DE GRUYTER Forum Math. 2019; 31(1): 1–18

Research Article

Yan Lin, Guozhen Lu* and Shanzhen Lu

Sharp maximal estimates for multilinear commutators of multilinear strongly singular Calderón–Zygmund operators and applications

https://doi.org/10.1515/forum-2018-0008
Received January 4, 2018; revised May 27, 2018

Abstract: In this paper, we aim to establish the sharp maximal pointwise estimates for the multilinear commutators generated by multilinear strongly singular Calderón–Zygmund operators and BMO functions or Lipschitz functions, respectively. As applications, the boundedness of these multilinear commutators on product of weighted Lebesgue spaces are obtained. It is interesting to note that there is no size condition assumption for the kernel of the multilinear strongly singular Calderón–Zygmund operator. Due to the stronger singularity for the kernel of the multilinear strongly singular Calderón–Zygmund operator, we need to be more careful in estimating the mean oscillation over the small balls to get the sharp maximal function estimates.

Keywords: Multilinear strongly singular Calderón–Zygmund operator, multilinear commutator, BMO function, Lipschitz function

MSC 2010: 42B20, 42B35

Communicated by: Christopher D. Sogge

1 Introduction

The strongly singular integral operator originated from a class of multiplier operator is defined by

$$(T_{\alpha,\beta}f)^\wedge(\xi) = \theta(\xi)\frac{e^{i|\xi|^\alpha}}{|\xi|^\beta}\hat{f}(\xi),$$

where $0 < \alpha < 1$, $0 < \beta \le n\alpha/2$ and $\theta(\xi)$ is a standard smooth cut-off function near the origin. Its convolution form can roughly be written as follows:

$$T_{\alpha,\beta}(f)(x) = \text{p.v.} \int \frac{e^{i|x-y|^{-\alpha'}}}{|x-y|^{n+\lambda}}\chi(|x-y|)f(y)\,dy,$$

where $\lambda = \frac{n\alpha/2-\beta}{1-\alpha}$ and $\alpha' = \frac{\alpha}{1-\alpha}$. Fefferman in [13] gave this operator the name weakly-strongly singular Calderón–Zygmund operator.

Yan Lin, School of Sciences, China University of Mining and Technology, Beijing 100083, P. R. China,
e-mail: linyan@cumtb.edu.cn
***Corresponding author: Guozhen Lu,** Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA,
e-mail: guozhen.lu@uconn.edu
Shanzhen Lu, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P. R. China,
e-mail: lusz@bnu.edu.cn

For the bounded properties on Lebesgue spaces, Hirschman in [25] and Stein in [52] showed that $T_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^n)$ when $|\frac{1}{2}-\frac{1}{p}|<\frac{\beta}{n}[\frac{n/2+\lambda}{\beta+\lambda}]$, and Wainger [54] proved that $T_{\alpha,\beta}$ is not bounded on $L^p(\mathbb{R}^n)$ when $|\frac{1}{2}-\frac{1}{p}|>\frac{\beta}{n}[\frac{n/2+\lambda}{\beta+\lambda}]$. In the critical case where p_0 satisfies $\frac{1}{p_0}-\frac{1}{2}=\frac{\beta}{n}[\frac{n/2+\lambda}{\beta+\lambda}]$, Fefferman in [13] obtained the boundedness of $T_{\alpha,\beta}$ from $L^{p_0}(\mathbb{R}^n)$ to the Lorentz space $L^{p_0,p_0'}(\mathbb{R}^n)$, where p_0' is the dual exponent of p_0. The weighted estimates for the classical linear strongly singular Calderón–Zygmund operators were given by Chanillo in [6]. Li and Lu in [30] gave a new proof to deal with the L^p boundedness by the scale changing method introduced by Carleson and Sjölin in [5].

For the situation when $\beta=n\alpha/2$, by means of the duality relationship between the Hardy space H^1 and the BMO space, Fefferman and Stein in [14] established the sharp endpoint estimate for this strongly singular integral.

Another kind of strongly singular non-convolution operator was introduced by Alvarez and Milman in [1]. Its properties behaved similarly to those of the standard Calderón–Zygmund operator. However, the kernel will be more singular near the diagonal than that of the standard case. The following is the specific definition of the strongly singular non-convolution operator.

Definition 1.1. Let $T:\mathcal{S}\to\mathcal{S}'$ be a bounded linear operator. T is called a *strongly singular Calderón–Zygmund operator* if the following conditions are satisfied:
(1) T can be extended into a continuous operator from $L^2(\mathbb{R}^n)$ into itself.
(2) There exists a function $K(x,y)$ continuous away from the diagonal $\{(x,y):x=y\}$ such that
$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|\le C\frac{|y-z|^{\delta}}{|x-z|^{n+\delta/\alpha}}$$
if $2|y-z|^{\alpha}\le|x-z|$ for some $0<\delta\le 1$ and $0<\alpha<1$, and
$$\langle Tf,g\rangle=\int\int K(x,y)f(y)g(x)\,dy\,dx$$
for $f,g\in\mathcal{S}$ with disjoint supports.
(3) For some $n(1-\alpha)/2\le\beta<n/2$, both T and its conjugate operator T^* can be extended into continuous operators from L^q to L^2, where $1/q=1/2+\beta/n$.

Alvarez and Milman disclosed the relationship between the pseudo-differential operator and the strongly singular Calderón–Zygmund operator. They verified that a class of pseudo-differential operators with symbols in the Hörmander's class $S^{-\beta}_{\alpha,\delta}$, where $0<\delta\le\alpha<1$ and $n(1-\alpha)/2\le\beta<n/2$, is actually included in the strongly singular Calderón–Zygmund operator. This relationship indicates that strongly singular Calderón–Zygmund operators have their importance not only in the theory of singular integrals in harmonic analysis but also in other related subjects in PDE.

The boundedness of the strongly singular Calderón–Zygmund operator on Lebesgue spaces was established by Alvarez and Milman in [1, 2]. Lin [33], and Lin and S. Lu [37] gave the sharp maximal estimates and endpoint estimates for the strongly singular Calderón–Zygmund operator, respectively. Furthermore, one can refer to [33, 35–37, 39] for other boundedness properties involving strongly singular Calderón–Zygmund operators and their commutators.

In this paper, we will pay attention to the multilinear form of the strongly singular Calderón–Zygmund operator.

Following the works of Coifman and Meyer in [9–11], in recent years, the topic of multilinear singular integrals has received increasing attention. In particular, the theory of multilinear Calderón–Zygmund operators has been developed systemically by Grafakos and Torres in [20, 21], and Grafakos and Kalton in [18], and the theory of multilinear fractional integrals has been treated by Kenig and Stein in [28]. There has been extensive research in the multilinear theory of Fourier multipliers and singular integrals since then, and we refer the reader to, e.g., [3, 4, 7, 8, 16, 17, 22–24, 26, 27, 29, 31, 40, 41, 43, 44, 46, 47, 49, 50]. The multilinear commutators generated by multilinear Calderón–Zygmund singular integrals or multilinear fractional integrals have also been extensively studied, for example, in [12, 32, 38, 42, 48, 51, 53, 55, 56].

Now we give a brief review to the definition of the multilinear Calderón–Zygmund operator. Let $m\in\mathbb{N}_+$ and let $K(y_0,y_1,\dots,y_m)$ be a function defined away from the diagonal $y_0=y_1=\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$. Let also T represent an m-linear operator, defined on a product of test function spaces related to the kernel function K, such that the following integral representation is valid:

$$T(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} K(x,y_1,\dots,y_m)\prod_{j=1}^m f_j(y_j)\,dy_1\cdots dy_m, \tag{1.1}$$

where f_j $(j=1,\dots,m)$ are smooth functions with compact support and $x\notin\bigcap_{j=1}^m \operatorname{supp} f_j$. In particular, we call K a *standard m-linear Calderón–Zygmund kernel* if it satisfies the following size and smoothness conditions:

$$|K(y_0,y_1,\dots,y_m)| \le \frac{C}{(\sum_{k,l=0}^m |y_k-y_l|)^{mn}} \tag{1.2}$$

for some $C>0$ and all $(y_0,y_1,\dots,y_m)\in(\mathbb{R}^n)^{m+1}$ away from the diagonal, and

$$\big|K(y_0,\dots,y_j,\dots,y_m)-K(y_0,\dots,y_j',\dots,y_m)\big| \le \frac{C|y_j-y_j'|^{\varepsilon}}{(\sum_{k,l=0}^m |y_k-y_l|)^{mn+\varepsilon}} \tag{1.3}$$

for some $\varepsilon>0$, whenever $0\le j\le m$ and $|y_j-y_j'|\le\frac12\max_{0\le k\le m}|y_j-y_k|$.

