|_{L^n}^n}\int_{\mathbb{H}^n}\frac{\Phi_n(\alpha(|u_k|)^{\frac{n}{n-1}})}{(1+|u_k|)^p}dV\to\infty,$$

for any $p\geq 0$.

This result sharpens the earlier work of the authors Lu and Tang (Adv Nonlinear Stud 13(4):1035–1052, 2013) on best constants for the Moser–Trudinger inequalities on hyperbolic spaces.

Keywords Hyperbolic spaces · Sharp Moser–Trudinger inequalities · Best constants · Exact growth condition

Mathematics Subject Classification 42B35 · 42B37

1 Introduction

The Moser–Trudinger inequalities can be considered as the limiting case of Sobolev inequalities. They were established independently by Yudovič [30], Pohožaev [25] and Trudinger [28]. In 1971, Moser [23], sharpening Trudinger's inequality, proved that

Theorem A *Let Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$, $n\geq 2$. Then there exist a positive constant $C(n)>0$ and a sharp constant $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$ (where ω_{n-1} is the area of the surface of the unit n-ball) such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha|f|^{\frac{n}{n-1}}\right)dx\leq C(n)<\infty \tag{1.1}$$

for any $\alpha\leq\alpha_n$, any $f\in C_0^\infty(\Omega)$ with $\int_\Omega|\nabla f|^n dx\leq 1$. This constant α_n is sharp in the sense that if $\alpha>\alpha_n$, then the above inequality can no longer hold with some $C(n)$ independent of f.

This result has been generalized in many directions. For instance, the singular Moser–Trudinger inequality was proved in [3], and the best constants for Moser–Trudinger inequalities on domains of finite measure on the Heisenberg group were established in [7,15]. There has also been substantial progress for the Moser–Trudinger inequalities on spheres, CR spheres, or compact Riemannian manifolds. We refer the interested reader to [5,8,9,17], etc. Moser–Trudinger inequalities have found many applications in geometric analysis and PDEs; see, e.g., [12,15,18,23,24,27], the survey articles [6] and [13], etc.

When Ω has infinite volume, the sharp version of Moser–Trudinger type inequalities for unbounded domains was established by Adachi and Tanaka [1] in order to determine the best constant. They proved that

Theorem B *Let $0<\alpha<\alpha_n$. There exists a constant $C(\alpha)>0$ such that*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}|\nabla u|^n dx\leq 1}\int_{\mathbb{R}^n}\Phi_n(\alpha|u|^{\frac{n}{n-1}})dx\leq C(\alpha)\|u\|_{L^n(\mathbb{R}^n)}^n,$$

where $\Phi_n(t) := e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}$. *Moreover, the constant* α_n *is sharp in the sense that if* $\alpha \geq \alpha_n$, *the supremum will become infinite.*

The method used in [1] is the symmetrization argument by reducing the problem to the radial case. However, such a symmetrization argument does not work on the Heisenberg group. Using an entirely different method of dividing the entire Heisenberg group into two parts using the level sets of the functions under consideration, such a sharp subcritical Moser–Trudinger inequality on the Heisenberg group has been established in [16].

Ruf [24] (for the case $n = 2$), Li and Ruf [18] (for the general case $n \geq 2$) established a critical Moser–Trudinger type inequality for unbounded domains in Euclidean spaces. They obtain the following theorem.

Theorem C *There exists a constant* $C(n) > 0$ *such that for any domain* $\Omega \subset \mathbb{R}^n$,

$$\sup_{u \in W_0^{1,n}(\Omega), \|u\|_{W^{1,n}(\Omega)} \leq 1} \int_\Omega \Phi_n(\alpha_n |u|^{\frac{n}{n-1}}) dx \leq C(n).$$

Moreover, the constant α_n *is sharp in the sense that if* α_n *is replaced by any* $\alpha > \alpha_n$, *the supremum will become infinite.*

Such a sharp Moser–Trudinger inequality at the critical case has also been established on the entire Heisenberg group in [11] where a symmetrization argument is not available.

We note that there is a sharp difference between the inequalities in Theorems B and C. In Theorem B, the inequality only holds for $\alpha < \alpha_n$ while the inequality in Theorem C holds for all $\alpha \leq \alpha_n$. The reason behind this is that the restriction on the class of functions in Theorem B is for all with the L^n norm of their gradients being less than or equal to 1 while the function class in Theorem C is for those with the full Sobolev $W^{1,n}$ norm less than or equal to 1.

In short, the failure of the original Moser–Trudinger inequality (1.1) on the entire $\mathbb{R}^n$ can be recovered either by weakening the exponent $\alpha = n\omega_{n-1}^{\frac{1}{n-1}}$ or by strengthening the Dirichlet norm $\|\nabla u\|_{L^n}$. Then a natural question arises: Can we still achieve the best constant $\alpha = n\omega_{n-1}^{\frac{1}{n-1}}$ when we only require the restriction on the norm $\|\nabla u\|_{L^n} \leq 1$?

Ibrahim, Masmoudi and Nakanishi [10] answered the question in the two-dimensional case. They set up the following theorem.

Theorem D *There exists a constant* $C > 0$, *such that*

$$\int_{\mathbb{R}^2} \frac{e^{4\pi u^2} - 1}{(1 + |u|)^2} dx \leq C \|u\|^2_{L^2(\mathbb{R}^2)},$$

for any $u \in W^{1,2}(\mathbb{R}^2)$ *with* $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. *Moreover, this fails if the power* 2 *in the denominator is replaced with any* $p < 2$.

In this paper, we will consider the Moser–Trudinger inequalities with exact growth condition on the hyperbolic spaces. The hyperbolic space $\mathbb{H}^n$ ($n \geq 2$) is a complete and simply connected Riemannian manifold having constant sectional curvature equal to -1, and for a given dimensional number, any two such spaces are isometric [29]. There are several models for $\mathbb{H}^n$, the most important model being the half-space model, the ball model, and the hyperboloid or Lorentz model, with the ball model being especially useful for questions involving rotational symmetry. We will only use the ball model in this paper.

Let $B^n = \{x \in R^n : |x| < 1\}$ denote the unit open ball in the Euclidean space $\mathbb{R}^n$. The space B^n endowed with the Riemannian metric $g_{ij} = (\frac{1}{1-|x|^2})^2\delta_{ij}$ is called the ball model of the hyperbolic space $\mathbb{H}^n$. Denote the associated hyperbolic volume by $dV = (\frac{2}{1-|x|^2})^n dx$. For any measurable set $E \subset \mathbb{H}^n$, set $|E| = \int_E dV$. Let $d(0, x)$ denote the hyperbolic distance between the origin and x. It is known that $d(0, x) = \ln \frac{1+|x|}{1-|x|}$ for $x \in \mathbb{H}^n$. The hyperbolic gradient ∇_g is given by $\nabla_g = (\frac{1-|x|^2}{2})^2\nabla$, where ∇ is the Euclidean gradient.

Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. Denote $\|f\|_{n,\Omega} = (\int_\Omega |f|^n dV)^{\frac{1}{n}}$. Then we have the following:

$$\|\nabla_g f\|_{n,\Omega} = \left(\int_\Omega \langle \nabla_g f, \nabla_g f\rangle_g^{n/2} dV\right)^{\frac{1}{n}} = \left(\int_\Omega |\nabla f|^n dx\right)^{\frac{1}{n}}.$$

Let $\|f\|_n = (\int_{\mathbb{H}^n} |f|^n dV)^{\frac{1}{n}}$. Then we have

$$\|\nabla_g f\|_n = \left(\int_{\mathbb{H}^n} \langle \nabla_g f, \nabla_g f\rangle_g^{n/2} dV\right)^{\frac{1}{n}} = \left(\int_{B^n} |\nabla f|^n dx\right)^{\frac{1}{n}}.$$

We use $W_0^{1,n}(\Omega)$ to express the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,n}(\Omega)} = \left(\int_\Omega |f|^n dV + \int_\Omega |\nabla f|^n dx\right)^{\frac{1}{n}}.$$

We will also use $W^{1,n}(\mathbb{H}^n)$ to express the completion of $C_0^\infty(\mathbb{H}^n)$ under the norm

$$\|u\|_{W^{1,n}(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |f|^n dV + \int_{\mathbb{H}^n} |\nabla f|^n dx\right)^{\frac{1}{n}}.$$

It is known that the symmetrization argument is the key tool in the proof of the classical Moser–Trudinger inequalities. Now, let us recall some facts about the rearrangement in the hyperbolic spaces [4].

Let $f : \mathbb{H}^n \to \mathbb{R}$ be such that

$$|\{x \in \mathbb{H}^n : |f(x)| > t\}| = \int_{\{x\in\mathbb{H}^n:|f(x)|>t\}} dV < +\infty$$

for every $t > 0$. Its distribution function is defined by

$$\mu_f(t) = |\{x \in \mathbb{H}^n : |f(x)| > t\}.$$

Then its decreasing rearrangement f^* is defined by

$$f^*(s) = \sup\{t > 0, \mu_f(t) > s\}.$$

Now, define $f^\sharp : \mathbb{H}^n \to \mathbb{R}$ by

$$f^\sharp(x) = f^*(|B(0, d(0, x))|),$$

where $B(0, d(0, x))$ is the ball centered at the origin and with radius $d(0, x)$ in the hyperbolic space. Then, for every continuous increasing function $\Phi : [0, \infty) \to [0, \infty)$, we have from [4] that

$$\int_{\mathbb{H}^n} \Phi(|f|) dV = \int_{\mathbb{H}^n} \Phi(f^\sharp) dV.$$

Moreover, for any Lipschitz continuous function f, when $p \geq 1$,

$$\|\nabla_g f^\sharp\|_p \leq \|\nabla_g f\|_p.$$

Moser–Trudinger inequalities on a hyperbolic space were considered by Mancini and Sandeep [21] in the two-dimensional case; they established the Moser–Trudinger inequalities on a conformal disc. Recently, the authors [19] established sharp constants for Moser–Trudinger inequalities on high dimensional hyperbolic spaces. They first proved the sharp singular Moser–Trudinger inequality on bounded domains in the hyperbolic space of any high dimension.

Theorem E *Let $\Omega \subset \mathbb{H}^n$ be a domain with $|\Omega| = \int_\Omega dV < +\infty$ and $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$. Then there exists a constant $C > 0$ such that*

$$\sup_{u \in C_0^\infty(\Omega), \|\nabla_g u\|_{n,\Omega} \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha_n |u|^{\frac{n}{n-1}}) dV \leq C.$$

The result is sharp in the sense that if α_n is replaced by any $\alpha > \alpha_n$, the supremum will become infinite.

Then they set up the following sharp subcritical Moser–Trudinger type inequality on the entire hyperbolic space in the spirit of Adachi–Tanaka [1].

Theorem F *For any $\alpha \in (0, \alpha_n)$, there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{H}^n} \Phi_n(\alpha |u|^{\frac{n}{n-1}}) dV \leq C_\alpha \|u\|_n^n, \tag{1.2}$$

for $u \in W^{1,n}(\mathbb{H}^n)$ *with* $\|\nabla_g u\|_n \le 1$*, where* $\Phi_n(x) = e^x - \sum_{j=0}^{n-2} \frac{x^j}{j!}$*. Moreover, the result is sharp in the sense that if* $\alpha \ge \alpha_n$*, there exists a sequence* $\{u_k\}_{k=1}^{\infty} \subset W^{1,n}(\mathbb{H}^n)$ *such that* $\|\nabla_g u_k\|_n = 1$ *and*

$$\frac{1}{\|u_k\|_n^n} \int_{\mathbb{H}^n} \Phi_n(\alpha(|u_k|)^{\frac{n}{n-1}}) dV \to \infty.$$

Furthermore, the authors established in [19] the following sharp critical singular Moser–Trudinger inequality on the entire hyperbolic space when we restrict the norms of functions to full hyperbolic Sobolev norm.

Theorem G *Let* $0 \le \beta < n, \tau > 0$*. For any* $\alpha \in (0, \alpha_n(1 - \frac{\beta}{n})]$*, there exists a constant* $C_{\alpha,\tau} > 0$ *such that*

$$\sup_{u \in W^{1,n}(\mathbb{H}^n), \|\nabla_g u\|_n^n + \tau \|u\|_n^n \le 1} \int_{\mathbb{H}^n} \frac{\Phi_n(\alpha |u|^{\frac{n}{n-1}})}{[d(0,x)]^{\beta}} dV \le C_{\alpha,\tau}.$$

The constant $\alpha_n(1 - \frac{\beta}{n})$ *is sharp in the sense that if* $\alpha_n(1 - \frac{\beta}{n})$ *is replaced by any* α *bigger than* α_n*, the supreme will become infinite.*

Motivated by the work [10] and our Theorems F and G, we naturally want to know what will happen to the inequality (1.2) if we keep the condition $\alpha = \alpha_n$ and $\|\nabla_g u\|_n \le 1$. In this paper, we will prove the exact growth condition for the sharp Moser–Trudinger type inequality on the hyperbolic space with the restriction on the norms of functions is only imposed on the gradient. This answers our question.

Theorem 1.1 *For any* $u \in W^{1,n}(\mathbb{H}^n)$ *satisfying* $\|\nabla_g u\|_n \le 1$*, there exists a constant* $C(n) > 0$ *such that*

$$\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n |u|^{\frac{n}{n-1}})}{(1 + |u|)^{\frac{n}{n-1}}} dV \le C(n) \|u\|_n^n.$$

We remark that both the power $\frac{n}{n-1}$ and the constant α_n are optimal. These can be justified by the following theorem.

Theorem 1.2 *If the power* $\frac{n}{n-1}$ *in the denominator is replaced by any* $p < \frac{n}{n-1}$*, there exists a sequence of functions* $\{u_k\}$ *such that* $\|\nabla_g u_k\|_n \le 1$*, but*

$$\frac{1}{\|u_k\|_n^n} \int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n(|u_k|)^{\frac{n}{n-1}})}{(1 + |u_k|)^p} dV \to \infty.$$

Moreover, if $\alpha > \alpha_n$*, there exists a sequence of functions* $\{u_k\}$ *such that* $\|\nabla_g u_k\|_n \le 1$*, but*

$$\frac{1}{\|u_k\|_n^n} \int_{\mathbb{H}^n} \frac{\Phi_n(\alpha(|u_k|)^{\frac{n}{n-1}})}{(1 + |u_k|)^p} dV \to \infty,$$

for any $p \ge 0$.

