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Duality theory of weighted Hardy spaces with arbitrary number of parameters

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Abstract. In this paper, we use the discrete Littlewood–Paley–Stein analysis to get the duality result of the weighted product Hardy space for arbitrary number of parameters under a rather weak condition on the product weight $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$. We will show that for any $k \geq 2$, $(H_w^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}))^* = \mathrm{CMO}_w^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ (a generalized Carleson measure), and obtain the boundedness of singular integral operators on BMO_w. Our theorems even when the weight function $w = 1$ extend the H^1-BMO duality of Chang–R. Fefferman for the non-weighted two-parameter Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ to $H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ for all $0 < p \leq 1$ and our weighted theory extends the duality result of Krug–Torchinsky on weighted Hardy spaces $H_w^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $w \in A_r(\mathbb{R}^n \times \mathbb{R}^m)$ with $1 \leq r \leq 2$ and $r/2 < p \leq 1$ to $H_w^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ with $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ for all $0 < p \leq 1$.

Keywords. Duality, weighted Hardy space, discrete Littlewood-Paley-Stein analysis, Caderón's identity, Max–Min type inequality, multiparameter A_∞ weight, multiparameter singular integrals.

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1 Introduction and statement of results

In the classical one-parameter setting, a celebrated theorem of C. Fefferman and Stein ([5, 6]) says that the space of functions of bounded mean oscillation on $\mathbb{R}^n$, $\mathrm{BMO}(\mathbb{R}^n)$, is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. In the multiparameter setting, the Hardy space was introduced by Gundy–Stein in the 1970's in [14] and was satisfactorily developed by S.-Y. Chang and R. Fefferman in [2, 3]. Chang and R. Fefferman proved that the dual space of the product $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the product $\mathrm{BMO}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ by using the bi-Hilbert transform.

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Using a new version of Journé's covering lemma ([18, 19, 27]), Ferguson and Lacey in [11] gave a new characterization of the product BMO($\mathbb{R} \times \mathbb{R}$) by bicommutator of Hilbert transforms (see also Lacey and Terwilleger [24]). Furthermore, Lacey, Petermichl, Pipher and Wick established in [23] such a characterization of product BMO($\mathbb{R}^n \times \mathbb{R}^m$) using multiparameter commutators of Riesz transforms.

Han and the first author have established in [15] the theory of boundedness of singular integrals and its duality of the multiparameter Hardy space $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ associated with the flag singular integrals, where the L^p theory has been developed by Müller–Ricci–Stein [25] and Nagel–Ricci–Stein [26]. The Carleson measure space $\mathrm{CMO}^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ associated with the multiparameter flag structures was introduced for all $0 < p \leq 1$, and the duality between $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ with $\mathrm{CMO}^p_F(\mathbb{R}^n \times \mathbb{R}^m)$. Such CMO^p_F spaces when $p = 1$ play the same role as BMO space. Using the method of discrete Littlewood–Paley–Stein analysis initially developed in [15], the theory of multiparameter Hardy spaces in several different settings have been established. We refer the reader to the recent expository article [16].

Motivated by the work [15], the characterization of the dual space of the multiparameter Hardy space $H^p(X \times Y)$ has been recently established by the first author with Han and Li in [17] for the product of two homogeneous spaces X and Y without weight in the sense of Coifman–Weiss when p, $0 < p \leq 1$, is close to 1.

The main purpose of this paper is to characterize the dual spaces of the multiparameter weighted Hardy spaces

$$(H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}))^* = \mathrm{CMO}^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$$

for any arbitrary number of parameters $k \geq 2$ and obtain the boundedness of the singular integral operator on BMO_w for all $0 < p \leq 1$ and $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$, the A_∞ product weights. This requirement on the weight A_∞ is considerably weaker than the commonly used condition $w \in A_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ in dealing with weighted L^p boundedness of singular integrals for $p > 1$ considered by R. Fefferman and Stein [10] (see also R. Fefferman [8], R. Fefferman–J. Pipher [9]). One should point out here that in general weighted H^p_w and L^p_w spaces ($p > 1$) are different spaces as demonstrated by J. Stromberg and R. Wheeden in [31] and [32]. Our theorems when the weight function $w = 1$ extend the H^1-BMO duality of Chang–R. Fefferman for the non-weighted two-parameter Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ to $H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ for all $0 < p \leq 1$.

R. Fefferman in [7] established the criterion of the H^p-L^p boundedness of singular integral operators in Journé's class by considering its action only on rectangle atoms. However, Journé in [19] provided a counter-example in the three-parameter setting of singular integral operators such that Fefferman's criterion does not hold. Journé's works show the sharp difference between the two and three parameters.

It thus motivates us to consider the characterization of the dual spaces of the multiparameter weighted Hardy spaces

$$(H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}))^* = \mathrm{CMO}^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$$

for any arbitrary number of parameters $k \geq 2$ for all $0 < p \leq 1$ and all weights $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$. As it turns out, our work involves more complicated geometric considerations than those in two parameters when characterizing the dual spaces. Theorem 1.2 demonstrates such a complicated nature when dealing with three parameters or more.

Weighted Hardy spaces have been studied extensively in the one-parameter setting (see for example [13,22,29–32] and many other references therein), where the weighted Hardy space was characterized by the non-tangential maximal functions and atomic decompositions. The relationship between L^p_w and H^p_w for $p > 1$ were considered in both one- and multiparameter cases (e.g., Strömberg and R. Wheeden in [31, 32]). The atomic decomposition of weighted multiparameter Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ was carried out by Krug [20]. The dual spaces for weighted Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ (defined by the maximal function) were characterized by Krug and Torchinsky [21] when the weights w are in some A_r $(1 \leq r \leq 2)$ and $r/2 < p < 1$. The method employed in [21] applies the atomic decomposition of the weighted Hardy spaces.

The weighted Hardy space estimates for singular integrals in both one-parameter and two-parameter cases using discrete Littlewood–Paley–Stein theory were recently established in [4] under the hypothesis on the weight $w \in A_\infty$. For the case of arbitrary number of parameters, this has been done in [28].

In this paper, we apply the discrete multiparameter Littlewood–Paley–Stein analysis to derive the duality results of the weighted Hardy spaces for arbitrary number of k parameters, $k \geq 2$, that is,

$$(H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}))^* = \mathrm{CMO}^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}),$$

where $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$. When $w = 1$, this extends the Chang–Fefferman [2] duality result in the two-parameter case for $p = 1$ to $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times R^{n_k})$ for all $0 < p \leq 1$ and with arbitrary number of k parameters. This also extends the duality result of Krug and Torchinsky [21] to $H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ for all $0 < p \leq 1$ and $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$. We will also get the boundedness of singular integral operators on $\mathrm{BMO}_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$.

We remark that the space CMO^p_w introduced in this paper is a generalized weighted Carleson measure space. The steps of the proofs of our main results are as follows. We first show the functions space CMO^p_w is well defined by the Max–Min type inequality (Theorem 1.2). The proof of Theorem 1.2 requires a careful analysis of some geometric properties of multiparameter rectangles. Discrete Calderón's identity and almost orthogonality estimates play a key role here. Especially due to the arbitrary choice of M and L in (2.6) below, $w \in A_\infty$ works out. Next, we get the duality results between the sequences spaces s_w^p and c_w^p (Theorem 1.4). Finally, relying on the projection and lifting between H_w^p and s_w^p and between CMO_w^p and c_w^p (Lemma 4.1), we get our desired results. The method we use here avoids the atomic decomposition and provides another proof of some of the known duality results for non-weighted Hardy spaces in the two parameter pure product settings by Chang and R. Fefferman (see e.g. [2, 3]).

We first recall the definitions of product weights in arbitrary number of parameters setting. For $1 < p < \infty$, a nonnegative locally integrable function w belongs to $A_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|R|}\int_R w(x)dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx\right)^{p-1} \le C,$$

for any rectangle $R = I_1 \times \cdots \times I_k$, where $I_i \subset \mathbb{R}^{n_i}$ are cubes, $1 \le i \le k$. We say w belongs to $A_1(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ if there exists a constant $C > 0$ such that

$$\mathcal{M}_s w(x) \le C w(x),$$

for almost every $x \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$, where $\mathcal{M}_s$ is the strong maximal function defined as

$$\mathcal{M}_s f(x) = \sup_{x \in R} \frac{1}{|R|}\int_R |f(y)|dy,$$

for any rectangle R on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ whose sides are parallel to the axes. We define $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ by

$$A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}) = \bigcup_{1 \le p < \infty} A_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}).$$

Notice that if $w \in A_\infty$, then $w \in A_{q_w}$, where $q_w = \min\{q : w \in A_q\}$.

In the following, we denote by D_x^α the high order derivatives $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_k}^{\alpha_k}$ for the multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$.

For $i = 1, \ldots, k$, let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ with

$$\operatorname{supp} \widehat{\psi^{(i)}} \subseteq \left\{\xi_i \in \mathbb{R}^{n_i} : \frac{1}{2} \le |\xi_i| \le 2\right\}$$

and satisfy

$$\sum_{j_i \in \mathbb{Z}} |\widehat{\psi^{(i)}}(2^{-j_i}\xi_i)|^2 = 1 \quad \text{for all } \xi_i \in \mathbb{R}^{n_i} \backslash \{0\}.$$

Denote

$$\psi_{j_1,\dots,j_k}(x_1,\dots,x_k) = \psi^{(1)}_{j_1}(x_1)\cdots\psi^{(k)}_{j_k}(x_k), \tag{1.1}$$

where

$$\psi^{(i)}_{j_i}(x_i) = 2^{j_i n_i}\psi^{(i)}(2^{j_i}x_i), \quad i = 1,\dots,k.$$

Denote by $\mathcal{S}_\infty(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})$ the functions $f \in \mathcal{S}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})$ such that for every i, $1 \le i \le k$,

$$\int_{\mathbb{R}^{n_i}} f(x_1,\dots,x_k)x_i^{\alpha_i}\,dx_i = 0 \quad \text{for any } |\alpha_i| \ge 0.$$

Let $w \in A_\infty$. For $0 < p \le 1$, define the weighted Hardy spaces on the product $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$ as $H^p_w = \{f \in (\mathcal{S}_\infty)' : \mathcal{G}^d(f) \in L^p_w\}$ with the norm defined by $\|f\|_{H^p_w} = \|\mathcal{G}^d(f)\|_{L^p_w}$, where

$$\mathcal{G}^d(f)(x_1,\dots,x_k) := \Big\{\sum_{j_1,\dots,j_k}\sum_{I_1,\dots,I_k}|\psi_{j_1,\dots,j_k} * f(2^{-j_1}\ell_1,\dots,2^{-j_k}\ell_k)|^2 \times \prod_{i=1}^{k}\chi_{I_i}(x_i)\Big\}^{\frac{1}{2}},$$

and I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i \in \mathbb{Z}^{n_i}$, $i = 1,\dots,k$, respectively.