According to [21], if an m-linear operator T, defined by (1.1), is related to a standard m-linear Calderón–Zygmund kernel K, and satisfies either of the following two conditions:
(C1) T maps $L^{t_1,1}\times\cdots\times L^{t_m,1}$ into $L^{t,\infty}$ if $t>1$,
(C2) T maps $L^{t_1,1}\times\cdots\times L^{t_m,1}$ into L^1 if $t=1$,
where $t_1,t_2,\dots,t_m,t$ are given numbers satisfying $1\le t_1,t_2,\dots,t_m,t<\infty$ and $1/t=1/t_1+1/t_2+\dots+1/t_m$, and $L^{t_1,1},\dots,L^{t_m,1},L^{t,\infty}$ are Lorentz spaces, then T will be called a *standard m-linear Calderón–Zygmund operator*.

Let T be an m-linear operator defined by (1.1). Given a collection of locally integrable functions $\vec b=(b_1,\dots,b_m)$, the m-linear commutator of T with $\vec b$ is defined by

$$T_{\vec b}(f_1,\dots,f_m)=\sum_{j=1}^m T_{\vec b}^j(\vec f),$$

where

$$T_{\vec b}^j(\vec f)=b_jT(f_1,\dots,f_m)-T(f_1,\dots,f_{j-1},b_jf_j,f_{j+1},\dots,f_m).$$

The notation $\vec b\in\mathrm{BMO}^m$ will stand for $b_j\in\mathrm{BMO}(\mathbb{R}^n)$ for $j=1,\dots,m$, and $\vec b\in\mathrm{Lip}_\beta^m$ will stand for $b_j\in\mathrm{Lip}_\beta(\mathbb{R}^n)$ for $j=1,\dots,m$. We set $\|\vec b\|_{\mathrm{BMO}^m}=\max_{1\le j\le m}\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}$ and $\|\vec b\|_{\mathrm{Lip}_\beta^m}=\max_{1\le j\le m}\|b_j\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}$, respectively.

In this paper, we will focus on the *multilinear strongly singular* Calderón–Zygmund operator defined as follows.

Definition 1.2. Let T be an m-linear operator defined by (1.1). Then T is called an *m-linear strongly singular Calderón–Zygmund operator* if the following conditions are satisfied:
(1) For some $\varepsilon>0$ and $0<\alpha\le1$,

$$\big|K(x,y_1,\dots,y_m)-K(x',y_1,\dots,y_m)\big| \le \frac{C|x-x'|^{\varepsilon}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon/\alpha}}, \tag{1.4}$$

whenever $|x-x'|^{\alpha}\le\frac12\max_{1\le j\le m}|x-y_j|$.
(2) For some given numbers $1\le r_1,\dots,r_m<\infty$, with $1/r=1/r_1+\cdots+1/r_m$, T maps $L^{r_1}\times\cdots\times L^{r_m}$ into $L^{r,\infty}$.
(3) For some given numbers $1\le l_1,\dots,l_m<\infty$, with $1/l=1/l_1+\cdots+1/l_m$, T maps $L^{l_1}\times\cdots\times L^{l_m}$ into $L^{q,\infty}$, where $0<l/q\le\alpha$.

To compare the differences between the standard multilinear Calderón–Zygmund operator and the multilinear strongly singular Calderón–Zygmund operator, we give the following remarks.

Remark 1.3. For the multilinear strongly singular Calderón–Zygmund operator in the special case $\alpha = 1$, condition (1.3) implies condition (1.4), and we can take $l_j = r_j$, $j = 1, \dots, m$, and $q = l = r$ in (3) of Definition 1.2. Then condition (3) of Definition 1.2 is consistent with condition (2), thus we can remove condition (3) in this situation. In this sense, we can say that the multilinear strongly singular Calderón–Zygmund operator indeed generalizes the standard case.

Remark 1.4. More attention should be paid to the case $0 < \alpha < 1$. In this situation, the kernel of the multilinear strongly singular Calderón–Zygmund operator, defined by Definition 1.2, is more singular near the diagonal than that of the standard one. This fact force us to search for new techniques to overcome the stronger singularities.

Remark 1.5. It also should be pointed out that there is no size condition like (1.2) needed for the kernel of the multilinear strongly singular Calderón–Zygmund operator. Thus, our results sharpen the known ones on the multilinear commutators.

Recently, the first two authors of this paper established in [34] the sharp maximal pointwise estimate for the multilinear strongly singular Calderón–Zygmund operator and the boundedness of this operator on product of weighted Lebesgue spaces and product of variable exponent Lebesgue spaces, respectively. They also obtained its boundedness of $L^\infty \times \cdots \times L^\infty \to \mathrm{BMO}$, $\mathrm{BMO} \times \cdots \times \mathrm{BMO} \to \mathrm{BMO}$ and $\mathrm{LMO} \times \cdots \times \mathrm{LMO} \to \mathrm{LMO}$ types, respectively.

In this paper, we are interested in the multilinear commutators generated by multilinear strongly singular Calderón–Zygmund operators and BMO functions or Lipschitz functions. We will discuss the sharp maximal pointwise estimates for these two kinds of multilinear commutators and establish their boundedness on products of weighted Lebesgue spaces, respectively.

Here and in what follows, for $1 \le p \le \infty$, p' will stand for the dual index of p, which means $1/p + 1/p' = 1$. The letter C will denote constants which are independent of the main parameters and may change from one occurrence to another. $E^c = \mathbb{R}^n \setminus E$ will stand for the complementary set of E. Denote by $B(x, R)$ the ball with center x and radius $R > 0$, $|B(x, R)|$ the Lebesgue measure of $B(x, R)$, $CB(x, R) = B(x, CR)$ for $C > 0$, and $f_{B(x,R)} = \frac{1}{|B(x,R)|} \int_{B(x,R)} f(y)dy$.

For any locally integrable function f, the sharp maximal function is defined by

$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B|\, dy \sim \sup_{B \ni x} \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(y) - a|\, dx,$$

where the supremum is taken over all balls B containing x. It is easy to check that the above definition is equivalent to the one of taking the supremum over all balls B centered at x. Let $M^\sharp_\delta(f)(x) = [M^\sharp(|f|^\delta)(x)]^{1/\delta}$ for $0 < \delta < \infty$.

M will denote the Hardy–Littlewood maximal operator and, for $0 < p < \infty$, $M_p(f)(x) = [M(|f|^p)(x)]^{1/p}$. The fractional maximal operator is defined by

$$M_{\alpha,l}(f)(x) = \sup_{r>0} \bigg(\frac{1}{|B(x,r)|^{1-\alpha l/n}} \int_{B(x,r)} |f(y)|^l\, dy \bigg)^{1/l}$$

for $\alpha, l > 0$. It is easy to see that $M_{\alpha,l}(f)(x) = [M_{\alpha l,1}(|f|^l)(x)]^{1/l}$.

We say that a non-negative measurable function w on $\mathbb{R}^n$ is in the Muckenhoupt class A_p, with $1 < p < \infty$, if there exists a constant $C > 0$ such that for any cube Q in $\mathbb{R}^n$ with the side parallel to the coordinate axes,

$$\bigg(\frac{1}{|Q|} \int_Q \omega(x)\, dx \bigg) \bigg(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\, dx \bigg)^{p-1} \le C.$$

And for the case $p = 1$, we say that a non-negative measurable function w on $\mathbb{R}^n$ belongs to A_1, if there exists a constant $C > 0$ such that for any cube Q in $\mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q w(y)\, dy \le Cw(x) \quad \text{for a.e. } x \in Q.$$

A locally integrable non-negative function w on $\mathbb{R}^n$ is said to belong to the weight class $A(p,q)$, $1 < p, q < \infty$, if there exists a constant $C > 0$ such that for any cube Q,

$$\bigg(\frac{1}{|Q|}\int_Q w(x)^q\,dx\bigg)^{1/q}\bigg(\frac{1}{|Q|}\int_Q w(x)^{-p'}\,dx\bigg)^{1/p'} \le C.$$

Let $A_\infty = \bigcup_{p\ge 1} A_p$. It is well known that if $w \in A_p$, with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$.