Finally, we end this Introduction by commenting on some recent works on Adams's inequalities on high order Sobolev spaces. Sharp Adams's inequalities on Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ were proved by Ruf and Sani [26] for all even m and established by Lam and the first author for all odd and fractional m in [12,14]. Motivated by the work of Masmoudi and Sani on Adams's inequality with exact growth in dimension four [22], the authors and M. Zhu have established the following Adams's inequalities with exact growth condition in general dimension [20].

Theorem 1.3 *There exists a constant $C > 0$ such that for all $f \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ $(n \geq 4)$ satisfying $\|\Delta f\|_{\frac{n}{2}} \leq 1$,*

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta_n |f|^{\frac{n}{n-2}})}{(1+|f|)^{\frac{n}{n-2}}} dx \leq C \|f\|_{\frac{n}{2}}^{\frac{n}{2}},$$

where $\Phi(t) = \exp(t) - \sum_{j=0}^{j_{\frac{n}{2}}-2} \frac{t^j}{j!}$, $j_{\frac{n}{2}} = \min\{j \in \mathbb{R} : j \geq \frac{n}{2}\} \geq n/2$ and $\beta_n = \beta(n,2) = \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 4}{\Gamma(n/2-1)}\right]^{\frac{n}{n-2}}$.

Moreover, both the power $\frac{n}{n-2}$ in the denominator and the constant β_n are optimal in the following sense:

(i) If the power $\frac{n}{n-2}$ in the denominator is replaced by any $p < \frac{n}{n-2}$, there exists a sequence of functions $\{f_k\} \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ such that $\|\Delta f_k\|_{\frac{n}{2}} \leq 1$, but

$$\frac{1}{\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\Phi(\beta_n(|f_k|)^{\frac{n}{n-2}})}{(1+|f_k|)^p} dx \to \infty.$$

(ii) If $\alpha > \beta_n$, there exists a sequence of functions $\{f_k\} \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ such that $\|\Delta f_k\|_{\frac{n}{2}} \leq 1$, but

$$\frac{1}{\|f_k\|_{\frac{n}{2}}^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\Phi(\alpha(|f_k|)^{\frac{n}{n-1}})}{(1+|f_k|)^p} dx \to \infty,$$

for any $p \geq 0$.

This extends Masmoudi and Sani's result in [22] when $n = 4$ to all dimensions $n \geq 4$.

The organization of the paper is as follows. In Sect. 2, we will establish an important lemma (Lemma 2.1) which plays a key role in the proof of our main result (Theorem 1.1). Section 3 will give the subcritical Moser–Trudinger inequality with exact growth condition on the hyperbolic space (Theorem 1.1). In Sect. 4, we will prove the sharpness of the inequality (Theorem 1.2).

2 A Crucial Lemma

In this section, we give the following lemma, which plays an important role in the proof of Theorem 1.1.

Lemma 2.1 *There exists a constant $C(n) > 0$ such that for any nonnegative decreasing function u satisfying $u(R) > K^{\frac{1}{n}}$ and $\omega_{n-1}\int_R^{+\infty}|u'|^n t^{n-1}dt \leq K$ for some $R, K > 0$, then we have*

$$\frac{\exp\left(\frac{\alpha_n}{K^{\frac{1}{n-1}}}u^{\frac{n}{n-1}}(R)\right)}{u^{\frac{n}{n-1}}(R)}R^n \leq C(n)\frac{\int_R^{+\infty}|u|^n t^{n-1}dt}{K^{\frac{n}{n-1}}}.$$

Such a lemma in dimension two ($n = 2$) was proved in [10] to establish Theorem D. In order to prove our Theorem 1.1 in the high dimensional case $n \geq 2$, we need to prove a high dimensional version.

To prove Lemma 2.1, we need the following useful lemma.

Lemma 2.2 *Given any sequence $a = \{a_k\}_{k\geq 1}$, let $\|a\|_1 = \sum_{k=0}^{+\infty}|a_k|$, $\|a\|_n = (\sum_{k=0}^{+\infty}|a_k|^n)^{1/n}$, $\|a\|_{(e)} = (\sum_{k=0}^{+\infty}|a_k|^n e^k)^{1/n}$ and $\mu(h) = \inf\{\|a\|_{(e)} : \|a\|_1 = h, \|a\|_n \leq 1\}$. Then for $h > 1$, we have*

$$\mu(h) \sim \frac{e^{\frac{h^{\frac{n}{n-1}}}{n}}}{h^{\frac{1}{n-1}}}.$$

Proof Since $\mu(h)$ is increasing in h, it suffices to show that $\mu(N^{1-\frac{1}{n}}) \sim \frac{e^{\frac{N}{n}}}{N^{1/n}}$. Choose $a_k = \{\frac{1}{N^{1/n}},\}$ when $k \leq N - 1$ and $a_k = 0$, when $k \geq N$. Obviously,

$$\|a\|_n = 1, \|a\|_1 = N^{1-\frac{1}{n}} \quad \text{and} \quad \|a\|_{(e)} \lesssim \frac{e^{\frac{N}{n}}}{N^{1/n}}.$$

So

$$\mu(N^{1-\frac{1}{n}}) \lesssim \frac{e^{\frac{N}{n}}}{N^{1/n}}$$

Now we only need to prove that $\mu(N^{1-\frac{1}{n}}) \gtrsim \frac{e^{\frac{N}{n}}}{N^{1/n}}$. By contradiction, suppose that for any $\varepsilon \ll 1$ and a sequence a, we have

$$\|a\|_n \leq 1, \ \|a\|_1 = N^{1-\frac{1}{n}}, \ \|a\|_{(e)} \leq \varepsilon\frac{e^{\frac{N}{n}}}{N^{1/n}}.$$

From the last condition, we know that when $k \geq N$,

$$|a_k| \lesssim \frac{\varepsilon}{N^{1/n}}e^{\frac{N-k}{n}}.$$

Now set $a'_k = a_k$, for $k \leq N-1$ and $a'_k = 0$ for $k \geq N$, then

$$\|a'\|_1 = \|a\|_1 - \sum_{k \geq N} |a_k| = N^{1-\frac{1}{n}} - \sum_{k \geq N} |a_k| \geq N^{1-\frac{1}{n}} - \frac{C\varepsilon}{N^{1/n}}. \tag{2.1}$$