In order to study the dual space of H^p_w on $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$, we need to introduce the weighted Carleson measure space $\mathrm{CMO}^p_w = \mathrm{CMO}^p_w(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})$.

Definition 1.1. Let $w \in A_\infty$. For $0 < p \le 1$, we call $f \in \mathrm{CMO}^p_w$ if $f \in (\mathcal{S}_\infty)'$ with the finite norm defined by

$$\sup_\Omega\Big\{\frac{1}{w(\Omega)^{\frac{2}{p}-1}}\sum_{j_1,\dots,j_k}\sum_{I_1\times\cdots\times I_k\subseteq\Omega}|\psi_{j_1,\dots,j_k} * f(2^{-j_1}\ell_1,\dots,2^{-j_k}\ell_k)|^2 \times \frac{|I_1\times\cdots\times I_k|^2}{w(I_1\times\cdots\times I_k)}\Big\}^{\frac{1}{2}}, \tag{1.2}$$

for all open sets Ω in $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$ with finite weighted measures, here I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i \in \mathbb{Z}^{n_i}$, $i = 1,\dots,k$, respectively.

We use BMO_w to denote CMO^1_w. The following theorem tells us that the definition of the space CMO^p_w is independent of the choice of functions $\psi_{j_1,\dots,j_k}$, thus the space CMO^p_w is well defined. One of the main theorems of our paper is the following theorem.

Theorem 1.2. *Let $w \in A_\infty$. Suppose $\psi_{j_1,\dots,j_k}$ and $\phi_{j_1,\dots,j_k}$ satisfy the conditions in* (1.1). *Then for $f \in (\mathcal{S}_\infty)'$,*

$$\sup_\Omega \left\{ \frac{1}{w(\Omega)^{\frac{2}{p}-1}} \sum_{j_1,\dots,j_k} \sum_{I_1\times\cdots\times I_k\subseteq\Omega} |\psi_{j_1,\dots,j_k} * f(2^{-j_1}\ell_1,\dots,2^{-j_k}\ell_k)|^2 \times \frac{|I_1\times\cdots\times I_k|^2}{w(I_1\times\cdots\times I_k)} \right\}^{\frac{1}{2}}$$

$$\approx \sup_\Omega \left\{ \frac{1}{w(\Omega)^{\frac{2}{p}-1}} \sum_{j_1,\dots,j_k} \sum_{I_1\times\cdots\times I_k\subseteq\Omega} |\phi_{j_1,\dots,j_k} * f(2^{-j_1}\ell_1,\dots,2^{-j_k}\ell_k)|^2 \times \frac{|I_1\times\cdots\times I_k|^2}{w(I_1\times\cdots\times I_k)} \right\}^{\frac{1}{2}},$$

for all open sets Ω in $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$ with finite weighted measures, here I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i \in \mathbb{Z}^{n_i}$, $i = 1,\dots,k$, respectively.

We may use the following weighted sequence space to derive that the space CMO_w^p is exactly the dual space of H_w^p on $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$.

Definition 1.3. Let $w \in A_\infty$ and $0 < p \le 1$. Set s_w^p to be the set of all sequences $s = \{s_{I_1\times\cdots\times I_k}\}$ such that

$$\|s\|_{s_w^p} := \left\| \left\{ \sum_{j_1,\dots,j_k} \sum_{I_1,\dots,I_k} |s_{I_1\times\cdots\times I_k}|^2 \prod_{i=1}^k |I_i|^{-1}\chi_{I_i} \right\}^{\frac{1}{2}} \right\|_{L_w^p} < \infty,$$

where the sum runs over all dyadic cubes I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i \in \mathbb{Z}^{n_i}$, $i = 1,\dots,k$, and χ_{I_i} are characteristic functions of I_i.

Let c_w^p be the set of all sequences $t = \{t_{I_1\times\cdots\times I_k}\}$ with the finite norm defined by

$$\|t\|_{c_w^p} := \sup_\Omega \left\{ \frac{1}{w(\Omega)^{\frac{2}{p}-1}} \sum_{j_1,\dots,j_k} \sum_{I_1\times\cdots\times I_k\subseteq\Omega} |t_{I_1\times\cdots\times I_k}|^2 \frac{|I_1\times\cdots\times I_k|}{w(I_1\times\cdots\times I_k)} \right\}^{\frac{1}{2}},$$

where Ω are all open sets in $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$ with finite weighted measures and I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i \in \mathbb{Z}^{n_i}$, $i = 1,\dots,k$.

Then we have the following theorem.

Theorem 1.4. *Let $w \in A_\infty$. Then for $0 < p \le 1$, $(s_w^p)^* = c_w^p$. To be precise, for each $t \in c_w^p$, the map which maps $s = \{s_{I_1\times\cdots\times I_k}\}$ to*

$$\langle s, t\rangle \equiv \sum_{j_1,\ldots,j_k} \sum_{I_1\times\cdots\times I_k} s_{I_1\times\cdots\times I_k}\, \bar{t}_{I_1\times\cdots\times I_k}$$

defines a continuous linear functional on s_w^p with $\|t\|_{(s_w^p)^*} \approx \|t\|_{c_w^p}$, and every $\ell \in (s_w^p)^*$ is of this form for some $t \in c_w^p$.

By the aid of this duality result above, we can get the duality of the space H_w^p for $0 < p \le 1$ which is another main theorem of our paper.

Theorem 1.5. *Let $w \in A_\infty$. Then for $0 < p \le 1$, $(H_w^p)^* = \mathrm{CMO}_w^p$. To be precise, if $g \in \mathrm{CMO}_w^p$, the map ℓ_g given by $\ell_g(f) = \langle f, g\rangle$, defined initially for $f \in \mathcal{S}_\infty$, extends to a continuous linear functional on H_w^p with $\|\ell_g\| \approx \|g\|_{\mathrm{CMO}_w^p}$. Conversely, for every $\ell \in (H_w^p)^*$ there exists some $g \in \mathrm{CMO}_w^p$ so that $\ell = \ell_g$. In particular, $(H_w^1)^* = \mathrm{BMO}_w$.*

Applying Theorem 1.5 above and Proposition 2.3 blow, we can obtain the boundedness of singular integral operators T with the product kernel $\mathcal{K}$ on BMO_w. The product kernel $\mathcal{K}$ in Definition 1.6 below has been studied by many researchers, e.g. Nagel, Ricci and Stein in [26].

Definition 1.6. Let $\mathcal{K}$ be a distribution on $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$, which coincides with a C^∞ function away from the coordinate subspace $x_i = 0$, $1 \le i \le k$, and satisfies

(i) (differential inequalities) for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{N}^{n_i}$, there exists a constant $C_\alpha > 0$ such that

$$|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_k}^{\alpha_k}\mathcal{K}(x_1,\ldots,x_k)| \le C_\alpha \prod_{1\le i\le k} |x_i|^{-n_i-|\alpha_i|}, \tag{1.3}$$

(ii) (cancellation conditions) for every $r_i > 0$ and every normalized bump function φ_i on $\mathbb{R}^{n_i}$, that is, φ_i is smooth, supported in the unit ball and satisfies

$$|D_x^\alpha \varphi_i(x)| \le 1, \quad 0 \le |\alpha| \le N,$$

where N is some fixed positive integer, there exist $A_i, C > 0$ such that

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^{n_1}\times\cdots\times\widehat{\mathbb{R}^{n_i}}\times\cdots\times\mathbb{R}^{n_k}} &\partial_{x_i}^{\alpha_i}\mathcal{K}(x_1,\ldots,x_k)\\ &\times \prod_{1\le \ell\le k,\ell\ne i}\varphi_\ell(r_\ell x_\ell)\,dx_1\cdots\widehat{dx_i}\cdots dx_k\Bigg|\\ \le\ A_i\,|x_i|^{-n_i-|\alpha_i|},\quad i = 1,\ldots,k,&\end{aligned} \tag{1.4}$$

and

$$\left|\int_{\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}} \mathcal{K}(x_1,\ldots,x_k)\prod_{1\le i\le k}\varphi_i(r_ix_i)dx_1\cdots dx_k\right|\le C. \tag{1.5}$$

Theorem 1.7. *Suppose that T is a convolution singular integral operator with the product kernel $\mathcal{K}$ satisfying the conditions in* (1.3)–(1.5). *Then T is bounded on* BMO_w.

The organization of this paper is as follows. In Section 2, we obtain the discrete Calderón's identity on $\mathcal{S}_\infty$ and its dual space $(\mathcal{S}_\infty)'$ and list some known results on the boundedness of the singular integral operators on H_w^p. In Section 3, we prove the Max–Min principle related to the space CMO_w^p, namely Theorem 1.2. This theorem guarantees that the CMO_w^p norm given in Definition 1.1 is well defined. Section 4 contains the proofs of Theorems 1.4 (the duality of the weighted sequence spaces s_w^p and c_w^p), Theorem 1.5 (duality of H_w^p and CMO_w^p), and Theorem 1.7 (the boundedness of singular integrals on the space BMO_w).

2 Some preliminaries

We now give the following necessary estimates and results in the setting of arbitrary number of parameters.