This paper will be organized as follows. The sharp maximal pointwise estimates and the boundedness on products of weighted Lebesgue spaces for the multilinear commutators of multilinear strongly singular Calderón–Zygmund operators will be established as main results in Section 2. Before proving them, some necessary lemmas will be given in Section 3. Finally, the details of the proof of our main results will appear in Section 4.

2 Main results

Firstly, we will give the pointwise estimates for the sharp maximal functions of multilinear commutators generated by the multilinear strongly singular Calderón–Zygmund operators and BMO functions or Lipschitz functions, respectively.

Theorem 2.1. *Let T be an m-linear strongly singular Calderón–Zygmund operator and $0 < l/q < \alpha$ in* (3) *of Definition* 1.2. *Let $s_0 = \max\{r_1,\dots,r_m,l_1,\dots,l_m\}$, where r_j and l_j are given as in Definition* 1.2, *$j = 1,\dots,m$. If $\vec{b} \in \mathrm{BMO}^m$, $0 < \delta < 1/m$, $\delta < t < \infty$ and $s_0 < s < \infty$, then*

$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \le C\|\vec{b}\|_{\mathrm{BMO}^m}\bigg(M_t(T(\vec{f}))(x) + \prod_{j=1}^m M_s(f_j)(x)\bigg)$$

for all m-tuples $\vec{f} = (f_1,\dots,f_m)$ of bounded measurable functions with compact support.

Theorem 2.2. *Let T be an m-linear strongly singular Calderón–Zygmund operator and let $s_0 = \max\{r_1,\dots,r_m, l_1,\dots,l_m\}$, where r_j and l_j are given as in Definition* 1.2, *$j = 1,\dots,m$. If $\vec{b} \in \mathrm{Lip}_\beta^m$, $0 < \beta < 1$ and $0 < \delta < 1/m$, then*

$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \le C\|\vec{b}\|_{\mathrm{Lip}_\beta^m}\sum_{j=1}^m\bigg(M_{\beta,\delta}(T(\vec{f}))(x) + M_{\beta,s_0}(f_j)(x)\prod_{i=1,i\ne j}^m M_{s_0}(f_i)(x)\bigg)$$

for all m-tuples $\vec{f} = (f_1,\dots,f_m)$ of bounded measurable functions with compact support.

Then, as applications of the maximal function estimates, we can establish the boundedness of multilinear commutators generated by multilinear strongly singular Calderón–Zygmund operators and BMO functions or Lipschitz functions on products of weighted Lebesgue spaces, respectively.

Theorem 2.3. *Let T be an m-linear strongly singular Calderón–Zygmund operator and $0 < l/q < \alpha$ in* (3) *of Definition* 1.2. *Let $s_0 = \max\{r_1,\dots,r_m,l_1,\dots,l_m\}$, where r_j and l_j are given as in Definition* 1.2, *$j = 1,\dots,m$. If $\vec{b} \in \mathrm{BMO}^m$, then for any $s_0 < p_1,\dots,p_m < \infty$, with $1/p = 1/p_1 + \cdots + 1/p_m$, $T_{\vec{b}}$ can be extended into a bounded operator from $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ into $L^p(w)$, where $(w_1,\dots,w_m) \in (A_{p_1/s_0},\dots,A_{p_m/s_0})$ and $w = \prod_{j=1}^m w_j^{p/p_j}$.*

Theorem 2.4. *Let T be an m-linear strongly singular Calderón–Zygmund operator and let $s_0 = \max\{r_1,\dots,r_m, l_1,\dots,l_m\}$, where r_j and l_j are given as in Definition* 1.2, *$j = 1,\dots,m$. Suppose $\vec{b} \in \mathrm{Lip}_\beta^m$, $0 < \beta < \min\{1, n/s_0\}$, $s_0 < p_j < n/\beta$, $1/q_j = 1/p_j - \beta/n$, $j = 1,\dots,m$, $p > 1$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \beta/n$. Then $T_{\vec{b}}$ can be extended into a bounded operator from $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ into $L^q(w)$, where $w_j \in A_{p_j/s_0}$, $w_j^{s_0/p_j} \in A(p_j/s_0, q_j/s_0)$, $j = 1,\dots,m$, $w = \prod_{j=1}^m w_j^{q/p_j}$ and $w^{1/q} \in A(p,q)$.*

3 Necessary lemmas

Before proving our main results, we need some necessary lemmas.

Lemma 3.1 ([37]). *Let f be a function in* BMO. *Suppose $1 \le p < \infty$, $x \in \mathbb{R}^n$ and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p\,dy\Big)^{1/p} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)\|f\|_{\mathrm{BMO}},$$

where $C > 0$ is independent of f, x, r_1 and r_2.

Lemma 3.2 ([15, 29]). *Let $0 < p < q < \infty$, then there exists a positive constant $C = C_{p,q}$ such that for any measurable function f, one has*

$$|Q|^{-1/p}\|f\|_{L^p(Q)} \le C|Q|^{-1/q}\|f\|_{L^{q,\infty}(Q)}.$$

Lemma 3.3. *Given $\varepsilon > 0$, we have*

$$\ln x \le \frac{1}{\varepsilon}x^\varepsilon \quad \textit{for all } x \ge 1.$$

Let $\varphi(x) = \ln x - \frac{1}{\varepsilon}x^\varepsilon$, $x \ge 1$. The above result can be deduced from the monotone property of the function φ.

Lemma 3.4. *Let $\delta > 0$, $x \in \mathbb{R}^n$, and let f be a locally integrable function. Then for any ball $B = B(x_0, r)$ containing x with $r > 0$, we have*

$$\int_{B^c} \frac{|f(y)|}{|x_0 - y|^{n+\delta}}\,dy \le Cr^{-\delta}M(f)(x),$$

where C is a positive constant independent of f, x, x_0 and r.

We omit the proof of Lemma 3.4, since it is conventional.

Lemma 3.5 ([29]). *Let $0 < p, \delta < \infty$ and $w \in A_\infty$. Then there exists a constant $C > 0$ depending only on the A_∞ constant of w such that*

$$\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p w(x)\,dx \le C\int_{\mathbb{R}^n} [M_\delta^\sharp(f)(x)]^p w(x)\,dx$$

for every function f such that the left-hand side is finite.

Lemma 3.6 ([19]). *For $(w_1, \dots, w_m) \in (A_{p_1}, \dots, A_{p_m})$, with $1 \le p_1, \dots, p_m < \infty$, and for $0 < \theta_1, \dots, \theta_m < 1$ such that $\theta_1 + \dots + \theta_m = 1$, we have $w_1^{\theta_1}\cdots w_m^{\theta_m} \in A_{\max\{p_1,\dots,p_m\}}$.*

Lemma 3.7 ([34]). *Let T be an m-linear strongly singular Calderón–Zygmund operator and let $s_0 = \max\{r_1, \dots, r_m, l_1, \dots, l_m\}$, where r_j and l_j are given as in Definition 1.2, $j = 1, \dots, m$. If $0 < \delta < 1/m$, then*

$$M_\delta^\sharp(T(\vec{f}))(x) \le C\prod_{j=1}^m M_{s_0}(f_j)(x)$$

for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support.

Lemma 3.8 ([34]). *Let T be an m-linear strongly singular Calderón–Zygmund operator and let $s_0 = \max\{r_1, \dots, r_m, l_1, \dots, l_m\}$, where r_j and l_j are given as in Definition 1.2, $j = 1, \dots, m$. Then for any $s_0 < p_1, \dots, p_m < \infty$, with $1/p = 1/p_1 + \dots + 1/p_m$, T can be extended into a bounded operator from $L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m)$ into $L^p(w)$, where $(w_1, \dots, w_m) \in (A_{p_1/s_0}, \dots, A_{p_m/s_0})$ and $w = \prod_{j=1}^m w_j^{p/p_j}$.*

Lemma 3.9. *For $1 < p, q < \infty$, $w \in A(p,q)$ if and only if $w^q \in A_{q/p'+1}$.*

The result of Lemma 3.9 directly comes from the definitions of the two kinds of weights.

Lemma 3.10 ([45]). *If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $w \in A(p,q)$, then there exists a constant $C > 0$, independent of f, such that*

$$\Big(\int_{\mathbb{R}^n} [M_{\alpha,1}(f)(x)w(x)]^q\,dx\Big)^{1/q} \le C\Big(\int_{\mathbb{R}^n} |f(x)w(x)|^p\,dx\Big)^{1/p}.$$

4 Proof of main results

Proof of Theorem 2.1. Without loss of generality, we will only consider the case $m = 2$ and omit all other situations, since there are similarities.

