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Viscosity solutions to inhomogeneous Aronsson's equations involving Hamiltonians $\langle A(x)p, p\rangle$

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Abstract

We consider inhomogeneous Aronsson's equation

$$\langle D\langle ADu, Du\rangle, ADu\rangle = f \quad \text{in } U, \tag{0.1}$$

where U is a bounded domain of $\mathbb{R}^n$ with $n \geq 2$, $A \in C^1(U; \mathbb{R}^{n\times n})$ is symmetric and uniformly elliptic, and $f \in C(U)$. First, we establish the existence and uniqueness of viscosity solutions for the corresponding Dirichlet problem on subdomains. Then we obtain the local Lipschitz regularity and the linear approximation property of viscosity solutions to (0.1). Moreover, under additional assumptions that $A \in C^{1,1}(U; \mathbb{R}^{n\times n})$ and $f \in C^{0,1}(U)$, we prove the everywhere differentiability of viscosity solutions to (0.1).

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Contents

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1 Introduction

Let $n \geq 2$ and U be a bounded domain (open connected subset) of $\mathbb{R}^n$. In 1960's, Aronsson [3–6] initiated the study of the infinity Laplace equation

$$\Delta_\infty u := \langle D^2 u Du, Du\rangle = 0 \quad \text{in} \quad U \tag{1.1}$$

by deriving it as the Euler–Lagrange equation of absolute minimizers for the L^∞-functional $\operatorname{esssup}_U |Du|^2$. Obviously, Δ_∞ is a highly degenerated nonlinear second order differential operator. Viscosity solutions to (1.1) are called infinity harmonic functions. In 1993, Jensen in the seminal paper [19] identified absolute minimizers with infinity harmonic functions, and further obtained their uniqueness under Dirichlet boundary; see also [1,7,11,13,26] for different proofs. The regularity of infinity harmonic functions is a challenge problem. In 2001, Crandall et al. [9,10] first obtained the linear approximation property (see (1.5) below). Based on this, when $n = 2$, the interior C^1-regularity was proved by Savin [27] in 2005, the interior $C^{1,\alpha}$-regularity by Evans–Savin [14] and the boundary C^1-regularity by Wang–Yu [29] later. When $n \geq 3$, the interior everywhere differentiability was proved by Evans–Smart [15,16] and the boundary everywhere differentiability by Wang–Yu [29] recently; but the C^1- and $C^{1,\alpha}$-regularity are still open.

In 2008, Lu–Wang [24] considered inhomogeneous infinity Laplace equation

$$\Delta_\infty u = f \quad \text{in} \quad U. \tag{1.2}$$

When $f \in C(U)$ is bounded and $|f| > 0$, they [24] obtained the existence and uniqueness of viscosity solutions to (1.2) under Dirichlet boundary. Counter-example was constructed there to show that the uniqueness may fail if f changes sign. Meanwhile, similar results were also established for inhomogeneous normalized infinity Laplace equation by Lu–Wang [23], Peres et al. [26] and also Armstrong–Smart [2]. Note that, under $f \geq 0$ or $f \leq 0$, the uniqueness for Dirichlet problems corresponding to (1.2) or the normalized equation is open. Recently, when $f \in C^1(U)$, Lindgren [21] proved everywhere differentiability of viscosity solutions to (1.2); but the C^1-regularity is unknown even when $n = 2$.

We are interested in the Aronsson's equation

$$\mathscr{A}_H[u] := \frac{1}{2}\langle D_x[H(x, Du)], D_p H(x, Du)\rangle = f \quad \text{in } U. \tag{1.3}$$

As always, we assume unless specified otherwise, that $f \in C(U)$ and the Hamiltonian $H(x, p) = \langle A(x)p, p\rangle$ with $A \in C^1(U; \mathbb{R}^{n\times n})$ being symmetric and uniformly elliptic, that is,

$$\frac{1}{L}|p|^2 \leq H(x, p) \leq L|p|^2 \quad \forall x \in U \text{ and } p \in \mathbb{R}^n \tag{1.4}$$

for some constant $L \geq 1$. Note that $A \in C^1(U; \mathbb{R}^{n\times n})$ and $f \in C(U)$ are the most natural (minimal in some sense) regularity on A and f required to define viscosity solutions to (1.3), see Sect. 2. If $A = I_n$, then $\mathscr{A}_H$ is exactly the same as $2\Delta_\infty$.

The homogeneous Aronsson's equation $\mathscr{A}_H[u] = 0$ in U (that is, (1.3) with $f \equiv 0$) has been studied in the literature. Indeed, viscosity solutions in this case are identified with absolute minimizers for the L^∞-functional $\operatorname{esssup}_U H(\cdot, Du)$ as proved by Barron et al. [8] and Yu [30] (see Sect. 2 below). The existence and uniqueness of absolute minimizers, and hence viscosity solutions, under Dirichlet boundary were established in [8,20,26]; the linear approximation property in [20,31]. Recently, under $A \in C^{1,1}(U; \mathbb{R}^{n\times n})$, viscosity solutions are differentiable everywhere as shown in [25]; but under merely $A \in C^1(U; \mathbb{R}^{n\times n})$, everywhere differentiability is unknown.

This paper focuses on the inhomogeneous Aronsson's equation (1.3) with $f \not\equiv 0$. First we have the following existence and uniqueness.

Theorem 1.1 *Suppose that $A \in C^1(U)$ is symmetric and uniformly elliptic. Let $V \Subset U$ and $f \in C(V)$ be bounded and satisfy $|f| > 0$ in V. For arbitrary $g \in C(\partial V)$, there exists a unique viscosity solution $u \in C(\overline{V})$ to the Dirichlet problem:*

$$\mathscr{A}_H[u] = f \quad \text{in} \quad V; \ u = g \quad \text{on } \partial V.$$

Next we prove the following local Lipschitz regularity and linear approximation property. By the linear approximation property, we mean that for every $x \in U$ and every sequence $\{r_j\}_{j\in\mathbb{N}}$ that converges to 0, there exist a subsequence $\mathbf{r} = \{r_{j_k}\}_{k\in\mathbb{N}}$ and a vector $\mathbf{e}_{x,\mathbf{r}}$ such that $H(x, \mathbf{e}_{x,\mathbf{r}}) = \operatorname{Lip}_{d_A} u(x)$ and

$$\lim_{k\to\infty} \max_{y\in K} \left| \frac{u(x + r_{j_k} y) - u(x)}{r_{j_k}} - \langle \mathbf{e}_{x,\mathbf{r}}, y\rangle \right| = 0 \quad \forall \text{ compact set } K \subset U. \tag{1.5}$$

See Sect. 3 for the intrinsic distance d_A and the pointwise Lipschitz constant $\operatorname{Lip}_{d_A} u(x)$.

Theorem 1.2 *Suppose that $A \in C^1(U)$ is symmetric and uniformly elliptic, and $f \in C(U)$. If $u \in C(U)$ is a viscosity solution to (1.3), then $u \in C^{0,1}(U)$ and enjoys the linear approximation property.*

Finally, we obtain the everywhere differentiability. Observe that everywhere differentiability always implies the linear approximation property; but the converse is not necessarily true even when $A = I_n$.

Theorem 1.3 *Suppose that $A \in C^{1,1}(U)$ is symmetric and uniformly elliptic and $f \in C^{0,1}(U)$. If $u \in C(U)$ is a viscosity solution to (1.3), then u is differentiable everywhere.*

The proofs of Theorems 1.1–1.3 heavily rely on some careful analysis of the intrinsic distance d_A determined by A and uniform estimates of solutions to approximation equations $\mathscr{A}_H[u] + \epsilon \operatorname{div}(ADu) = f$. In particular, when $A \neq I_n$, since the intrinsic distance d_A loses some important properties which hold for the Euclidean distance and play crucial roles in the case $A = I_n$ (that is, $\Delta_\infty u = f$), new ideas are required. The proofs are organized as below.

Section 3 is devoted to the analysis of the intrinsic distance d_A. Set $d_{A,x^0} = d_A(x^0, \cdot)$ for $x^0 \in U$. For $\lambda > 0$ and $x^0 \in U$, let $\mathcal{L}^\lambda_{A,x^0}$ be some viscosity solution to the Hamilton–Jacobi equation

$$\langle ADu, Du\rangle + \lambda u = 1 \quad \text{in } U\backslash\{x^0\}; \quad u(x^0) = 0.$$

The following properties obtained in Lemmas 3.1–3.3 will be useful below:

(i) $\lim_{\lambda\to 0} \mathcal{L}^\lambda_{A,x^0} = d_{A,x^0}$ locally uniformly in U,

(ii) $e^{-4\lambda d_{A,x^0}} d_{A,x^0} \le \mathcal{L}^\lambda_{A,x^0} \le d_{A,x^0}$ if $\lambda d_{A,x^0} < \ln\sqrt{2}$,

(iii) $\mathscr{A}_H[-\mathcal{L}^{\lambda}_{A,x^0}] \ge \frac{\lambda}{2}$ in $V \Subset U$ in viscosity sense if $x^0 \in \partial V$ and $\lambda\, \mathrm{diam}_A V \le 1/2$,
(iv) $\mathscr{A}_H[d_{A,x^0}] \le 0$ in $U\backslash\{x^0\}$ in viscosity sense.

In Sects. 4 and 5, we prove Theorem 1.1 under $f > 0$ (and hence under $f < 0$). The uniqueness is proved by using some ideas from [12,24], see Theorem 4.1 and Lemma 4.2. Note that $A \in C^1(U)$ and $f \in C(U)$ is the minimal regularity required here. To prove the existence (see Theorem 5.1), Lemmas 3.1–3.3 allow us to use Perron's approach. Indeed, the existence of viscosity sub-solutions follows from $\mathscr{A}_H[-\mathcal{L}^{\lambda}_{A,x^0}] \ge \frac{\lambda}{2}$ for large $\lambda > 0$. Moreover, to show that boundaries of the supremum of all sub-solutions and the infimum of all sup-solutions are the same as the given boundary, we need some barrier functions v, w so that

$$\mathscr{A}_H[v] \le 0 \quad \text{and} \quad \mathscr{A}_H[w] \ge 1 \quad \text{in } V$$

in viscosity sense. By Lemmas 3.1–3.3, we may take $v = d_{A,x^0}$ and $w = -\mathcal{L}^{\lambda}_{A,x^0}$ for some large $\lambda > 0$. Recall that in the case $A = I_n$ (that is, $\Delta_\infty u = f$), Lu and Wang [24] take $w(x) = C|x-x^0|^{4/3}$ since $\Delta_\infty[|x-x^0|^{4/3}] = 4^3/3^4$. But when $A \ne I_n$, $\mathscr{A}_H[d^{4/3}_{A,x^0}] \ge 4^3/3^4$ is not available.

Theorem 1.2 (that is, Theorem 6.1 below) is proved in Sect. 6. The proof relies on a key monotonicity of maps $r \to S^{\pm}_{\mathcal{L}^{\lambda}_A,r}(u)(x)$ for large $\lambda > 0$, see Lemma 6.2 for details. The idea here is that, instead of the slope $S^{\pm}_{A,r}(u)(x)$ with respect to d_A, we consider $S^{\pm}_{\mathcal{L}^{\lambda}_A,r}(u)(x)$ which is defined in the same way as $S^{\pm}_{A,r}(u)(x)$ by replacing d_A there with $\mathcal{L}^{\lambda}_A$ above. This monotonicity follows from Lemmas 3.1–3.3 ($\mathscr{A}_H[\mathcal{L}^{\lambda}_{A,x^0}] \le -\lambda/2$) and the comparison principle in Lemma 4.2. Comparing with the monotonicity of maps $r \to S^{\pm}_{A,r}(u)(x)$ in the case $f \equiv 0$ (that is $\mathcal{A}_H[u] = 0$, see [20]), we see that $\mathcal{L}^{\lambda}_A$ plays the role of d_A in some sense. We also recall the monotonicity of maps $r \to S^{\pm}_{A,r}(u)(x) + r$ in the case $A_n = I_n$ (that is, $\Delta_\infty u = f$ see [21]), whose proof relies on the fact that

$$\Delta_\infty |x|^\gamma \le \gamma^3(\gamma-1)|x|^{3\gamma-4} < 0 \quad \text{for} \quad \gamma \in (0,1)$$

in viscosity sense. When $A \ne I_n$, similar properties for d^{γ}_{A,x^0} with $\gamma \in (0,1)$, and hence the monotonicity of the maps $r \to S^{\pm}_{A,r}(u)(x) + r$, are not available.

Sections 7 and 8 are contributed to the proof of Theorem 1.3 (that is, Theorem 8.1 below). With the aid of Theorem 1.2, we can use the approach in [16] (see also [21,28]) by overcoming several technical difficulties. Firstly, under $A \in C^{1,1}(U;\mathbb{R}^{n\times n})$ and $f \in C^{0,1}(U)$ with $f > 0$, with the aid of uniqueness in Sect. 4 we approximate the viscosity solution u to (1.3) in $V = B(0,3) \Subset U$ by u^ϵ —smooth solutions to

$$\mathscr{A}_{H^\epsilon}[u^\epsilon] + \epsilon\, \mathrm{div}\,(A^\epsilon Du^\epsilon) = f^\epsilon \text{ in } V; \quad u^\epsilon|_{\partial V} = g^\epsilon,$$

where $A^\epsilon, f^\epsilon, g^\epsilon$ are smooth approximations of A, f, u and $H^\epsilon(x,p) = \langle A^\epsilon(x)p, p\rangle$; see Lemma 8.2. Note that the required smoothness of u^ϵ, uniform estimates and uniform boundary regularity estimates of $|u^\epsilon|$, and locally uniform estimates of $|Du^\epsilon|$ are established in Lemmas 7.1–7.3. Secondly, observe that, after some suitable scaling, we may assume that $\|u(x) - u(0) - x_n\|_{L^\infty(B(0,2))}$, $A(0) = I_n$ and $\|DA\|_{L^\infty(V)} + \|D^2A\|_{L^\infty(V)} + \|Df\|_{L^\infty(V)}$ are sufficiently small. This allows us to build up a uniform flat estimate for $|Du^\epsilon|^2 - u^\epsilon_n$ as did in Lemma 7.4. Finally, via such flat estimates and the linear approximation property in Theorem 1.2, an argument similar to [16,21,28] leads to everywhere differentiability of u.

2 Viscosity solutions

We first recall the notion of viscosity (sub-/sup-)solutions.

Let U be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. For continuous functions $F : U \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$, we consider equations

$$F(\cdot, u, Du, D^2u) = 0 \quad \text{in } U. \tag{2.1}$$

Definition 2.1 (i) A function u is called a *viscosity sub-solution* to (2.1) if for every $x^0 \in U$, we have

$$F(x^0, \varphi(x^0), D\phi(x^0), D^2\phi(x^0)) \geq 0$$

whenever $\phi \in C^2(U)$ and $u - \phi$ attains its local maximum at x^0.

(ii) A function u is called a *viscosity sup-solution* to (2.1) if for every $x^0 \in U$, we have

$$F(x^0, \varphi(x^0), D\phi(x^0), D^2\phi(x^0)) \leq 0$$

whenever $\phi \in C^2(U)$ and $u - \phi$ attains its local minimal at x^0.

(iii) A function u is called a viscosity solution to (2.1) if it is a viscosity sub-solution and also a viscosity sup-solution.

As always, we assume without otherwise specified, that $A = (a^{ij})_{i,j=1}^n \in C^1(U; \mathbb{R}^{n\times n})$ is symmetric and uniformly elliptic, and $f \in C(U)$. Write $H(x, p) := \langle A(x)p, p\rangle$ for $x \in U$ and $p \in \mathbb{R}^n$, and the Aronsson operator

$$\mathscr{A}_H[u](x) := \frac{1}{2}\langle D_x H(x, Du), D_p H(x, Du)\rangle = \langle D\langle A(x)Du, Du\rangle, A(x)Du\rangle$$

being as in (1.3). For $\epsilon \geq 0$, consider equations

$$\mathscr{A}_H[u] + \epsilon \operatorname{div}(ADu) = f \quad \text{in } U. \tag{2.2}$$

If $\epsilon = 0$, this is exactly the Aronsson equation (1.3); if $\epsilon > 0$, we call them as the approximation equations of (1.3).

The viscosity (sub-/sup-)solutions to (2.1) are defined via Definition 2.1. Indeed, for $\epsilon \geq 0$, set

$$F_\epsilon(x, p, X) = 2[a^{ik}(x)p_k a^{j\ell}(x)p_\ell + \epsilon a^{ij}]X_{ij} + a_s^{ik}(x)p_k a^{i\ell}(x)p_\ell p_s + \epsilon a_i^{ij} p_j - f(x)$$

where $p = (p_i)_{i=1}^n$, $X = (X_{ij})_{i,j=1}^n$ and $a_k^{ij} = \frac{\partial}{\partial x_k} a^{ij}$. Here and below, to simplify the presentation, we will use the Einstein summation convention, that is, $a_i b^i = \sum_{i=1}^n a^i b_i$. Note that $A \in C^1(U; \mathbb{R}^{n\times n})$ and $f \in C(U)$ are the minimal regularity on A and f required to guarantee the continuity of $F_\epsilon : U \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ for $\epsilon \geq 0$, in particular $\epsilon = 0$. Observe that

$$\mathscr{A}_H[u](x) + \epsilon \operatorname{div}(A(x)Du(x)) - f(x) = F_\epsilon(x, Du(x), D^2u(x)) \quad \text{in } U$$

whenever $u \in C^2(U)$. Thereby, we define the viscosity (sub-/sup-)solutions to (2.1) as those of equations $F_\epsilon(\cdot, Du, D^2u) = 0$ as in Definition 2.1 correspondingly.

In a similar way, for $\lambda \geq 0$ we define the viscosity (sub-/sup-)solutions to the Hamilton–Jacobi equation

$$H(x, Du(x)) + \lambda u(x) = 1 \quad \text{in } U$$

as those of $\widetilde{F}_\lambda(x, u, Du) = 0$ correspondingly, where

$$\widetilde{F}_\lambda(x, r, p) = H(x, p) + \lambda r - 1.$$

Observe that $A \in C^1(U; \mathbb{R}^{n\times n})$ guarantees the continuity of $\widetilde{F}_\lambda : U \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ for $\lambda > 0$, and that

$$H(x, Du(x)) + \lambda u - 1 = \widetilde{F}_\lambda(x, u(x), Du(x))$$

whenever $u \in C^1(U)$.

Finally, we recall the identification between viscosity (sub-/sup-)solutions to $\mathscr{A}_H[u] = 0$ with absolute (sub-/sup-)minimizers of L^∞-functional $\operatorname{esssup}_U H(\cdot, Du)$.

Definition 2.2 (i) A function $u \in C^{0,1}(U)$ is called an absolute sub-minimizer in U for H, if for each $V \Subset U$, $v \in C^{0,1}(V) \cap C(\overline{V})$ satisfies $v \le u$ in V, and $v = u$ on ∂V, then

$$\operatorname{esssup}_{x\in V} H(x, Du(x)) \le \operatorname{esssup}_{x\in V} H(x, Dv(x)).$$

(ii) A function $u \in C^{0,1}(U)$ is called an absolute sup-minimizer in U for H if $-u$ is an absolute sub-minimizer in U for H.