Using the fundamental inequality: $(1-x)^p \geq 1 - px$, when $p > 1$ and $0 \leq x < 1$, we can obtain

$$\begin{aligned}
\|a'\|_1^{\frac{n}{n-1}} &\geq \left(N^{1-1/n} - \frac{C\varepsilon}{N^{1/n}}\right)^{\frac{n}{n-1}} \\
&= N\left(1 - \frac{C\varepsilon}{N}\right)^{\frac{n}{n-1}} \\
&\geq N - C\varepsilon.
\end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned}
\|a'\|_1^{\frac{n}{n-1}} &= \left(N \sum_{0 \leq j \leq N-1} |a_j|^2 - \sum_{0 \leq j,k \leq N-1} \frac{(a_j - a_k)^2}{2}\right)^{\frac{n}{2(n-1)}} \\
&= N^{\frac{n}{2(n-1)}} \left(\sum_{0 \leq j \leq N-1} |a_j|^2\right)^{\frac{n}{2(n-1)}} \left(1 - \frac{1}{N} \frac{\sum_{0 \leq j,k \leq N-1} \frac{(a_j-a_k)^2}{2}}{\sum_{0 \leq j \leq N-1} |a_j|^2}\right)^{\frac{n}{2(n-1)}} \\
&\leq N^{\frac{n}{2(n-1)}} \left(\sum_{0 \leq j \leq N-1} |a_j|^2\right)^{\frac{n}{2(n-1)}} \left(1 - \frac{n}{2(n-1)} \frac{\sum_{0 \leq j,k \leq N-1} \frac{(a_j-a_k)^2}{2}}{N\left(\sum_{0 \leq j \leq N-1} |a_j|^2\right)}\right) \\
&\leq N^{\frac{n}{2(n-1)}} \left(N^{\frac{n}{2}-1}\left(\sum_{0 \leq j \leq N-1} |a_j|^n\right)\right)^{\frac{1}{n-1}} \left(1 - \frac{n}{2(n-1)} \frac{\sum_{0 \leq j,k \leq N-1} \frac{(a_j-a_k)^2}{2}}{N(\sum_{0 \leq j \leq N-1} |a_j|^2)}\right) \\
&\leq N\left(1 - \frac{n}{2(n-1)} \frac{\sum_{0 \leq j,k \leq N-1} \frac{(a_j-a_k)^2}{2}}{N^{1+1-\frac{2}{n}}}\right) \\
&= N - \frac{n}{2(n-1)} \frac{\sum_{0 \leq j,k \leq N-1} \frac{(a_j-a_k)^2}{2}}{N^{1-\frac{2}{n}}},
\end{aligned} \tag{2.3}$$

where the first inequality uses the fundamental inequality:

$$(1-x)^q \leq 1 - qx, \quad \text{when} \quad 0 < q < 1 \quad \text{and} \quad 0 \leq x < 1,$$

and the second inequality uses the fundamental inequality:

$$\left(\sum_{j=1}^{N} |c_j|\right)^p \leq N^{p-1} \sum_{j=1}^{N} |c_j|^p \textit{ for } p \geq 1.$$

Then, by (2.2) and (2.3),

$$\sum_{0\le j,k\le N-1} \frac{(a_j-a_k)^2}{2} \le C\varepsilon N^{1-\frac{2}{n}}.$$

Now choose $m < N-1$ so that $\min_{0\le j\le N-1}|a_j| = a_m$. Then

$$\begin{aligned}\|a'\|_1 - N|a_m| &\le \|a_j - a_m\|_{l^1(j\le N-1)}\\ &\le \sqrt{N}\|a_j - a_m\|_{l^2(j\le N-1)}\\ &\le C\varepsilon N^{1-\frac{1}{n}}\end{aligned}$$

If ε is small enough, combining it with (2.1), we get

$$|a_m| \gtrsim \frac{1}{N^{1/n}}.$$

Hence

$$\|a\|_{(e)} \gtrsim \left(\sum_{j\le N-1}\frac{e^j}{N}\right)^{\frac{1}{n}} \gtrsim \frac{e^{N/n}}{N^{1/n}},$$

which yields a contradiction. So, we complete the proof of Lemma 2.2. □

Now, let us prove Lemma 2.1. By scaling, it suffices to show that for any nonnegative decreasing function u satisfying $u(1) > 1$ and $\omega_{n-1}\int_1^{+\infty}|u'(r)|^n t^{n-1}dt \le 1$,

$$\frac{\exp \alpha_n u^{\frac{n}{n-1}}(1)}{u^{\frac{n}{n-1}}(1)} \le C\int_1^{+\infty}|u|^n t^{n-1}dt.$$

Set $h_k = \alpha_n^{\frac{n-1}{n}} u(e^{k/n})$, $a_k = h_k - h_{k+1}$ and $a = \{a_k\}$. Then $a_k \ge 0$ and

$$\sum_{k\ge 0}|a_k| = h_0 = \alpha_n^{\frac{n-1}{n}} u(1).$$

Since

$$\begin{aligned}h_k - h_{k+1} &= \alpha_n^{\frac{n-1}{n}}\left(u\left(e^{\frac{k}{n}}\right) - u\left(e^{\frac{k+1}{n}}\right)\right)\\ &= \alpha_n^{\frac{n-1}{n}}\int_{e^{\frac{k+1}{n}}}^{e^{\frac{k}{n}}} u'(t)dt\end{aligned}$$

$$\leq \alpha_n^{\frac{n-1}{n}} \left(\int_{e^{\frac{k}{n}}}^{e^{\frac{k+1}{n}}} |u'|^n t^{n-1} dt \right)^{1/n} \left(\int_{e^{\frac{k}{n}}}^{e^{\frac{k+1}{n}}} \frac{1}{t} dt \right)^{\frac{n-1}{n}}$$

$$= \left(\omega_{n-1} \int_{e^{\frac{k}{n}}}^{e^{\frac{k+1}{n}}} |u'|^n t^{n-1} dt \right)^{\frac{1}{n}},$$

then

$$\|a\|_n = \left(\sum_{j \geq 0} |a_k|^n \right)^{1/n} = \left(\sum_{j \geq 0} |h_k - h_{k+1}|^n \right)^{1/n} \leq 1.$$

At the same time,

$$\begin{aligned}
\int_1^\infty |u|^n t^{n-1} dt &= \sum_{k \geq 0} \int_{e^{\frac{k}{n}}}^{e^{\frac{k+1}{n}}} |u|^n t^{n-1} dt \\
&\geq \sum_{k \geq 0} \left(u\left(e^{\frac{k+1}{n}} \right) \right)^n \int_{e^{\frac{k}{n}}}^{e^{\frac{k+1}{n}}} t^{n-1} dt \\
&\gtrsim \sum_{k \geq 0} \left(u\left(e^{\frac{k+1}{n}} \right) \right)^n e^{k+1} \gtrsim \sum_{k \geq 0} (h_{k+1})^n e^{k+1} \\
&= \sum_{k \geq 1} (h_k)^n e^k \geq \sum_{k \geq 1} (a_k)^n e^k.
\end{aligned}$$

Therefore

$$\|a\|_{(e)}^n = a_0^n + \sum_{k \geq 1} (a_k)^n e^k \leq h_0^n + \sum_{k \geq 1} (a_k)^n e^k \leq h_0^n + \int_1^{+\infty} u^n t^{n-1} dt \quad (2.4)$$

Next, let us estimate h_0. Set $1 < r < e^{1/4n}$, since

$$\begin{aligned}
h_0 - \alpha_n^{\frac{n-1}{n}} u(r) &= \alpha_n^{\frac{n-1}{n}} \int_1^r |u'(t)| dt \\
&\leq \alpha_n^{\frac{n-1}{n}} \left(\int_1^r |u'(t)|^n t^{n-1} dt \right)^{1/n} (\ln r)^{\frac{n-1}{n}} \\
&\leq n^{\frac{n-1}{n}} \left(\omega_{n-1} \int_1^r |u'(t)|^n t^{n-1} dt \right)^{1/n} (\ln r)^{\frac{n-1}{n}} \\
&< 1/2 \leq h_0/2,
\end{aligned}$$

then we have

$$\int_1^{+\infty} u^n t^{n-1} dt \geq \int_1^{e^{1/4n}} u^n(t) t^{n-1} dt \gtrsim h_0^n. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\|a\|_{(e)}^n \lesssim \int_1^{+\infty} u^n t^{n-1} dt.$$

By Lemma 2.2, thus

$$\int_1^{+\infty} u^n t^{n-1} dt \gtrsim \left(\frac{e^{\frac{h_0^{\frac{n}{n-1}}}{n}}}{h_0^{\frac{1}{n-1}}} \right)^n = \frac{\exp(\alpha_n u^{\frac{n}{n-1}}(1))}{\alpha_n u^{\frac{n}{n-1}}(1)}.$$

This completes the proof of Lemma 2.1.