Let f_1, f_2 be bounded measurable functions with compact support. Then for any ball $B = B(x_0, r_B)$ containing x, with $r_B > 0$, we consider two cases.

Case 1: $r_B \geq 1$. Write

$$\begin{aligned} f_1 &= f_1\chi_{2B} + f_1\chi_{(2B)^c} := f_1^1 + f_1^2, \\ f_2 &= f_2\chi_{2B} + f_2\chi_{(2B)^c} := f_2^1 + f_2^2 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} T_{\vec{b}}^1(\vec{f})(z) = (b_1(z) - b_B^1)T(f_1, f_2)(z) &- T((b_1 - b_B^1)f_1^1, f_2^1)(z) \\ &- T((b_1 - b_B^1)f_1^1, f_2^2)(z) - T((b_1 - b_B^1)f_1^2, f_2^1)(z) - T((b_1 - b_B^1)f_1^2, f_2^2)(z), \end{aligned} \tag{4.2}$$

where

$$b_B^1 = \frac{1}{|B|}\int_B b_1(z)\,dz.$$

Take

$$c_1 = T((b_1 - b_B^1)f_1^1, f_2^2)(x_0) + T((b_1 - b_B^1)f_1^2, f_2^1)(x_0) + T((b_1 - b_B^1)f_1^2, f_2^2)(x_0).$$

Then

$$\begin{aligned} \Big(\frac{1}{|B|}\int_B |T_{\vec{b}}^1(\vec{f})(z) + c_1|^\delta\,dz\Big)^{1/\delta} &\leq C\Big(\frac{1}{|B|}\int_B \big|(b_1(z) - b_B^1)T(f_1, f_2)(z)\big|^\delta\,dz\Big)^{1/\delta} \\ &\quad + C\Big(\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)f_1^1, f_2^1)(z)\big|^\delta\,dz\Big)^{1/\delta} \\ &\quad + C\Big(\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)f_1^1, f_2^2)(z) - T((b_1 - b_B^1)f_1^1, f_2^2)(x_0)\big|^\delta\,dz\Big)^{1/\delta} \\ &\quad + C\Big(\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)f_1^2, f_2^1)(z) - T((b_1 - b_B^1)f_1^2, f_2^1)(x_0)\big|^\delta\,dz\Big)^{1/\delta} \\ &\quad + C\Big(\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)f_1^2, f_2^2)(z) - T((b_1 - b_B^1)f_1^2, f_2^2)(x_0)\big|^\delta\,dz\Big)^{1/\delta} \\ &:= \sum_{j=1}^5 I_j. \end{aligned} \tag{4.3}$$

Since $0 < \delta < 1/m$ and $\delta < t < \infty$, there exists u such that $1 < u < \min\{\frac{t}{\delta}, \frac{1}{1-\delta}\}$. Then $\delta u < t$ and $\delta u' > 1$. By Hölder's inequality, we have

$$\begin{aligned} I_1 &\leq C\Big(\frac{1}{|B|}\int_B |b_1(z) - b_B^1|^{\delta u'}\,dz\Big)^{1/(\delta u')}\Big(\frac{1}{|B|}\int_B |T(f_1, f_2)(z)|^{\delta u}\,dz\Big)^{1/(\delta u)} \\ &\leq C\|b_1\|_{\mathrm{BMO}}\Big(\frac{1}{|B|}\int_B |T(f_1, f_2)(z)|^t\,dz\Big)^{1/t} \\ &\leq C\|b_1\|_{\mathrm{BMO}} M_t(T(\vec{f}))(x). \end{aligned}$$

Set $v = s/s_0$. Since $s_0 < s < \infty$, we have $1 < v < \infty$. Notice that $0 < \delta < r < \infty$, where r is given as in Definition 1.2. From Definition 1.2 (2) and Lemmas 3.1–3.2, it follows that

$$\begin{aligned}
I_2 &\le C|B|^{-1/\delta}\|T((b_1 - b_B^1)f_1^1, f_2^1)\|_{L^\delta(B)} \le C|B|^{-1/r}\|T((b_1 - b_B^1)f_1^1, f_2^1)\|_{L^{r,\infty}(B)} \\
&\le C\Big(\frac{1}{|2B|}\int_{2B}|b_1(y_1) - b_B^1|^{r_1}|f_1(y_1)|^{r_1}\,dy_1\Big)^{1/r_1}\Big(\frac{1}{|2B|}\int_{2B}|f_2(y_2)|^{r_2}\,dy_2\Big)^{1/r_2} \\
&\le C\Big(\frac{1}{|2B|}\int_{2B}|b_1(y_1) - b_B^1|^{r_1 v'}\,dy_1\Big)^{1/(r_1 v')}\Big(\frac{1}{|2B|}\int_{2B}|f_1(y_1)|^{r_1 v}\,dy_1\Big)^{1/(r_1 v)} M_{r_2}(f_2)(x) \\
&\le C\|b_1\|_{\mathrm{BMO}} M_{r_1 v}(f_1)(x) M_{r_2}(f_2)(x) \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x) M_s(f_2)(x).
\end{aligned}$$

For $z \in B$ and $y_2 \in (2B)^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le r_B^\alpha \le r_B \le \frac{1}{2}|y_2 - x_0|$. By Hölder's inequality, the condition of the kernel in Definition 1.2 (1) and Lemma 3.4, we have

$$\begin{aligned}
I_3 &\le C\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)f_1^1, f_2^2)(z) - T((b_1 - b_B^1)f_1^1, f_2^2)(x_0)\big|\,dz \\
&\le C\frac{1}{|B|}\int_B\int_{(2B)^c}\int_{2B}|K(z, y_1, y_2) - K(x_0, y_1, y_2)||b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le C\frac{1}{|B|}\int_B\int_{(2B)^c}\int_{2B}\frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le Cr_B^\varepsilon\Big(\int_{2B}|b_1(y_1) - b_B^1||f_1(y_1)|\,dy_1\Big)\Big(\int_{(2B)^c}\frac{|f_2(y_2)|}{|x_0 - y_2|^{2n+\varepsilon/\alpha}}\,dy_2\Big) \\
&\le Cr_B^\varepsilon\Big(\frac{1}{|2B|}\int_{2B}|b_1(y_1) - b_B^1|^{s'}\,dy_1\Big)^{1/s'}\Big(\frac{1}{|2B|}\int_{2B}|f_1(y_1)|^s\,dy_1\Big)^{1/s}|B|M(f_2)(x)r_B^{-(n+\varepsilon/\alpha)} \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x)M(f_2)(x)r_B^{\varepsilon-\varepsilon/\alpha} \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x)M_s(f_2)(x).
\end{aligned}$$

For $z \in B$ and $y_1 \in (2B)^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le \frac{1}{2}|y_1 - x_0|$. From Hölder's inequality, Definition 1.2 (1) and Lemma 3.1, it follows that

$$\begin{aligned}
I_4 &\le C\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)f_1^2, f_2^1)(z) - T((b_1 - b_B^1)f_1^2, f_2^1)(x_0)\big|\,dz \\
&\le C\frac{1}{|B|}\int_B\int_{2B}\int_{(2B)^c}\frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le Cr_B^\varepsilon\Big(\int_{(2B)^c}\frac{|b_1(y_1) - b_B^1||f_1(y_1)|}{|x_0 - y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)\Big(\int_{2B}|f_2(y_2)|\,dy_2\Big) \\
&\le Cr_B^\varepsilon\Big(\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^k B}\frac{|b_1(y_1) - b_B^1||f_1(y_1)|}{|x_0 - y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)M(f_2)(x)|B| \\
&\le Cr_B^\varepsilon\sum_{k=1}^\infty(2^k r_B)^{-(n+\varepsilon/\alpha)}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y_1) - b_B^1|^{s'}\,dy_1\Big)^{1/s'} \\
&\qquad\times\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|^s\,dy_1\Big)^{1/s}|B|M(f_2)(x) \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x)M(f_2)(x)r_B^{\varepsilon-\varepsilon/\alpha}\sum_{k=1}^\infty k2^{-k(n+\varepsilon/\alpha)} \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x)M_s(f_2)(x).
\end{aligned}$$