Proposition 2.1 (Discrete Calderón identity). *Let $\psi_{j_1,\ldots,j_k}$ be the same as in* (1.1). *Then*

$$\begin{aligned} f(x_1,\ldots,x_k) = \sum_{j_1,\ldots,j_k}\sum_{I_1,\ldots,I_k}\prod_{i=1}^{k}|I_i|\psi_{j_1,\ldots,j_k}(x_1-2^{-j_1}\ell_1,\ldots,x_k-2^{-j_k}\ell_k)\\ \times\psi_{j_1,\ldots,j_k}*f(2^{-j_1}\ell_1,\ldots,2^{-j_k}\ell_k), \end{aligned} \tag{2.1}$$

where I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i\in\mathbb{Z}^{n_i}$, $i=1,\ldots,k$, and the series in (2.1) *converges in L^2, $\mathcal{S}_\infty$ and $(\mathcal{S}_\infty)'$.*

Proof. Our proof is similar to that in [12]. For the sake of completeness, we give the details here. Denote x_{I_i} by $2^{-j_i}\ell_i$, $\ell_i\in\mathbb{Z}^{n_i}$, $i=1,\ldots,k$. Let I_i be dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are x_{I_i}, $i=1,\ldots,k$.

By taking the Fourier transform, we can get the following continuous version of Calderón's identity:

$$f(x_1,\ldots,x_k)=\sum_{j_1,\ldots,j_k\in\mathbb{Z}}\psi_{j_1,\ldots,j_k}*\psi_{j_1,\ldots,j_k}*f(x_1,\ldots,x_k), \tag{2.2}$$

where the series converge in L^2, $\mathcal{S}_\infty$ and $(\mathcal{S}_\infty)'$. Set

$$g = \psi_{j_1,\dots,j_k} * f \quad \text{and} \quad h = \psi_{j_1,\dots,j_k}.$$

Then

$$\widehat{g}(\xi_1,\dots,\xi_k) = \prod_{i=1}^{k} \widehat{\psi^{(i)}}(2^{-j_i}\xi_i)\widehat{f}(\xi_1,\dots,\xi_k),$$

$$\widehat{h}(\xi_1,\dots,\xi_k) = \prod_{i=1}^{k} \widehat{\psi^{(i)}}(2^{-j_i}\xi_i).$$

Note that $\operatorname{supp}\widehat{g}, \operatorname{supp}\widehat{h} \subseteq R_{j_1,\dots,j_k}$, where

$$R_{j_1,\dots,j_k} = \{(\xi_1,\dots,\xi_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} : |\xi_i| \le 2^{j_i}\pi,\ i = 1,\dots,k\},$$

which imply that we can expand $\widehat{g}$ in a Fourier series first on the rectangle $R_{j_1,\dots,j_k}$ and then replace $R_{j_1,\dots,j_k}$ on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$

$$\begin{aligned}\widehat{g}(\xi_1,\dots,\xi_k) = &\sum_{I_1,\dots,I_k} (2\pi)^{-(n_1+\cdots+n_k)} \prod_{j=1}^{k} |I_j|\, e^{-i(2^{-j_1}\ell_1\cdot\xi_1+\cdots+2^{-j_k}\ell_k\cdot\xi_k)} \\ &\times \int_{R_{j_1,\dots,j_k}} \widehat{g}(\eta_1,\dots,\eta_k) e^{i(2^{-j_1}\ell_1\cdot\eta_1+\cdots+2^{-j_k}\ell_k\cdot\eta_k)} d\eta_1\cdots d\eta_k \\ = &\sum_{I_1,\dots,I_k} \prod_{j=1}^{k} |I_j|\, e^{-i(2^{-j_1}\ell_1\cdot\xi_1+\cdots+2^{-j_k}\ell_k\cdot\xi_k)} g(x_{I_1},\dots,x_{I_k}).\end{aligned}$$

Multiplying $\widehat{h}(\xi_1,\dots,\xi_k)$ from both sides and applying the identity $g*h = (\widehat{g}\widehat{h})^\vee$, we get

$$(g * h)(x_1,\dots,x_k) = \sum_{I_1,\dots,I_k} \prod_{i=1}^{k} |I_i|\, g(x_{I_1},\dots,x_{I_k}) h(x_1 - x_{I_1},\dots,x_k - x_{I_k}). \tag{2.3}$$

Substituting g by $\psi_{j,k} * f$ and h by $\psi_{j,k}$ into Calderón's identity in (2.2) gives the discrete Calderón's identity in (2.1) and the convergence of the series in L^2.

We now prove that the series in (2.1) converges in $\mathcal{S}_\infty$. It suffices to show that

$$\begin{aligned}\sum_{(j_1,\dots,j_k)\in W^c} \sum_{I_1,\dots,I_k} \prod_{i=1}^{k} |I_i|\, (\psi_{j_1,\dots,j_k} * f)(x_{I_1},\dots,x_{I_k}) \\ \times \psi_{j_1,\dots,j_k}(x_1 - x_{I_1},\dots,x_k - x_{I_k})\end{aligned} \tag{2.4}$$

tend to zero in $\mathcal{S}(\mathbb{R}^{n_1+\cdots+n_k})$ as $N_1,\dots,N_k$ tend to infinity, where

$$W := \{|j_i| \le N_i,\ i = 1,\dots,k\}.$$

Claim. *For any fixed $j_1, \ldots, j_k$, and any given integer $M > 0$, $|\alpha| \geq 0$, there exists a constant $C = C(M, \alpha) > 0$, which is independent of $j_1, \ldots, j_k$, such that*

$$\Big| \sum_{I_1,\ldots,I_k} \prod_{i=1}^{k} |I_i| (\psi_{j_1,\ldots,j_k} * f)(x_{I_1}, \ldots, x_{I_k}) \times (D_x^\alpha \psi_{j_1,\ldots,j_k})(x_1 - x_{I_1}, \ldots, x_k - x_{I_k}) \Big| \leq C\, 2^{-(|j_1|+\cdots+|j_k|)} (1 + |x_1| + \cdots + |x_k|)^{-M}. \tag{2.5}$$

Using the classical almost orthogonality argument, that is, for any given positive integers L and M, there exists a constant $C = C(L, M) > 0$ such that

$$|\psi_{j_i}^{(i)} * \psi_{j_i'}^{(i)}(x_i)| \leq C \frac{2^{-|j_i - j_i'|L}\, 2^{n_i (j_i \wedge j_i')}}{(1 + 2^{(j_i \wedge j_i')} |x_i|)^M}, \tag{2.6}$$

we have that

$$|(\psi_{j_1,\ldots,j_k} * f)(x_1, \ldots, x_k)| \leq C \prod_{i=1}^{k} \left(2^{-|j_i|L} (1 + |x_i|)^{-M} \right). \tag{2.7}$$

From the size conditions of the functions $\psi^{(i)}$, $i = 1, \ldots, k$, we have that for any fixed large M,

$$|D_u^\alpha \psi_{j_1,\ldots,j_k}(u_1, \ldots, u_k)| \leq C \prod_{i=1}^{k} 2^{|j_i|(M + n_i + |\alpha|)} (1 + |u_i|)^{-M}. \tag{2.8}$$

Estimates in (2.7) and (2.8) yield

$$\begin{aligned}
&\Big| \sum_{I_1,\ldots,I_k} \prod_{i=1}^{k} |I_i| (D_x^\alpha \psi_{j_1,\ldots,j_k})(x_1 - x_{I_1}, \ldots, x_k - x_{I_k}) \times (\psi_{j_1,\ldots,j_k} * f)(x_{I_1}, \ldots, x_{I_k}) \Big| \\
&\leq C \prod_{i=1}^{k} 2^{-|j_i|(L - M - n_i - |\alpha|)} \\
&\qquad \times \sum_{I_1,\ldots,I_k} \int_{I_1 \times \cdots \times I_k} \prod_{j=1}^{k} (1 + |x_{I_j}|)^{-M} (1 + |x_j - x_{I_j}|)^{-M} du_1 \cdots du_k \\
&\leq C \prod_{i=1}^{k} \left(2^{-|j_i|(L - 5M - n_i - |\alpha|)} (1 + |x_i|)^{-M} \right).
\end{aligned}$$

Choosing $L = 5M + \sum_{i=1}^{k} n_i + |\alpha|$, we derive the estimates in (2.5) and hence the series in (2.4) converges to zero as $N_1, \dots, N_k$ tend to infinity. Therefore, the series in (2.1) converges in $\mathcal{S}_\infty$. By the duality argument, we obtain the series in (2.1) converges in $(\mathcal{S}_\infty)'$. □

Proposition 2.2 ([28]). *Given any positive integers $M_1, \dots, M_k$ and $N_1, \dots, N_k$, let I_i, I_i' be dyadic cubes in $\mathbb{R}^{n_i}$ with the side-lengths $\ell(I_i) = 2^{-j_i}, \ell(I_i') = 2^{-j_i'}$, $i = 1, \dots, k$. Then for any $u_i, u_i^* \in I_i$, $i = 1, \dots, k$, we have*

$$\sum_{I_1', \dots, I_k'} \left(\prod_{i=1}^{k} \frac{2^{-(j_i \wedge j_i')N_i} |I_i'|}{(2^{-(j_i \wedge j_i')} + |u_i - x_{I_i'}|)^{n_i + N_i}} \right) \times |\phi_{j_1', j_2', \dots, j_k'} * f(x_{I_1'}, x_{I_2'}, \dots, x_{I_k'})|$$
$$\leq C_0 \left\{ \mathcal{M}_s \left(\sum_{I_1', \dots, I_k'} |\phi_{j_1', j_2', \dots, j_k'} * f(x_{I_1'}, x_{I_2'}, \dots, x_{I_k'})| \times \prod_{i=1}^{k} \chi_{I_i'} \right)^{\delta} (u_1^*, \dots, u_k^*) \right\}^{\frac{1}{\delta}},$$

where $\mathcal{M}_s$ is the strong maximal function on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$, $0 < \delta \leq \min\{1, \frac{p}{q_w}\}$ and

$$C_0 = C \prod_{i=1}^{k} 2^{(1-\frac{1}{\delta})(j_i - j_i')_+}.$$

In [28], we have obtained the H_w^p boundedness of the convolution singular integral operator T with the product kernel $\mathcal{K}$ on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$. We can extend this result below by applying similar arguments.