(iii) A function $u \in C^{0,1}(U)$ is called an absolute minimizer in U for H, if it is both an absolute sub-minimizer and an absolute sup-minimizer in U for H.

Denote by $USC(U)$ (resp. $LSC(U)$) the collection of all upper (resp. lower) semi-continuous functions u on U.

Lemma 2.3 *The following are equivalent:*

(i) $u \in C(U)$ ($u \in USC(U)$/$u \in LSC(U)$) is a viscosity (sub-/sup-)solution to $\mathscr{A}_H[u] = 0$ in U

(ii) $u \in C^{0,1}(U)$ is an absolute (sub-/sup-)minimizer in U for H.

The proof of (ii)⇒(i) was given by Crandall et al. [13]. When $A \in C^2(U)$, Yu [30] clearly proved (i)⇒(ii); when $A \in C^1(U)$, (i)⇒(ii) also follows from the arguments in [30] as informed by Yifeng Yu (personal communication).

As a consequence of Lemma 2.3, we obtain the following result.

Lemma 2.4 *If $u \in C(U)$ ($u \in USC(U)$/$u \in LSC(U)$) is a viscosity (sub-/sup-)solution to $\mathscr{A}_H[u] = f$ in U, then $u \in C^{0,1}_{\mathrm{loc}}(U)$.*

Proof Consider $\widetilde{u}(\widetilde{x}) = u(x) + C|x_{n+1}|^{4/3}$ for $\widetilde{x} = (x, x_{n+1}) \in U \times \mathbb{R}$ (see e.g. [18, Theorem 1]). Then $u \in C(U)$ ($u \in USC(U)$/$u \in LSC(U)$) implies that $\widetilde{u} \in C(U \times \mathbb{R})$ ($\widetilde{u} \in USC(U\times\mathbb{R})$/$\widetilde{u} \in LSC(U\times\mathbb{R})$). Moreover, since u is a viscosity (sub-/sup-)solution to $\mathscr{A}_H[u] = f$ in U, we know that $\widetilde{u}$ is a viscosity (sub-/sup-)solution to $\mathscr{A}_{\widetilde{H}}[\widetilde{u}] = \widetilde{f}$ in $U\times\mathbb{R}^n$, where $\widetilde{f}(\widetilde{x}) = f(x) + C^3 4^3/3^4$ and $\widetilde{H}(\widetilde{x}, p) = \langle \widetilde{A}(\widetilde{x})p, p\rangle$ with $\widetilde{A}(\widetilde{x}) = \operatorname{diag}\{A(x), 1\}$ for all $\widetilde{x} \in U\times\mathbb{R}$ and $p \in \mathbb{R}^{n+1}$. For any $V \Subset U$, if $4C/3^{4/3} > \|f\|_{C(\overline{V})}$, then $\widetilde{f} > 0$ in $V\times\mathbb{R}$, and hence by Lemma 2.3, $\widetilde{u} \in C^{0,1}(V\times\mathbb{R})$. This implies that $u \in C^{0,1}(U)$ as desired. □

3 Intrinsic distance

We always assume that $A \in C^1(U; \mathbb{R}^{n\times n})$ is symmetric and uniformly elliptic in this section. Define the intrinsic distance d_A by

$$d_A(x, y) := \inf\left\{\left(\int_0^1 \langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\, ds\right)^{1/2} \Big| \xi \in \mathcal{C}(0, 1; x, y; U)\right\} \quad \forall x,\ y \in \overline{U}. \tag{3.1}$$

Here and below, for $t > 0$, denote by $\mathcal{C}(0, t; x, y; U)$ all rectifiable curves $\xi : [0, t] \to U$ joining x, y; and by $\mathcal{C}^1(0, t; x, y; U)$ all $\xi \in C^1([0, t]) \cap \mathcal{C}(0, t; x, y; U)$. The uniform ellipticity implies that d_A is a distance and comparable with the Euclidean distance locally. For all $x \in U$, set

$$\operatorname{dist}_A(x, \partial U) := \min\{d_A(x, y) | y \in \partial U\}$$

and

$$B_A(x, r) := \{y \in U | d_A(x, y) < r\} \quad \text{if } r < \operatorname{dist}_A(x, \partial U).$$

For $K \subset U$, write

$$\operatorname{diam}_A K := \sup\{d_A(x, y) | x, y \in K\}.$$

Denote by $\operatorname{Lip}_{d_A} u(x)$ the pointwise Lipschitz constant, that is,

$$\operatorname{Lip}_{d_A} u(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_A(x, y)}.$$

When d_A is the Euclidean distance $|\cdot - \cdot|$, we define $\operatorname{dist}(x, \partial U)$, $B(x, r)$, $\operatorname{diam} K$ and $\operatorname{Lip} u$ correspondingly. Note that when $A = I_n$, one has $d_A(x, y) = |x - y|$ whenever $|x - y| \le \operatorname{dist}(x, \partial U)$, but $d_A(x, y)$ may be strictly larger than $|x - y|$ when $|x - y| > \operatorname{dist}(x, \partial U)$.

Below we consider an approximation of the intrinsic distance, which has several nice properties. For $\lambda \ge 0$, define

$$\mathcal{L}_A^\lambda(x, y) := \inf\left\{\int_0^t \left[1 + \frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] e^{-\lambda(t-s)}\, ds \,\middle|\, t > 0, \xi \in \mathcal{C}(0, t; x, y; U)\right\}$$

for all $x, y \in \overline{U}$. The following Lemmas 3.1–3.3 are crucial in this paper.

Lemma 3.1 *For all $\lambda > 0$ and $x, y \in U$, we have*

$$0 \le \mathcal{L}_A^\lambda(x, y) \le \mathcal{L}_A^0(x, y) = d_A(x, y) \tag{3.2}$$

and

$$d_A(x, y) \le \mathcal{L}_A^\lambda(x, y) e^{4\lambda \mathcal{L}_A^\lambda(x, y)} \quad \text{whenever } \lambda \mathcal{L}_A^\lambda(x, y) < \ln \sqrt{2}. \tag{3.3}$$

Proof *Proof of* (3.2). Obviously, $0 \le \mathcal{L}_A^\lambda \le \mathcal{L}_A^0$ for all $\lambda > 0$. To see (3.2), it suffices to prove $\mathcal{L}_A^0 = d_A$. By the change of variables we have

$$\frac{1}{t} d_A^2(x, y) = \inf\left\{\int_0^t \langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\, ds \,\middle|\, \xi \in \mathcal{C}(0, t; x, y; U)\right\} \quad \forall t > 0, x, y \in U.$$

Thus,

$$\mathcal{L}_A^0(x, y) \le \inf_{t > 0}\left\{t + \frac{d_A^2(x, y)}{4t}\right\} \le d_A(x, y) \quad \forall x, y \in U,$$

where we choose $t = d_A(x, y)/2$.

On the other hand, we claim that

$$d_A(x, y) = \inf\left\{\int_0^t \sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}\, ds \,\middle|\, t > 0, \xi \in \mathcal{C}(0, t; x, y; U)\right\} \; \forall x, y \in U. \tag{3.4}$$

The claim (3.4) is known to be true by a standard reparametrization argument; for reader's convenience we give the details at the end of the proof of Lemma 3.1. Assume that (3.4) holds for the moment. Observe that for all $x \in U$, $q \in \mathbb{R}^n$ and $\sigma > 0$, we have

$$\sup_{\langle A(x)p,p\rangle \le \sigma} p \cdot q = \sup_{|p|\le\sqrt{\sigma}} \langle p, A(x)^{-1/2}q\rangle = \sqrt{\sigma}|A(x)^{-1/2}q|$$

and hence

$$\begin{aligned}\frac{1}{4}\langle A^{-1}(x)q, q\rangle &= \sup_{p\in\mathbb{R}^n}\{p\cdot q - \langle A(x)p, p\rangle\}\\ &= \sup_{\sigma\ge 0}\sup_{\langle A(x)p,p\rangle\le\sigma}\{p\cdot q-\sigma\}\\ &= \sup_{\sigma\ge 0}\{\sqrt{\sigma}|A(x)^{-1/2}q| - \sigma\}\\ &\ge |A(x)^{-1/2}q| - 1,\end{aligned}$$

that is, $\sqrt{\langle A^{-1}(x)q, q\rangle} \le \frac{1}{4}\langle A^{-1}(x)q, q\rangle + 1$. Therefore, by (3.4) we have

$$d_A(x, y) \le \inf\left\{\int_0^t \left[\frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle + 1\right] \mathrm{d}s \middle| t > 0,\ \xi \in \mathcal{C}(0, t; x, y; U)\right\} \tag{3.5}$$

which gives $d_A(x, y) \le \mathcal{L}_A^0(x, y)$ for all $x, y \in U$, as desired.

Proof of (3.3). Assume that $0 < \lambda\mathcal{L}_A^\lambda(x, y) < \ln\sqrt{2}$. For any $\epsilon > 0$ with $(1+\epsilon)\lambda\mathcal{L}_A^\lambda(x, y) \le \ln\sqrt{2}$, there exists $\xi \in \mathcal{C}(0, T; x, y, U)$ for some $T > 0$ such that

$$(1+\epsilon)\mathcal{L}_A^\lambda(x, y) \ge \int_0^T \left[1 + \frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] e^{-\lambda(T-s)}ds.$$

This implies that

$$(1+\epsilon)\mathcal{L}_A^\lambda(x, y) \ge \int_0^T e^{-\lambda(T-s)}ds,$$

which together with $(1+\epsilon)\lambda\mathcal{L}_A^\lambda(x, y) \le \ln\sqrt{2}$ gives

$$T \le \frac{-1}{\lambda}\ln[1-\lambda(1+\epsilon)\mathcal{L}_A^\lambda(x, y)] \le 2(1+\epsilon)\mathcal{L}_A^\lambda(x, y).$$

Hence, for all $s \in (0, T)$,

$$e^{4\lambda(1+\epsilon)\mathcal{L}_A^\lambda(x,y)}e^{-\lambda(T-s)} \ge e^{\lambda(T-s)} \ge 1,$$

which together with (3.5) leads to that

$$e^{4\lambda(1+\epsilon)\mathcal{L}_A^\lambda(x,y)}(1+\epsilon)\mathcal{L}_A^\lambda(x, y) \ge \int_0^T \left[1 + \frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] ds \ge d_A(x, y).$$

Sending $\epsilon \to 0$, we have

$$e^{4\lambda\mathcal{L}_A^\lambda(x,y)}\mathcal{L}_A^\lambda(x, y) \ge d_A(x, y),$$

that is, (3.3) holds.

Proof of the claim (3.4). Let

$$\widetilde{d}_A(x, y) := \inf\left\{\int_0^1 \sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}\, ds \middle| \xi \in \mathcal{C}(0, 1; x, y; U)\right\} \quad \forall x, y \in U.$$

By a change of variable, we have

$$\widetilde{d}_A(x,y) = \inf\left\{ \int_0^t \sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}\, ds \,\middle|\, t>0, \xi \in \mathcal{C}(0,t;x,y;U) \right\}.$$

Thus, to prove the claim (3.4), we only need to prove that $\widetilde{d}_A(x,y) = d_A(x,y)$ for all $x, y \in U$. By Hölder's inequality, we see that

$$\int_0^1 \sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}\, ds \le \left(\int_0^1 \langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\, ds \right)^{1/2} \quad \forall \xi \in \mathcal{C}(0,1;x,y;U)$$

and hence, $\widetilde{d}_A(x,y) \le d_A(x,y)$. To see $d_A(x,y) \le \widetilde{d}_A(x,y)$, for any $\epsilon > 0$ let $\xi \in \mathcal{C}(0,1;x,y;U)$ such that

$$L = \int_0^1 \sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}\, ds \le \widetilde{d}_A(x,y) + \epsilon.$$

Up to a standard smooth modification, we may assume that $\xi \in \mathcal{C}^1(0,1;x,y;U)$. It then suffices to find a reparametrization $\eta \in \mathcal{C}(0,1;x,y;U)$ of ξ so that

$$\langle A^{-1}(\eta(s))\dot{\eta}(s), \dot{\eta}(s)\rangle = L \quad \text{for almost all } s \in [0,1].$$

Indeed, this implies that

$$[d_A(x,y)]^2 \le \int_0^1 \langle A^{-1}(\eta(s))\dot{\eta}(s), \dot{\eta}(s)\rangle\, ds = L^2 \le [\widetilde{d}_A(x,y) + \epsilon]^2.$$

Letting $\epsilon \to 0$, we obtain $d_A(x,y) \le \widetilde{d}_A(x,y)$ as desired.

Finally, we find the reparametrization $\eta \in \mathcal{C}(0,1;x,y;U)$ of ξ required as above. If $|\dot{\xi}| > 0$ almost everywhere in $[0,1]$, then define

$$\psi(r) = \frac{1}{L}\int_0^r \sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}\, ds \quad \forall r \in (0,1].$$

Obviously, ψ is a strictly increasing continuous function from $[0,1]$ to $[0,1]$. Set $\eta(t) = \xi(\psi^{-1}(t))$ for $t \in [0,1]$. One has $\eta \in \mathcal{C}(0,1;x,y;U)$ and

$$\dot{\eta}(t) = \dot{\xi}(\psi^{-1}(t))(\psi^{-1})'(t) = \frac{\dot{\xi}(\psi^{-1}(t))}{\dot{\psi}(\psi^{-1}(t))} \text{ for almost all } t \in [0,1].$$

Since $\dot{\psi}(s) = \frac{1}{L}\sqrt{\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle}$ for all $s \in [0,1]$, we attain

$$\sqrt{\langle A^{-1}(\eta(t))\dot{\eta}(t), \dot{\eta}(t)\rangle} = L \text{ for almost all } t \in [0,1]$$

as desired.

In general, $\dot{\xi}$ may vanish in a set with positive measure in $[0,1]$. By an argument similar to above, it suffices to find a reparametrization $\widetilde{\xi} \in \mathcal{C}^1(0,a;x,y;U)$ of ξ for some $a > 0$ and $\dot{\widetilde{\xi}} > 0$ almost everywhere in $[0,a]$. This is done by removing all open sub-intervals of $[0,1]$ where $\dot{\xi}$ vanishes. Precisely, since $\dot{\xi}$ is continuous, the set $I = \{s \in [0,1] : |\dot{\xi}(s)| > 0\}$ is open (relative to $[0,1]$). The open set $(0,1)\backslash\overline{I}$ is the union of at most countable many open intervals $I_j = (a_j, b_j)$ so that $a_j < b_j < a_{j+1}$ for all possible j. For each j, we know that $\dot{\xi}$ vanishes, and hence ξ is a constant, in I_j. Define a function $\varphi : [0,1] \to [0, 1 - \sum_j |I_j|]$ by $\varphi(s) = s - \sum_{j, b_j \le s} |I_j|$ for $s \in [0,1]\backslash(\cup_j I_j)$, and $\varphi(s) = \varphi(a_j)$ whenever $s \in I_j$ for some

j. Set $\widetilde{\xi}(t) = \xi(\varphi^{-1}\{t\})$ for $t \in [0, 1 - \sum_j |I_j|]$. We have $\widetilde{\xi} \in \mathcal{C}^1(0, 1 - \sum_j |I_j|; x, y; U)$. Indeed, letting s_+ be the maximum of $\varphi^{-1}(t)$, one has

$$\frac{1}{h}[\widetilde{\xi}(t+h) - \widetilde{\xi}(t)] = \frac{1}{h}[\xi(s_+ + h) - \xi(s_+)] \to \varphi'(s_+) \quad \text{as } h \to 0+;$$

similarly, letting s be the minimum of $\varphi^{-1}(t)$, one has $\frac{1}{-h}[\widetilde{\xi}(t-h) - \widetilde{\xi}(t)] \to \varphi'(s_-)$ as $h \to 0-$. If $\varphi^{-1}(\{t\})$ contains a single point s, we have $s_\pm = s$ and $\dot{\widetilde{\xi}}(t) = \dot{\xi}(s)$; otherwise $\varphi^{-1}(\{t\}) = [s_-, s_+] = [a_j, b_j]$ for some j, and hence $\dot{\xi}(s) = 0$ in $[a_j, b_j]$, that is, $\dot{\widetilde{\xi}}(t) = 0$. The continuity of $\widetilde{\xi}$ comes from that of ξ. Moreover, $\dot{\widetilde{\xi}} > 0$ almost everywhere in $[0, 1 - \sum_j |I_j|]$ as desired. This completes the proof of Lemma 3.1. □

Lemma 3.2 *For any compact set $K \subset U$, there exists a constant $C > 0$ depending on L, K such that*

$$\sup_{\lambda>0} \operatorname{Lip}(\mathcal{L}_A^\lambda; K \times K) \le C.$$

Consequently, $\lim_{\lambda\to 0} \mathcal{L}_A^\lambda = d_A$ *locally uniformly in $U \times U$.*

Proof Let $x, y, z \in K$. If $|y - z| \ge \frac{1}{2}\operatorname{dist}(K, \partial U)$, by Lemma 3.1, we have

$$|\mathcal{L}_A^\lambda(x, z) - \mathcal{L}_A^\lambda(x, y)| \le 2\operatorname{diam}_A K \le 4\frac{\operatorname{diam}_A K}{\operatorname{dist}(K, \partial U)}|y - z|.$$

If $|y - z| < \frac{1}{2}\operatorname{dist}(K, \partial U)$, choose $\xi \in \mathcal{C}(0, t; x, y; U)$ for some $t > 0$ such that

$$\mathcal{L}_A^\lambda(x, y) + |y - z| \ge \int_0^t \left[1 + \frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] e^{-\lambda(t-s)}\, ds.$$

Let $\eta(s) = \xi(s)$ for $s \in (0, t]$ and $\eta(s) = y + (s - t)\frac{z-y}{|y-z|}$ for $s \in (t, t + |y - z|)$. Then $\eta \in \mathcal{C}(0, t + |y - z|; x, z; U)$, and we have

$$\begin{aligned}
\mathcal{L}_A^\lambda(x, z) &\le \int_0^{t+|y-z|} \left[1 + \frac{1}{4}\langle A^{-1}(\eta(s))\dot{\eta}(s), \dot{\eta}(s)\rangle\right] e^{-\lambda(t+|y-z|-s)}\, ds \\
&= e^{-\lambda|y-z|}\int_0^t \left[1 + \frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] e^{-\lambda(t-s)}\, ds \\
&\quad + \int_t^{t+|y-z|} \left[1 + \frac{1}{4}\langle A^{-1}(\eta(s))\frac{z-y}{|y-z|}, \frac{z-y}{|y-z|}\rangle\right] e^{-\lambda(t+|y-z|-s)}\, ds \\
&\le e^{-\lambda|y-z|}(\mathcal{L}_A^\lambda(x, y) + |y - z|) + (1 + L)|y - z| \\
&\le \mathcal{L}_A^\lambda(x, y) + (L + 2)|y - z|.
\end{aligned}$$

Changing the roles of y, z, we have $\mathcal{L}_A^\lambda(x, y) \le \mathcal{L}_A^\lambda(x, z) + (L + 2)|y - z|$ and hence

$$|\mathcal{L}_A^\lambda(x, z) - \mathcal{L}_A^\lambda(x, y)| \le (L + 2)|y - z|.$$

By symmetry, we have

$$|\mathcal{L}_A^\lambda(z, x) - \mathcal{L}_A^\lambda(y, x)| \le C(L, K)|y - z|.$$

Therefore, for all $x, y, z, w \in K$, we have

$$\begin{aligned}
|\mathcal{L}_A^\lambda(x, z) - \mathcal{L}_A^\lambda(y, w)| &\le |\mathcal{L}_A^\lambda(x, z) - \mathcal{L}_A^\lambda(x, w)| + |\mathcal{L}_A^\lambda(x, w) - \mathcal{L}_A^\lambda(y, w)| \\
&\le C(L, K)[|w - z| + |x - y|].
\end{aligned}$$

Combining this with the fact that $\lim_{\lambda\to 0}\mathcal{L}_A^\lambda(x,y)=d_A(x,y)$ for all $x,y\in U$ given by Lemma 3.1, we conclude that $\lim_{\lambda\to 0}\mathcal{L}_A^\lambda=d_A$ locally uniformly in $U\times U$. This completes the proof of Lemma 3.2. □

Below, for convenience, we set $d_{A,x^0}=d_A(x^0,\cdot)$ and $\mathcal{L}^\lambda_{A,x^0}=\mathcal{L}^\lambda_A(x^0,\cdot)$ for $x^0\in U$ and $\lambda\ge 0$.