3 Sharp Moser–Trudinger Inequalities with Exact Growth: Theorem 1.1

In this section, we will prove Theorem 1.1. To do so, we will apply Lemma 2.1 together with some ideas used in [10] in the Euclidean case. Nevertheless, in our hyperbolic spaces, it is considerably more difficult to carry out the argument.

To prove Theorem 1.1, we will use the arrangement argument. By means of symmetrization, it suffices to show the desired inequality for functions $u(x) = u_0(d(0, x))$, which are radially symmetric, nonnegative, smooth, compactly supported and $u_0(t) : [0, +\infty) \to \mathbb{R}$ is decreasing.

Following Moser's argument of the classical inequality [23], we set $w(t) = u_0(t)$, $|x| = \tanh t/2$, then $w(t) \geq 0$, $w' \leq 0$ and $w(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Then, we have

$$\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n |u|^{\frac{n}{n-1}})}{(1+|u|)^{\frac{n}{n-1}}} dV = \omega_{n-1} \int_0^\infty \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(1+w(t))^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt,$$
$$\|\nabla_g u\|_n^n = \omega_{n-1} \int_0^\infty |w'|^n (\sinh t)^{n-1} dt,$$
$$\int_{\mathbb{H}^n} |u|^n dV = \omega_{n-1} \int_0^\infty |w|^n (\sinh t)^{n-1} dt.$$

Thus, to prove the theorem, it suffices to show that there exists a constant C such that

$$\int_0^\infty \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(1+w(t))^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \leq C \int_0^\infty |w|^n (\sinh t)^{n-1} dt$$

for any w satisfying $w(t) \geq 0$, $w' \leq 0$, $w(t_0) = 0$ for some $t_0 \in \mathbb{R}$ and

$$\omega_{n-1} \int_0^{\infty} |w'|^n (\sinh t)^{n-1} dt \leq 1.$$

Set $R_0 = \inf\{t \in R : w(t) < 1\}$, and we know that for $t > R_0$, $0 \leq w(t) < 1$ and $w(R_0) = 1$.

For $t \in (R_0, \infty)$, we have $w(t) \in [0, 1)$. Since for $x \in [0, n)$ we can find a constant C_n such that $\Phi_n(x) \leq C_n x^{n-1}$, thus we have

$$\begin{aligned} &\int_{R_0}^{+\infty} \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \\ &\quad \leq \int_{R_0}^{+\infty} \Phi_n(\alpha_n |w|^{\frac{n}{n-1}}) (\sinh t)^{n-1} dt \\ &\quad \leq C_n \int_{R_0}^{+\infty} |w|^n (\sinh t)^{n-1} dt. \end{aligned} \tag{3.1}$$

Next, we consider the integral over $(0, R_0]$. Fix $0 < \varepsilon_0 < 1$. And let $R_1(u) > 0$ such that

$$\omega_{n-1} \int_0^{R_1} |w'|^n (\sinh t)^{n-1} dt \leq \beta(1-\varepsilon_0) \text{ and } \omega_{n-1} \int_{R_1}^{\infty} |w'|^n (\sinh t)^{n-1} dt \leq \beta\varepsilon_0,$$

where $0 < \beta \leq 1$.

In order to estimate the integral over $(0, R_0]$, we need to consider two cases: $R_1 \geq R_0$ and $R_1 < R_0$.

First we consider the case $R_1 \geq R_0$. When $0 < t \leq R_0$, we have

$$\begin{aligned} w(t) &= w(R_0) + \int_{R_0}^{t} w'(s) ds \\ &\leq w(R_0) + \left(\int_t^{R_1} |w'(s)|^n (\sinh s)^{n-1} ds \right)^{1/n} \left(\int_t^{R_0} \frac{1}{\sinh s} ds \right)^{\frac{n-1}{n}} \\ &\leq 1 + \left(\frac{\beta(1-\varepsilon_0)}{\omega_{n-1}} \right)^{1/n} \left(\ln \left(\frac{e^{R_0}-1}{e^{R_0}+1} \frac{e^t+1}{e^t-1} \right) \right)^{\frac{n-1}{n}}. \end{aligned}$$

It is well known that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon = (1 - \frac{1}{(1+\varepsilon)^{n-1}})^{\frac{1}{1-n}} > 0$ s.t.

$$1 + s^{\frac{n-1}{n}} \leq ((1+\varepsilon)s + C_\varepsilon)^{\frac{n-1}{n}}.$$

Then

$$|w(t)|^{\frac{n}{n-1}} \leq (1+\varepsilon)\left(\frac{\beta(1-\varepsilon_0)}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \ln\left(\frac{e^{R_0}-1}{e^{R_0}+1}\frac{e^t+1}{e^t-1}\right) + C_\varepsilon.$$

Set $\varepsilon = (1+\varepsilon_0)^{\frac{1}{n-1}} - 1$, so

$$|w(t)|^{\frac{n}{n-1}} \leq \left(\frac{\beta(1-\varepsilon_0^2)}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \ln\left(\frac{e^{R_0}-1}{e^{R_0}+1}\frac{e^t+1}{e^t-1}\right) + C_{\varepsilon_0}.$$

Denote $c_0 = n((1-\varepsilon_0^2))^{\frac{1}{n-1}}$, then $0 < c_0 < n$ and

$$\begin{aligned}
&\int_0^{R_0} \frac{\Phi_n(\alpha_n|w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}}(\sinh t)^{n-1}dt \\
&\leq \int_0^{R_0} \Phi_n(\alpha_n|w|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \leq \int_0^{R_0} e^{\alpha_n|w|^{\frac{n}{n-1}}}(\sinh t)^{n-1}dt \\
&\leq \int_0^{R_0} e^{\alpha_n C_{\varepsilon_0}}\left[\exp\left(\ln\left(\frac{e^{R_0}-1}{e^{R_0}+1}\frac{e^t+1}{e^t-1}\right)\right)\right]^{n(\beta(1-\varepsilon_0^2))^{\frac{1}{n-1}}} (\sinh t)^{n-1}dt \\
&\leq e^{\alpha_n C_{\varepsilon_0}}\left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{c_0}\int_0^{R_0}\frac{(e^t+1)^{c_0+n-1}}{(e^t-1)^{c_0-n+1}}\frac{dt}{(2e^t)^{n-1}}.
\end{aligned}$$