For $z \in B$ and $y_1, y_2 \in (2B)^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \leq \frac{1}{2}|y_1 - x_0|$ and $|z-x_0|^\alpha \leq \frac{1}{2}|y_2 - x_0|$. By Hölder's inequality, Definition 1.2 (1), and Lemmas 3.1 and 3.4, we have

$$\begin{aligned}
I_5 &\leq C\frac{1}{|B|}\int_B |T((b_1 - b_B^1)f_1^2, f_2^2)(z) - T((b_1 - b_B^1)f_1^2, f_2^2)(x_0)|\, dz \\
&\leq C\frac{1}{|B|}\int_B \int_{(2B)^c} \int_{(2B)^c} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}} |b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\, dy_1\, dy_2\, dz \\
&\leq Cr_B^\varepsilon\Big(\int_{(2B)^c} \frac{|b_1(y_1) - b_B^1||f_1(y_1)|}{|x_0 - y_1|^{n+\varepsilon/(2\alpha)}}\, dy_1\Big)\Big(\int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{n+\varepsilon/(2\alpha)}}\, dy_2\Big) \\
&\leq Cr_B^\varepsilon\Big(\sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^kB} \frac{|b_1(y_1) - b_B^1||f_1(y_1)|}{|x_0 - y_1|^{n+\varepsilon/(2\alpha)}}\, dy_1\Big) M(f_2)(x) r_B^{-\varepsilon/(2\alpha)} \\
&\leq Cr_B^\varepsilon \sum_{k=1}^{\infty} (2^k r_B)^{-\varepsilon/(2\alpha)}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} |b_1(y_1) - b_B^1|^{s'}\, dy_1\Big)^{1/s'} \\
&\qquad \times \Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} |f_1(y_1)|^s\, dy_1\Big)^{1/s} r_B^{-\varepsilon/(2\alpha)} M(f_2)(x) \\
&\leq C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x) M(f_2)(x) r_B^{\varepsilon-\varepsilon/\alpha} \sum_{k=1}^{\infty} k2^{-k\varepsilon/(2\alpha)} \\
&\leq C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x) M_s(f_2)(x).
\end{aligned}$$

Case 2: $0 < r_B < 1$. Since $0 < l/q < \alpha$, there exists θ such that $l/q < \theta < \alpha$. Set $\tilde{B} = B(x_0, r_B^\theta)$ and write

$$f_1 = f_1\chi_{2\tilde{B}} + f_1\chi_{(2\tilde{B})^c} := \tilde{f}_1^1 + \tilde{f}_1^2, \quad f_2 = f_2\chi_{2\tilde{B}} + f_2\chi_{(2\tilde{B})^c} := \tilde{f}_2^1 + \tilde{f}_2^2 \tag{4.4}$$

and

$$\begin{aligned}
T_{\vec{b}}^1(\vec{f})(z) &= (b_1(z) - b_B^1)T(f_1, f_2)(z) - T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^1)(z) \\
&\quad - T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^2)(z) - T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^1)(z) - T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^2)(z).
\end{aligned} \tag{4.5}$$

Take

$$\tilde{c}_1 = T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^2)(x_0) + T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^1)(x_0) + T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^2)(x_0).$$

Then

$$\begin{aligned}
\Big(\frac{1}{|B|}\int_B |T_{\vec{b}}^1(\vec{f})(z) + \tilde{c}_1|^\delta\, dz\Big)^{1/\delta} &\leq C\Big(\frac{1}{|B|}\int_B |(b_1(z) - b_B^1)T(f_1, f_2)(z)|^\delta\, dz\Big)^{1/\delta} \\
&\quad + C\Big(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^1)(z)|^\delta\, dz\Big)^{1/\delta} \\
&\quad + C\Big(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^2)(z) - T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^2)(x_0)|^\delta\, dz\Big)^{1/\delta} \\
&\quad + C\Big(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^1)(z) - T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^1)(x_0)|^\delta\, dz\Big)^{1/\delta} \\
&\quad + C\Big(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^2)(z) - T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^2)(x_0)|^\delta\, dz\Big)^{1/\delta} \\
&=: \sum_{j=1}^{5} \tilde{I}_j.
\end{aligned} \tag{4.6}$$

For the same estimate of I_1, we have

$$\tilde{I}_1 \leq C\|b_1\|_{\mathrm{BMO}} M_t(T(\vec{f}))(x).$$

Set $v = s/s_0$ and $\varepsilon_1 = n(\theta/l - 1/q)$. Then $v > 1$ and $\varepsilon_1 > 0$. Since $0 < \delta < q < \infty$, from Definition 1.2 (3), and Lemmas 3.1–3.3, it follows that

$$
\begin{aligned}
\tilde{I}_2 &\le C|B|^{-1/\delta}\|T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^1)\|_{L^\delta(B)} \\
&\le C|B|^{-1/q}\|T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^1)\|_{L^{q,\infty}(B)} \\
&\le C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |b_1(y_1) - b_B^1|^{l_1}|f_1(y_1)|^{l_1}\,dy_1\Big)^{1/l_1}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_2(y_2)|^{l_2}\,dy_2\Big)^{1/l_2} \\
&\le C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |b_1(y_1) - b_B^1|^{l_1 v'}\,dy_1\Big)^{1/(l_1 v')}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_1(y_1)|^{l_1 v}\,dy_1\Big)^{1/(l_1 v)} M_{l_2}(f_2)(x) \\
&\le C r_B^{n(\theta/l - 1/q)}\|b_1\|_{\mathrm{BMO}}\Big(1 + (1-\theta)\ln\frac{1}{r_B}\Big) M_{l_1 v}(f_1)(x) M_{l_2}(f_2)(x) \\
&\le C r_B^{n(\theta/l - 1/q) - \varepsilon_1}\|b_1\|_{\mathrm{BMO}} M_{l_1 v}(f_1)(x) M_{l_2}(f_2)(x) \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x) M_s(f_2)(x).
\end{aligned}
$$

For $z \in B$ and $y_2 \in (2\tilde{B})^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le r_B^\alpha \le r_B^\theta \le \frac{1}{2}|y_2 - x_0|$. Set $\varepsilon_2 = \varepsilon(\alpha - \theta)/\alpha$. Then $\varepsilon_2 > 0$. By Hölder's inequality, Definition 1.2 (1), and Lemmas 3.1, 3.3 and 3.4, we have

$$
\begin{aligned}
\tilde{I}_3 &\le C\frac{1}{|B|}\int_B |T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^2)(z) - T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^2)(x_0)|\,dz \\
&\le C\frac{1}{|B|}\int_B \int_{(2\tilde{B})^c}\int_{2\tilde{B}} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le C r_B^\varepsilon\Big(\int_{2\tilde{B}} |b_1(y_1) - b_B^1||f_1(y_1)|\,dy_1\Big)\Big(\int_{(2\tilde{B})^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n+\varepsilon/\alpha}}\,dy_2\Big) \\
&\le C r_B^\varepsilon\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |b_1(y_1) - b_B^1|^{s'}\,dy_1\Big)^{1/s'}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_1(y_1)|^s\,dy_1\Big)^{1/s}|\tilde{B}|M(f_2)(x)(r_B^\theta)^{-(n+\varepsilon/\alpha)} \\
&\le C r_B^{(\varepsilon/\alpha)(\alpha-\theta)}\|b_1\|_{\mathrm{BMO}}\Big(1 + (1-\theta)\ln\frac{1}{r_B}\Big) M_s(f_1)(x) M(f_2)(x) \\
&\le C r_B^{(\varepsilon/\alpha)(\alpha-\theta)-\varepsilon_2}\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x) M(f_2)(x) \\
&\le C\|b_1\|_{\mathrm{BMO}} M_s(f_1)(x) M_s(f_2)(x).
\end{aligned}
$$