Lemma 3.3 *(i) For all $x^0\in U$, $\mathscr{A}_H[d_{A,x^0}]\le 0$ in $U\backslash\{x^0\}$ in viscosity sense.*
(ii) If $V\Subset U$, $x^0\in\partial V$ and $0<\lambda\le\frac{1}{2\,\mathrm{diam}_A V}$, then $\mathscr{A}_H[-\mathcal{L}^\lambda_{A,x^0}]\ge\frac{\lambda}{2}$ in V in viscosity sense.

To prove Lemma 3.3, we need an approximation for $\mathcal{L}_A^\lambda$ via smoothing A. Let $\phi\in C^\infty(\mathbb{R}^n)$, $0\le\phi\le 1$, $\int_{\mathbb{R}^n}\phi(x)\,dx=1$ and $\mathrm{supp}\,\phi\subset B(0,2)$. For $\epsilon>0$, set $\phi_\epsilon(x)=\epsilon^{-n}\phi(\epsilon^{-1}x)$ for all $x\in\mathbb{R}^n$. Set $U_\epsilon=\{x\in U,\ \mathrm{dist}\,(x,\partial U)>\epsilon\}$ for $\epsilon>0$. For every $\epsilon>0$ and $x\in U$,

$$(A^\epsilon)^{-1}(x)=\int_{\mathbb{R}^n}\left[A^{-1}(y)\chi_{U_{3\epsilon}}(y)+LI_n\chi_{U_{3\epsilon}^{\complement}}(y)\right]\phi_\epsilon(x-y)\,dy;$$

in particular, for $x\in U_{5\epsilon}$, $A^\epsilon(x)=(A^{-1}*\phi_\epsilon(x))^{-1}$. Then $(A^\epsilon)^{-1}$, $A^\epsilon\in C^2(U,\mathbb{R}^{n\times n})$ are uniformly elliptic with the same constant L as A. For $\epsilon>0$ and $\lambda>0$, define $\mathcal{L}^\lambda_{A^\epsilon}$ in the same way of $\mathcal{L}^\lambda_A$.

Lemma 3.4 *For every $\lambda>0$,*

$$\lim_{\epsilon\to 0}\mathcal{L}^\lambda_{A^\epsilon}=\mathcal{L}^\lambda_A\quad\text{locally uniformly in } U\times U.$$

Proof Due to Lemma 3.2, it suffices to prove $\lim_{\epsilon\to 0}\mathcal{L}^\lambda_{A^\epsilon}(x,y)=\mathcal{L}^\lambda_A(x,y)$ for all $x,y\in U$. We first show $\liminf_{\epsilon\to 0}\mathcal{L}^\lambda_{A^\epsilon}(x,y)\ge\mathcal{L}^\lambda_A(x,y)$. At each $z\in U_\epsilon$, we have

$$\begin{aligned}\langle (A^\epsilon)^{-1}(z)p,p\rangle&=\int_{\mathbb{R}^n}\left[\langle A^{-1}(y)p,p\rangle\chi_{U_{3\epsilon}}+L|p|^2\chi_{U_{3\epsilon}^{\complement}}\right]\phi_\epsilon(z-y)\,dy\\&\ge\int_{\mathbb{R}^n}\langle A^{-1}(z)p,p\rangle\phi_\epsilon(z-y)\,dy\\&\quad+\int_{\mathbb{R}^n}\langle (A^{-1}(y)-A^{-1}(z))p,p\rangle\chi_{U_{3\epsilon}}\phi_\epsilon(z-y)\,dy\\&\ge\langle A^{-1}(z)p,p\rangle-2\epsilon\|A^{-1}\|_{C^{0,1}(\overline{U}_\epsilon)}|p|^2,\end{aligned}$$

which implies that

$$\langle (A^\epsilon)^{-1}(z)p,p\rangle\ge(1-2L\epsilon\|A^{-1}\|_{C^{0,1}(\overline{U}_\epsilon)})\langle A^{-1}(z)p,p\rangle.$$

At each $z\in U\backslash U_\epsilon$, we have

$$\langle (A^\epsilon)^{-1}(z)p,p\rangle=\int_{\mathbb{R}^n}L|p|^2(y)\phi_\epsilon(z-y)\,dy\ge\langle A^{-1}(z)p,p\rangle.$$

Therefore, for $\xi\in\mathcal{C}(0,t;x,y;U)$, we have

$$\begin{aligned}&\int_0^t\left[1+\frac14\langle (A^\epsilon)^{-1}(\xi(s))\dot\xi(s),\dot\xi(s)\rangle\right]e^{-\lambda(t-s)}\,ds\\&\ge(1-2L\epsilon\|A^{-1}\|_{C^{0,1}(\overline{U}_\epsilon)})\int_0^t\left[1+\frac14\langle A^{-1}(\xi(s))\dot\xi(s),\dot\xi(s)\rangle\right]e^{-\lambda(t-s)}\,ds.\end{aligned}$$

Taking infimum over all $\xi \in \mathcal{C}(0, t; x, y; U)$, we deduce that

$$\mathcal{L}^{\lambda}_{A^{\epsilon}}(x, y) \geq (1 - 2L\epsilon \|A^{-1}\|_{C^{0,1}(\overline{U}_{\epsilon})})\mathcal{L}^{\lambda}_{A}(x, y),$$

that is, $\liminf_{\epsilon \to 0} \mathcal{L}^{\lambda}_{A^{\epsilon}}(x, y) \geq \mathcal{L}^{\lambda}_{A}(x, y)$ as desired.

To see $\limsup_{\epsilon \to 0} \mathcal{L}^{\lambda}_{A^{\epsilon}}(x, y) \leq \mathcal{L}^{\lambda}_{A}(x, y)$, for any $\delta \in (0, 1)$, choose $\xi \in \mathcal{C}(0, t; x, y; U)$ such that

$$(1 + \delta)\mathcal{L}^{\lambda}_{A}(x, y) \geq \int_0^t \left[1 + \frac{1}{4}\langle A^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] e^{-\lambda(t-s)}\, ds.$$

Observe that if $\epsilon < \epsilon_0 = \min\{\delta, \frac{1}{5}\operatorname{dist}(\xi, \partial U)\}$, then $\cup_{s\in[0,t]} B(\xi(s), 2\epsilon) \subset U_{3\epsilon} \subset U_{3\epsilon_0}$. Thus $(A^{\epsilon})^{-1}(\xi(s)) = A^{-1} * \phi_{\epsilon}(\xi(s))$ and hence, at $\xi(s)$,

$$|\langle A^{-1}p, p\rangle - \langle (A^{\epsilon})^{-1}p, p\rangle| = |\langle (A^{-1} - A^{-1} * \phi_{\epsilon})p, p\rangle| \leq 2L\epsilon \|A^{-1}\|_{C^{0,1}(\overline{U}_{3\epsilon_0})} \langle A^{-1}p, p\rangle$$

for all $p \in \mathbb{R}^n$, that is,

$$(1 + 2L\epsilon \|A^{-1}\|_{C^{0,1}(\overline{U}_{3\epsilon_0})})\langle A^{-1}p, p\rangle \geq \langle (A^{\epsilon})^{-1}p, p\rangle.$$

Thus,

$$\begin{aligned}
&(1 + 2L\epsilon \|A^{-1}\|_{C^{0,1}(\overline{U}_{3\epsilon_0})})(1 + \delta)\mathcal{L}^{\lambda}_{A}(x, y) \\
&\geq \int_0^t \left[1 + \frac{1}{4}\langle (A^{\epsilon})^{-1}(\xi(s))\dot{\xi}(s), \dot{\xi}(s)\rangle\right] e^{-\lambda(t-s)}\, ds \\
&\geq \mathcal{L}^{\lambda}_{A^{\epsilon}}(x, y).
\end{aligned}$$

Sending $\epsilon \to 0$ and $\delta \to 0$ in order, we conclude that $\limsup_{\epsilon \to 0} \mathcal{L}^{\lambda}_{A^{\epsilon}}(x, y) \leq \mathcal{L}^{\lambda}_{A}(x, y)$ as desired. □

We also need the following fact, which was used in [30]. For convenience, we give the details.

Lemma 3.5 *Suppose that $u \in C(U)$ is semi-concave and is a viscosity solution to*

$$\langle ADu, Du\rangle + \lambda u = 1 \quad \text{in } U \tag{3.6}$$

for some $\lambda > 0$. Then u is a viscosity sup-solution to

$$\mathscr{A}_H[u] + \lambda\langle ADu, Du\rangle = 0 \text{ in } U. \tag{3.7}$$

Proof *Step 1.* We first prove that for almost all $x \in U$ with $Du(x)$ and $D^2u(x)$ existing,

$$\mathscr{A}_H[u](x) + \lambda\langle A(x)Du(x), Du(x)\rangle \leq 0 \tag{3.8}$$

holds in pointwise way and hence in viscosity sense.

Note that the semi-concavity of u implies that $u \in C^{0,1}(U)$, differentiable almost everywhere and

$$\langle ADu(y), Du(y)\rangle + \lambda u(y) = 1 \tag{3.9}$$

whenever u is differentiable at $y \in U$. Moreover, the semi-concavity guarantees that there exists $E \subset U$ with full measure such that Du, D^2u exist in E and for all $x \in E$,

$$Du(y) = Du(x) + D^2u(x) \cdot (y - x) + o(|y - x|) \quad \text{for } y \in E. \tag{3.10}$$

Without loss of generality, we may let $U = [0, 1]^n$. Applying the Fubini Theorem, there exists $E_2 \subset [0, 1]^{n-1}$ with $(n-1)$-Lebesgue measure $|E_2| = 1$ such that $E_{y'} = E \cap [0, 1] \times$

$\{y'\}$ has length 1 for all $y' \in E_2$. For each $y' \in E_2$, let $\widetilde{E}_{y'}$ be the set of all density points of $E_{y'}$. Notice that $\widetilde{E} = \{(t, y')|y' \in E_2,\ t \in \widetilde{E}_{y'}\} \subset E$ satisfies $|\widetilde{E}| = 1$. If $(t, x') \in \widetilde{E}$, there exists a family of points $\{t_m\}_{m\geq 1} \in E_{y'}$ such that $(t_m, x') \in E$ and $t_m \to t$. Observe that (3.9) holds whenever y is given by $x_m := (t_m, x')$ or $x := (t, x')$. By (3.10),

$$
\begin{aligned}
&\frac{1}{t_m - t}[\langle A(x_m)Du(x_m), Du(x_m)\rangle - \langle A(x)Du(x), Du(x)\rangle]\\
&= \frac{1}{t_m - t}\langle [A(x_m) - A(x)]Du(x_m), Du(x_m)\rangle\\
&\quad + \frac{1}{t_m - t}\langle A(x)[Du(x_m) - Du(x)], [Du(x_m) + Du(x)]\rangle\\
&= \langle [D_{x_1}A(x)]Du(x_m), Du(x_m)\rangle + \langle A(x)D^2u(x)\mathbf{e}_1, [Du(x_m) + Du(x)]\rangle + o(1)\\
&\quad \to \langle [D_{x_1}A(x)]Du(x), Du(x)\rangle + 2\langle A(x)D^2u(x)\mathbf{e}_1, Du(x)\rangle
\end{aligned}
$$

as $m \to \infty$. On the other hand,

$$
\frac{1}{t_m - t}[\lambda u(x) - \lambda u(x_m)] = \lambda u_1(x) + o(1) \to \lambda u_1(x), \quad m \to \infty.
$$

Thus,

$$
\langle [D_{x_1}A(x)]Du(x), Du(x)\rangle + 2\langle A(x)D^2u(x)\mathbf{e}_1, Du(x)\rangle + \lambda u_1(x) = 0.
$$

Here and below, for $k \in \{1, \ldots, n\}$ we write $\mathbf{e}_k$ as the vector whose kth element is 1 and others are 0.

Similarly, we can show that there exists a set $\widetilde{E}^{(n)} \subset \widetilde{E}^{(1)} \subset E$ such that $|\widetilde{E}^{(n)}| = 1$, D^2u and Du exist on $\widetilde{E}^{(n)}$ and for each $x \in \widetilde{E}^{(n)}$ and $k \in \{1, \ldots, n\}$, we have

$$
\langle [D_{x_k}A(x)]Du(x), Du(x)\rangle + 2\langle A(x)D^2u(x)\mathbf{e}_k, Du(x)\rangle + \lambda u_k(x) = 0
$$

which times $ADu(x)$ yields that (3.8) as desired.

Step 2. Now we prove that u is a viscosity sup-solution to (3.7), that (3.8) holds for all $x \in U$ in viscosity sense.

Suppose that $\phi \in C^2(U)$ and $u - \phi$ attains strictly minimal at $\hat{x} \in U$. Since $u - \phi$ is semi-concave, owing to Lemma A.3 in [12], for any $r, \delta > 0$, there exists $x_{r,\delta} \in B(\hat{x}, r)$ and $p_{r,\delta} \in B(0, \delta)$ such that $u - \phi - \langle p_{r,\delta}, x\rangle$ has a local minimal at $x_{r,\delta}$, u is twice differentiable at $x_{r,\delta}$, and

$$
\mathscr{A}_H[u](x_{r,\delta}) + \lambda\langle A(x_{r,\delta})Du(x_{r,\delta}), Du(x_{r,\delta})\rangle \leq 0
$$

in the viscosity sense.

Obviously, we have $D\phi(x_{r,\delta}) = Du(x_{r,\delta}) - p_{r,\delta}$ and $D^2\phi(x_{r,\delta}) \leq D^2u(x_{r,\delta})$. So due to the ellipticity of A, we have

$$
\mathscr{A}_H[\phi + \langle p_{r,\delta}, x\rangle](x_{r,\delta}) + \lambda\langle A(x_{r,\delta})[D\phi(x_{r,\delta}) + p_{r,\delta}], [D\phi(x_{r,\delta} + p_{r,\delta})]\rangle \leq 0.
$$

Sending $r = \delta$ to 0 and noting $(x_{r,\delta}, p_{r,\delta}) \to (\hat{x}, 0)$, we arrive at (3.8) with $x = \hat{x}$ as desired. $\square$

Proof of Lemma 3.3 Thanks to Lemma 3.2, we know that (i) follows from (ii). Below we show (ii). Let $\{A^\epsilon\}_{\epsilon>0}$ be as in Lemma 3.3. Denote by $H^\epsilon(x, p) = \langle A^\epsilon p, p\rangle$. Then $\|D^2 A^\epsilon\|_{C(U)} \lesssim \frac{1}{\epsilon^2}$. Thanks to Lions [22, pp. 134–135], $\mathcal{L}^\lambda_{A^\epsilon, x^0}$ is semi-concave and a viscosity solution to

$$\langle A^\epsilon Du, Du\rangle - 1 + \lambda u = 0 \quad \text{in } U\backslash\{x^0\},$$

and, by Lemma 3.5, is a viscosity sup-solution to

$$\mathscr{A}_{H^\epsilon}[u] + \lambda\langle A^\epsilon Du, Du\rangle = 0 \quad \text{in } U\backslash\{x^0\}.$$

Note that $\lim_{\epsilon\to 0} \mathcal{L}^\lambda_{A^\epsilon} = \mathcal{L}^\lambda_A$ locally uniformly in U as given by Lemma 3.3. Sending $\epsilon \to 0$, we know that $\mathcal{L}^\lambda_{A,x^0}$ is a viscosity solution to

$$\langle ADu, Du\rangle - 1 + \lambda u = 0 \quad \text{in } U\backslash\{x^0\}, \tag{3.11}$$

and also is a viscosity sup-solution to

$$\mathscr{A}_H[u] + \lambda\langle ADu, Du\rangle = 0 \quad \text{in } U\backslash\{x^0\}. \tag{3.12}$$

Assume that $\mathcal{L}^\lambda_{A,x^0} - \phi$ attains a minimum at $\bar{x} \in V\backslash\{x^0\}$ for some $\phi \in C^2(U)$. Since u is a viscosity solution to (3.11), we know that

$$\langle A(\bar{x})D\phi(\bar{x}), D\phi(\bar{x})\rangle - 1 + \lambda\mathcal{L}^\lambda_A(x^0, \bar{x}) \ge 0.$$

If $\lambda \le \frac{1}{2\,\mathrm{diam}_A V}$, since $\mathcal{L}^\lambda_A(x^0, \bar{x}) \le d_A(x^0, \bar{x}) \le \mathrm{diam}_A V$ by Lemma 3.1, we have

$$\langle A(\bar{x})D\phi(\bar{x}), D\phi(\bar{x})\rangle \ge 1 - \lambda d_A(x^0, \bar{x}) \ge 1/2,$$

Considering (3.12), we conclude that $\mathscr{A}_H[\phi](\bar{x}) \le -\frac{\lambda}{2}$ in V in viscosity sense as desired. $\square$

4 Uniqueness

We always assume that $f \in C(U)$ with $|f| > 0$ and $A \in C^1(U; \mathbb{R}^{n\times n})$ is symmetric and uniformly elliptic.

Theorem 4.1 *For any $g \in C(\partial U)$ there exists at most one viscosity solution $u \in C(\overline{U})$ to the Dirichlet problem:*

$$\mathscr{A}_H[u] = f \quad \text{in } U; \quad u|_{\partial U} = g.$$

To prove Theorem 4.1, we need a comparison principle as below.