Proposition 2.3. *If T is a singular integral with the kernel $\mathcal{K}$, where $\mathcal{K}$ satisfies the conditions in Definition 1.6, then for $w \in A_\infty$, there exists a constant $C > 0$ such that for $0 < p < \infty$,*

$$\|T(f)\|_{H_w^p} \leq C \|f\|_{H_w^p}.$$

3 Proof of Theorem 1.2

The purpose of this section is to get the Max–Min principle to ensure that the space CMO_w^p we introduce in this paper is well defined. The proof follows from the discrete Calderón identity, almost orthogonality estimates and some geometric properties. Here we only consider the case $n_i = 1, 1 \leq i \leq k$, since the extension to general n_i, $i = 1, \dots, k$, is straightforward.

For simplicity, we denote $f_{j_1,\dots,j_k} = f_R$, where $R = I_1 \times \cdots \times I_k \subseteq \mathbb{R}^k$, I_i are dyadic intervals with the side lengths $|I_i| = 2^{-j_i}$, and the left lower points of I_i are x_{I_i}, $i = 1, \dots, k$. Then we can rewrite the discrete Calderón identity on $(S_\infty)'$,

$$f(x_1, \dots, x_k) = \sum_R \prod_{i=1}^k |I_i| \, \phi_R(x_1 - x_{I_1}, \dots, x_k - x_{I_k}) \, \phi_R * f(x_{I_1}, \dots, x_{I_k}).$$

Thus for all $(u_1, \dots, u_k) \in R$,

$$\begin{aligned}\psi_R * f(u_1, \dots, u_k) = \sum_{R' = I_1' \times \cdots \times I_k'} \prod_{i=1}^k |I_i'| \, \psi_R * \phi_{R'}(u_1 - x_{I_1'}, \dots, u_k - x_{I_k'}) \\ \times \phi_{R'} * f(x_{I_1'}, \dots, x_{I_k'}),\end{aligned}$$

where I_i' are dyadic intervals with the side lengths $|I_i'| = 2^{-j_i'}$, and the left lower points of I_i' are $x_{I_i'}$, $i = 1, \dots, k$.

The almost orthogonality estimate in (2.6) implies that

$$\begin{aligned}&|\psi_R * f(u_1, \dots, u_k)|^2 \\ &\quad \le C \sum_{R' = I_1' \times \cdots \times I_k'} \prod_{i=1}^k \left(\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|} \right)^L \frac{|I_i'|^{N+1}}{(|I_i'| + |u_i - x_{I_i'}|)^{N+1}} \\ &\qquad \times |\phi_{R'} * f(x_{I_1'}, \dots, x_{I_k'})|^2, \end{aligned} \tag{3.1}$$

where N, L are any positive integers such that $L > q_w + \frac{2}{p} + 2$ and $N > \frac{2}{p}$, the constant C depends only on N, L and functions ψ and ϕ.

Let

$$M_R = |\psi_R * f(2^{-j_1}\ell_1, \dots, 2^{-j_k}\ell_k)|^2,$$
$$m_R = |\phi_R * f(2^{-j_1}\ell_1, \dots, 2^{-j_k}\ell_k)|^2.$$

Then

$$\begin{aligned}&\sum_{R \subseteq \Omega} \frac{|I_1 \times \cdots \times I_k|^2}{w(I_1 \times \cdots \times I_k)} M_R \\ &\quad \le C \sum_{R \subseteq \Omega} \sum_{R'} \widetilde{r}(R, R') P(R, R') \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} m_{R'},\end{aligned}$$

where

$$\widetilde{r}(R, R') = \prod_{i=1}^k \left(\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|} \right)^{L-2} \frac{w(I_1' \times \cdots \times I_k')}{w(I_1 \times \cdots \times I_k)}$$

and

$$P(R,R') = \prod_{i=1}^{k} \frac{1}{(1+\frac{\operatorname{dist}(I_i,I_i')}{|I_i'|})^{1+N}}.$$

Since $w \in A_\infty$ and $I_i \subseteq I_i \cup I_i'$, we have that $w \in A_{q_w}$, and

$$\begin{aligned}\frac{w(I_1' \times \cdots \times I_k')}{w(I_1 \times \cdots \times I_k)} &\le \frac{w((I_1' \cup I_1) \times \cdots \times (I_k' \cup I_k))}{w(I_1 \times \cdots \times I_k)} \\ &\le C \prod_{i=1}^{k} \Big(1 + \frac{|I_i'|}{|I_i|}\Big)^{q_w} \\ &\le C \prod_{i=1}^{k} \Big\{ \sum_{0 \le r_i \le q_w} \Big(\frac{|I_i'|}{|I_i|}\Big)^{r_i} \Big\}.\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{r}(R,R') &\le C \prod_{i=1}^{k} \Big\{ \Big(\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|}\Big)^{L-2-q_w} \sum_{0 \le r_i \le q_w} \Big(\frac{|I_i'|}{|I_i|}\Big)^{r_i} \Big(\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|}\Big)^{q_w} \Big\} \\ &\le C \prod_{i=1}^{k} \Big(\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|}\Big)^{L-2-q_w} \\ &:= r(R,R'),\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{R \subseteq \Omega} \frac{|I_1 \times \cdots \times I_k|^2}{w(I_1 \times \cdots \times I_k)}\, M_R \\ &\qquad \le C \sum_{R \subseteq \Omega} \sum_{R'} r(R,R') P(R,R') \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} m_{R'}.\end{aligned} \tag{3.2}$$

In the following we estimate the right-hand side of inequality (3.2).

Define

$$\Omega^{j_1,\dots,j_k} = \bigcup_{I_1 \times \cdots \times I_k \subseteq \Omega} 3(2^{j_1} I_1 \times \cdots \times 2^{j_k} I_k) \quad \text{for } j_1,\dots,j_k \ge 0.$$

Given $1 \le d \le k-1$, for any $j_1,\dots,j_k \ge 0$, $1 \le m_r \le k$, $m_r < m_{r+1}$, with $r = 1,\dots,d$, denote

$$\begin{aligned}\Omega^{j_1,\dots,j_k}_{\widehat{m_1},\dots,\widehat{m_d}} = \{(x_1,\dots,x_k) \in \Omega : x_{m_i} \in 3I_{m_i},\ 1 \le i \le d,& \\ x_\ell \in 3(2^{j_\ell} I_\ell),\ 1 \le \ell \le k,\ \ell \ne m_1,\dots,m_d\},&\end{aligned}$$

and

$$\widetilde{R}^{j_1,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}} = \{(x'_1,\ldots,x'_k) \in \mathbb{R}^k : x'_{m_i} \in 3I'_{m_i}, 1 \le i \le d,$$
$$x'_\ell \in 3(2^{j_\ell} I'_\ell), 1 \le \ell \le k, \ell \ne m_1,\ldots,m_d\}.$$

For $j_1,\ldots,j_k \ge 1$, let $B^{j_1,\ldots,j_k}$ be a set of dyadic rectangle R' such that

$$B^{j_1,\ldots,j_k} = \{R' = I'_1 \times \cdots \times I'_k : 3(2^{j_1} I'_1 \times \cdots \times 2^{j_k} I'_k) \cap \Omega^{j_1,\ldots,i_k} \ne \emptyset,$$
$$3(2^{j_1-1} I'_1 \times \cdots \times 2^{j_k-1} I'_k) \cap \Omega^{j_1,\ldots,j_k} = \emptyset\},$$

and for any $1 \le d \le k-1$ and $j_\ell \ge 1$, $1 \le \ell \le k, \ell \ne m_1,\ldots,m_d$,

$$B^{j_1,\ldots,j_s,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}} = \{R' = I'_1 \times \cdots \times I'_k : 3(\widetilde{R}^{j_1,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}}) \cap \Omega^{j_1,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}} \ne \emptyset,$$
$$3(\widetilde{R}^{j_1-1,\ldots,j_k-1}_{\widehat{m_1},\ldots,\widehat{m_d}}) \cap \Omega^{j_1,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}} = \emptyset\},$$

and $B^{0,0,\ldots,0} = \{R' = I'_1 \times \cdots \times I'_k : 3(I'_1 \times \cdots \times I'_k) \cap \Omega^{0,0,\ldots,0} \ne \emptyset\}$.

Since each $(j_1,\ldots,j_k, I_1,\ldots,I_k)$ belongs to precisely one rectangle $B^{j_1,\ldots,j_k}$ or $B^{j_1,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}}$, we write

$$\begin{aligned}
&\sum_{R \subseteq \Omega} \sum_{R'} \frac{|I'_1 \times \cdots \times I'_k|^2}{w(I'_1 \times \cdots \times I'_k)} r(R,R') P(R,R') m_{R'} \\
&\quad \le \Bigg(\sum_{j_1,\ldots,j_k \ge 1} \sum_{R' \in B^{j_1,\ldots,j_k}} + \sum_{R' \in B^{0,0,\ldots,0}} \Bigg) \\
&\qquad\qquad \times \sum_{R \subseteq \Omega} \frac{|I'_1 \times \cdots \times I'_k|^2}{w(I'_1 \times \cdots \times I'_k)} r(R,R') P(R,R') m_{R'} \\
&\qquad + \sum_{d=1}^{k-1} \sum_{j_\ell \ge 1, \ell \ne m_1,\ldots,m_d, 1 \le \ell \le k} \sum_{R' \in B^{j_1,\ldots,j_k}_{\widehat{m_1},\ldots,\widehat{m_d}}} \\
&\qquad\qquad \times \sum_{R \subseteq \Omega} \frac{|I'_1 \times \cdots \times I'_k|^2}{w(I'_1 \times \cdots \times I'_k)} r(R,R') P(R,R') m_{R'}.
\end{aligned}$$

We first consider the case for $R' \in B^{0,0,\ldots,0}$. In this case, $3R' \cap \Omega^{0,0,\ldots,0} \ne \emptyset$.