For $z \in B$ and $y_1 \in (2\tilde{B})^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le \frac{1}{2}|y_1 - x_0|$. From Hölder's inequality, Definition 1.2 (1), and Lemmas 3.1 and 3.3, it follows that

$$
\begin{aligned}
\tilde{I}_4 &\le C\frac{1}{|B|}\int_B \big|T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^1)(z) - T((b_1 - b_B^1)\tilde{f}_1^2, \tilde{f}_2^1)(x_0)\big|\,dz \\
&\le C\frac{1}{|B|}\int_B \int_{2\tilde{B}}\int_{(2\tilde{B})^c} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le C r_B^\varepsilon\Big(\int_{(2\tilde{B})^c} \frac{|b_1(y_1) - b_B^1||f_1(y_1)|}{|x_0 - y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)\Big(\int_{2\tilde{B}} |f_2(y_2)|\,dy_2\Big) \\
&\le C r_B^\varepsilon\Big(\sum_{k=1}^\infty \int_{2^{k+1}\tilde{B}\setminus 2^k\tilde{B}} \frac{|b_1(y_1) - b_B^1||f_1(y_1)|}{|x_0 - y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big) M(f_2)(x)|\tilde{B}| \\
&\le C r_B^\varepsilon \sum_{k=1}^\infty (2^k r_B^\theta)^{-(n+\varepsilon/\alpha)}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}} |b_1(y_1) - b_B^1|^{s'}\,dy_1\Big)^{1/s'}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}} |f_1(y_1)|^s\,dy_1\Big)^{1/s}|\tilde{B}|M(f_2)(x)
\end{aligned}
$$

$$\begin{aligned}
&\leq Cr_B^{(\varepsilon/\alpha)(\alpha-\theta)}\|b_1\|_{\mathrm{BMO}}\Big(1+(1-\theta)\ln\frac{1}{r_B}\Big)M_s(f_1)(x)M(f_2)(x)\sum_{k=1}^{\infty}k2^{-k(n+\varepsilon/\alpha)}\\
&\leq Cr_B^{(\varepsilon/\alpha)(\alpha-\theta)-\varepsilon_2}\|b_1\|_{\mathrm{BMO}}M_s(f_1)(x)M(f_2)(x)\\
&\leq C\|b_1\|_{\mathrm{BMO}}M_s(f_1)(x)M_s(f_2)(x).
\end{aligned}$$

For $z\in B$ and $y_1,y_2\in(2\tilde{B})^c$, and for the center of the ball x_0, we have $|z-x_0|^\alpha\leq\frac{1}{2}|y_1-x_0|$ and $|z-x_0|^\alpha\leq\frac{1}{2}|y_2-x_0|$. By Hölder's inequality, Definition 1.2 (1), Lemmas 3.1, 3.3 and 3.4, we have

$$\begin{aligned}
\tilde{I}_5&\leq C\frac{1}{|B|}\int_B\big|T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^2)(z)-T((b_1-b_B^1)\tilde{f}_1^2,\tilde{f}_2^2)(x_0)\big|\,dz\\
&\leq C\frac{1}{|B|}\int_B\int_{(2\tilde{B})^c}\int_{(2\tilde{B})^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz\\
&\leq Cr_B^\varepsilon\Big(\int_{(2\tilde{B})^c}\frac{|b_1(y_1)-b_B^1||f_1(y_1)|}{|x_0-y_1|^{n+\varepsilon/(2\alpha)}}\,dy_1\Big)\Big(\int_{(2\tilde{B})^c}\frac{|f_2(y_2)|}{|x_0-y_2|^{n+\varepsilon/(2\alpha)}}\,dy_2\Big)\\
&\leq Cr_B^\varepsilon\Big(\sum_{k=1}^{\infty}\int_{2^{k+1}\tilde{B}\setminus 2^k\tilde{B}}\frac{|b_1(y_1)-b_B^1||f_1(y_1)|}{|x_0-y_1|^{n+\varepsilon/(2\alpha)}}\,dy_1\Big)M(f_2)(x)(r_B^\theta)^{-\varepsilon/(2\alpha)}\\
&\leq Cr_B^\varepsilon\sum_{k=1}^{\infty}(2^kr_B^\theta)^{-\varepsilon/(2\alpha)}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|b_1(y_1)-b_B^1|^{s'}\,dy_1\Big)^{1/s'}\\
&\qquad\times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^s\,dy_1\Big)^{1/s}r_B^{-\varepsilon\theta/(2\alpha)}M(f_2)(x)\\
&\leq Cr_B^{(\varepsilon/\alpha)(\alpha-\theta)}\|b_1\|_{\mathrm{BMO}}\Big(1+(1-\theta)\ln\frac{1}{r_B}\Big)M_s(f_1)(x)M(f_2)(x)\sum_{k=1}^{\infty}k2^{-k\varepsilon/(2\alpha)}\\
&\leq Cr_B^{(\varepsilon/\alpha)(\alpha-\theta)-\varepsilon_2}\|b_1\|_{\mathrm{BMO}}M_s(f_1)(x)M(f_2)(x)\\
&\leq C\|b_1\|_{\mathrm{BMO}}M_s(f_1)(x)M_s(f_2)(x).
\end{aligned}$$

Similarly, to estimate $T_{\vec{b}}^1(\vec{f})(z)$, if $r_B\geq 1$, then

$$\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^2(\vec{f})(z)+c_2|^\delta\,dz\Big)^{1/\delta}\leq C\|b_2\|_{\mathrm{BMO}}(M_t(T(\vec{f}))(x)+M_s(f_1)(x)M_s(f_2)(x)),$$

where

$$c_2=T(f_1^1,(b_2-b_B^2)f_2^2)(x_0)+T(f_1^2,(b_2-b_B^2)f_2^1)(x_0)+T(f_1^2,(b_2-b_B^2)f_2^2)(x_0),$$

and if $0<r_B<1$, then

$$\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^2(\vec{f})(z)+\tilde{c}_2|^\delta\,dz\Big)^{1/\delta}\leq C\|b_2\|_{\mathrm{BMO}}(M_t(T(\vec{f}))(x)+M_s(f_1)(x)M_s(f_2)(x)),$$

where

$$\tilde{c}_2=T(\tilde{f}_1^1,(b_2-b_B^2)\tilde{f}_2^2)(x_0)+T(\tilde{f}_1^2,(b_2-b_B^2)\tilde{f}_2^1)(x_0)+T(\tilde{f}_1^2,(b_2-b_B^2)\tilde{f}_2^2)(x_0).$$

If $r_B\geq 1$, then

$$\begin{aligned}
\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B|}\int_B\big||T_{\vec{b}}(\vec{f})(z)|^\delta-a\big|\,dz\Big)^{1/\delta}&\leq C\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^1(\vec{f})(z)+c_1|^\delta\,dz\Big)^{1/\delta}+C\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^2(\vec{f})(z)+c_2|^\delta\,dz\Big)^{1/\delta}\\
&\leq C\|\vec{b}\|_{\mathrm{BMO}^2}(M_t(T(\vec{f}))(x)+M_s(f_1)(x)M_s(f_2)(x)),
\end{aligned}$$

and if $0<r_B<1$, then

$$\begin{aligned}
\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B|}\int_B\big||T_{\vec{b}}(\vec{f})(z)|^\delta-a\big|\,dz\Big)^{1/\delta}&\leq C\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^1(\vec{f})(z)+\tilde{c}_1|^\delta\,dz\Big)^{1/\delta}+C\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^2(\vec{f})(z)+\tilde{c}_2|^\delta\,dz\Big)^{1/\delta}\\
&\leq C\|\vec{b}\|_{\mathrm{BMO}^2}(M_t(T(\vec{f}))(x)+M_s(f_1)(x)M_s(f_2)(x)).
\end{aligned}$$

Combining the above two cases, we have

$$\begin{aligned} M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) &\sim \sup_{B\ni x}\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B|}\int_B \big||T_{\vec{b}}(\vec{f})(z)|^\delta - a\big|\,dz\Big)^{1/\delta} \\ &\le C\|\vec{b}\|_{\mathrm{BMO}^2}(M_t(T(\vec{f}))(x) + M_s(f_1)(x)M_s(f_2)(x)). \end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. In order to simplify the proof, we only consider the case $m = 2$. Actually, a similar procedure works for all other situations.

Let f_1, f_2 be bounded measurable functions with compact support. Then for any ball $B = B(x_0, r_B)$ with center x_0 and radius $r_B > 0$, we consider two cases.

Case 1: $r_B \ge 1$. Using the same decompositions as in (4.1)–(4.2) and taking the same c_1, we can also dominate

$$\Big(\frac{1}{|B|}\int_B |T_{\vec{b}}^1(\vec{f})(z) + c_1|^\delta\,dz\Big)^{1/\delta}$$

by five terms, still denoted by I_j, $j = 1, \dots, 5$, as in (4.3).