Lemma 4.2 *Let $\epsilon \ge 0$. Suppose that $f_1, f_2 \in C(U)$ satisfy $f_1 < f_2$, and that $u_1 \in USC(\overline{U})$ is a viscosity sup-solution to*

$$\mathscr{A}_H[u] + \epsilon\,\mathrm{div}(ADu) = f_1 \tag{4.1}$$

and $u_2 \in LSC(\overline{U})$ is a viscosity sub-solution to

$$\mathscr{A}_H[u] + \epsilon\,\mathrm{div}(ADu) = f_2. \tag{4.2}$$

If either $u_1 \in C^{0,1}(U)$ or $u_2 \in C^{0,1}(U)$, then

$$\max_{\overline{U}}[u_2 - u_1] = \max_{\partial U}[u_2 - u_1].$$

Proof of Theorem 4.1 Let $u, v \in C(\overline{U})$ be viscosity solutions to $\mathscr{A}_H[u] = f$ with $u|_{\partial U} = g$. We may assume that $f > 0$ up to considering $-u$,$-v$. For any $\epsilon > 0$, set $u_\epsilon = (1+\epsilon)u - \epsilon\|g\|_{L^\infty(\partial U)}$ on $\overline{U}$. Then

$$\mathscr{A}_H[u_\epsilon] = (1+\epsilon)^3 f > f = \mathscr{A}_H[v]$$

in U in viscosity sense and $u_\epsilon \le u = v$ on ∂U. Since $\mathscr{A}_H[u_\epsilon] \ge 0$ in U in viscosity sense, by Lemma 2.3, we know that $u_\epsilon \in C^{0,1}(U)$. Applying Lemma 4.2, we have $u_\epsilon \le v$ in U for all $\epsilon > 0$. By sending $\epsilon \to 0$, it follows that $u \le v$ in U. Similarly, we have $u \ge v$. Therefore $u = v$ as desired. □

To prove Lemma 4.2, we recall the notion of jets in [12, Section 2]. Define the second-order superjet $J_U^{2,+}u(x^0)$ of a function u at x^0 as the collection of all $(D\phi(x^0), D^2\phi(x^0))$ satisfying that $\phi \in C^2(U)$ and $u - \phi$ taking its local maximum at x^0. Denote by $\overline{J}_U^{2,+}u(x^0)$ its closure, that is, the collection of (p, X), for which there exists $x_m \in U$ and $(p_m, X_m) \in J_U^{2,+}u(x^0)$ such that $(x_m, u(x_m), p_m, X_m) \to (x^0, u(x^0), p, X)$. Similarly, define the second-order subjet $J_U^{2,-}u(x^0)$ and its closure $\overline{J}_U^{2,-}u(x^0)$ in the same manner with the local maximum replaced by the local minimum.

Proof of Lemma 4.2 We may assume that $\max_{\partial U}[u_2 - u_1] = 0$ up to considering $u_1 - \max_{\partial U}[u_2 - u_1]$ instead of u_1. It suffices to prove $u_2 \le u_1$ in U. Suppose that this is not correct. Then

$$M_0 := \sup_{x \in \overline{U}}[u_2(x) - u_1(x)] > 0.$$

For any small $\delta > 0$, define

$$w_\delta(x, y) = u_2(x) - u_1(y) - \frac{1}{2\delta}|x - y|^2 \quad \forall\,(x, y) \in \overline{U} \times \overline{U}$$

and let

$$M_\delta = \sup_{x,\, y \in \overline{U}} w_\delta(x, y) = w_\delta(x_\delta, y_\delta)$$

for some $x_\delta, y_\delta \in \overline{U}$.

Obviously, $M_\delta \ge M_0$ for all $\delta > 0$. By Lemma 3.1 of [12], $M_0 = \lim_{\delta\to 0} M_\delta$ and $x_\delta, y_\delta \in U_1 \Subset U$ for all $\delta > 0$ sufficient small. Moreover,

$$|x_\delta - y_\delta| \le C(U_1)\delta. \tag{4.3}$$

Indeed, if $u_2 \in C^{0,1}(U)$, by $M_\delta \ge M_0$, we have

$$u_2(x_\delta) - u_1(y_\delta) - \frac{1}{2\delta}|x_\delta - y_\delta|^2 \ge u_2(y_\delta) - u_1(y_\delta),$$

which leads to that

$$|x_\delta - y_\delta| \le 2\delta \frac{u_2(y_\delta) - u_2(x_\delta)}{|x_\delta - y_\delta|} \le \|u_2\|_{C^{0,1}(\overline{U}_1)}\delta,$$

that is, (4.3). If $u_1 \in C^{0,1}(U)$, similarly, we have (4.3). By [12, Lemma 3.1] again, there exist $X, Y \in \mathscr{S}^{n\times n}$ such that

$$\left(\frac{1}{\delta}(x_\delta - y_\delta), X\right) \in \overline{J}^{2,+}_{U_1} u_2(x_\delta), \quad \left(\frac{1}{\delta}(x_\delta - y_\delta), Y\right) \in \overline{J}^{2,-}_{U_1} u_1(y_\delta)$$

and

$$-\frac{3}{\delta}\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\delta}\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}. \tag{4.4}$$

Let $p = \frac{1}{\delta}(x_\delta - y_\delta)$. Since $(p, X) \in \overline{J}^{2,+}_{U_1} u_2(x_\delta)$, there exists a sequence (z_m, p_m, X_m) with $(p_m, X_m) \in J^{2,+}_{U_1} u_2(z_m)$ approximating (x_δ, p, X). For each $(p_m, X_m) \in J^{2,+}_{U_1} u_2(z_m)$ we can find $\phi_m \in C^2$ such that $p_m = D\phi_m(z_m)$, $X_m = D^2\phi_m(z_m)$ and $u_m - \phi_m$ attaining its local maximum at z_m. From the definition of viscosity sub-solution, we deduce that

$$\mathscr{A}_H[\phi_m](z_m) + \epsilon \operatorname{div}(A(z_m)D\phi_m(z_m)) \ge f_2(z_m).$$

Sending $m \to \infty$, by $(z_m, p_m, X_m) \to (x_\delta, p, X)$ and the continuity of DA, A and f, we obtain

$$f_2(x_\delta) \le \langle XA(x_\delta)p, A(x_\delta)p\rangle + \langle\langle DA(x_\delta)p, p\rangle, A(x_\delta)p\rangle + \epsilon a^{ij}(x_\delta)X_{ij} + \epsilon a^{ij}_i(x_\delta)p_j.$$

Similarly, we also have

$$f_1(y_\delta) \ge \langle YA(y_\delta)p, A(y_\delta)p\rangle + \langle\langle DA(y_\delta)p, p\rangle, A(y_\delta)p\rangle + \epsilon a^{ij}(y_\delta)Y_{ij} + \epsilon a^{ij}_i(y_\delta)p_j.$$

Below we show that for arbitrary $\eta > 0$, $f_2(x_\delta) \le f_1(y_\delta) + C\eta$ whenever $\delta \in (0, \eta)$ is sufficiently small. If this is true, sending $\eta \to 0$, we have $f_2(x^0) \le f_1(x^0)$ for some $x^0 \in \overline{U}_1$ which is contradiction with $f_1(x^0) < f_2(x^0)$, as desired.

To see $f_2(x_\delta) \le f_1(y_\delta) + C\eta$, by (4.3) and (4.4) we have

$$\begin{aligned}
&\langle XA(x_\delta)p, A(x_\delta)p\rangle - \langle YA(y_\delta)p, A(y_\delta)p\rangle \\
&= \frac{3}{\delta}(A(x_\delta)p - A(y_\delta)p)^T(A(x_\delta)p - A(y_\delta)p) \\
&= \frac{3}{\delta}|A(x_\delta)p - A(y_\delta)p|^2 \\
&\le \frac{3}{\delta}|A(x_\delta) - A(y_\delta)|^2 |\frac{x_\delta - y_\delta}{\delta}|^2 \\
&\le C(U_1, A)\delta \\
&\le C(U_1, A)\eta
\end{aligned}$$

whenever $\delta < \eta$. Let $A^{1/2} = (b^{ij})^n_{i,j=1}$. For each k, the same argument leads to

$$b^{ki}(x_\delta)X_{ij}b^{jk}(x_\delta) - b^{ki}(y_\delta)Y_{ij}b^{jk}(y_\delta) \le C(U_1, A)\delta \le C(U_1, A)\eta$$

when $\delta < \eta$, and hence

$$a^{ij}(x_\delta)X_{ij} - a^{ij}(y_\delta)Y_{ij} = \sum_{k=1}^n [b^{ki}(x_\delta)X_{ij}b^{jk}(x_\delta) - b^{ki}(y_\delta)Y_{ij}b^{jk}(y_\delta)] \le C(U_1, A)\eta.$$

Moreover, using (4.3) again, we have

$$\begin{aligned}&\langle\langle DA(x_\delta)p, p\rangle, A(x_\delta)p\rangle - \langle\langle DA(y_\delta)p, p\rangle, A(y_\delta)p\rangle\\ &\quad\le |\langle[DA(x_\delta) - DA(y_\delta)]p, p\rangle||A(x_\delta)p|\\ &\qquad + |\langle DA(y_\delta)p, p\rangle||[A(x_\delta) - A(y_\delta)]p|\\ &\quad\le C|DA(x_\delta) - DA(y_\delta)| + C\delta,\end{aligned}$$

which, by the continuity of DA, will be less than $C\eta$ when $\delta \in (0, \eta)$ is small enough. Similar arguments lead to that

$$a_i^{ij}(x_\delta)p_j - a_i^{ij}(y_\delta)p_j \le C\eta$$

when $\delta \in (0, \eta)$ is small enough. Combining all above estimates, we conclude for arbitrary $\eta > 0$, $f_2(x_\delta) \le f_1(y_\delta) + C\eta$ whenever $\delta \in (0, \eta)$ is small enough. This completes the proof. □

5 Existence

We always assume that $A \in C^1(U; \mathbb{R}^{n\times n})$ is symmetric and uniformly elliptic in this section.

Theorem 5.1 *Let $V \Subset U$ and $f \in C(V)$ be bounded. For arbitrary $g \in C(\partial V)$, there exists a viscosity solution $u \in C(\overline{V})$ to the Dirichlet problem:*

$$\mathscr{A}_H[u] = f \quad \text{in } V; \quad u|_{\partial V} = g.$$

Proof Denote by $\mathscr{S}^+_{f,g}$ the set of all viscosity sup-solutions $v \in C(\overline{V})$ to the Dirichlet problem:

$$\mathscr{A}_H[u] = f \quad \text{in } V; \quad u|_{\partial V} \ge g.$$

Notice that $\mathscr{S}^+_{f,g} \ne \emptyset$. Indeed, for any $x_0 \in V$, $0 < \lambda < \frac{1}{2\,\mathrm{diam}_A V}$ and $C > (2\|f\|_{C(V)}/\lambda)^{1/3}$, by Lemma 3.3, we have

$$\mathscr{A}_H[C\mathcal{L}^\lambda_{A,x^0} + \|g\|_{C(\partial V)}] = C^3\mathscr{A}_H[\mathcal{L}^\lambda_{A,x^0}] \le -C^3\lambda/2 \le -\|f\|_{C(V)} \le f \quad \text{in } V$$

in viscosity sense. Thus $C\mathcal{L}^\lambda_{A,x^0} + \|g\|_{C(\partial V)} \in \mathscr{S}^+_{f,g}$. Set

$$u(x) = \inf\{v(x)| v \in \mathscr{S}^+_{f,g}\} \quad \forall x \in \overline{V}.$$

We claim that u is the desired viscosity solution.

To prove the claim, we observe that $u \in USC(\overline{V})$, $u \ge g$ on ∂V and by [12, Lemma 4.2], $\mathscr{A}_H[u] \le f$ in V in the viscosity sense. Moreover, similarly define

$$\hat{u}(x) = \sup\{w(x)| w \in \mathscr{S}^-_{f,g}\} \quad \forall x \in \overline{V},$$

where $\mathscr{S}^-_{f,g}$ is the set of all viscosity sub-solutions $v \in C(\overline{V})$ to the Dirichlet problem:

$$\mathscr{A}_H[u] = f \quad \text{in } V; \quad u|_{\partial V} \le g.$$

Note that $\mathscr{S}^-_{f,g} \ne \emptyset$. Indeed, for any $x^0 \in \partial V$, $0 < \lambda < \frac{1}{2\,\mathrm{diam}_A V}$ and $C > (2\|f\|_{C(V)}/\lambda)^{1/3}$, letting $b \in \mathbb{R}$ with $-C\mathcal{L}^\lambda_{A,x^0} + b \le \|g\|_{C(\partial V)}$, by Lemma 3.3, we have

$$\mathscr{A}_H[-C\mathcal{L}^\lambda_{A,x^0} + b] = C^3\mathscr{A}_H[-\mathcal{L}^\lambda_{A,x^0}] \ge C^3\lambda/2 \ge \|f\|_{C(V)} \ge f \quad \text{in } V$$

in viscosity sense. Thus $-C\mathcal{L}^{\lambda}_{A,x^0}+b\in\mathscr{S}^-_{f,g}$. Note that $\hat{u}\in LSC(\overline{V})$, $\hat{u}\le g$ on ∂V and by [12, Lemma 4.2], $\mathscr{A}_H[\hat{u}]\ge f$ in V in the viscosity sense.

Now we are ready to prove the claim by 3 steps.

Step 1. $\mathscr{A}_H[u]=f$ in V in the viscosity sense.

It suffices to prove $\mathscr{A}_H[u]\ge f$ in V in the viscosity sense. Suppose this is not true. Then there exist $\varphi\in C^2(V)$ and a point $x^0\in V$ such that $u-\varphi$ attains its local maximum at x^0, but $\mathscr{A}_H[\varphi](x^0)<f(x^0)$. Without loss of generality, we may assume that $u(x^0)=\varphi(x^0)$.

For any small $\epsilon>0$, we define

$$\varphi_\epsilon(x)=\varphi(x)+\epsilon|x-x^0|^2.$$

Since $\mathscr{A}_H[\varphi](x^0)<f(x^0)$, if ϵ is small enough, we have $\mathscr{A}_H[\varphi_\epsilon](x^0)<f(x^0)$, and hence, by the continuity of f and DA, we have

$$\mathscr{A}_H[\varphi_\epsilon](x)<f(x)$$

for all x in some small open neighborhood of x^0, say $V(x^0)$. Moreover, x^0 is a strict local maximum point of $u-\varphi_\epsilon$; indeed, this follows from the fact that $u-\varphi$ attains its local maximum at x^0 and $\varphi-\varphi_\epsilon$ attains its strictly local maximum at x^0. Observing $u(x^0)=\varphi(x^0)=\varphi_\epsilon(x^0)$, we know that $\varphi_\epsilon(x)>u(x)$ for $x\in V_1(x^0)\backslash\{x^0\}$, where $V_1(x^0)\subset V(x^0)$ is some open neighborhood of x^0.

Let $\delta>0$ be sufficiently small so that the closure of

$$V_2(x^0):=\{x\in V_1(x_0)|\varphi_\epsilon(x)-\delta<u(x)\}$$

is contained in $V_1(x^0)$, and hence, $\varphi_\epsilon-\delta\ge u$ in $V_1(x^0)\backslash V_2(x^0)$. Set

$$\hat{v}=\min(\varphi_\epsilon-\delta,u)=\begin{cases}\hat{\varphi}(x) & x\in V_2(x^0)\\ u(x) & x\in\overline{V}\backslash V_2(x^0).\end{cases}$$

Then $\hat{v}=u\ge g$ on ∂V. Since $\mathscr{A}_H[u]\le f$ in V and $\mathscr{A}_H[\hat{\varphi}]<f$ in $V_1(x^0)\supseteq V_2(x^0)$ in viscosity sense, we conclude that $\mathscr{A}_H[\hat{v}]\le f$ in V in the viscosity sense. Therefore, $\hat{v}\in\mathscr{S}^+_{f,g}$. However, $\hat{v}=\varphi_\epsilon-\delta<u$ in $V_2(x^0)$, which is a contradiction with $u\le\hat{v}$ by definition.

Step 2. $u=g=\hat{u}$ on ∂V.

Let $x^0\in\partial V$. For any $\epsilon>0$, there exists $r\in(0,\epsilon)$ such that $|g(x)-g(x^0)|<\epsilon$ for all $x\in B_A(x^0,r)\cap\partial V$. Let $C_1>\frac{2}{r}\|g\|_{C(\partial V)}$ and define

$$v=g(x^0)+\epsilon+C_1d_{A,x^0}.$$

Then

$$v(x)\ge g(x^0)+\epsilon\ge g(x)\quad\forall x\in B_A(x^0,r)\cap\partial V$$

and

$$v(x)\ge g(x^0)+\epsilon+C_1r\ge\|g\|_{C(\partial V)}\ge g(x)\quad\forall x\in\partial V\backslash B_A(x^0,r).$$

By Lemma 3.3, $\mathscr{A}_H[v]\le 0\le f$ in V in viscosity sense, and hence $v\in\mathscr{S}^+_{f,g}$. Thus

$$g(x^0)\le u(x^0)\le v(x^0)=g(x^0)+\epsilon,$$

which together with the arbitrariness of $\epsilon>0$ yields that $u(x^0)=g(x^0)$.

On the other hand, by Lemma 3.2 and $V \Subset U$, for all sufficiently small $\lambda > 0$ we have

$$\mathcal{L}^{\lambda}_{A,x^0}(x) \geq \frac{1}{2} d_A(x^0, x) \geq r/2 \quad \forall x \in V \backslash B_A(x^0, r).$$

Define

$$w = g(x^0) - \epsilon - C_2 \mathcal{L}^{\lambda}_{A,x^0}$$

where C_2 satisfying $C_2 r/2 \geq 2\|g\|_{C(\partial V)}$ and $C_2^3 \lambda/2 \geq \|f\|_{C(V)}$. If $\lambda > 0$ small enough, Lemma 3.3 leads to that

$$\mathscr{A}_H[w] \geq C_2^3 \lambda/2 \geq \|f\|_{C(V)} \geq f \quad \text{in } V$$

in viscosity sense. Observe that

$$w(x) \leq g(x^0) - \epsilon < g(x) \quad \forall x \in \partial V \cap B_A(x^0, r)$$

and

$$w(x) \leq -\|g\|_{C(\partial V)} - \epsilon < g(x) \quad \forall x \in \partial V \backslash B_A(x^0, r).$$

We know that $w \in \mathscr{S}^-_{f,g}$. Therefore,

$$g(x^0) \geq \hat{u}(x^0) \geq w(x^0) = g(x^0) - \epsilon,$$

which together with the arbitrariness of $\epsilon > 0$ implies $\hat{u}(x^0) = g(x^0)$.
Step 3. We prove $u \in C(\overline{V})$.