When $n > c_0 > n-1$,

$$\begin{aligned}
&\left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{c_0}\int_0^{R_0}\frac{(e^t+1)^{c_0+n-1}}{(e^t-1)^{c_0-n+1}}\frac{dt}{(2e^t)^{n-1}} \\
&\leq 2\left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{c_0}\int_0^{R_0}\frac{(2e^t)^{c_0-1}}{(e^t-1)^{c_0-n+1}}de^t \\
&\leq 2\left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{c_0}(2e^{R_0})^{c_0-1}\int_0^{R_0}\frac{1}{(e^t-1)^{c_0-n+1}}de^t \\
&\leq \frac{2^{c_0}}{n-c_0}\frac{(e^{R_0}-1)^n}{e^{R_0}}.
\end{aligned}$$

When $c_0 \leq n-1$,

$$\begin{aligned}
&\int_0^{R_0} \frac{\Phi_n(\alpha_n|w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}}(\sinh t)^{n-1}dt \\
&\leq 2\left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{n-1}\int_0^{R_0}\frac{(2e^t)^{n-1-1}}{(e^t-1)^{n-1-n+1}}de^t
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{n-1}\left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{n-1}\int_0^{R_0}(e^t)^{n-2}de^t\\
&\leq C\frac{(e^{R_0}-1)^n}{e^{R_0}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^{R_0}|w(t)|^n(\sinh t)^{n-1}dt\geq\int_0^{R_0}(\sinh t)^{n-1}dt\\
&=\int_0^{R_0}\frac{(e^t-1)^{n-1}(e^t+1)^{n-1}}{2^{n-1}(e^t)^{n-1}}dt\\
&\geq\frac{1}{2^{n-1}}\int_0^{R_0}\frac{(e^t-1)^{n-1}}{e^t}de^t\\
&\geq\frac{1}{2^{n-1}}\frac{1}{e^{R_0}}\frac{(e^{R_0}-1)^n}{n}.
\end{aligned}
$$

Then

$$
\int_0^{T_0}\Phi_n(\alpha_n|w|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt\leq C_n\int_0^{T_0}|w|^n(\sinh t)^{n-1}dt. \tag{3.2}
$$

Therefore, by (3.1) and (3.2), we get the desired inequality of Theorem 1.1 when $R_1\geq R_0$.

Now, let us consider the case $R_1<R_0$. First, we consider the integral over (R_1,R_0). Since $\omega_{n-1}\int_{R_1}^{\infty}|w'|^n(\sinh t)^{n-1}dt\leq\beta\varepsilon_0$, then when $R_1<t<R_0$,

$$
\begin{aligned}
w(t)&=w(R_0)+\int_{R_0}^{t}w'(s)ds\\
&\leq w(R_0)+\left(\int_{R_1}^{+\infty}|w'(s)|^n(\sinh s)^{n-1}ds\right)^{1/n}\left(\int_t^{R_0}\frac{1}{\sinh s}ds\right)^{\frac{n-1}{n}}\\
&\leq 1+\left(\frac{\beta\varepsilon_0}{\omega_{n-1}}\right)^{1/n}\left(\ln\left(\frac{e^{R_0}-1}{e^{R_0}+1}\frac{e^t+1}{e^t-1}\right)\right)^{\frac{n-1}{n}}.
\end{aligned}
$$

Setting $\varepsilon=1-(\varepsilon_0)^{\frac{1}{n-1}}$ and using the inequality

$$
1+s^{\frac{n-1}{n}}\leq((1+\varepsilon)s+C_\varepsilon)^{\frac{n-1}{n}},
$$

we have

$$
|w(t)|^{\frac{n}{n-1}}\leq\left(1-(1-(\varepsilon_0)^{\frac{1}{n-1}})^2\right)\left(\frac{1}{\omega_{n-1}}\right)^{\frac{1}{n-1}}\ln\left(\frac{e^{R_0}-1}{e^{R_0}+1}\frac{e^t+1}{e^t-1}\right)+C_{\varepsilon_0}.
$$

Denote $c_1 = n(1-(1-(\varepsilon_0)^{\frac{1}{n-1}})^2)$, then $0 < c_1 < n$ and

$$\begin{aligned}
&\int_{R_1}^{R_0} \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \\
&\le \int_{R_1}^{R_0} \Phi_n(\alpha_n |w|^{\frac{n}{n-1}})(\sinh t)^{n-1} dt \le \int_{R_1}^{R_0} e^{\alpha_n |w|^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \\
&\le \int_0^{R_0} e^{\alpha_n C_{\varepsilon_0}} \left[\exp\left(\ln\left(\frac{e^{R_0}-1}{e^{R_0}+1}\frac{e^t+1}{e^t-1}\right)\right)\right]^{c_1} (\sinh t)^{n-1} dt \\
&\le e^{\alpha_n C_{\varepsilon_0}} \left(\frac{e^{R_0}-1}{e^{R_0}+1}\right)^{c_1} \int_0^{R_0} \frac{(e^t+1)^{c_1+n-1}}{(e^t-1)^{c_1-n+1}} \frac{dt}{(2e^t)^{n-1}}.
\end{aligned}$$

Using the same calculation as we did in the case $R_1 \ge R_0$, we can obtain

$$\int_{R_1}^{R_0} \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \le C \int_0^{R_0} |w(t)|^n (\sinh t)^{n-1} dt.$$

Now we only need to consider the integral on $[0, R_1)$. Set $v(t) = w(t) - w(R_1)$, since

$$\omega_{n-1} \int_0^{R_1} |w'|^n (\sinh t)^{n-1} dt \le \beta(1-\varepsilon_0),$$

then

$$\omega_{n-1} \int_0^{R_1} |v'|^n (\sinh t)^{n-1} dt \le \beta(1-\varepsilon_0).$$

And

$$|w(t)|^{\frac{n}{n-1}} = (v(t) + w(R_1))^{\frac{n}{n-1}} \le (1+\varepsilon)|v|^{\frac{n}{n-1}} + C_\varepsilon w(R_1)^{\frac{n}{n-1}}.$$

Thus

$$\begin{aligned}
&\int_0^{R_1} \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \\
&\le \int_0^{R_1} \frac{\Phi_n(\alpha_n |w|^{\frac{n}{n-1}})}{(w(R_1))^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \le \int_0^{R_1} \frac{e^{\alpha_n |w|^{\frac{n}{n-1}}} (\sinh t)^{n-1}}{(w(R_1))^{\frac{n}{n-1}}} dt \\
&\le \frac{\exp(\alpha_n C_\varepsilon w(R_1)^{\frac{n}{n-1}})}{(w(R_1))^{\frac{n}{n-1}}} \int_0^{R_1} \exp(\alpha_n (1+\varepsilon)|v|^{\frac{n}{n-1}})(\sinh t)^{n-1} dt. \qquad (3.3)
\end{aligned}$$

Set $\varepsilon = (1-\varepsilon_0)^{\frac{1}{1-n}} - 1$, then

$$C_\varepsilon = \left(1 - \frac{1}{(1+\varepsilon)^{n-1}}\right)^{\frac{1}{1-n}} = \frac{1}{\varepsilon_0^{\frac{1}{n-1}}}.$$