For each integer $\lambda \ge 1$, let

$$\mathcal{F}_\lambda = \left\{ R' \in B^{0,0,\ldots,0} : |3(I'_1 \times \cdots \times I'_k) \cap \Omega^{0,0,\ldots,0}| \ge \frac{1}{2^\lambda} |3(I'_1 \times \cdots \times I'_k)| \right\}.$$

Let

$$\mathcal{D}_\lambda = \mathcal{F}_\lambda \backslash \mathcal{F}_{\lambda-1} \quad \text{and} \quad \Omega_\lambda = \bigcup_{R' \in \mathcal{D}_\lambda} R'.$$

Without loss of generality we may assume that for any open set $\Omega \subset \mathbb{R}^k$,

$$\sum_{R = I_1 \times \cdots \times I_k \subseteq \Omega} \frac{|I_1 \times \cdots \times I_k|^2}{w(I_1 \times \cdots \times I_k)} m_R \le C w(\Omega)^{\frac{2}{p}-1}. \tag{3.3}$$

Since $B^{0,0,\ldots,0} = \bigcup_{\lambda \ge 1} \mathcal{D}_\lambda$ and for each $R' \in B^{0,0,\ldots,0}$, $P(R,R') \le 1$, we have

$$\begin{aligned} &\sum_{R' \in B^{0,0,\ldots,0}} \sum_{R \subseteq \Omega} \frac{|I'_1 \times \cdots \times I'_k|^2}{w(I'_1 \times \cdots \times I'_k)} r(R,R') P(R,R') m_{R'} \\ &\quad \le \sum_{\lambda \ge 1} \sum_{R' \subseteq \Omega_\lambda} \sum_{R \subseteq \Omega} \frac{|I'_1 \times \cdots \times I'_k|^2}{w(I'_1 \times \cdots \times I'_k)} r(R,R') m_{R'}. \end{aligned}$$

For each $\lambda \ge 1$ and $R' \subseteq \Omega_\lambda$, decompose $\{R : R \subseteq \Omega\}$ into

$$A^{0,0,\ldots,0}(R') = \big\{R \subseteq \Omega : \operatorname{dist}(I_i, I'_i) \le |I_i| \vee |I'_i|,\ 1 \le i \le k\big\},$$

and

$$\begin{aligned} A^{j'_1,\ldots,j'_k}_{\widehat{m_1},\ldots,\widehat{m_d}}(R') = \big\{R \subseteq \Omega : {}& \operatorname{dist}(I_{m_r}, I'_{m_r}) \le |I_{m_r}| \vee |I'_{m_r}|,\ 1 \le r \le d, \\ & 2^{j'_i-1}(|I_i| \vee |I'_i|) < \operatorname{dist}(I_i, I'_i) \le 2^{j'_i}(|I_i| \vee |I'_i|), \\ & 1 \le i \le k, i \ne m_1, \ldots, m_d\big\}, \end{aligned}$$

where $1 \le d \le k-1$, $j'_i \ge 1$ for $1 \le i \le k$ and $i \ne m_1, \ldots, m_d$, and

$$\begin{aligned} A^{j'_1,j'_2,\ldots,j'_k}(R') = \big\{R \subseteq \Omega : 2^{j'_i-1}(|I_i| \vee |I'_i|) &< \operatorname{dist}(I_i, I'_i) \\ &\le 2^{j'_i}(|I_i| \vee |I'_i|),\ 1 \le i \le k\big\} \end{aligned}$$

for $j'_1, j'_2, \ldots, j'_k \ge 1$, where

$$|I_i| \vee |I'_i| = \max\{|I_i|, |I'_i|\}.$$

Split

$$\sum_{\lambda \ge 1} \sum_{R' \subseteq \Omega_\lambda} \sum_{R \subseteq \Omega} \frac{|I'_1 \times \cdots \times I'_k|^2}{w(I'_1 \times \cdots \times I'_k)} r(R,R') P(R,R') m_{R'}$$

into three parts

$$\Bigg(\sum_{\lambda\ge 1}\sum_{R'\in\Omega_\lambda}\sum_{R\in A^{0,0,\dots,0}(R')}$$
$$+\sum_{\lambda\ge 1}\sum_{R'\in\Omega_\lambda}\sum_{d=1}^{k-1}\sum_{j_i'\ge 1, 1\le i\le k, i\ne m_1,\dots,m_d}\sum_{R\in A^{j_1',j_2',\dots,j_k'}_{\widehat{m_1},\dots,\widehat{m_d}}(R')}$$
$$+\sum_{\lambda\ge 1}\sum_{R'\in\Omega_\lambda}\sum_{j_1',\dots,j_k'\ge 1}\sum_{R\in A^{j_1',\dots,j_k'}(R')}\Bigg)\frac{|I_1'\times\cdots\times I_k'|^2}{w(I_1'\times\cdots\times I_k')}\,r(R,R')P(R,R')m_{R'}$$
$$:= I_1 + I_2 + I_3.$$

To estimate the term I_1, since $P(R,R')\le 1$, we only need to estimate

$$\sum_{R\in A^{0,0,\dots,0}(R')} r(R,R').$$

Note that when $R\in A^{0,0,\dots,0}(R')$, we have $3R\cap 3R'\ne\emptyset$. For such R, there are three cases:

Case 1: $|I_i'|\ge |I_i|$, $1\le i\le k$,

Case 2: $|I_i'|\le |I_i|$, $1\le i\le k$,

Case 3: $|I_{d_i}'|\ge |I_{d_i}|$ and $|I_\ell'|\le |I_\ell|$, where $\ell\ne d_i$, $d_i<d_{i+1}$, $1\le d_i\le k$, and $1\le i\le r\le k-1$.

In case 1, since $3R\cap 3R'\ne\emptyset$ and $R\in A^{0,0,\dots,0}(R')$, we can deduce that

$$|I_i|\le |3I_i\cap 3I_i'|\le 3|I_i|,\quad 1\le i\le k,$$

then

$$|R|\le |3R\cap 3R'|\le |3R'\cap\Omega^{0,0,\dots,0}|\le \frac{1}{2^{\lambda-1}}|3R'|.$$

Thus, $|R'| = 2^{\lambda-1-\gamma_k+\eta}|R|$ for some $\eta>0$, where $2^{\gamma_k-1}<3^k<2^{\gamma_k}$. For each fixed $\eta>0$, the fact that $3R\cap 3R'\ne\emptyset$ implies that the number of such R must be less than $5^k 2^{\lambda+\eta-\gamma_k-1}$, therefore,

$$\sum_{R\in\text{case }1} r(R,R')\le 5^k\sum_{\eta>0}2^{\lambda+\eta}\left(\frac{1}{2^{\lambda-1-\gamma_k+\eta}}\right)^{L-q_w-2}\le C2^{-\lambda(L-q_w-3)}.$$

In case 2,

$$|R'|\le |3R\cap 3R'|\le |3R'\cap\Omega^{0,0,\dots,0}|\le \frac{1}{2^{\lambda-1}}|3R'|,$$

which implies that $2^{\lambda-1} \le 3^k$. Since $|R'| \le |R|$, we have that $|R'| = 2^{\theta}|R|$ for some $\theta > 0$. For each fixed $\theta > 0$, the number of such R must be less than $5^k 2^{\theta}$. Hence,

$$\sum_{R \in \text{case } 2} r(R, R') \le 5^k \sum_{\theta > 0} 2^{\theta} \left(\frac{1}{2^{\theta}}\right)^{L-q_w-2} \le C.$$

In case 3, for $1 \le r \le k-1$,

$$\prod_{i=1}^{r} \frac{|I_{d_i}|}{3^k |I'_{d_i}|} |3R'| \le |3R \cap 3R'| \le |3R' \cap \Omega^{0,0,\ldots,0}| \le \frac{1}{2^{\lambda-1}} |3R'|.$$

Thus,

$$\prod_{i=1}^{r} |I'_{d_i}| = 2^{\lambda-1-\gamma_k+\mu} \prod_{i=1}^{r} |I_{d_i}|,$$

for some $\mu > 0$. For fixed μ,

$$3(I_{d_1} \times \cdots \times I_{d_r}) \cap 3(I'_{d_1} \times \cdots \times I'_{d_r}) \ne \emptyset$$

implies that the product number of such $I_{d_1}, \ldots, I_{d_r}$ must be less than $5^r\ 2^{\lambda+\mu}$. As for $|I_j| = 2^{\nu_j}|I'_j|$, for some $\nu_j \ge 0$, where $j \ne d_1, \ldots, d_r$. For fixed ν_j, since $3I_j \cap 3I'_j \ne \emptyset$, we have that the number of such I_j must be less than 5. Therefore,

$$\begin{aligned} \sum_{R \in \text{case } 3} r(R, R') &\le 5^k \sum_{\mu, \nu_j > 0} 2^{\lambda+\mu} \left(\frac{1}{2^{\lambda-1-\gamma_k+\mu} \prod_{j \ne d_1, \ldots, d_r} 2^{\nu_j}}\right)^{L-q_w-2} \\ &\le C 2^{-\lambda(L-q_w-3)}. \end{aligned}$$

Hence, we have

$$\sum_{R \in A^{0,0,\ldots,0}(R')} r(R, R') \le C 2^{-\lambda(L-q_w-3)}.$$

Since $|\Omega_\lambda| \le C\lambda 2^{\lambda}|\Omega^{0,0,\ldots,0}|$, $|\Omega^{0,0,\ldots,0}| \le C|\Omega|$ and w which belongs to A_∞ is a doubling measure, together with (3.3) and $L > q_w + \frac{2}{p} + 1 \ge q_w + 3$, we have that I_1 is bounded by

$$\begin{aligned} \sum_{\lambda \ge 1} 2^{-\lambda(L-q_w-3)L} w(\Omega_\lambda)^{\frac{2}{p}-1} &\le \sum_{\lambda \ge 1} 2^{-\lambda(L-q_w-3)} w(C\lambda^{1/k} 2^{\lambda/k} \Omega^{0,0,\ldots,0})^{\frac{2}{p}-1} \\ &\le C w(\Omega^{0,0,\ldots,0})^{\frac{2}{p}-1} \le C w(\Omega)^{\frac{2}{p}-1}. \end{aligned}$$

In the following, we only estimate the term I_3 since estimates of I_2 can be concluded by applying the same techniques as for I_1 and I_3.