Since $u \in USC(\overline{V})$ and $\mathscr{A}_H[u] \geq f$ in V in the viscosity sense, by Lemma 2.4, $u \in C^{0,1}(V)$ and hence $u \in C(V)$. It suffices to prove that u is continuous up to ∂V. Since $u \in USC(\overline{V})$ and $u|_{\partial V} = g$, we only need to show that $u \in LSC(\overline{V})$. To this, applying Lemma 4.1 to every pair of $v \in \mathscr{S}^-_{f,g}$ and $w \in \mathscr{S}^+_{f,g}$, we have $w \leq v$ on $\overline{V}$, which yields that $u \leq \hat{u}$ on $\overline{V}$. Since $\hat{u}|_{\partial V} = g$ given in Step 2, we conclude that

$$\liminf_{\overline{V} \ni x \to x^0} u(x) \leq \liminf_{\overline{V} \ni x \to x^0} \hat{u}(x) \leq g(x^0)$$

for every point $x^0 \in \partial V$, that is, $u \in LSC(\overline{V})$ as desired. □

6 Linear approximation property

We always assume that $f \in C(U)$ and $A \in C^1(U; \mathbb{R}^{n \times n})$ is symmetric and uniformly elliptic in this section.

Theorem 6.1 *If $u \in C(U)$ is a viscosity solution to (1.3), then $u \in C^{0,1}(U)$ and enjoys the linear approximation property.*

Instead of u, we consider the function $\widetilde{u}(\widetilde{x}) = u(x) + 2x_{n+1}$ for $\widetilde{x} = (x, x_{n+1}) \in \widetilde{U} = U \times \mathbb{R}$. Then the local Lipschitz regularity and linear approximation property of u will follow from those of $\widetilde{u}$. Observe that $\mathscr{A}_{\widetilde{H}}[\widetilde{u}] = \widetilde{f}$ in $\widetilde{U}$ in viscosity sense, where $\widetilde{f}(\widetilde{x}) = f(x)$ and $\widetilde{H}(\widetilde{x}, p) = \langle \widetilde{A}(\widetilde{x})p, p\rangle$ with $\widetilde{A}(\widetilde{x}) = \mathrm{diag}\{A(x), 1\}$ for all $\widetilde{x} \in \widetilde{U}$ and $p \in \mathbb{R}^{n+1}$.

Moreover, $\widetilde{u}$ has the following property

$$S^{\pm}_{A,r}(\widetilde{u})(\widetilde{x}) := \sup \left\{ \frac{\pm[\widetilde{u}(\widetilde{y}) - \widetilde{u}(\widetilde{x})]}{r} \,\middle|\, d_{\widetilde{A},\widetilde{x}}(\widetilde{y}) \leq r \right\} \geq 2,$$

for all $\widetilde{x} \in \widetilde{U}$ and all possible $r > 0$, which is required in the following lemmas. Below we write $\widetilde{u}, \widetilde{A}, \widetilde{f}, \widetilde{U}$ as u, A, f, U correspondingly.

For $\lambda \geq 0$ and $x \in U$ and $r \in (0, \operatorname{dist}_A(x, \partial U))$, define

$$S^{\pm}_{\mathcal{L}^{\lambda}_A, r}(u)(x) := \sup\left\{\frac{\pm[u(y) - u(x)]}{r} \,\middle|\, \mathcal{L}^{\lambda}_{A,x}(y) \leq r\right\}.$$

When $\lambda = 0$, we have $S^{\pm}_{\mathcal{L}^{0}_A, r}(u)(x) = S^{\pm}_{A,r}(u)(x)$. For $\epsilon > 0$, set

$$U_\epsilon := \{x \in U, d_A(x, \partial U) > \epsilon\}.$$

Lemma 6.2 *For any $\epsilon > 0$ and $\lambda > 2\|f\|_{C(U_\epsilon)}$, there exists $r_{\epsilon,\lambda} \in (0, \epsilon)$ such that for all $x \in U_{2\epsilon}$, the maps $r \in (0, r_{\epsilon,\lambda}) \to S^{\pm}_{\mathcal{L}^{\lambda}_A, r}(u)(x)$ are increasing.*

Proof Let

$$r_{\epsilon,\lambda} = \min\{\epsilon/2, \eta^{-1}(\epsilon), \eta^{-1}(1/4\lambda), (\ln\sqrt{2})/\lambda\},$$

where $\eta(t) = e^{4\lambda t}t$. By Lemma 3.1, we have

$$d_{A,x}(y) \leq \eta(\mathcal{L}^{\lambda}_{A,x}(y))$$

whenever $\mathcal{L}^{\lambda}_{A,x}(y) < (\ln\sqrt{2})/\lambda$. Thus for all $x \in U_{2\epsilon}$ and $0 < r \leq r_{\epsilon,\lambda}$, we have

$$\{y \in U : \mathcal{L}^{\lambda}_{A,x}(y) < r\} \subset B_A(x, \eta(r)) \subset B_A(x, \epsilon) \subset U_\epsilon.$$

Given $x \in U_{2\epsilon}$ and $0 < r \leq r_{\epsilon,\lambda}$, set

$$v^{\pm}(y) = \pm S^{\pm}_{\mathcal{L}^{\lambda}_A, r}(u)(x)\mathcal{L}^{\lambda}_{A,x}(y).$$

Then

$$-v^-(y) \leq u(y) - u(x) \leq v^+(y) \quad \text{when } \mathcal{L}^{\lambda}_{A,x}(y) = r \text{ or } y = x.$$

By $\mathcal{L}^{\lambda}_{A,x}(y) \leq d_{A,x}(y)$ for all $x, y \in U$, we have

$$S^{\pm}_{\mathcal{L}^{\lambda}_A, r}(u)(x) \geq S^{\pm}_{A,r}(u)(x) \geq 2 \quad \forall x \in U \text{ and } r \in (0, \operatorname{dist}_A(x, \partial U)).$$

Since $r_{\epsilon,\lambda} \leq \eta^{-1}(1/4\lambda)$ implies that

$$\operatorname{diam}_A\{y :\ \mathcal{L}^{\lambda}_{A,x}(y) < r\} \leq 2\eta(r) \leq 1/2\lambda,$$

applying Lemma 3.3 we have

$$\mathscr{A}_H[v^+] \leq -\lambda/2 < -f \text{ and } \mathscr{A}_H[v^-] \geq \lambda/2 > f \text{ in } \{y :\ \mathcal{L}^{\lambda}_{A,x}(y) < r\}.$$

Notice that $\mathscr{A}_H[u - u(x)] = f$ in $\{y :\ \mathcal{L}^{\lambda}_{A,x}(y) < r\}$. By Lemma 4.2, we have

$$v^- \leq [u - u(x)] \leq v^+ \quad \text{in} \quad \{y :\ \mathcal{L}^{\lambda}_{A,x}(y) \leq r\}.$$

In particular, when $\mathcal{L}^{\lambda}_{A,x}(y) \leq s < r$, we have

$$-sS^{-}_{\mathcal{L}^{\lambda}_A, r}(u)(x) \leq u(y) - u(x) \leq sS^{+}_{\mathcal{L}^{\lambda}_A, r}(u)(x),$$

which implies that

$$S^{\pm}_{\mathcal{L}^{\lambda}_A, s}(u)(x) \leq S^{\pm}_{\mathcal{L}^{\lambda}_A, r}(u)(x)$$

as desired. □

Corollary 6.3 *We have* $u \in C^{0,1}(U)$ *and*

$$S_A^{\pm}(u)(x) := \lim_{r\to 0} S_{A,r}^{\pm}(u)(x) = \lim_{r\to 0} S_{\mathcal{L}_A^{\mu},r}^{\pm}(u)(x) \quad \forall x \in U,\ \mu > 0.$$

Moreover, $\mathrm{Lip}_{d_A} u = S_A^{\pm}(u) \in USC(U)$.

Proof Note that $u \in C^{0,1}(U)$ is given by Lemma 2.4. Here we would like to give a different proof via Lemma 6.2. For $\epsilon > 0$ sufficiently small, let λ and $r_{\epsilon,\lambda}$ be as in Lemma 6.2. Let $x, y \in U_{2\epsilon}$ with $d_A(x, y) = r$ for $r < r_{\epsilon,\lambda}$. Then $\mathcal{L}_A^{\lambda}(x, y) \le d_A(x, y) \le r$ and hence, by Lemma 6.2,

$$|u(y) - u(x)| \le r[S^{+}_{\mathcal{L}_A^{\lambda},r}(u)(x) + S^{-}_{\mathcal{L}_A^{\lambda},r}(u)(x)] \le r[S^{+}_{\mathcal{L}_A^{\lambda},r_{\epsilon,\lambda}}(u)(x) + S^{-}_{\mathcal{L}_A^{\lambda},r_{\epsilon,\lambda}}(u)(x)]$$

Since

$$S^{+}_{\mathcal{L}_A^{\lambda},r_{\epsilon,\lambda}}(u)(x) + S^{-}_{\mathcal{L}_A^{\lambda},r_{\epsilon,\lambda}}(u)(x) \le \frac{4}{r_{\epsilon,\lambda}}\|u\|_{C(\overline{U}_\epsilon)},$$

we have

$$|u(y) - u(x)| \le d_A(x, y)\frac{4}{r_{\epsilon,\lambda}}\|u\|_{C(\overline{U}_\epsilon)}.$$

This holds trivially when $x, y \in U_{2\epsilon}$ with $d_A(x, y) \ge r_{\epsilon,\lambda}$. Thus $u \in C^{0,1}(\overline{U}_{2\epsilon})$.

Moreover, by Lemma 3.1,

$$B_A(x, r) \subset \{z \in U : \mathcal{L}^{\lambda}_{A,x}(z) < r\} \subset B_A(x, \eta(r)) \subset B_A(x, \epsilon) \subset U_\epsilon$$

whenever $x \in U_{2\epsilon}$ and $r < r_{\epsilon,\lambda}$. This implies that

$$\{z : \mathcal{L}^{\lambda}_{A,x}(z) \le \eta^{-1}(r)\} \subset \overline{B_A(x, r)} \subset \{z : \mathcal{L}^{\lambda}_{A,x}(z) \le r\}$$

and hence

$$\frac{\eta^{-1}(r)}{r} S^{\pm}_{\mathcal{L}_A^{\lambda},\eta^{-1}(r)}(u)(x) \le S^{\pm}_{A,r}(u)(x) \le S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u)(x).$$

Recall that $\eta(t) = e^{4\lambda t}t$. By $\lim_{r\to 0} \eta^{-1}(r) = 0$, we have

$$\lim_{r\to 0} \frac{\eta^{-1}(r)}{r} = \lim_{r\to 0} e^{-4\lambda\eta^{-1}(r)} = 1,$$

and hence

$$\liminf_{r\to 0} S^{\pm}_{\mathcal{L}_A^{\lambda},\eta^{-1}(r)}(u)(x) \le \liminf_{r\to 0} S^{\pm}_{A,r}(u)(x) \le \limsup_{r\to 0} S^{\pm}_{A,r}(u)(x) \le \limsup_{r\to 0} S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u)(x).$$

Since the map $r \mapsto S^{\pm}_{\mathcal{L}_A^{\lambda},r}u(x)$ is increasing as given by Lemma 6.2, we have

$$\limsup_{r\to 0} S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u)(x) = \liminf_{r\to 0} S^{\pm}_{\mathcal{L}_A^{\lambda},\eta^{-1}(r)}(u)(x) = \inf_{r\in(0,r_{\epsilon,\lambda})} S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u)(x),$$

which yields that

$$S^{\pm}_A(u)(x) := \lim_{r\to 0} S^{\pm}_{A,r}(u)(x) = \lim_{r\to 0} S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u)(x) = \inf_{r\in(0,r_{\epsilon,\lambda})} S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u)(x).$$

This together with $S^{\pm}_{\mathcal{L}_A^{\lambda},r}(u) \in C(U)$ tells that $S^{\pm}_{A,r}(u) \in USC(\overline{U}_\epsilon)$.

For $0<\mu<\lambda$, by $\mathcal{L}_A^\lambda \le \mathcal{L}_A^\mu \le d_A$ we have

$$\overline{B_A(x,r)} \subset \{y : \mathcal{L}_{A,x}^\mu(y) \le r\} \subset \{y : \mathcal{L}_{A,x}^\lambda(y) \le r\},$$

and hence

$$S_{A,r}^\pm(u)(x) \le S_{\mathcal{L}_A^\mu,r}^\pm(u)(x) \le S_{\mathcal{L}_A^\lambda,r}^\pm(u)(x).$$

Therefore $\lim_{r\to 0} S_{\mathcal{L}_A^\mu,r}^\pm(u)(x) = S_A^\pm(u)(x)$.

Finally, we show that $S_A^\pm(u)(x) = \mathrm{Lip}_{d_A} u(x)$ for $x \in U_{2\epsilon}$. Obviously $S_A^\pm(u)(x) \le \mathrm{Lip}_{d_A} u(x)$. On the other hand, for any $t \in (0, r_{\epsilon,\lambda})$, by Lemma 3.1, Lemma 6.2 and the continuity of $S_{\mathcal{L}_A^\lambda,t}^\pm(u)$, we have

$$\begin{aligned}
\mathrm{Lip}_{d_A} u(x) &\le \lim_{r\to 0} \sup\left\{\frac{|u(z)-u(w)|}{d_A(z,w)} | z, w \in B_A(x,r)\right\} \\
&\le \lim_{r\to 0} \sup\left\{\frac{|u(z)-u(w)|}{\mathcal{L}_A^\lambda(z,w)} | z, w \in B_A(x,r)\right\} \\
&\le \lim_{r\to 0} \sup\left\{\sup_{s\in(0,2r)} S_{\mathcal{L}_A^\lambda,s}^\pm(u)(w) | w \in B_A(x,r)\right\} \\
&\le \lim_{r\to 0} \sup\left\{S_{\mathcal{L}_A^\lambda,t}^\pm(u)(w) | w \in B_A(x,r)\right\} \\
&\le S_{\mathcal{L}_A^\lambda,t}^\pm(u)(x).
\end{aligned}$$

Therefore,

$$\mathrm{Lip}_{d_A} u(x) \le \lim_{t\to 0} S_{\mathcal{L}_A^\lambda,t}^\pm(u)(x) = S_A^\pm(u)(x)$$

as desired. □

Lemma 6.4 *Assume that $0 \in U$ and let $A_r(x) = A(rx)$ and $u_r(x) = \frac{u(rx)}{r}$ for all possible $r > 0$ and $x \in \frac{1}{r}U$. For all possible $r > 0$, $s > 0$ and $x \in U$, we have*

$$S_{A,sr}^\pm(u)(rx) = S_{A_r,s}^\pm(u_r)(x) = S_{A_{sr},1}^\pm(u_{rs})(x/s).$$

Proof Let d_{A_r} be the intrinsic distance determined by A_r. Note that

$$d_A(rx, ry) = r d_{A_r}(x,y) \quad \forall x, y \in \frac{1}{r}U. \tag{6.1}$$

Indeed, by $(A^{-1})_r(z)=(A_r)^{-1}(z)$ for $z\in\frac{1}{r}U$, we have

$$
\begin{aligned}
&d_A(rx,ry)\\
&=\inf\left\{\left(\int_0^1\langle A^{-1}(\xi(s))\dot\xi(s),\dot\xi(s)\rangle\,ds\right)^{1/2}\Big|\xi\in\mathcal{C}(0,1;rx,ry;U)\right\}\\
&=r\inf\left\{\left(\int_0^1\langle (A^{-1})_r((\frac{1}{r}\xi)(s))(\frac{1}{r}\xi)'(s),(\frac{1}{r}\xi)'(s)\rangle\,ds\right)^{1/2}\Big|(\frac{1}{r}\xi)\in\mathcal{C}(0,1;x,y;\frac{1}{r}U)\right\}\\
&=r\inf\left\{\left(\int_0^1\langle (A_r)^{-1}(\eta(s))\eta'(s),\eta'(s)\rangle\,ds\right)^{1/2}\Big|\eta\in\mathcal{C}(0,1;x,y;\frac{1}{r}U)\right\}\\
&=rd_{A_r}(x,y)\quad\forall x,y\in\frac{1}{r}U.
\end{aligned}
$$

By (6.1), we know that $\frac{1}{r}B_A(x,s)=B_{A_r}(x/r,s/r)$ for all possible x,y,r,s.

By the definition,

$$
S^{\pm}_{A_r,s}(u_r)(x)=\max_{y\in B_{A_r}(x,s)}\frac{\pm[u_r(y)-u_r(x)]}{s}=\max_{y\in B_{A_r}(x,s)}\pm[u_{rs}(y/s)-u_{rs}(x/s)].
$$

By (6.1), we have

$$
d_{A_r}(x,y)=sd_{A_{rs}}(x/s,y/s).
$$

Hence, $d_{A_r}(x,y)\le s$ implies that $d_{A_{rs}}(x/s,y/s)\le 1$. So

$$
S^{\pm}_{A_r,s}(u_r)(x)=\max_{z\in B_{A_{rs}}(x/s,1)}\pm[u_{rs}(z)-u_{rs}(x/s)]=S^{\pm}_{A_{rs},1}(u_{rs})(x/s).
$$

Similarly, we have $S^{\pm}_{A_r,s}(u_r)(x)=S^{\pm}_{A,rs}(u)(rx)$. □

We also need the following result, which can be found in [10]. When $A=I_n$ and $U=\mathbb{R}^n$ (in this case, d_A is the Euclidean distance), we write $S^{\pm}_{A,r}(u)$ as $S^{\pm}_r(u)$ respectively.

Lemma 6.5 *Suppose that u is a viscosity solution to $\Delta_\infty u=0$ in $\mathbb{R}^n$ and*

$$
S^{\pm}_r(u)(0)=S^{\pm}(u)(0),\ S^{\pm}_r(u)(y)\le S^{\pm}(u)(0)\quad\forall y\in\mathbb{R}^n\text{ and }r>0.
$$

Then u is a linear function.

With those lemmas above, we are ready to prove that all blow-ups are linear.

Proof of Theorem 6.1 Fix $x^0\in U_{2\epsilon}$ for any $\epsilon>0$. Up to dilations and translations, we may assume that $x^0=0$, $A(0)=I_n$ and $u(x^0)=0$. Let $A_r(x)=A(rx)$ and $u_r(x)=\frac{u(rx)}{r}$ for all $x\in\frac{1}{r}U$ and $r<r_{\epsilon,\lambda}$

Let $x,y\in B_{A_r}(0,\frac{r_{\epsilon,\lambda}}{2r})$. By (6.1), $rx,ry\in B_A(0,r_{\epsilon,\lambda}/2)\subset U_\epsilon$. Hence, by Lemma 6.3,

$$
\begin{aligned}
|u_r(x)|&=\frac{|u(rx)|}{r}\le C\frac{1}{r}d_A(rx,0)=Cd_{A_r}(x,0)\le CL|x|,\\
|u_r(x)-u_r(y)|&=\frac{|u(rx)-u(ry)|}{r}\le C\frac{1}{r}d_A(rx,ry)\le Cd_{A_r}(x,y)\le CL|x-y|.
\end{aligned}
$$

For each sequence $\{r_j\}$ with $r_j\to 0$ as $j\to\infty$, by the Arzela–Ascoli lemma, there is a subsequence $\{r_{j_k}\}$ and v such that $u_{r_{j_k}}\to v$ locally uniformly in $\mathbb{R}^n$ as $k\to\infty$. For short we write $\{r_{j_k}\}$ as $\{r_j\}$ below. Obviously, $v(0)=0$. By the compactness of viscosity solutions and $A(0)=I_n$, we have $\Delta_\infty v=0$ in $\mathbb{R}^n$ in viscosity sense.