In this theorem, $\vec{b} \in \mathrm{Lip}_\beta^m$. We can estimate the five terms as follows. By the definition of the Lipschitz function, we have

$$\begin{aligned} I_1 &\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\beta\Big(\frac{1}{|B|}\int_B |T(f_1, f_2)(z)|^\delta\,dz\Big)^{1/\delta} \\ &= C\|b_1\|_{\mathrm{Lip}_\beta}\Big(\frac{1}{|B|^{1-\beta\delta/n}}\int_B |T(f_1, f_2)(z)|^\delta\,dz\Big)^{1/\delta} \\ &\le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,\delta}(T(\vec{f}))(x_0). \end{aligned}$$

Note that $0 < \delta < r < \infty$, where r is given as in Definition 1.2. Then, by Lemma 3.2 and Definition 1.2 (2), we have

$$\begin{aligned} I_2 &\le C|B|^{-1/\delta}\|T((b_1 - b_B^1)f_1^1, f_2^1)\|_{L^\delta(B)} \\ &\le C|B|^{-1/r}\|T((b_1 - b_B^1)f_1^1, f_2^1)\|_{L^{r,\infty}(B)} \\ &\le C\Big(\frac{1}{|2B|}\int_{2B} |b_1(y_1) - b_B^1|^{r_1}|f_1(y_1)|^{r_1}\,dy_1\Big)^{1/r_1}\Big(\frac{1}{|2B|}\int_{2B} |f_2(y_2)|^{r_2}\,dy_2\Big)^{1/r_2} \\ &\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\beta\Big(\frac{1}{|2B|}\int_{2B} |f_1(y_1)|^{r_1}\,dy_1\Big)^{1/r_1} M_{r_2}(f_2)(x_0) \\ &\le C\|b_1\|_{\mathrm{Lip}_\beta}\Big(\frac{1}{|2B|^{1-\beta s_0/n}}\int_{2B} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M_{r_2}(f_2)(x_0) \\ &\le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0). \end{aligned}$$

For $z \in B$ and $y_2 \in (2B)^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le \frac{1}{2}|y_2 - x_0|$. From Hölder's inequality, Definition 1.2 (1) and Lemma 3.4, it follows that

$$\begin{aligned} I_3 &\le C\frac{1}{|B|}\int_B\int_{(2B)^c}\int_{2B} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\ &\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^{\varepsilon+\beta}\Big(\int_{2B} |f_1(y_1)|\,dy_1\Big)\Big(\int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n+\varepsilon/\alpha}}\,dy_2\Big) \\ &\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^{\varepsilon+\beta}\Big(\frac{1}{|2B|}\int_{2B} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0}|B|M(f_2)(x_0)r_B^{-(n+\varepsilon/\alpha)} \end{aligned}$$

$$= C\|b_1\|_{\mathrm{Lip}_\beta} r_B^{\varepsilon-\varepsilon/\alpha}\Big(\frac{1}{|2B|^{1-\beta s_0/n}}\int_{2B}|f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M(f_2)(x_0)$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0).$$

For $z \in B$ and $y_1 \in (2B)^c$, and for the center of the ball x_0, we have $|z-x_0|^\alpha \le \frac{1}{2}|y_1-x_0|$. By Hölder's inequality and Definition 1.2 (1), we have

$$I_4 \le C\frac{1}{|B|}\int_B\int_{2B}\int_{(2B)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon\Big(\int_{(2B)^c}\frac{|x_0-y_1|^\beta|f_1(y_1)|}{|x_0-y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)\Big(\int_{2B}|f_2(y_2)|\,dy_2\Big)$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon\Big(\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\frac{|x_0-y_1|^\beta|f_1(y_1)|}{|x_0-y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)M(f_2)(x_0)|B|$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon\sum_{k=1}^\infty(2^kr_B)^{-(n+\varepsilon/\alpha)}\Big(\frac{1}{|2^{k+1}B|^{1-\beta/n}}\int_{2^{k+1}B}|f_1(y_1)|\,dy_1\Big)|B|M(f_2)(x_0)$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^{\varepsilon-\varepsilon/\alpha}\sum_{k=1}^\infty 2^{-k(n+\varepsilon/\alpha)}\Big(\frac{1}{|2^{k+1}B|^{1-\beta s_0/n}}\int_{2^{k+1}B}|f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M(f_2)(x_0)$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0).$$

For $z \in B$ and $y_1, y_2 \in (2B)^c$, and for the center of the ball x_0, we have $|z-x_0|^\alpha \le \frac{1}{2}|y_1-x_0|$ and $|z-x_0|^\alpha \le \frac{1}{2}|y_2-x_0|$. From Hölder's inequality, Definition 1.2 (1) and Lemma 3.4, it follows that

$$I_5 \le C\frac{1}{|B|}\int_B\int_{(2B)^c}\int_{(2B)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon\Big(\int_{(2B)^c}\frac{|x_0-y_1|^\beta|f_1(y_1)|}{|x_0-y_1|^{n+\varepsilon/(2\alpha)}}\,dy_1\Big)\Big(\int_{(2B)^c}\frac{|f_2(y_2)|}{|x_0-y_2|^{n+\varepsilon/(2\alpha)}}\,dy_2\Big)$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon\Big(\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\frac{|x_0-y_1|^\beta|f_1(y_1)|}{|x_0-y_1|^{n+\varepsilon/(2\alpha)}}\,dy_1\Big)M(f_2)(x_0)r_B^{-\varepsilon/(2\alpha)}$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon\sum_{k=1}^\infty(2^kr_B)^{-\varepsilon/(2\alpha)}\Big(\frac{1}{|2^{k+1}B|^{1-\frac{\beta}{n}}}\int_{2^{k+1}B}|f_1(y_1)|\,dy_1\Big)M(f_2)(x_0)r_B^{-\varepsilon/(2\alpha)}$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^{\varepsilon-\varepsilon/\alpha}\sum_{k=1}^\infty 2^{-k\varepsilon/(2\alpha)}\Big(\frac{1}{|2^{k+1}B|^{1-\beta s_0/n}}\int_{2^{k+1}B}|f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M(f_2)(x_0)$$
$$\le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0).$$

Case 2: $0 < r_B < 1$. Set $\tilde{B} = B(x_0, r_B^\alpha)$. Using the same decompositions as in (4.4)–(4.5) and taking the same $\tilde{c}_1$, we can also dominate

$$\Big(\frac{1}{|B|}\int_B|T_{\vec{b}}^1(\vec{f})(z)+\tilde{c}_1|^\delta\,dz\Big)^{1/\delta}$$

by five terms, still denoted by $\tilde{I}_j$, $j = 1, \dots, 5$, as in (4.6).

In this theorem, $\vec{b} \in \mathrm{Lip}_\beta^m$. Then we can estimate the five terms as follows. As the same estimate of I_1, we have

$$\tilde{I}_1 \le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,\delta}(T(\vec{f}))(x_0).$$

Note that $0 < \delta < q < \infty$ and $0 < l/q \le \alpha$, where l and q are given as in Definition 1.2. Then, by Lemma 3.2 and Definition 1.2 (3), we have

$$\begin{aligned}
\tilde{I}_2 &\le C|B|^{-1/\delta}\|T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^1)\|_{L^\delta(B)} \\
&\le C|B|^{-1/q}\|T((b_1 - b_B^1)\tilde{f}_1^1, \tilde{f}_2^1)\|_{L^{q,\infty}(B)} \\
&\le C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |b_1(y_1) - b_B^1|^{l_1}|f_1(y_1)|^{l_1}\,dy_1\Big)^{1/l_1}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_2(y_2)|^{l_2}\,dy_2\Big)^{1/l_2} \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}|B|^{-1/q}|\tilde{B}|^{1/l}r_B^{\alpha\beta}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_1(y_1)|^{l_1}\,dy_1\Big)^{1/l_1} M_{l_2}(f_2)(x_0) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}r_B^{n(\alpha/l-1/q)}\Big(\frac{1}{|2\tilde{B}|^{1-\beta s_0/n}}\int_{2\tilde{B}} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M_{l_2}(f_2)(x_0) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0).
\end{aligned}$$