For each $j_1, j_2, \ldots, j_k \geq 1$, if $R \in A^{j_1,j_2,\ldots,j_k}(R')$, then

$$P(R, R') \leq \prod_{i=1}^{k} 2^{-(j_i'-1)(1+N)}. \tag{3.4}$$

Similarly, we only need to estimate the sum

$$\sum_{R\in A^{j_1',j_2',\ldots,j_k'}(R')} r(R, R').$$

Note that for $R \in A^{j_1',j_2',\ldots,j_k'}(R')$,

$$3(2^{j_1'} I_1 \times \cdots \times 2^{j_k'} I_k) \cap 3(2^{j_1'} I_1' \times \cdots \times 2^{j_k'} I_k') \neq \emptyset.$$

To estimate this sum, we also split the sum above into three cases:

Case 1: $|2^{j_i'} I_i'| \geq |2^{j_i'} I_i|$, $1 \leq i \leq k$,

Case 2: $|2^{j_i'} I_i'| \leq |2^{j_i'} I_i|$, $1 \leq i \leq k$,

Case 3: $|2^{j_{d_t}'} I_{d_t}'| \geq |2^{j_{d_t}'} I_{d_t}|$ and $|2^{j_\ell'} I_\ell'| \leq |2^{j_\ell'} I_\ell|$, where $\ell \neq d_t, d_t < d_{t+1}$, $1 \leq d_t \leq k$, $1 \leq t \leq r \leq k-1$.

Following a similar argument to that in I_1, we can conclude that

$$\sum_{R\in A^{j_1',j_2',\ldots,j_k'}(R')} r(R, R') \leq C2^{-\lambda(L-q_w-3)},$$

which combined with the estimate of $P(R, R')$ in (3.4) implies that

$$I_3 \leq C \sum_{\lambda\geq 1} \sum_{j_1,j_2,\ldots,j_k\geq 1} \sum_{R'\subseteq\Omega_\lambda} 2^{-\lambda(L-q_w-3)} \prod_{i=1}^{k} 2^{-j_i'(1+N)} \frac{|I_1' \times\cdots\times I_k'|^2}{w(I_1' \times\cdots\times I_k')} m_{R'}.$$

Therefore I_3 is bounded by

$$\sum_{\lambda\geq 1} 2^{-\lambda(L-q_w-3)} w(\Omega_\lambda)^{\frac{2}{p}-1} \leq C w(\Omega)^{\frac{2}{p}-1}.$$

Combining I_1, I_2 and I_3, we have

$$\begin{aligned} &\frac{1}{w(\Omega)^{\frac{2}{p}-1}} \sum_{R'\in B^{0,0,\ldots,0}} \sum_{R\subseteq\Omega} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} r(R, R') P(R, R') m_{R'} \\ &\quad \leq C \sup_{\bar{\Omega}} \frac{1}{w(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subseteq\bar{\Omega}} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} m_{R'}. \end{aligned}$$

Now we consider

$$\sum_{j_1,j_2,\ldots,j_k\geq 1}\ \sum_{R'\in B^{j_1,j_2,\ldots,j_k}}\ \sum_{R\subseteq\Omega} \frac{|I_1'\times\cdots\times I_k'|^2}{w(I_1'\times\cdots\times I_k')}\, r(R,R')P(R,R')m_{R'}.$$

Note that for $R'\in B^{j_1,j_2,\ldots,j_k}$, $3(2^{j_1}I_1'\times\cdots\times 2^{j_k}I_k')\cap\Omega^{j_1,j_2,\ldots,j_k}\neq\emptyset$. Let

$$\mathcal{F}_\lambda^{j_1,j_2,\ldots,j_k} = \Big\{R'\in B^{j_1,j_2,\ldots,j_k} : |3(2^{j_1}I_1'\times\cdots\times 2^{j_k}I_k')\cap\Omega^{j_1,j_2,\ldots,j_k}| \geq \frac{1}{2^\lambda}|3(2^{j_1}I_1'\times\cdots\times 2^{j_k}I_k')|\Big\},$$

$$\mathcal{D}_\lambda^{j_1,j_2,\ldots,j_k} = \mathcal{F}_\lambda^{j_1,j_2,\ldots,j_k}\setminus\mathcal{F}_{\lambda-1}^{j_1,j_2,\ldots,j_k},$$

and

$$\Omega_\lambda^{j_1,j_2,\ldots,j_k} = \bigcup_{R'\in\mathcal{D}_\lambda^{j_1,j_2,\ldots,j_k}} R'.$$

Since $B^{j_1,j_2,\ldots,j_k}=\bigcup_{\lambda\geq 1}\mathcal{D}_\lambda^{j_1,j_2,\ldots,j_k}$, we have

$$\begin{aligned}&\sum_{j_1,j_2,\ldots,j_k\geq 1}\ \sum_{R'\in B^{j_1,j_2,\ldots,j_k}}\ \sum_{R\subseteq\Omega} \frac{|I_1'\times\cdots\times I_k'|^2}{w(I_1'\times\cdots\times I_k')} r(R,R')P(R,R')m_{R'}\\ &= \sum_{j_1,j_2,\ldots,j_k\geq 1}\ \sum_{\lambda\geq 1}\ \sum_{R'\in\mathcal{D}_\lambda^{j_1,j_2,\ldots,j_k}}\ \sum_{R\subseteq\Omega} \frac{|I_1'\times\cdots\times I_k'|^2}{w(I_1'\times\cdots\times I_k')} r(R,R')P(R,R')m_{R'}.\end{aligned}$$

We first estimate

$$\sum_{R'\in\mathcal{D}_\lambda^{j_1,j_2,\ldots,j_k}}\ \sum_{R\subseteq\Omega} \frac{|I_1'\times\cdots\times I_k'|^2}{w(I_1'\times\cdots\times I_k')} r(R,R')P(R,R')m_{R'}$$

for some $j_1,j_2,\ldots,j_k\geq 1$.

Note that for each $R'\in\mathcal{D}_\lambda^{j_1,j_2,\ldots,j_k}$,

$$3(2^{j_1-1}I_1'\times\cdots\times 2^{j_k-1}I_k')\cap\Omega^{j_1,j_2,\ldots,j_k}=\emptyset.$$

So for any $R\subseteq\Omega$, we have $2^{j_i}(|I_i|\vee|I_i'|)\leq \operatorname{dist}(I_i,I_i')$, $1\leq i\leq k$. Thus

$$\{R:\ R\subseteq\Omega\} = \sum_{j_1',j_2',\ldots,j_k'\geq 1} \widetilde{A}_{j_1',j_2',\ldots,j_k'}(R'),$$

where

$$\widetilde{A}_{j_1',j_2',\dots,j_k'}(R') = \{R \subset \Omega : 2^{j_i'-1} 2^{j_i}(|I_i| \vee |I_i'|) \le \operatorname{dist}(I_i, I_i') \\ \le 2^{j_i'} 2^{j_i}(|I_i| \vee |I_i'|),\ 1 \le i \le k\}.$$

Then we rewrite

$$\sum_{R' \in \mathcal{D}_\lambda^{j_1,j_2,\dots,j_k}} \sum_{R \subseteq \Omega} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} r(R,R') P(R,R') m_{R'} \\ = \sum_{j_1',j_2',\dots,j_k' \ge 1} \sum_{R' \in \mathcal{D}_\lambda^{j_1,j_2,\dots,j_k}} \sum_{R \in \widetilde{A}_{j_1',j_2',\dots,j_k'}(R')} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} \\ \times r(R,R') P(R,R') m_{R'}.$$

For $R' \in B^{j_1,j_2,\dots,j_k}$ and $R \in \widetilde{A}_{j_1',j_2',\dots,j_k'}(R')$,

$$P(R,R') \le \prod_{i=1}^{k} 2^{-j_i(1+N)} 2^{-(j_i'-1)(1+N)}.$$

Following the same proof with $B^{0,0,\dots,0}$ replaced by $B^{j_1,j_2,\dots,j_k}$, we have

$$\sum_{R \in \widetilde{A}_{j_1',j_2',\dots,j_k'}(R')} r(R,R') \le C 2^{-\lambda(L-q_w-3)},$$

and since

$$\sum_{R' \in B^{j_1,j_2,\dots,j_k}} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} m_{R'} \le C \prod_{i=1}^{k} (j_i 2^{j_i})^{\frac{2}{p}-1} w(\Omega)^{\frac{2}{p}-1},$$

we have

$$\sum_{R' \in \mathcal{D}_\lambda^{j_1,j_2,\dots,j_k}} \sum_{R \in A_{j_1',j_2',\dots,j_k'}(R')} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} r(R,R') P(R,R') m_{R'} \\ \le C \prod_{i=1}^{k} 2^{-j_i(M-\frac{2}{p})} 2^{-j_i'(1+M)} j_i^{\frac{2}{p}-1} \lambda^{\frac{2}{p}-1} 2^{-\lambda(L-q_w-2-\frac{2}{p})} \\ \times \sup_{\bar{\Omega}} \frac{1}{w(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subseteq \bar{\Omega}} \prod_{i=1}^{k} |I_i'|\, m_{R'}.$$

Summing all $j_1, j_2, \ldots, j_k, j_1', j_2', \ldots, j_k' \geq 1$,

$$\frac{1}{w(\Omega)^{\frac{2}{p}-1}} \sum_{j_1, j_2, \ldots, j_k \geq 1} \sum_{R' \in B^{j_1, j_2, \ldots, j_k}} \sum_{R \subseteq \Omega} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} \times r(R, R') P(R, R') m_{R'}$$

$$\leq C \sup_{\bar{\Omega}} \frac{1}{w(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subseteq \bar{\Omega}} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} m_{R'}.$$

Similar estimates hold for

$$\sum_{d=1}^{k-1} \sum_{j_\ell \geq 1, \ell \neq m_1, \ldots, m_d} \sum_{R' \in B^{j_1, j_2, \ldots, j_k}_{\widehat{m_1}, \ldots, \widehat{m_d}}} \sum_{R \subseteq \Omega} \frac{|I_1' \times \cdots \times I_k'|^2}{w(I_1' \times \cdots \times I_k')} \times r(R, R') P(R, R') m_{R'}.$$

We complete the proof of Theorem 1.2. □

4 Proofs of Theorem 1.4, Theorem 1.5 and Theorem 1.7

In this section, we use the lifting and projection between H_w^p and s_w^p and between CMO_w^p and c_w^p respectively to get that the space CMO_w^p is exactly the dual space of the weighted Hardy space H_w^p. We also obtain the boundedness of the convolution singular integral operator T on BMO_w.