We claim that $S_s^{\pm}(v)(0) = S^{\pm}(v)(0) = S_A^{\pm}(u)(0)$ and $S_s^{\pm}(v)(x) \le S^{\pm}(v)(0)$ for all $s > 0$ and $x \in \mathbb{R}^n$. If this is true, then Lemma 6.5 implies that v is linear. Theorem 6.1 then follows from this and Corollary 6.3.

To see the claim, observe that $\lim_{j\to\infty} u_{r_j} = v$ and $\lim_{r\to 0} d_{A_r} = |\cdot - \cdot|$ locally uniformly in $\mathbb{R}^n$ (see [20]). This implies that

$$\lim_{j\to\infty} S^{\pm}_{A_{r_j},s}(u_{r_j})(x) = \lim_{j\to\infty} \sup_{x\in B_{A_{r_j}}(x,s)} \frac{\pm u_{r_j}(x)}{s} = \sup_{x\in B(x,s)} \frac{\pm v(x)}{s} = S_s^{\pm}(v)(x).$$

By Corollary 6.3, for every $s > 0$ we have

$$\lim_{j\to\infty} S^{\pm}_{A_{r_j},s}(u_{r_j})(0) = \lim_{j\to\infty} S^{\pm}_{A,sr_j}(u)(0) = S^{\pm}_A(u)(0).$$

Hence, $S_s^{\pm}(v)(0) = S_A^{\pm}(u)(0)$ for all $s > 0$. Moreover, for all $x \in \mathbb{R}^n$, if $R \in (0, r_{\epsilon,\lambda})$ and $r_j < R/s$, since the maps $r \to S^{\pm}_{\mathcal{L}^{\lambda}_A,r}(u)(r_j x)$ are increasing when $\lambda > 2\|f\|_{C(U_\epsilon)}$, we have

$$S^{\pm}_{A_{r_j},s}(u_{r_j})(x) = S^{\pm}_{A,sr_j}(u)(r_j x) \le S^{\pm}_{\mathcal{L}^{\lambda}_A,sr_j}(u)(r_j x) \le S^{+}_{\mathcal{L}^{\lambda}_A,R}(u)(r_j x).$$

Letting $j \to \infty$ first and $R \to 0$ later, by Corollary 6.3 we arrive at

$$S_s^{\pm}(v)(x) \le S_A^{\pm}(u)(0) = S^{\pm}(v)(0)$$

as desired. □

7 Approximation equations

In this section, we always let $f, g \in C^\infty(U)$, and $A \in C^\infty(U; \mathbb{R}^{n\times n})$ being symmetric and uniformly elliptic. Assume that $V = B(0,3) \Subset U$ and $f > 0$ in V. For $\epsilon \in (0,\infty)$, we consider the approximation equations:

$$\mathscr{A}_H[v] + \epsilon \operatorname{div}(ADv) = f \quad \text{in} \quad V; \quad v = g \quad \text{on} \quad \partial V. \tag{7.1}$$

Lemma 7.1 *For each $\epsilon \in (0, \infty)$, there exists a classical solution $u^\epsilon \in C^\infty(V) \cap C(\overline{V})$ solves (7.1).*

Assume that $\{u^\epsilon\}_{\epsilon>0}$ are (viscosity) solutions to (7.1) as given in Lemma 7.1. We have the following uniform estimates for u^ϵ, locally uniform estimates for Du^ϵ and locally uniform flat estimates for $|Du^\epsilon|^2 - u^\epsilon_n$. Write $L_V \ge 1$ as the elliptic constant of A in V, that is,

$$\frac{1}{L_V}|p|^2 \le \langle A(x)p, p\rangle \le L_V|p| \quad \forall x \in V \ p \in \mathbb{R}^n.$$

Lemma 7.2 *Assume that $1 \le L_V < 2^{1/4}$.*

(i) There exists $\delta_0 > 0$ such that if $\|DA\|_{C(V)} \le \delta_0$, then

$$\sup_{\epsilon\in(0,1]} \max_{\overline{V}} |u^\epsilon| \le C,$$

where $C \ge 1$ depends on $\|g\|_{C(\partial V)}$, $\|f\|_{C(V)}$.

(ii) Moreover, for any $\gamma \in (0,1)$, there exists δ_γ such that if $\|DA\|_{C(V)} \le \delta_\gamma$, then

$$\sup_{\epsilon\in(0,1]} |u^\epsilon(x) - g(x^0)| \le C|x - x^0|^\gamma, \ \forall\, x^0 \in \partial V \text{ and } x \in V, \tag{7.2}$$

where $C \ge 1$ depends on γ, $\|g\|_{C^{0,1}(\overline{V})}$, $\|f\|_{C(V)}$.

Lemma 7.3 *Assume that* $1 \le L_V < 2^{1/4}$. *For each* $W \Subset V$, *there exists a constant* $C \ge 1$ *depending on* $\|g\|_{C(\partial V)}$, $\|f\|_{C(V)}$, $\|Df\|_{C(V)}$, $\|A\|_{C(V)}$, $\|DA\|_{C(V)}$, $\|D^2 A\|_{C(V)}$ *and* $\mathrm{dist}\,(W, \partial V)$ *such that*

$$\sup_{\epsilon\in(0,1]} \max_{\overline{W}} |Du^\epsilon| \le C.$$

Lemma 7.4 *Assume that* $1 \le L_V < 2^{\frac{1}{4}}$. *and* $A(0) = I_n$. *Suppose that, for some small constant* $\lambda > 0$,

$$\|DA\|_{C(V)} + \|D^2 A\|_{C(V)} + \|Df\|_{C(V)} \le \lambda$$

and

$$\max_{x\in B(0,2)} |u^\epsilon(x) - x_n| \le \lambda.$$

Then there exists a constant $C > 0$ *depending* $\|f\|_{C(V)}$, $\|g\|_{C(\partial V)}$ *but independent of* λ *such that*

$$|Du^\epsilon(x)|^2 \le u^\epsilon_n(x) + C\lambda^{1/2} \quad \forall\, x \in B(0,1) \text{ and } \epsilon \in (0,1].$$

Lemma 7.1 follows from the elliptic theory (see [17, Chapters 13&14]).

Proof of Lemma 7.1 To show that (7.1) has a solution $u \in C^\infty(V) \cap C(\overline{V})$, due to the elliptic theory, it suffices to show this equation has a solution $u \in C^{2,\alpha}(V) \cap C(\overline{V})$ for some $\alpha \in (0,1)$. Indeed, if Du is bounded locally in V, and hence the above equation is a uniform elliptic equation in each subdomain $W \Subset V$, then the elliptic theory yields that $u \in C^\infty(W)$ as desired.

For convenience, we only consider the case $\epsilon = 1$; the case $\epsilon \ne 1$ is similar. Rewrite (7.1) with $\epsilon = 1$ as

$$a^{ij}(x, Dv)v_{ij} + b(x, Dv) = 0 \quad \text{in} \quad V; v = g \quad \text{on} \quad \partial V$$

where

$$\begin{aligned} a^{ij}(x,p) &= 2a^{ik}(x)p_k a^{j\ell}(x)p_\ell + a^{ij}(x), \\ b(x,p) &= -f(x) + a^{ij}_k(x)p_i p_j a^{k\ell}(x)p_\ell + a^{ij}_i(x)p_i. \end{aligned}$$

We always use the Einstein summation convention and also write $v_i = \frac{\partial}{\partial x_i} v$ and $v_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} v$. Obviously $a^{ij}(x,p) \in C^\infty(\overline{V} \times \mathbb{R}^n)$ and $b(x,p) \in C^\infty(\overline{V} \times \mathbb{R}^n)$. Set

$$\Lambda(x,p) = L(1 + 2L|p|^2)$$

and

$$\mathcal{E}(x,p) = a^{ij}(x,p)p_i p_j = a^{ij}(x)p_i p_j + 2[a^{ij}(x)p_i p_j]^2$$

for all $x \in V$ and $p \in \mathbb{R}^n$. Then

$$\frac{1}{L}|p|^2 \le \mathcal{E}(x,p) \le \Lambda(x,p)|p|^2.$$

By [17, Theorem 13.8], the existence of a solution $u \in C^{2,\alpha}(V) \cap C(\overline{V})$ is reduced to proving that if $v^\sigma \in C^{2,\alpha}(\overline{V})$ are a solution to the equation

$$\sum_{i,j=1}^n a^{ij}(x, Dv)v_{ij} + \sigma b(x, Dv) = 0 \quad \text{in} \quad V; v = \sigma g \quad \text{on} \quad \partial V$$

for each $\sigma \in [0,\ 1]$, then there exists a constant $C \geq 1$ independent of σ such that

$$\sup_{\overline{V}} |v^\sigma| + \sup_{\overline{V}} |Dv^\sigma| \leq C.$$

Note that $\sup_{\overline{V}} |v^\sigma| \leq C$ follows from the maximum principle (see [17, Theorem 10.3]) due to

$$\frac{|\sigma b(x,p)|}{\mathcal{E}(x,p)} \leq \frac{\mu_1|p| + \mu_2}{|p|^2} \quad \forall x \in \overline{V} \text{ and } p \in \mathbb{R}^n$$

for some constant $\mu_1, \mu_2 \geq 1$ independent of σ.

Moreover, $\sup_{\partial V} |Dv^\sigma| \leq C$ follows from [17, Theorem 14.1] since there exists a constant $\mu \geq 1$ independent of σ such that

$$\begin{aligned}
&|p|\Lambda(x,p) + |\sigma b(x,p)| \\
&\quad \leq L|p|(1 + 2L|p|^2) + \|f\|_{C(V)} + n^2\|DA\|_{C(V)}|p|(1 + n\|A\|_{C(V)}|p|^2) \\
&\quad \leq \mu \frac{1}{L}|p|^2(1 + \frac{1}{L}|p|^2) \\
&\quad \leq \mu\mathcal{E}(x,p)
\end{aligned}$$

whenever $|p| \geq \mu$, $x \in \overline{V}$ and $\sigma \in [0, 1]$.

Finally, we prove that $\sup_V |Dv^\sigma| \leq C$. We consider the following quantities

$$\begin{aligned}
\alpha(x,p) &= \frac{1}{\mathcal{E}(x,p)}\left\{(\sum_{i=1}^n p_i D_{p_i} - 1)\mathcal{E}(x,p)\right\} \\
\beta_A(x,p) &= \frac{1}{\mathcal{E}(x,p)}\sum_{i=1}^n \{|p|^{-2} p_i D_{x_i}\mathcal{E}(x,p) + \sigma(p_i D_{p_i} - 1)b(x,p)\} \\
\gamma_A(x,p) &= \frac{1}{\mathcal{E}(x,p)}\left\{\frac{|p|^2}{4/L}\sum_{k=1}^n [|p|^{-2} p_k a_k^{ij}(x)] + \sigma\sum_{i=1}^n p_i D_{p_i} b(x,p)\right\}
\end{aligned}$$

that is, $r = -1$, $s = 0$, $\delta = |p|^{-2}\sum_{i=1}^n p_i D_{x_i}$, $\overline{\delta} = \sum_{i=1}^n p_i D_{p_i}$, $a_*^{ij}(x,p) = a^{ij}(x)$ and $\lambda^* = 1/L$ in [17, (15.27)]. With the aid of $\sup_{\partial V} |Dv^\sigma| \leq C$, $\sup_V |Dv^\sigma| \leq C$ follows from Theorem 15.2 of [17] if we can show that $\limsup_{p\to\infty} \alpha(x,p)$ and $\limsup_{p\to\infty} \beta_A(x,p)$ are uniformly in $x \in V$ and are uniformly bounded in $\sigma \in [0, 1]$, and $\limsup_{p\to\infty} \gamma_A(x,p) = 0$. Observing that

$$\begin{aligned}
p_i D_{p_i}\mathcal{E}(x,p) &= \sum_{k=1}^n p_k D_{p_k}\{a^{ij}(x)p_i p_j + 2[a^{ij}(x)p_i p_j]^2\} \\
&= 2a^{ij}(x)p_i p_j + 8[a^{ij}(x)p_i p_j]^2,
\end{aligned}$$

we know that $\limsup_{p\to\infty} \alpha(x,p) = 3$ uniformly in $x \in V$. Moreover, by $\mathcal{E}(x,p) = O(|p|^4)$, $A \in C^1(\overline{V}; \mathbb{R}^{n\times n})$, we have

$$\begin{aligned}
\sum_{i=1}^n p_i D_{x_i}\mathcal{E}(x,p) &= \sum_{k=1}^n p_k D_{x_k}\{a^{ij}(x)p_i p_j + 2[a^{ij}(x)p_i p_j]^2\} \\
&= \sum_{k=1}^n p_k a_k^{ij}(x)p_i p_j[1 + 2a^{k\ell}(x)p_k p_\ell] \\
&\leq O(|p|^5),
\end{aligned}$$

and

$$\sum_{i=1}^{n} p_i D_{p_i} b(x,p) = 3a_k^{ij}(x) p_i p_j a^{k\ell}(x) p_\ell + a_i^{ij}(x) p_j \le O(|p|^3),$$

we obtain $\limsup_{p\to\infty} \beta_A(x,p) = \limsup_{p\to\infty} \gamma_A(x,p) = 0$ uniformly in $x \in V$ and $\sigma \in [0,1]$ as desired. □

Lemma 7.2 follows from Lemma 4.2.

Proof of Lemma 7.2 *Proof of (i)* Let $v = \|g\|_{C(\partial V)}$ be a constant function. Since $f > 0$, by Lemma 4.2, we know that $u^\epsilon \le \|g\|_{C(\partial V)}$ in V.

To get the lower bound of u^ϵ in V, it suffices to find $w \in C(\overline{V})$ such that $w \le g$ on ∂V,

$$w \ge -C(\|f\|_{C(V)}, \|g\|_{C(\partial V)}) \quad \text{in } V, \tag{7.3}$$

and

$$\mathscr{A}_H[w] + \epsilon \operatorname{div}(ADw) \ge \|f\|_{C(V)} \quad \text{in} \quad V. \tag{7.4}$$

Indeed, if such an w exists, then by Lemma 4.2, $w \le u^\epsilon$ in V and hence

$$u^\epsilon \ge w \ge -C(\|g\|_{C(\partial V)}, \|f\|_{C(V)}) \quad \text{in } V$$

as desired.

We take $w(x) = -\lambda|x-x^0|^\gamma - \|g\|_{C(\partial V)}$ where $\gamma \in (0,1)$ and $x^0 \in \partial V$, but the value of $\lambda > 1/\gamma$ will be determined later. Then $w \le g$ on ∂V. It is easy to see that

$$2a^{ik}(x)w_k(x)a^{j\ell}(x)w_\ell(x) = 2\lambda^2\gamma^2|x-x^0|^{2\gamma-4}a^{ik}(x)(x_k-x_k^0)a^{j\ell}(x)(x_\ell-x_\ell^0)$$

and

$$-w_{ij}(x) = \lambda\gamma(\gamma-2)|x-x^0|^{\gamma-4}(x_i-x_i^0)(x_j-x_j^0) + \lambda\gamma|x-x^0|^{\gamma-2}\delta_{ij}.$$

Then

$$\begin{aligned}
&-2a^{ik}(x)w_k(x)a^{j\ell}(x)w_\ell(x)w_{ij}(x) - \epsilon a^{ij}(x)w_{ij}(x)\\
&\quad = 2\lambda^3\gamma^3(\gamma-2)|x-x^0|^{3\gamma-8}[a^{ik}(x)(x_k-x_k^0)(x_i-x_i^0)]^2\\
&\qquad + \epsilon\lambda\gamma(\gamma-2)|x-x^0|^{\gamma-4}a^{ij}(x)(x_j-x_j^0)(x_i-x_i^0)\\
&\qquad + 2\lambda^3\gamma^3|x-x^0|^{3\gamma-6}a^{ik}(x)a^{i\ell}(x)(x_k-x_k^0)(x_\ell-x_\ell^0)]\\
&\qquad + \epsilon\lambda\gamma|x-x^0|^{\gamma-2}a^{ij}\delta_{ij}\\
&\quad \le 2\lambda^3\gamma^3(\gamma-2)|x-x^0|^{3\gamma-4}\frac{1}{L^2} + \epsilon\lambda\gamma(\gamma-2)|x-x^0|^{\gamma-2}\\
&\qquad + L^2\lambda^3\gamma^3|x-x^0|^{3\gamma-4} + nL\lambda\epsilon\gamma|x-x^0|^{\gamma-2}.
\end{aligned}$$

By $L^4 < 2$, we have $\frac{2}{L^2}(2-\gamma) + L^2 \le \frac{2}{L^2}(\gamma-1)$, and moreover, we can choose λ large enough such that

$$nL \le \frac{1}{L^2}\lambda^2\gamma^2(1-\gamma)6^{2\gamma-2}.$$

Thus

$$-2a^{ik}(x)w_k(x)a^{j\ell}(x)w_\ell(x)w_{ij}(x) - \epsilon a^{ij}(x)w_{ij}(x) \le \frac{1}{L^2}\lambda^3\gamma^3(\gamma-1)|x-x^0|^{3\gamma-4}.$$

Moreover,

$$\begin{aligned}
&|-f(x)+a_k^{ij}(x)w_i(x)w_j(x)a^{k\ell}(x)w_\ell(x)+\epsilon a_i^{ij}(x)w_i(x)|\\
&\le \|f\|_{C(V)}+n^2\|DA\|_{C(V)}L|Dw|^3+\epsilon n^2\|DA\|_{C(V)}|Dw|\\
&\le \|f\|_{C(V)}+n^2\delta_0L\lambda^3\gamma^3|x-x^0|^{3\gamma-3}+n^2\delta_0\lambda\gamma|x-x^0|^{\gamma-1}.
\end{aligned}$$

If $\delta_0\le(1-\gamma)/8n^2 6^{2\gamma-3}$, then

$$n^2\delta_0L\lambda^3\gamma^3|x-x^0|^{3\gamma-3}+\epsilon n^2\delta_0\lambda\gamma|x-x^0|^{\gamma-1}\le\frac{1}{2L^2}\lambda^3\gamma^3(\gamma-1)|x-x^0|^{3\gamma-4}.$$

Combining these estimates, we arrive at

$$\mathscr{A}_H[w]+\epsilon\,\mathrm{div}\,(ADw)\ge\frac{1}{2L^2}\lambda^3\gamma^3(1-\gamma)|x-x^0|^{3\gamma-4}-\|f\|_{C(V)}\ge\|f\|_{C(V)}$$

if we let λ be large enough such that

$$\frac{1}{L^2}\lambda^3\gamma^3(1-\gamma)6^{3\gamma-4}\ge4\|f\|_{C(V)}.$$

This gives (7.4).