Since $\omega_{n-1}\int_{R_1}^{+\infty}|w'|^n(\sinh t)^{n-1}dt \le \beta\varepsilon_0 \le \varepsilon_0$, using Lemma 2.1 we have

$$\frac{\exp\left(\alpha_n C_\varepsilon w(R_1)^{\frac{n}{n-1}}\right)}{(w(R_1))^{\frac{n}{n-1}}} = \frac{\exp\left(\frac{\alpha_n}{\varepsilon_0^{\frac{1}{n-1}}} w(R_1)^{\frac{n}{n-1}}\right)}{(w(R_1))^{\frac{n}{n-1}}} \le C\frac{\int_{R_1}^{+\infty}|w|^n t^{n-1}dt}{R_1^n(\varepsilon_0)^{\frac{n}{n-1}}}. \tag{3.4}$$

Let $\Omega = \{x : d(0,x) < R_1\}$ and $w_1(x) = (1+\varepsilon)^{\frac{n-1}{n}} v(d(0,x))$ in Ω , then

$$\|\nabla_g w_1\|_{n,\Omega} = \omega_{n-1}\int_0^{R_1}(1+\varepsilon)^{n-1}|v'|^n(\sinh t)^{n-1}dt \le \beta \le 1.$$

By **Theorem E** (the Moser–Trudinger inequality on bounded domain in hyperbolic space), we have

$$\int_\Omega \exp(\alpha_n|w_1|^{\frac{n}{n-1}})dV \le C|\Omega|.$$

That is

$$\int_0^{R_1}\exp\left(\alpha_n(1+\varepsilon)|v|^{\frac{n}{n-1}}\right)(\sinh t)^{n-1}dt \le C\int_0^{R_1}(\sinh t)^{n-1}dt. \tag{3.5}$$

Since $\frac{\sinh t}{t}$ is monotone increasing on $(0,+\infty)$, then by inequalities (3.3), (3.4) and (3.5) we have

$$\begin{aligned}
&\int_0^{R_1}\frac{\Phi_n(\alpha_n|w|^{\frac{n}{n-1}})}{(1+w)^{\frac{n}{n-1}}}(\sinh t)^{n-1}dt \\
&\le C\frac{\int_{R_1}^{+\infty}|w|^n t^{n-1}dt}{R_1^n(\varepsilon_0)^{\frac{n}{n-1}}}\int_0^{R_1}(\sinh t)^{n-1}dt \\
&= C\frac{\int_{R_1}^{+\infty}|w|^n \sinh t^{n-1}\left(\frac{t}{\sinh t}\right)^{n-1}dt}{R_1^n(\varepsilon_0)^{\frac{n}{n-1}}}\int_0^{R_1} t^{n-1}\left(\frac{\sinh t}{t}\right)^{n-1}dt \\
&\le C\frac{\int_{R_1}^{+\infty}|w|^n \sinh t^{n-1}dt}{(\varepsilon_0)^{\frac{n}{n-1}}}.
\end{aligned}$$

Thus we have completed the proof of Theorem 1.1.

4 Sharpness of Theorem 1.1: Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2, namely the sharpness of Theorem 1.1. We will show that the inequality in Theorem 1.1 is sharp in both senses as described in Theorem 1.2. First, we will show that the inequality in Theorem 1.1 does not hold if the power $\frac{n}{n-1}$ in the denominator is replaced by any $p < \frac{n}{n-1}$.

We choose $\{u_k\}_{k=1}^{\infty}$ as follows:

$$u_k(x) = \omega_{n-1}^{-\frac{1}{n}} C_k \begin{cases} k^{\frac{n-1}{n}}, & \text{if } 0 \le d(0,x) \le e^{-k}, \\ k^{\frac{n-1}{n}} \frac{-\ln[d(0,x)]}{k}, & \text{if } e^{-k} \le d(0,x) \le 1, \\ 0, & \text{if } 1 < d(0,x), \end{cases}$$

where $C_k = (k^{-1}\int_{e^{-k}}^{1} t^{-n}(\sinh t)^{n-1}dt)^{-\frac{1}{n}}$. Since $C_k \sim (\frac{(\sinh e^{-k})^{n-1}}{e^{-(n-1)k}})^{-\frac{1}{n}}$, as $k \to \infty$, then $C_k \to 1$ and $C_k^{\frac{n}{n-k}} k - k \to 0$, as $k \to \infty$. Then, by calculation

$$\|\nabla_g u_k\|_n^n = 1,$$

and

$$\int_{\mathbb{H}^n} |u_k|^n dV = O\left(\frac{1}{k}\right).$$

It follows that, as $k \to \infty$

$$\begin{aligned}
&\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n |u_k|^{\frac{n}{n-1}})}{(1+|u_k|)^p} dV \\
&\ge \int_{d(0,x)\le e^k} \frac{\Phi_n(\alpha_n |u_k|^{\frac{n}{n-1}})}{(1+|u_k|)^p} dV \\
&= \frac{\Phi_n\left(\alpha_n |\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-1}{n}}|^{\frac{n}{n-1}}\right)}{\left(1+|\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-1}{n}}|\right)^p} \int_0^{e^{-k}} (\sinh t)^{n-1} dt \\
&\sim \frac{\Phi_n\left(nkC_k^{\frac{n}{n-1}}\right)}{\left(1+|\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-1}{n}}|\right)^p} e^{-nk} \\
&\sim \frac{\exp\left(nkC_k^{\frac{n}{n-1}}\right)}{\left(1+|\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-1}{n}}|\right)^p} e^{-nk} \\
&\sim k^{-p\frac{n-1}{n}}.
\end{aligned}$$

Now, since $p < \frac{n}{n-1}$, we have

$$\begin{aligned}&\frac{1}{\|u_k\|_n^n}\int_{\mathbb{H}^n}\frac{[\Phi_n(\alpha_n(|u_k|)^{\frac{n}{n-1}})]}{(1+|u_k|)^p}dV\\&\geq \frac{k^{-p\frac{n-1}{n}}}{O(\frac{1}{k})}\sim k^{1-p\frac{n-1}{n}}\\&\to\infty.\end{aligned}$$

So we have proved that the power $\frac{n}{n-1}$ is optimal.

Next, let us show that the constant α_n is optimal. We only need to find a sequence of functions $\{u_k\}$ such that $\|\nabla_g u_k\|_n \leq 1$, but for any $p \geq 0$ and $\alpha > \alpha_n$,

$$\frac{1}{\|u_k\|_n^n}\int_{\mathbb{H}^n}\frac{\left[\Phi_n(\alpha(|u_k|)^{\frac{n}{n-1}})\right]}{(1+|u_k|)^p}dV\to\infty.$$

We still choose $\{u_k\}_{k=1}^{\infty}$ as follows:

$$u_k(x)=\omega_{n-1}^{-\frac{1}{n}}C_k\begin{cases}k^{\frac{n-1}{n}}, & \text{if } 0\leq d(0,x)\leq e^{-k},\\ k^{\frac{n-1}{n}}\frac{-\ln[d(0,x)]}{k}, & \text{if } e^{-k}\leq d(0,x)\leq 1,\\ 0, & \text{if } 1<d(0,x),\end{cases}$$

where $C_k = (k^{-1}\int_{e^{-k}}^{1} t^{-n}(\sinh t)^{n-1}dt)^{-\frac{1}{n}}$.