Proof of Theorem 1.4. We first show

$$c_w^p \subseteq (s_w^p)^*.$$

Suppose that $t = \{t_{I_1 \times \cdots \times I_k}\} \in c_w^p$ and set

$$h(x_1, \ldots, x_k) = \left\{ \sum_{j_1, \ldots, j_k} \sum_{I_1 \times \cdots \times I_k} |s_{I_1 \times \cdots \times I_k}|^2 \prod_{i=1}^{k} |I_i|^{-1} \chi_{I_i}(x_i) \right\}^{\frac{1}{2}},$$

and

$$\Omega_\ell = \{(x_1, \ldots, x_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} : h(x_1, \ldots, x_k) > 2^\ell\},$$

where I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i} \ell_i$, $1 \leq i \leq k$. Denote

$$\mathcal{B}_\ell = \left\{ R = I_1 \times \cdots \times I_k : w(R \cap \Omega_\ell) > \frac{1}{2} w(R), w(R \cap \Omega_{\ell+1}) \leq \frac{1}{2} w(R) \right\},$$

and then by Cauchy–Schwarz's inequality we have for any $0 < p \le 1$,

$$
\begin{aligned}
&\Big|\sum_{j_1,\dots,j_k}\sum_{I_1\times\cdots\times I_k} s_{I_1\times\cdots\times I_k}\,\bar{t}_{I_1\times\cdots\times I_k}\Big| \\
&\qquad = \Big|\sum_{j_1,\dots,j_k}\sum_{\ell}\sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} s_{I_1\times\cdots\times I_k}\,\bar{t}_{I_1\times\cdots\times I_k}\Big| \\
&\qquad \le \Bigg\{\sum_{\ell}\Bigg[\sum_{j_1,\dots,j_k}\sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |s_{I_1\times\cdots\times I_k}|^2\frac{w(I_1\times\cdots\times I_k)}{|I_1\times\cdots\times I_k|}\Bigg]^{\frac{p}{2}} \\
&\qquad\qquad \times\Bigg[\sum_{j_1,\dots,j_k}\sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |t_{I_1\times\cdots\times I_k}|^2\frac{|I_1\times\cdots\times I_k|}{w(I_1\times\cdots\times I_k)}\Bigg]^{\frac{p}{2}}\Bigg\}^{\frac{1}{p}} \\
&\qquad \le C\|t\|_{c_w^p}\Bigg\{\sum_{\ell} w(\Omega_\ell)^{1-\frac{p}{2}}\Bigg[\sum_{j_1,\dots,j_k}\sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |s_{I_1\times\cdots\times I_k}|^2 \\
&\qquad\qquad\qquad \times\frac{w(I_1\times\cdots\times I_k)}{|I_1\times\cdots\times I_k|}\Bigg]^{\frac{p}{2}}\Bigg\}^{\frac{1}{p}}, \qquad (4.1)
\end{aligned}
$$

where the last inequality follows from the fact that if $I_1\times\cdots\times I_k \in \mathcal{B}_\ell$, then there exists $0 < \theta < 1$ such that

$$I_1\times\cdots\times I_k \subseteq \widetilde{\Omega}_\ell := \{(x_1,\dots,x_k) : \mathcal{M}_s(\chi_{\Omega_\ell})(x_1,\dots,x_k) > \theta\},$$

which together with the fact $w(\widetilde{\Omega}_\ell) \le C\,w(\Omega_\ell)$ implies that

$$\Bigg\{\sum_{j_1,\dots,j_k}\sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |t_{I_1\times\cdots\times I_k}|^2\frac{|I_1\times\cdots\times I_k|}{w(I_1\times\cdots\times I_k)}\Bigg\}^{\frac{1}{2}} \le C\|t\|_{c_w^p} w(\Omega_\ell)^{\frac{1}{p}-\frac{1}{2}}.$$

We **claim** that

$$\sum_{j_1,\dots,j_k}\sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |s_{I_1\times\cdots\times I_k}|^2\frac{w(I_1\times\cdots\times I_k)}{|I_1\times\cdots\times I_k|} \le C2^{2\ell}w(\Omega_\ell).$$

Assume the claim for the moment; then

$$
\begin{aligned}
\Big|\sum_{j_1,\dots,j_k}\sum_{I_1\times\cdots\times I_k} s_{I_1\times\cdots\times I_k}\bar{t}_{I_1\times\cdots\times I_k}\Big| &\le C\|t\|_{c_w^p}\Big\{\sum_{\ell} w(\Omega_\ell)2^{\ell p}\Big\}^{\frac{1}{p}} \\
&\le C\|t\|_{c_w^p}\|h\|_{L_w^p} \le C\|t\|_{c_w^p}\|s\|_{s_w^p},
\end{aligned}
$$

and thus, $c_w^p \subseteq (s_w^p)^*$. To show the claim, we notice that

$$\int_{\widetilde{\Omega}_\ell \backslash \Omega_{\ell+1}} h^2\, w \le 2^{2(\ell+1)} w(\widetilde{\Omega}_\ell) \le C\, 2^{2\ell} w(\Omega_\ell),$$

so it is sufficient to show that

$$\sum_{j_1,\dots,j_k} \sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |s_{I_1\times\cdots\times I_k}|^2 \frac{w(I_1\times\cdots\times I_k)}{|I_1\times\cdots\times I_k|} \le C \int_{\widetilde{\Omega}_\ell \backslash \Omega_{\ell+1}} h^2\, w.$$

However,

$$\begin{aligned}
&\int_{\widetilde{\Omega}_\ell \backslash \Omega_{\ell+1}} h^2\, w \\
&\qquad = \int_{\tilde{\Omega}_\ell \backslash \Omega_{\ell+1}} \sum_{j_1,\dots,j_k} \sum_{I_1\times\cdots\times I_k} |s_{I_1\times\cdots\times I_k}|^2 \prod_{i=1}^k |I_i|^{-1}\, \chi_{I_i}\, w \\
&\qquad \ge \sum_{j_1,\dots,j_k} \sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |s_{I_1\times\cdots\times I_k}|^2 \frac{w((I_1\times\cdots\times I_k)\cap(\widetilde{\Omega}_\ell\backslash\Omega_{\ell+1}))}{|I_1\times\cdots\times I_k|} \\
&\qquad > \frac{1}{2} \sum_{j_1,\dots,j_k} \sum_{(I_1\times\cdots\times I_k)\in\mathcal{B}_\ell} |s_{I_1\times\cdots\times I_k}|^2 \frac{w(I_1\times\cdots\times I_k)}{|I_1\times\cdots\times I_k|},
\end{aligned}$$

where in the last inequality we follow from the fact that for $I_1\times\cdots\times I_k \in \mathcal{B}_\ell$,

$$w((I_1\times\cdots\times I_k)\cap\Omega_\ell) > \frac{1}{2} w(I_1\times\cdots\times I_k),$$
$$w((I_1\times\cdots\times I_k)\cap\Omega_{\ell+1}) \le \frac{1}{2} w(I_1\times\cdots\times I_k),$$

which yields that $I_1\times\cdots\times I_k \subseteq \tilde{\Omega}_\ell$, hence,

$$w((I_1\times\cdots\times I_k)\cap(\tilde{\Omega}_\ell\backslash\Omega_{\ell+1})) > \frac{1}{2} w(I_1\times\cdots\times I_k).$$

This proves the claim and thus we have showed that $c_w^p \subset (s_w^p)^*$.

Now we show that

$$(s_w^p)^* \subseteq c_w^p.$$

Let $\ell \in (s_w^p)^*$; then there exists some $t = \{t_{I_1\times\cdots\times I_k}\}$ such that

$$\ell(s) = \sum_{j_1,\dots,j_k} \sum_{I_1\times\cdots\times I_k} s_{I_1\times\cdots\times I_k} \bar{t}_{I_1\times\cdots\times I_k} \quad \text{for every } s = \{s_{I_1\times\cdots\times I_k}\} \in s_w^p,$$

For any open set $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ with finite weighted measure, and any sequence $s = \{s_{I_1 \times \cdots \times I_k}\} \in s_w^p$, we define

$$\|s\|_{s^p_{w,\Omega}} = \left\{ \int_\Omega \left(\sum_{j_1,\dots,j_k} \sum_{I_1 \times \cdots \times I_k \subseteq \Omega} |s_{I_1 \times \cdots \times I_k}|^2 \prod_{i=1}^k |I_i|^{-1} \chi_{I_i} \right)^{\frac{p}{2}} w(x)dx \right\}^{\frac{1}{p}}.$$

Then

$$\begin{aligned} \|s\|_{s^p_{w,\Omega}} &\le w(\Omega)^{\frac{1}{p}-\frac{1}{2}} \left\{ \int_\Omega \sum_{j_1,\dots,j_k} \sum_{I_1 \times \cdots \times I_k \subseteq \Omega} |s_{I_1 \times \cdots \times I_k}|^2 \right. \\ &\qquad\qquad \left. \times \prod_{i=1}^k |I_i|^{-1} \chi_{I_i} w(x)dx \right\}^{\frac{1}{2}} \\ &= w(\Omega)^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{j_1,\dots,j_k} \sum_{I_1 \times \cdots \times I_k \subseteq \Omega} |s_{I_1 \times \cdots \times I_k}|^2 \frac{w(I_1 \times \cdots \times I_k)}{|I_1 \times \cdots \times I_k|} \right)^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

To estimate the c_w^p norm of $t = \{t_{I_1 \times \cdots \times I_k}\}$, we introduce the notation $\ell^2_{w,\Omega}$ for $s = \{s_{I_1 \times \cdots \times I_k}\}$ and any open set $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ with finite weighted measure by

$$\|s\|_{\ell^2_{w,\Omega}} = \left(\sum_{j_1,\dots,j_k} \sum_{I_1 \times \cdots \times I_k \subseteq \Omega} |s_{I_1 \times \cdots \times I_k}|^2 \frac{w(I_1 \times \cdots \times I_k)}{|I_1 \times \cdots \times I_k|} \right)^{\frac{1}{2}}.$$