Proof of (ii). Take a point $x^0\in\partial V$. Define $w(x)=-\lambda|x-x^0|^\gamma$, the value of λ will be determined later. First, since $g\in C^{0,1}(\partial V)$, we can choose $\lambda>\|g\|_{C^{0,1}(\partial V)}$ such that

$$w+g(x^0)\le g\le g(x^0)-w\text{ on }\partial V.$$

Moreover, following the procedure in (i), if $\|DA\|_{C(V)}\le\delta_\gamma=(1-\gamma)/8n^2 6^{2\gamma-3}$, and λ is large enough (depending on $\|f\|_{C(V)}$), we have

$$\mathscr{A}_H[w]+\epsilon\,\mathrm{div}\,(ADw)\ge\|f\|_{C(V)}.$$

Applying Lemma 4.2, we conclude that

$$w+g(x^0)\le u^\epsilon\le g(x^0)-w\text{ in }V.$$

That is, $|u^\epsilon(x)-g(x^0)|\le C|x-x^0|^\gamma$ as desired. □

The proofs of Lemmas 7.3 and 7.4 are similar to those of [28, Theorem 3.1 and Theorem 3.3] respectively, where $f=0$. Here we only sketch it by omitting several details, but pointing out that the additional terms comes from $f\neq0$ can be controlled.

Proof of Lemma 7.3 We let all the notation be the same as in the proof of [28, Theorem 3.1] except that we write A^ϵ, H^ϵ, f^ϵ, g^ϵ, u^ϵ there as A, H, f, g, u here for simple.

Recall that

$$\mathscr{A}_H[u]=2a^{ik}u_ku_{ij}a^{j\ell}u_\ell+a_k^{ij}u_iu_ja^{k\ell}u_\ell.$$

We always use the Einstein summation convention. Taking $\frac{\partial}{\partial s}$ of the equation $\mathscr{A}_H[u]+\epsilon\,\mathrm{div}\,(ADu)=f$, we obtain

$$\begin{aligned}
&2a^{ik}u_ku_{ijs}a^{j\ell}u_\ell+4a_s^{ik}u_ku_{ij}a^{j\ell}u_\ell+4a^{ik}u_{ks}u_{ij}a^{j\ell}u_\ell+a_{ks}^{ij}u_iu_ja^{k\ell}u_\ell+2a_k^{ij}u_{is}u_ja^{k\ell}u_\ell\\
&\quad+a_k^{ij}u_iu_ja_s^{k\ell}u_\ell+a_k^{ij}u_iu_ja^{k\ell}u_{\ell s}+\epsilon\,\mathrm{div}(ADu_s)+\epsilon\,\mathrm{div}(A_sDu)=f_s.
\end{aligned}\tag{7.5}$$

Set

$$G_m:=4a^{im}u_{ij}a^{j\ell}u_\ell+2a_k^{mj}u_ja^{k\ell}u_\ell+a_k^{ij}u_iu_ja^{km},\tag{7.6}$$

and

$$F_s := 4a_s^{ik}u_k u_{ij}a^{j\ell}u_\ell + a_k^{ij}u_i u_j a_s^{k\ell}u_\ell + a_{ks}^{ij}u_i u_j a^{k\ell}u_\ell + \epsilon\,\mathrm{div}(A_s Du). \tag{7.7}$$

Define the operator L_ϵ by

$$L_\epsilon v := 2a^{ik}u_k v_{ij}a^{j\ell}u_\ell + \sum_{m=1}^{n} G_m v_m + \epsilon\,\mathrm{div}(ADv). \tag{7.8}$$

Then (7.5) can be written as

$$-L_\epsilon(u_s) = F_s + f_s. \tag{7.9}$$

Set $v := \frac{1}{2}|Du|^2$. Then, by (7.9) and an argument similar to [28, Theorem 3.1] we have

$$L_\epsilon v = 2|D^2uADu|^2 + \sum_{s=1}^{n}\left[\epsilon a^{ij}u_{si}u_{sj} - u_s F_s - u_s f_s\right]. \tag{7.10}$$

Set $z := \frac{1}{2}(u)^2$. Then, by $\mathscr{A}_H[u]+\epsilon$ div $(ADu) = f$ and an argument similar to [28, Theorem 3.1] we have

$$\begin{aligned} L_\epsilon z &= 2\langle Du, ADu\rangle^2 + \epsilon\langle ADu, Du\rangle + uf \\ &\quad + 4u\langle ADu, D^2uADu\rangle + 2u\langle\langle Du, DADu\rangle, ADu\rangle, \end{aligned}$$

where $\langle Du, DADu\rangle$ is interpreted as the vector $(\langle Du, A_k Du\rangle)_k$ with A_k being the element-wise derivative of A.

Choose $\phi \in C_0^\infty(V)$ such that $\phi = 1$ in V, $0 \le \phi \le 1$, and, for $\beta > 0$ to be determined later, define the auxiliary function w by

$$w := \phi^2 v + \beta z.$$

If w attains its maximum on ∂V, then

$$\sup_{\overline{V}} v \le \sup_{\overline{V}} w(x) \le \max_{\overline{V}} w = \max_{\partial V} w = \frac{\beta}{2}\max_{\partial V} u^2,$$

as desired. Thus we may assume w attains its maximum at an interior point $x^0 \in V$. This gives $Dw(x^0) = 0$ and $D^2w(x^0) \le 0$, so that

$$-L_\epsilon w(x^0) = -(2a^{ik}u_k a^{j\ell}u_\ell + \epsilon a^{ij})w_{ij}\Big|_{x=x^0} \ge 0. \tag{7.11}$$

On the other hand, by (7.10) and (7.11), similarly to the proof of [28, Theorem 3.1] we have that, at $x = x^0$,

$$\begin{aligned} 0 &\le -L_\epsilon w(x^0) = -L_\epsilon(\phi^2 v) - \beta L_\epsilon z \\ &= \left[-2\phi^2|D^2uADu|^2 - \epsilon\phi^2\sum_{s=1}^{n} a^{ij}u_{si}u_{sj} - 2\beta\langle Du, ADu\rangle^2 - \epsilon\beta\langle Du, ADu\rangle - \beta uf\right] \\ &\quad - \left[4\beta u\langle ADu, D^2uADu\rangle + 2\beta u a_k^{mj}u_j u_m a^{k\ell}u_\ell\right] \\ &\quad - \left[8\phi a^{ik}u_k a^{j\ell}u_\ell\phi_i\sum_{r=1}^{n} u_{rj}u_r + 4\epsilon\phi\sum_{m=1}^{n}\phi_i a^{ij}u_{mj}u_m\right] + \phi^2\sum_{s=1}^{n} u_s[F_s + f_s] - vL_\epsilon(\phi^2) \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Observe that the terms I_2, I_3, I_5 are exactly the same as in the proof of [28, Theorem 3.1]. So the same argument as there leads to that

$$I_2 \le \beta^{4/3}|D^2uADu|^{4/3} + C|Du|^4 + C(\beta),$$
$$I_3 \le \frac{1}{8}|D^2uADu|^2\phi^2 + \frac{\epsilon}{16L}|D^2u|^2\phi^2 + C|Du|^4 + C,$$
$$I_5 \le \frac{1}{8}|D^2uADu|^2\phi^2 + C|Du|^4 + C,$$

Comparing I_1 and I_4 with those in the proof of [28, Theorem 3.1], we get additional terms βuf in I_1 and $\phi^2\sum_{s=1}^n u_s f_s$ in I_4 here. But applying an argument similar to proof of [28, Theorem 3.1], we also have

$$I_1 \le -2\phi^2|D^2uADu|^2 - \frac{\epsilon}{L}\phi^2|D^2u|^2 - \frac{2\beta}{L^2}|Du|^4 + C(\beta)$$
$$I_4 \le \frac{1}{8}|D^2uADu|^2\phi^2 + C|Du|^4 + \frac{\epsilon}{16L}\phi^2|D^2u|^2 + C.$$

Above $C > 0$ denotes constants depending only on n, L, $\|A\|_{C^{1,1}(V)}$, $\|f\|_{C^1(V)}$, $\|u\|_{C(\overline{V})}$, $\|f\|_{C(V)}$ and $\mathrm{dist}(V, \partial V)$.

Combining all these estimates with (7.11) yields that, at $x = x^0$,

$$\begin{aligned}
&2\phi^2|D^2uADu|^2 + \frac{\epsilon}{L}\phi^2|D^2u|^2 + \frac{2}{L^2}\beta|Du|^4 \\
&\quad \le |D^2uADu|^2\phi^2 + C|Du|^4 + C\beta^{4/3}|D^2uADu|^{4/3} + \frac{\epsilon}{8L}\phi^2|D^2u|^2 + C(\beta),
\end{aligned}$$

so that

$$|D^2uADu|^2\phi^2 + \frac{2}{L^2}\beta|Du|^4 \le C|Du|^4 + C\beta^{4/3}|D^2uADu|^{4/3} + C(\beta).$$

We may choose $\beta > 1$ sufficiently large so that

$$|D^2uADu|^2\phi^2 + \frac{\beta}{L^2}|Du|^4 \le C\beta^{4/3}|D^2uADu|^{4/3} + C(\beta).$$

Multiplying both sides of this inequality by ϕ^4 and applying Young's inequality implies

$$\begin{aligned}
|D^2uADu|^2\phi^6 + \frac{\beta}{L^2}|Du|^4\phi^4 &\le C\beta^{4/3}|D^2uADu|^{4/3}\phi^4 + C(\beta) \\
&\le \frac{1}{2}|D^2uADu|^2\phi^6 + C(\beta).
\end{aligned}$$

Hence we have $|Du(x^0)|^4\phi(x^0)^4 \le C$.

This finishes the proof. □

Proof of Lemma 7.4 We let all the notation be the same as in the proof of [28, Theorem 3.3] except that we write $A^\epsilon, f^\epsilon, u^\epsilon$ as A, f, u for simplicity.

Set $\Phi(p) := (|p|^2 - p_n)_+^2 = \max\{|p|^2 - p_n, 0\}^2$. Let $\phi \in C_0^\infty(B(0,3))$ be such that

$$\phi = 1 \text{ in } B(0,1),\ \phi = 0 \text{ outside } B(0,2),\ 0 \le \phi \le 1, \text{ and } |D\phi| \le 2.$$

Define

$$v = \phi^2\Phi(Du) + \beta(u - x_n)^2 + \lambda|Du|^2,$$

where $\beta > 0$ is a sufficiently large number whose value will be determined later. Applying Lemmas 7.2 and 7.3, we have $|u| + |Du| \le C$ in $B(0,2)$. If $\max_{B(0,2)} v$ is attained on $\partial B(0,2)$, then by the same argument as in [28], we have the desired estimate. Therefore we may assume that v attains its maximum at an interior point $x^0 \in B(0,2)$. Moreover, we can also assume $\big(|Du|^2 - u_n\big)(x^0) > 0$.

To estimate $v(x^0)$, let L_ϵ and F_s be given by (7.8) and (7.7). We need to compute $L_\epsilon v$ at x^0. Using

$$\mathscr{A}_H[u] + \epsilon\,\mathrm{div}\,(ADu) = 2a^{ik}u_k u_{ij} a^{j\ell} u_\ell + a_k^{ij} u_i u_j a^{k\ell} u_\ell + \epsilon\,\mathrm{div}\,(ADu) = f,$$

similarly to [28, Theorem 3.3] we obtain

$$\begin{aligned} -L_\epsilon((u-x_n)^2) &= -4\big(\langle Du, ADu\rangle - a^{nk}u_k\big)^2 - 2\epsilon\langle Du - \mathbf{e}_n, A(Du - \mathbf{e}_n)\rangle \\ &\quad - 8a^{ik}(u_k - \delta_{kn})u_{ij}a^{j\ell}u_\ell(u - x_n) \\ &\quad - 4a_k^{ij}(u_i - \delta_{in})u_j a^{k\ell}u_\ell(u - x_n) \\ &\quad + 2a_k^{ij}u_i u_j a^{k\ell}\delta_{\ell n}(u - x_n) + 2\epsilon\sum_{i=1}^n a_i^{in}(u - x_n) - 2(u - x_n)f \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7, \end{aligned}$$

where we denote $\mathbf{e}_n = (0,\ldots,0,1)$. Comparing the formula of $-L_\epsilon((u-x_n)^2)$ as in that appeared in the proof of [28, Theorem 3.3], we will see that the terms J_1 through J_6 are the same and J_7 is new here due to $f \neq 0$. Regards of the terms J_1 to J_6, with the aid of Theorem 7.2 and by exactly the same argument as in the proof of [28, Theorem 3.3], we have

$$J_2 \le -\frac{\epsilon}{L}\big|Du - \mathbf{e}_n\big|^2, \quad |J_3| \le C\lambda|D^2uADu|, \quad |J_4| + |J_5| \le C\lambda, \quad |J_6| \le C\epsilon\lambda,$$

and

$$J_1 \le -4\big||Du|^2 - u_n\big|^2 + C\lambda,$$

where we use $\|DA\|_{L^\infty} \le \lambda$ and $A(0) = I_n$. It is easy to see that $|J_7| \le C\lambda$. Therefore, we arrive at

$$-L_\epsilon\big((u-x_n)^2\big) \le -4\big(|Du|^2 - u_n\big)^2 - \frac{2\epsilon}{L}|Du - \mathbf{e}_n|^2 + C\lambda(1 + |D^2uADu|) \tag{7.12}$$

Moreover, similarly to the proof of [28, Theorem 3.1], by (7.10) we have

$$\frac{1}{2}L_\epsilon\big(|Du|^2\big) \ge |D^2uADu|^2 + \frac{\epsilon}{L}|D^2u|^2 - C. \tag{7.13}$$

Next we need to estimate $L_\epsilon(\phi^2\Phi(Du))$. As explained earlier, we may assume $|Du|^2 > u_n$ at $x^0 \in B(0,2)$. As in the proof of [28, Theorem 3.3] for $L_\epsilon(\Phi(Du))$, we write, at $x = x^0$,

$$\begin{aligned}
L_\epsilon(\Phi(Du)) &= 4a^{ik}u_k a^{j\ell}u_\ell\Big(2\sum_{s=1}^n u_{sj}u_s - u_{nj}\Big)\Big(2\sum_{s=1}^n u_{si}u_s - u_{ni}\Big)\\
&\quad + 8\big(|Du|^2 - u_n\big)a^{ik}u_k a^{j\ell}u_\ell\Big(\sum_{s=1} u_{si}u_{sj}\Big)\\
&\quad + 2\epsilon a^{ij}\Big(2\sum_{s=1}^n u_{si}u_s - u_{ni}\Big)\Big(2\sum_{s=1}^n u_{sj}u_s - u_{nj}\Big)\\
&\quad + 4\epsilon a^{ij}\Big(|Du|^2 - u_n\Big)\Big(\sum_{s=1}^n u_{sj}u_{sj}\Big)\\
&\quad + 2\big(|Du|^2 - u_n\big)\Big(2\sum_{s=1}^n u_s L_\epsilon(u_s) - L_\epsilon(u_n)\Big)\\
&= K_1 + K_2 + K_3 + K_4 + K_5.
\end{aligned} \tag{7.14}$$

Here G_m is as defined in (7.6). The estimate of $K_1, \ldots, K_4$ are exactly the same as in the proof of [28, Theorem 3.3], that is,

$$\begin{aligned}
K_1 &= 4\Big[2\langle Du, D^2uADu\rangle - \langle (D^2u)^n, ADu\rangle\Big]^2,\\
K_2 &= 8(|Du|^2 - u_n)|D^2uADu|^2,\\
K_3 &\geq \frac{2\epsilon}{L}\sum_{i=1}^n\Big(2\sum_{s=1}^n u_{si}u_s - u_{ni}\Big)^2,\\
K_4 &\geq \frac{4\epsilon}{L}\big(|Du|^2 - u_n\big)\big|D^2u\big|^2,
\end{aligned}$$

where $(D^2u)^n$ denotes the nth-row of D^2u. Regards of K_5, by (7.5), we have

$$K_5 = 2\big(|Du|^2 - u_n\big)\Big(\sum_{s=1}^n 2u_sF_s + u_sf_s - F_n - f_n\Big).$$

Applying Lemma 7.3, we obtain

$$\big|K_5\big| \leq \big(|Du|^2 - u_n\big)\Big(C\lambda|D^2uADu| + \frac{\epsilon}{4L}|D^2u|^2 + C\lambda\Big).$$

Putting these estimates into (7.14) gives

$$\begin{aligned}
L_\epsilon(\Phi(Du)) \geq\ & 8\big(|Du|^2 - u_n\big)\Big(|D^2uADu|^2 + \frac{\epsilon}{4L}|D^2u|^2\Big)\\
&+ 4\Big[2\langle Du, D^2uADu\rangle - \langle (D^2u)^n, ADu\rangle\Big]^2\\
&+ \frac{2\epsilon}{L}\sum_{i=1}^n\Big(2\sum_{s=1}^n u_{si}u_s - u_{ni}\Big)^2\\
&- C\lambda(|Du|^2 - u_n)|D^2uADu| - C\lambda.
\end{aligned} \tag{7.15}$$

Applying this and using the arguments same as in the proof of [28, Theorem 3.3], we conclude that

$$L_\epsilon\big(\phi^2\Phi(Du)\big) \geq -C(|Du|^2 - u_n)^2 - C\lambda(|Du|^2 - u_n) - C\lambda. \tag{7.16}$$

Combining the estimates (7.12), (7.13), with (7.16) yields that, at $x = x^0$,

$$\begin{aligned}
0 &\leq -L_\epsilon(v) = -L_\epsilon\big(\phi^2\Phi(Du)\big) - \beta L_\epsilon\big((u - x_n)^2\big) - \lambda L_\epsilon\big(|Du|^2\big) \\
&\leq C\big(|Du|^2 - u_n\big)^2 + C\lambda\big(|Du|^2 - u_n\big) + C\lambda \\
&\quad - 4\beta\big(|Du|^2 - u_n\big)^2 - \frac{2\epsilon\beta}{L}\big|Du - \mathbf{e}_n\big|^2 + C\beta\lambda + C\beta\lambda\big|D^2uADu\big| \\
&\quad + 2\lambda\Big(-|D^2uADu|^2 - \frac{\epsilon}{L^2}|D^2u|^2 + C\Big).
\end{aligned}$$

Thus we have that, at $x = x^0$,

$$\begin{aligned}
&(4\beta - C)\big(|Du|^2 - u_n\big)^2 + 2\lambda\big|D^2uADu\big|^2 + \frac{2\lambda\epsilon}{L^2}\big|D^2u\big|^2 \\
&\leq C\lambda\big(|Du|^2 - u_n\big) + C(1+\beta)\lambda + C\beta\lambda\big|D^2uADu\big|.
\end{aligned}$$

Choosing $\beta > C$ and applying Young's inequality, we obtain

$$\beta\big(|Du|^2 - u_n\big)^2 \leq C\lambda + 2\beta^2\lambda.$$

Thus we conclude that $|Du(x^0)|^2 - u_n(x^0) \leq C\sqrt{\lambda}$ as desired. □

8 Everywhere differentiability

In this section we always assume that $A \in C^{1,1}(U; \mathbb{R}^{n\times n})$ is symmetric and uniform symmetric, and $f \in C^{0,1}(U)$.