We already know that $C_k^{\frac{n}{n-k}}k - k \to 0$, as $k \to \infty$,

$$\|\nabla_g u_k\|_n^n = 1,$$

and

$$\int_{\mathbb{H}^n}|u_k|^n dV = O(\frac{1}{k}).$$

Since $\alpha > \alpha_n$, it follows that

$$\begin{aligned}&\int_{\mathbb{H}^n}\frac{\Phi_n(\alpha|u_k|^{\frac{n}{n-1}})}{(1+|u_k|)^p}dV\\&\geq\int_{d(0,x)\leq e^k}\frac{\Phi_n(\alpha|u_k|^{\frac{n}{n-1}})}{(1+|u_k|)^p}dV\\&=\frac{\Phi_n\left(\frac{\alpha}{\alpha_n}\alpha_n|\omega_{n-1}^{-\frac{1}{n}}C_k k^{\frac{n-1}{n}}|^{\frac{n}{n-1}}\right)}{\left(1+|\omega_{n-1}^{-\frac{1}{n}}C_k k^{\frac{n-1}{n}}|\right)^p}\int_0^{e^{-k}}(\sinh t)^{n-1}dt\end{aligned}$$

$$\sim \frac{\Phi_n\left(\frac{\alpha}{\alpha_n} nk C_k^{\frac{n}{n-1}}\right)}{\left(1+|\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-1}{n}}|\right)^p} e^{-nk}$$
$$\sim \frac{\exp\left(\frac{\alpha}{\alpha_n} nk C_k^{\frac{n}{n-1}}\right)}{\left(1+|\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-1}{n}}|\right)^p} e^{-nk}$$
$$\sim e^{(\frac{\alpha}{\alpha_n}-1)nk} k^{-p\frac{n-1}{n}}$$
$$\to +\infty.$$

Therefore

$$\frac{1}{\|u_k\|_n^n} \int_{\mathbb{H}^n} \frac{[\Phi_n(\alpha(|u_k|)^{\frac{n}{n-1}})]}{(1+|u_k|)^p} dV \to \infty.$$

Thus, we have completed the proof of Theorem 1.2.

Acknowledgments The first author's research is supported partly by a US NSF Grant DMS #1301595 and the second author's research is partly supported by a NNSF Grant of China (No. 11371056).

References

1. Adachi, S., Tanaka, K.: Trudinger type inequalities in R^N and their best exponents. Proc. Am. Math. Soc. **128**, 2051–2057 (1999)
2. Adams, D.R.: A sharp inequality of J. Moser for higher order derivatives. Ann. of Math. (2) **128**(2), 385–398 (1988)
3. Adimurthi, Sandeep, K.: A singular Moser–Trudinger embedding and its applications. NoDEA Nonlinear Differ. Equ. Appl. **13**(5–6), 585–603 (2007)
4. Baernstein, A.: II.A unified approach to symmetrization. In: Partial Differential Equations of Elliptic Type (Cortona, 1992), Sympos. Math., XXXV, pp. 47–91. Cambridge Univ. Press, Cambridge (1994)
5. Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality. Ann. Math. (2) **138**(1), 213–242 (1993)
6. Chang, S.Y.A., Yang, P.: The inequality of Moser and Trudinger and applications to conformal geometry. Dedicated to the memory of Jürgen K. Moser. Comm. Pure Appl. Math. **56**(8), 1135–1150 (2003)
7. Cohn, W.S., Lu, G.: Best constants for Moser–Trudinger inequalities on the Heisenberg group. Indiana Univ. Math. J. **50**(4), 1567–1591 (2001)
8. Cohn, W.S., Lu, G.: Sharp constants for Moser–Trudinger inequalities on spheres in complex space C^n. Comm. Pure Appl. Math. **57**(11), 1458–1493 (2004)
9. Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. Comment. Math. Helv. **68**, 415–454 (1993)
10. Ibrahim, S., Masmoudi, N., Nakanishi, K.: Trudinger–Moser inequality on the whole plane with with the exact growth condition. Nonlinearity **25**(6), 1843–1849 (2012)
11. Lam, N., Lu, G.: Sharp Moser–Trudinger inequality on the Heisenberg group at the critical case and applications. Adv. Math. **231**(6), 3259–3287 (2012)
12. Lam, N., Lu, G.: Sharp Adams type inequalities in Sobolev spaces $W^{m,\frac{m}{n}}(\mathbb{R}^n)$for arbitrary integer m. J. Differ. Equ. **253**, 1143–1171 (2012)

13. Lam, N., Lu, G.: The Moser–Trudinger and Adams inequalities and elliptic and subelliptic equations with nonlinearity of exponential growth. Recent developments in geometry and analysis, pp. 179–251. Adv. Lect. Math. (ALM), 23, Int. Press, Somerville, MA (2012)
14. Lam, N., Lu, G.: A new approach to sharp Moser–Trudinger and Adams type inequalities: a rearrangement-free argument. J. Differ. Equ. **255**(3), 298–325 (2013)
15. Lam, N., Lu, G., Tang, H.: On nonuniformly subelliptic equations of Q-sub-Laplacian type with critical growth in H^n. Adv. Nonlinear Stud. **12**, 659–681 (2012)
16. Lam, N., Lu, G., Tang, H.: Sharp subcritical Moser–Trudinger inequalities on Heisenberg groups and subelliptic PDEs. Nonlinear Anal. **95**, 77–92 (2014)
17. Li, Y.X.: Extremal functions for the Moser–Trudinger inequalities on compact Riemannian manifolds. Sci. China Ser. A **48**(5), 618–648 (2005)
18. Li, Y.X., Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^n$. Indiana Univ. Math. J. **57**(1), 451–480 (2008)
19. Lu, G., Tang, H.: Best constants for Moser–Trudinger inequalities on high dimensional hyperbolic spaces. Adv. Nonlinear Stud. **13**(4), 1035–1052 (2013)
20. Lu, G., Tang, H., Zhu, M.: Best constants for Adams' inequalities with the exact growth condition in $\mathbb{R}^n$ (to appear)
21. Mancini, G., Sandeep, K.: Moser–Trudinger inequalities on conformal discs. Commun. Contemp. Math. **12**(6), 1055–1068 (2010)
22. Masmoudi, N., Sani, F.: Trudinger–Moser inequalities with the exact growth condition in $\mathbb{R}^n$. Comm. Pure Appl. Math. **67**(8), 1307–1335 (2014)
23. Moser, J.: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. **20**(1970/71), 1077–1092
24. Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$. J. Funct. Anal **219**(2), 340–367 (2005)
25. Pohožaev, S.I.: On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, (Russian) Dokl. Akad. Nauk SSSR **165**, 36–39 (1965)
26. Ruf, B., Sani, F.: Sharp Adams-type inequalities in R^n. Trans. Am. Math. Soc. **365**(2), 645–670 (2013)
27. Shaw, M.C.: Eigenfunctions of the nonlinear equation $\triangle u + vf(x,u) = 0$ in R^2. Pacific J. Math. **129**(2), 349–356 (1987)
28. Trudinger, N.S.: On imbeddings into Orlicz spaces and some applications. J. Math. Mech. **17**, 473–483 (1967)
29. Wolf, J.: Spaces of Constant Curvature. McGraw-Hill, New York (1967)
30. Yudovič, V.I.: Some estimates connected with integral operators and with solutions of elliptic equations, (Russian) Dokl. Akad. Nauk SSSR **138**, 805–808 (1961)