Thus,

$$\|s\|_{s^p_{w,\Omega}} \le C w(\Omega)^{\frac{1}{p}-\frac{1}{2}} \|s\|_{\ell^2_{w,\Omega}}.$$

Therefore,

$$\begin{aligned} &\left\{ \frac{1}{w(\Omega)^{\frac{2}{p}-1}} \sum_{j_1,\dots,j_k} \sum_{I_1 \times \cdots \times I_k \subseteq \Omega} |t_{I_1 \times \cdots \times I_k}|^2 \frac{|I_1 \times \cdots \times I_k|}{w(I_1 \times \cdots \times I_k)} \right\}^{\frac{1}{2}} \\ &\quad = \frac{1}{w(\Omega)^{\frac{1}{p}-\frac{1}{2}}} \sup_{\|s\|_{\ell^2_{w,\Omega}} \le 1} \left| \sum_{j_1,\dots,j_k} \sum_{I_1 \times \cdots \times I_k \subseteq \Omega} s_{I_1 \times \cdots \times I_k} \bar{t}_{I_1 \times \cdots \times I_k} \right| \\ &\quad \le \frac{1}{w(\Omega)^{\frac{1}{p}-\frac{1}{2}}} \|t\|_{(s_w^p)^*} \sup_{\|s\|_{\ell^2_{w,\Omega}} \le 1} \|s\|_{s^p_{w,\Omega}} \\ &\quad \le C \, \|t\|_{(s_w^p)^*}. \end{aligned}$$

Therefore, $t \in c_w^p$ and $\|t\|_{c_w^p} \le C \|t\|_{(s_w^p)^*}$. □

To show Theorem 1.5, we need Theorem 1.4 and the following lemma.

Lemma 4.1. *Let $w \in A_\infty$. For $0 < p < \infty$, define a map $\mathcal{L}$ on $(\mathcal{S}_\infty)'$ by*

$$\mathcal{L}(f) = \left\{ \prod_{i=1}^{k} |I_i|^{\frac{1}{2}} \psi_{j_1,\ldots,j_k} * f(2^{-j_1}\ell_1, \ldots, 2^{-j_k}\ell_k) \right\}_{I_1,\ldots,I_k}.$$

For any sequence $s = \{s_{I_1\times\cdots\times I_k}\}$, we also define the map $\mathcal{P}$ by

$$\mathcal{P}(s) = \sum_{I_1,\ldots,I_k} \prod_{i=1}^{k} |I_i|^{\frac{1}{2}} \psi_{j_1,\ldots,j_k}(2^{-j_1}\ell_1, \ldots, 2^{-j_k}\ell_k) s_{I_1\times\cdots\times I_k},$$

where I_i are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length 2^{-j_i} and the left lower corners of I_i are $2^{-j_i}\ell_i$, $\ell_i \in \mathbb{Z}^{n_i}$, $i = 1, \ldots, k$. Then the maps

$$\mathcal{L} : H^p_w \to s^p_w \quad \text{and} \quad \mathrm{CMO}^p_w \to c^p_w,$$

and

$$\mathcal{P} : s^p_w \to H^p_w \quad \text{and} \quad c^p_w \to \mathrm{CMO}^p_w$$

are bounded, and $\mathcal{P} \circ \mathcal{L}$ is the identity on H^p_w and CMO^p_w.

Proof. Obviously, there exists a constant $C > 0$ such that

$$\|\mathcal{L}(f)\|_{s^p_w} \le C\|f\|_{H^p_w}, \quad \|\mathcal{L}(f)\|_{c^p_w} \le C\|f\|_{\mathrm{CMO}^p_w}.$$

Let $s = \{s_{I_1\times\cdots\times I_k}\} \in s^p_w$; then

$$\|\mathcal{P}(s)\|_{H^p_w} \le C \left\| \left\{ \sum_{j_1,\ldots,j_k} \sum_{I_1,\ldots,I_k} |\psi_{j_1,\ldots,j_k} * \mathcal{P}(s)(2^{-j_1}\ell_1, \ldots, 2^{-j_k}\ell_k)|^2 \times \prod_{i=1}^{k} \chi_{I_i}(x_i) \right\}^{\frac{1}{2}} \right\|_{L^p_w}.$$

The almost orthogonality estimates in (2.6) and Proposition 2.2 imply that for any $0 < r < \min\{1, \frac{p}{q_w}\}$,

$$\begin{aligned}
&\left| \psi_{j_1,\ldots,j_k} * \mathcal{P}(s)(2^{-j_1}\ell_1, \ldots, 2^{-j_k}\ell_k) \prod_{i=1}^{k} \chi_{I_i}(x_i) \right|^2 \\
&= \Bigg| \sum_{I'_1,\ldots,I'_k} \prod_{i=1}^{k} |I'_i| \psi_{j_1,\ldots,j_k} * \psi_{j'_1,\ldots,j'_k}(2^{-j_1}\ell_1, \ldots, 2^{-j_k}\ell_k) s_{I'_1\times\cdots\times I'_k} \\
&\qquad \times \prod_{i=1}^{k} |I'_i|^{-\frac{1}{2}} \chi_{I_i}(x_i) \Bigg|^2
\end{aligned}$$

$$\leq C \sum_{j'_1,\ldots,j'_k} 2^{-|j_i-j'_i|L} \left\{ \mathcal{M}_s \left(\sum_{I'_1,\ldots,I'_k} |s_{I'_1\times\cdots\times I'_k} \prod_{i=1}^{k} |I'_i|^{-\frac{1}{2}} \chi_{I'_i} \right)^r \right\}^{\frac{2}{r}} (u_1^*,\ldots,u_k^*) \times \prod_{i=1}^{k} \chi_{I_i}(x_i),$$

where $u_i^* \in I_i$, $1 \leq i \leq k$. Since $w \in A_{q_w} \subseteq A_{p/r}$, by the $L_w^{p/r}(\ell^{2/r})$ boundedness of the strong maximal operator $\mathcal{M}_s$ and Hölder's inequality, we get the boundedness of the operator $\mathcal{P}$ from s_w^p to H_w^p. We can also obtain the boundedness of $\mathcal{P}$ from c_w^p to CMO_w^p by similar arguments to that in the proof of Theorem 1.2. The discrete Calderón identity in Proposition 2.1 could yield that $\mathcal{P} \circ \mathcal{L}$ is the identity on H_w^p and CMO_w^p. We omit the details here. □

Proof of Theorem 1.5. Let $f \in \mathcal{S}_\infty \cap H_w^p$ and $g \in \mathrm{CMO}_w^p$. For the map ℓ_g initially defined on $\mathcal{S}_\infty$, by the discrete Calderón reproducing formula in (2.1), Theorem 1.4 and Lemma 4.1,

$$\begin{aligned}
|\ell_g(f)| &= |\langle f, g\rangle| \\
&= \Bigg| \sum_{R=I_1\times\cdots\times I_k} \prod_{i=1}^{k} |I_i|^{\frac{1}{2}} \psi_R * f(2^{-j_1}\ell_1,\ldots,2^{-j_k}\ell_k) \\
&\qquad\qquad \times |I_i|^{\frac{1}{2}} \psi_R * g(2^{-j_1}\ell_1,\ldots,2^{-j_k}\ell_k) \Bigg| \\
&= |\langle \mathcal{L}(f), \mathcal{L}(g)\rangle| \\
&\leq \|\mathcal{L}(f)\|_{s_w^p} \|\mathcal{L}(g)\|_{c_w^p} \\
&\leq C\|f\|_{H_w^p} \|g\|_{\mathrm{CMO}_w^p}.
\end{aligned}$$

Since $\mathcal{S}_\infty$ is dense in H_w^p, we have from limiting arguments that the map ℓ_g can be extended to a continuous linear functional on the weighted Hardy spaces H_w^p and $\|\ell_g\| \leq C\|g\|_{\mathrm{CMO}_w^p}$.

Now we suppose $\ell \in (H_w^p)^*$ and set $\ell_1 = \ell \circ \mathcal{P}$. Then from Lemma 4.1,

$$|\ell_1(f)| = |\ell(\mathcal{P}(f))| \leq \|\ell\| \|\mathcal{P}(f)\|_{H_w^p} \leq C\|\ell\| \|f\|_{s_w^p} \quad \text{for } f \in s_w^p,$$

which implies $\ell_1 \in (s_w^p)^*$. Then by Theorem 1.4, there exists $t = \{t_{I_1\times\cdots\times I_k}\} \in c_w^p$ such that

$$\ell_1(s) = \sum_{I_1\times\cdots\times I_k} s_{I_1\times\cdots\times I_k} \bar{t}_{I_1\times\cdots\times I_k} \quad \text{for all } s = \{s_{I_1\times\cdots\times I_k}\} \in s_w^p,$$

and

$$\|t\|_{c_w^p} \approx \|\ell_1\| \leq C\|\ell\|.$$

Since $\mathcal{P} \circ \mathcal{L}$ is an identity on H^p_w from Lemma 4.1, we have $\ell = \ell \circ \mathcal{P} \circ \mathcal{L} = \ell_1 \circ \mathcal{L}$ and

$$\ell(f) = \ell_1(\mathcal{L}(f)) = \langle \mathcal{L}(f), t \rangle = \langle f, g \rangle,$$

where

$$g := \sum_{I_1 \times \cdots \times I_k} \prod_{i=1}^{k} |I_i|^{\frac{1}{2}} \, t_{I_1 \times \cdots \times I_k} \, \psi_R(x_{I_1} - x_1, x_{I_2} - x_2, x_{I_k} - x_k) = \mathcal{P}(t).$$

This shows that $\ell = \ell_g$ and by Lemma 4.1, $\|g\|_{\mathrm{CMO}^p_w} \leq C \|t\|_{c^p_w} \leq C \|\ell_g\|$. □

Proof of Theorem 1.7. Since H^1_w is a Banach space, we may use the duality argument to show our result by applying Theorem 1.5 and Proposition 2.3. □

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