Theorem 8.1 *If $u \in C(U)$ is a viscosity solution to $\mathscr{A}_H[u] = f$ in U, then u is differentiable everywhere in U.*

Assume that $B(0,3) \Subset U$ and $f > 0$ in $B(0,3)$. Write $V = B(0,3)$. Denote by L_V the ellipticity constant of A in V, and assume that $1 \leq L_V < 2^{\frac{1}{8}}$. It is a standard fact that there exist $\{A^\epsilon\}_{\epsilon>0} \subset C^\infty(U; \mathbb{R}^{n\times n})$, $\{f^\epsilon\}_{\epsilon>0}$, $\{g^\epsilon\}_{\epsilon>0} \subset C^\infty(U)$, and constant $\epsilon_0 \in (0,1)$ such that

- (A1) $A^\epsilon(0) = A(0)$, and A^ϵ is symmetric and uniformly elliptic with constant L_V^2 for all $\epsilon \in (0, \epsilon_0)$
- (A2) $\|DA^\epsilon\|_{C(\overline{V})} \leq 2\|DA\|_{C(\overline{V})}$ and $\|D^2A^\epsilon\|_{C(\overline{V})} \leq 2\|D^2A\|_{L^\infty(\overline{V})}$ for all $\epsilon \in (0, \epsilon_0)$
- (A3) for any $\alpha \in (0,1)$, $A^\epsilon \to A$ in $C^{1,\alpha}(\overline{V})$ as $\epsilon \to 0$,
- (A4) $f^\epsilon > 0$, $\|Df^\epsilon\|_{C(V)} \leq 2\|Df\|_{C(V)}$ and $\|Dg^\epsilon\|_{C(\overline{V})} \leq 2\|Du\|_{L^\infty(V)}$ for all $\epsilon \in (0, \epsilon_0)$,
- (A5) for any $\alpha \in (0,1)$, $f^\epsilon \to f$ in $C^{0,\alpha}(U)$ and $g^\epsilon \to u$ in $C^{0,\alpha}(\overline{V})$ as $\epsilon \to 0$.

For $\epsilon \in (0, \epsilon_0)$ let u^ϵ be the smooth solution to the approximation equation

$$\mathscr{A}_{H^\epsilon}[v] + \epsilon\,\mathrm{div}\,(A^\epsilon Dv) = f^\epsilon \quad \text{in} \quad V; \quad v = g^\epsilon \quad \text{on} \quad \partial V \tag{8.1}$$

as given in Lemma 7.1. We have the following approximation property.

Lemma 8.2 *There exists a constant $\hat{\delta} > 0$ such that if $\|DA\|_{C(\overline{V})} \le \hat{\delta}$, then $u^\epsilon \to u$ locally uniformly in V.*

Proof Fix a $\gamma \in (0, 1)$ and assume $\|DA\|_{C(V)} < \hat{\delta} = \min\{\delta_\gamma, \delta_0\}/2$ where δ_γ and δ_0 are the same as in Lemma 7.2. Then $\|DA^\epsilon\|_{C(V)} < 2\hat{\delta}$ for $\epsilon \in (0, \epsilon_0]$. Notice that A^ϵ has the same elliptic constant $L_V^2 < 2^{1/4}$. By Lemma 7.2, we have

$$\sup_{\epsilon \in (0,\epsilon_0]} \|u^\epsilon\|_{C(\overline{V})} \lesssim \|u\|_{C(\overline{V})} \tag{8.2}$$

and

$$\sup_{\epsilon \in (0,\epsilon_0]} |u^\epsilon(x) - u(x^0)| \le C|x - x^0|^\gamma, \ \forall\, x^0 \in \partial V \text{ and } x \in V. \tag{8.3}$$

Moreover, due to (A1)–(A5) again, applying Lemma 7.3 we know that for any compact subset $K \Subset V$, there exists a constant $C > 0$ such that

$$\sup_{\epsilon \in (0,\epsilon_0]} \left\|Du^\epsilon\right\|_{C(K)} \le C.$$

By this and (8.2) one has that, up to some subsequence, $u^\epsilon \to \hat{u}$ locally uniformly in V for some $\hat{u} \in C^{0,1}(V)$. From this and (8.3), it follows that

$$|\hat{u}(x) - u(x^0)| \le C|x - x^0|^\gamma, \ \forall\, x \in V \text{ and } x^0 \in \partial V.$$

Thus, $\hat{u} \in C(\overline{V})$ and $\hat{u} \equiv u$ on ∂V. By [12, Lemma 6.1], we know that $\hat{u} \in C(\overline{V})$ is a viscosity solution to the Aronsson equation (1.1). Since $\hat{u} \equiv u$ on ∂V and $f > 0$ in V, by Theorem 1.1, we have $\hat{u} = u$. Therefore, $u^\epsilon \to u$ locally uniformly in V as desired. □

With the aid of Lemma 8.2 and Lemma 7.4, Theorem 7.1 follows from an argument similar to those of [16, Theorem 1.1], [28, Theorem 1.1] and [21, Theorem1.2].

Proof of Theorem 8.1 For each fixed point $x^0 \in U$, we need to show the differentiability of u at x^0. Up to consider $\widetilde{u}(\widetilde{x}) = u(x) + C|x_{n+1}|^{4/3}$ for $\widetilde{x} = (x, x_{n+1}) \in U \times \mathbb{R}$ (see e.g. [18, Theorem 1]), we may assume that $f > 0$ in $B(x^0, \frac{1}{2}\operatorname{dist}_A(x^0, \partial U))$. Indeed, differentiability of u at x^0 follows from that of $\widetilde{u}$ at $(x^0, 0)$. Moreover, $\mathscr{A}_{\widetilde{H}}[\widetilde{u}] = \widetilde{f}$ in $U \times \mathbb{R}$, where $\widetilde{f}(\widetilde{x}) = f(x) + C^3 4^3/3^4$ and $\widetilde{H}(\widetilde{x}, p) = \langle \widetilde{A}(\widetilde{x})p, p\rangle$ with $\widetilde{A}(\widetilde{x}) = \operatorname{diag}\{A(x), 1\}$ for all $\widetilde{x} \in U \times \mathbb{R}$ and $p \in \mathbb{R}^{n+1}$. If $4C/3^{4/3} > \|f\|_{C(\overline{V})}$, then $\widetilde{f} > 0$ in V.

Up to some scaling, rotation and translation (see [28, Lemma 4.2]), we may assume that $x^0 = 0$, $u(x^0) = 0$, and $A(x^0) = I_n$. Moreover, we assume that $\operatorname{Lip}_{d_A} u(0) > 0$ otherwise (8.4) holds with $p_0 = 0$. Up to consider $u/\operatorname{Lip}_{d_A} u(0)$, we may further assume that $\operatorname{Lip}_{d_A} u(0) = 1$.

Now, it suffices to prove the existence of a vector $p_0 \in \mathbb{R}^n$ such that

$$|u(h) - \langle p_0, h\rangle| = o(|h|), \ \forall\, h \in \mathbb{R}^n. \tag{8.4}$$

By Theorem 6.1, we need to show that for every pair of sequences $\mathbf{r} = \{r_j\}$ and $\mathbf{s} = \{s_k\}$ that converge to 0, if

$$\lim_{j\to\infty} \max_{y \in B(0,\, 3r_j)} \frac{1}{r_j} |u(y) - \langle \mathbf{a}, y\rangle| = 0$$

and

$$\lim_{k\to\infty} \max_{y \in B(0,\, 3s_k)} \frac{1}{s_k} |u(y) - \langle \mathbf{b}, y\rangle| = 0$$

for some $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^n$, then $\mathbf{a} = \mathbf{b}$.

We prove the above claim by contradiction. Suppose that $\mathbf{a} \neq \mathbf{b}$. Recalling that

$$H(0, \mathbf{a}) = \langle \mathbf{a},\ \mathbf{a} \rangle = \mathrm{Lip}_{d_A} u(0) = \langle \mathbf{b},\ \mathbf{b} \rangle = H(0, \mathbf{b})$$

as given in Theorem 6.1, we have $|\mathbf{a}| = |\mathbf{b}| = 1$. Up to a rotation, we may assume that $\mathbf{a} = e_n$. Since $|\mathbf{b}| = 1$ and $\mathbf{b} \neq e_n$, we have

$$\theta := 1 - b_n > 0.$$

Let $C > 0$ be the constant in Lemma 7.4 and choose $\lambda > 0$ such that

$$C\lambda^{\frac{1}{2}} = \frac{\theta}{4}.$$

Choose $r \in \{r_j\}$ such that $f > 0$ in $B(0, 3r)$,

$$\max_{y \in B(0,3r)} \frac{1}{r} |u(y) - y_n| \leq \frac{\lambda}{4}, \tag{8.5}$$

and

$$\begin{cases} 2^{-1/8}|p|^2 \leq \langle A(x)p,\ p\rangle \leq 2^{1/8}|p|^2,\ x \in B(0,3r),\ p \in \mathbb{R}^n, \\ r\|DA\|_{C(B(0,3))} + r^2\|D^2A\|_{C(B(0,3))} + r\|Df\|_{C(B(0,3))} \leq \frac{1}{2}\min\{\hat{\delta}, \lambda\}, \end{cases} \tag{8.6}$$

where $\hat{\delta}$ is the constant given by Lemma 8.2.

For $x \in B(0,3) \subset \widetilde{U} = \frac{1}{r}U$, let $\widetilde{A}(x) = A(rx)$, $\widetilde{f}(x) = rf(rx)$ and $\widetilde{u}(x) = \frac{1}{r}u(rx)$. Then $\mathscr{A}_{\widetilde{H}}[\widetilde{u}] = \widetilde{f}$ in $B(0,3)$ in the viscosity sense. We also let $\{\widetilde{A}^\epsilon\}_{\epsilon>0}$, $\{\widetilde{f}^\epsilon\}_{\epsilon>0}$ and $\{\widetilde{g}^\epsilon\}$ be smooth approximations of $\widetilde{A}$, $\widetilde{f}$ and $\widetilde{u}$ in $\widetilde{U}$ as in the beginning of this section, and hence satisfy (A1)–(A5). Observe that $D\widetilde{A}(x) = r(DA)(rx)$, $D\widetilde{f}(x) = r^2Df(rx)$ and $D^2\widetilde{A}(x) = r^2(D^2A)(rx)$ for $x \in B(0,3)$. By (8.6), for $\epsilon < \epsilon_0$

$$\begin{cases} 2^{-1/4}|p|^2 \leq \langle \widetilde{A}^\epsilon(x)p,\ p\rangle \leq 2^{1/4}|p|^2,\ x \in B(0,3),\ p \in \mathbb{R}^n, \\ \|D\widetilde{A}^\epsilon\|_{C(B(0,3))} + \|D^2\widetilde{A}^\epsilon\|_{C(B(0,3))} + \|D\widetilde{f}^\epsilon\|_{C(B(0,3))} \leq \min\{\hat{\delta}, \lambda\}. \end{cases}$$

By Lemma 7.1, we denote by $\widetilde{u}^\epsilon \in C^\infty(B(0,3)) \cap C(\overline{B(0,3)})$ be smooth solutions to the Dirichlet problem:

$$\mathscr{A}_{\widetilde{H}^\epsilon}[v] + \epsilon\, \mathrm{div}\,(\widetilde{A}^\epsilon Dv) = \widetilde{f}^\epsilon \quad \text{in} \quad B(0,3); \quad v = \widetilde{g}^\epsilon \quad \text{on} \quad \partial B(0,3).$$

Lemma 8.2 implies that $\widetilde{u}^\epsilon \to \widetilde{u}$ uniformly in $B(0,2)$. From (8.5), we also have

$$\max_{y \in B(0,2)} |\widetilde{u}(y) - y_n| \leq \frac{\lambda}{4}.$$

Hence there exists $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\epsilon < \epsilon_1$,

$$\max_{y \in B(0,2)} \left|\widetilde{u}^\epsilon(y) - y_n\right| \leq \frac{\lambda}{2}.$$

Applying Lemma 7.4, we arrive at

$$\left|D\widetilde{u}^\epsilon\right|^2 \leq \widetilde{u}^\epsilon_n + C\lambda^{1/2} \quad \text{in } B(0,1).$$

On the other hand, set $\widetilde{s}_k = s_k/r$. Then

$$\lim_{k\to\infty} \max_{y \in B(0,3\widetilde{s}_k)} \frac{1}{\widetilde{s}_k} |\widetilde{u}(y) - \langle \mathbf{b},\ y\rangle| = 0.$$

Choose $\eta = \frac{\theta}{48}$ and pick $s \in \{\widetilde{s}_k\}$, with $0 < s < 1$, so that

$$\max_{y \in B(0,s)} \frac{1}{s} |\widetilde{u}(y) - \langle \mathbf{b}, y \rangle| \leq \frac{\eta}{2}.$$

By Lemma 8.2 again, there exists $\epsilon_2 > 0$ such that for all $\epsilon < \epsilon_2$,

$$\max_{y \in B(0,s)} \frac{1}{s} \left|\widetilde{u}^{\epsilon}(y) - \langle \mathbf{b}, y \rangle\right| \leq \eta.$$

Applying [28, Lemma 4.3] to $\frac{1}{s}\widetilde{u}^{\epsilon}(s\cdot)$, we can find a point $x^0 \in B(0, s)$ such that

$$\left|D\widetilde{u}^{\epsilon}(x^0) - \mathbf{b}\right| \leq 4\eta,$$

which, combined with $|\mathbf{b}| = 1$, yields

$$\begin{cases} \widetilde{u}^{\epsilon}_n(x^0) \leq b_n + 4\eta \leq 1 - \theta + 4\eta, \\ \left|D\widetilde{u}^{\epsilon}(x^0)\right| \geq 1 - 4\eta. \end{cases}$$

Thus

$$(1 - 4\eta)^2 \leq \left|D\widetilde{u}^{\epsilon}(x^0)\right|^2 \leq \widetilde{u}^{\epsilon}_n(x^0) + C\lambda^{1/2} \leq \widetilde{u}^{\epsilon}_n(x^0) + \frac{\theta}{4},$$

which gives

$$\theta \leq 12\eta + \frac{\theta}{4} \leq \frac{\theta}{2},$$

this is impossible. The proof is complete. □

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References

1. Armstrong, S.N., Smart, C.K.: An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions. Calc. Var. Part. Differ. Equ. **37**, 381–384 (2010)
2. Armstrong, S.N., Smart, C.K.: A finite difference approach to the infinity laplace equation and tug-of-war games. Trans. Am. Math. Soc. **364**, 595–636 (2012)
3. Aronsson, G.: Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. Ark. Mat. **6**, 33–53 (1965)
4. Aronsson, G.: Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. II. Ark. Mat. **6**, 409–431 (1966)
5. Aronsson, G.: Extension of functions satisfying Lipschitz conditions. Ark. Mat. **6**, 551–561 (1967)
6. Aronsson, G.: Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. III. Ark. Mat. **7**, 509–512 (1969)
7. Barles, G., Busca, J.: Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term. Commun. Part. Differ. Equ. **26**, 2323–2337 (2001)
8. Barron, E.N., Jensen, R.R., Wang, C.Y.: The Euler equation and absolute minimizers of L^{∞} functionals. Arch. Ration. Mech. Anal. **157**, 255–283 (2001)
9. Crandall, M.G., Evans, L.C., Gariepy, R.F.: Optimal Lipschitz extensions and the infinity Laplacian. Calc. Var. Part. Differ. Equ. **13**, 123–139 (2001)
10. Crandall, M.G., Evans, L.C.: A remark on infinity harmonic functions. In: Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000), 123–129, Electron. J. Differ. Equ. Conf., 6, Southwest Texas State Univ., San Marcos, TX (2001)

11. Crandall, M.G., Gunnarsson, G., Wang, P.: Uniqueness of ∞-harmonic functions and the eikonal equation. Commun. Part. Differ. Equ. **32**, 1587–1615 (2007)
12. Crandall, M.G., Ishii, H., Lions, P.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. (N.S.) **27**, 1–67 (1992)
13. Crandall, M.G., Wang, C.Y., Yu, Y.F.: Derivation of the Aronsson equation for C^1-Hamiltonians. Trans. Am. Math. Soc. **361**, 103–124 (2009)
14. Evans, L.C., Savin, O.: $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions. Calc. Var. Part. Differ. Equ. **32**, 325–347 (2008)
15. Evans, L.C., Smart, C.K.: Everywhere differentiability of infinity harmonic functions. Calc. Var. Part. Differ. Equ. **42**, 289–299 (2011)
16. Evans, L.C., Smart, C.K.: Adjoint methods for the infinity Laplacian partial differential equation. Arch. Ration. Mech. Anal. **201**, 87–113 (2011)
17. Gilbarg, D., Turdinger, N.: Elliptic Partial Differential Euqations of Sectiond Order, 2nd edn. Grundlehren der Mathematischen Wissenschaften, vol. 224. Springer, Berlin (1983)
18. Hong, G., Feng, X.: Superposition principle on the viscosity solutions of infinity Laplace equations. Nonlinear Anal. **171**, 32–40 (2018)
19. Jensen, R.: Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. Arch. Ration. Mech. Anal. **123**, 51–74 (1993)
20. Koskela, P., Shanmugalingam, N., Zhou, Y.: Intrinsic geometry and analysis of diffusion process and L^∞-variational problem. Arch. Ration. Mech. Anal. **214**, 99–142 (2014)
21. Lindgren, E.: On the regularity of solutions of the inhomogeneous infinity Laplace equation. Proc. Am. Math. Soc. **142**, 277–288 (2014)
22. Lions, P.: Generalized Solutions of Hamilton–Jacobi Equations. Res. Notes Math. 69. Pitman, Boston (1982)
23. Lu, G., Wang, P.: A PDE perspective of the normalized infinity Laplacian. Commun. Part. Differ. Equ. **33**, 1788–1817 (2008)
24. Lu, G., Wang, P.: Inhomogeneous infinity Laplace equation. Adv. Math. **217**, 1838–1868 (2008)
25. Miao, Q., Wang, C.Y., Zhou, Y.: Uniqueness of absolute minimizers for L^∞-variational problems involving Hamiltonians $H(x, p)$. Arch. Ration. Mech. Anal. **223**(1), 141–198 (2017)
26. Peres, Y., Schramm, O., Sheffield, S., Wilson, D.B.: Tug-of-war and the infinity Laplacian. J. Am. Math. Soc. **22**, 167–210 (2009)
27. Savin, O.: C^1 regularity for infinity harmonic functions in two dimensions. Arch. Ration. Mech. Anal. **176**, 351–361 (2005)
28. Siljander, J., Wang, C., Zhou, Y.: Everywhere differentiability of viscosity solutions to a class of Aronsson's equations. AIH Poincaré Nonlinear Anal. **34**, 119–138 (2017)
29. Wang, C.Y., Yu, Y.F.: C^1-boundary regularity of planar infinity harmonic functions. Math. Res. Lett. **19**, 823–835 (2012)
30. Yu, Y.F.: L^∞ variational problems and Aronsson equations. Arch. Ration. Mech. Anal. **182**, 153–180 (2006)
31. Yu, Y.F.: L^∞ variational problems and weak KAM theory. Commun. Pure Appl. Math. **60**, 1111–1147 (2007)