For $z \in B$ and $y_2 \in (2\tilde{B})^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le \frac{1}{2}|y_2 - x_0|$. From Hölder's inequality, Definition 1.2 (1) and Lemma 3.4, it follows that

$$\begin{aligned}
\tilde{I}_3 &\le C\frac{1}{|B|}\int_B\int_{(2\tilde{B})^c}\int_{2\tilde{B}} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}r_B^{\varepsilon+\alpha\beta}\Big(\int_{2\tilde{B}} |f_1(y_1)|\,dy_1\Big)\Big(\int_{(2\tilde{B})^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n+\varepsilon/\alpha}}\,dy_2\Big) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}r_B^{\varepsilon+\alpha\beta}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0}|\tilde{B}|M(f_2)(x_0)(r_B^\alpha)^{-(n+\varepsilon/\alpha)} \\
&= C\|b_1\|_{\mathrm{Lip}_\beta}\Big(\frac{1}{|2\tilde{B}|^{1-\beta s_0/n}}\int_{2\tilde{B}} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M(f_2)(x_0) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0).
\end{aligned}$$

For $z \in B$ and $y_1 \in (2\tilde{B})^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le \frac{1}{2}|y_1 - x_0|$. By Hölder's inequality and Definition 1.2 (1), we have

$$\begin{aligned}
\tilde{I}_4 &\le C\frac{1}{|B|}\int_B\int_{2\tilde{B}}\int_{(2\tilde{B})^c} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}r_B^{\varepsilon}\Big(\int_{(2\tilde{B})^c} \frac{|x_0 - y_1|^\beta|f_1(y_1)|}{|x_0 - y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)\Big(\int_{2\tilde{B}} |f_2(y_2)|\,dy_2\Big) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}r_B^{\varepsilon}\Big(\sum_{k=1}^{\infty}\int_{2^{k+1}\tilde{B}\setminus 2^k\tilde{B}} \frac{|x_0 - y_1|^\beta|f_1(y_1)|}{|x_0 - y_1|^{2n+\varepsilon/\alpha}}\,dy_1\Big)M(f_2)(x_0)|\tilde{B}| \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}r_B^{\varepsilon}\sum_{k=1}^{\infty}(2^k r_B^\alpha)^{-(n+\varepsilon/\alpha)}\Big(\frac{1}{|2^{k+1}\tilde{B}|^{1-\beta/n}}\int_{2^{k+1}\tilde{B}} |f_1(y_1)|\,dy_1\Big)|\tilde{B}|M(f_2)(x_0) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{\infty}2^{-k(n+\varepsilon/\alpha)}\Big(\frac{1}{|2^{k+1}\tilde{B}|^{1-\beta s_0/n}}\int_{2^{k+1}\tilde{B}} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M(f_2)(x_0) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta}M_{\beta,s_0}(f_1)(x_0)M_{s_0}(f_2)(x_0).
\end{aligned}$$

For $z \in B$ and $y_1, y_2 \in (2\tilde{B})^c$, and for the center of the ball x_0, we have $|z - x_0|^\alpha \le \frac{1}{2}|y_1 - x_0|$ and $|z - x_0|^\alpha \le \frac{1}{2}|y_2 - x_0|$. From Hölder's inequality, Definition 1.2 (1) and Lemma 3.4, it follows that

$$\begin{aligned}
\tilde{I}_5 &\le C\frac{1}{|B|}\int_B \int_{(2\tilde{B})^c}\int_{(2\tilde{B})^c} \frac{|z - x_0|^\varepsilon}{(|x_0 - y_1| + |x_0 - y_2|)^{2n+\varepsilon/\alpha}}|b_1(y_1) - b_B^1||f_1(y_1)||f_2(y_2)|\,dy_1\,dy_2\,dz \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon \Big(\int_{(2\tilde{B})^c} \frac{|x_0 - y_1|^\beta |f_1(y_1)|}{|x_0 - y_1|^{n+\varepsilon/(2\alpha)}}\,dy_1\Big)\Big(\int_{(2\tilde{B})^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{n+\varepsilon/(2\alpha)}}\,dy_2\Big) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon \Big(\sum_{k=1}^\infty \int_{2^{k+1}\tilde{B}\setminus 2^k\tilde{B}} \frac{|x_0 - y_1|^\beta |f_1(y_1)|}{|x_0 - y_1|^{n+\varepsilon/(2\alpha)}}\,dy_1\Big) M(f_2)(x_0)(r_B^\alpha)^{-\varepsilon/(2\alpha)} \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta} r_B^\varepsilon \sum_{k=1}^\infty (2^k r_B^\alpha)^{-\varepsilon/(2\alpha)}\Big(\frac{1}{|2^{k+1}\tilde{B}|^{1-\beta/n}}\int_{2^{k+1}\tilde{B}} |f_1(y_1)|\,dy_1\Big) M(f_2)(x_0) r_B^{-\varepsilon/2} \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta} \sum_{k=1}^\infty 2^{-k\varepsilon/(2\alpha)}\Big(\frac{1}{|2^{k+1}\tilde{B}|^{1-\beta s_0/n}}\int_{2^{k+1}\tilde{B}} |f_1(y_1)|^{s_0}\,dy_1\Big)^{1/s_0} M(f_2)(x_0) \\
&\le C\|b_1\|_{\mathrm{Lip}_\beta} M_{\beta,s_0}(f_1)(x_0) M_{s_0}(f_2)(x_0).
\end{aligned}$$

$T_{\vec{b}}^2(\vec{f})$ can be dealt with by using the same method. Finally, combining the above two cases, we have

$$\begin{aligned}
M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x_0) &\sim \sup_{r_B>0}\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B(x_0, r_B)|}\int_{B(x_0,r_B)} \big||T_{\vec{b}}(\vec{f})(z)|^\delta - a\big|\,dz\Big)^{1/\delta} \\
&\le C\|\vec{b}\|_{\mathrm{Lip}_\beta^2} \sum_{j=1}^2 \Big(M_{\beta,\delta}(T(\vec{f}))(x_0) + M_{\beta,s_0}(f_j)(x_0)\prod_{i=1, i\ne j}^2 M_{s_0}(f_i)(x_0)\Big).
\end{aligned}$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. It follows from Lemma 3.6 that $w \in A_{\max\{p_1/s_0,\dots,p_m/s_0\}} \subset A_\infty$. Take δ and t such that $0 < \delta < t < 1/m$. By Lemmas 3.5 and 3.7, we have

$$\|M_t(T(\vec{f}))\|_{L^p(w)} \le C\|M_t^\sharp(T(\vec{f}))\|_{L^p(w)} \le C\Big\|\prod_{j=1}^m M_{s_0}(f_j)\Big\|_{L^p(w)}.$$

For every $j = 1, \dots, m$, since $w_j \in A_{p_j/s_0}$, there exists t_j such that $1 < t_j < p_j/s_0$ and $w_j \in A_{t_j}$. It follows from $s_0 < p_j/t_j$ that there exists s_j such that $s_0 < s_j < p_j/t_j < p_j$. Let $s = \min_{1\le j\le m} s_j$. Then $s_0 < s < p_j$, $j = 1, \dots, m$.

Since $t_j < p_j/s_j \le p_j/s$, we have $w_j \in A_{t_j} \subset A_{p_j/s}$, and M is bounded on $L^{p_j/s}(w_j)$, $j = 1, \dots, m$. From Lemma 3.5, Theorem 2.1 and Hölder's inequality, it follows that

$$\begin{aligned}
\|T_{\vec{b}}(\vec{f})\|_{L^p(w)} &\le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^p(w)} \le C\|M_\delta^\sharp(T_{\vec{b}}(\vec{f}))\|_{L^p(w)} \\
&\le C\|\vec{b}\|_{\mathrm{BMO}^m}\Big(\|M_t(T(\vec{f}))\|_{L^p(w)} + \Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^p(w)}\Big) \\
&\le C\|\vec{b}\|_{\mathrm{BMO}^m}\Big(\Big\|\prod_{j=1}^m M_{s_0}(f_j)\Big\|_{L^p(w)} + \Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^p(w)}\Big) \\
&\le C\|\vec{b}\|_{\mathrm{BMO}^m}\Big\|\prod_{j=1}^m M_s(f_j)\Big\|_{L^p(w)} \le C\|\vec{b}\|_{\mathrm{BMO}^m}\prod_{j=1}^m \|M_s(f_j)\|_{L^{p_j}(w_j)} \\
&= C\|\vec{b}\|_{\mathrm{BMO}^m}\prod_{j=1}^m \|M(|f_j|^s)\|_{L^{p_j/s}(w_j)}^{1/s} \le C\|\vec{b}\|_{\mathrm{BMO}^m}\prod_{j=1}^m \||f_j|^s\|_{L^{p_j/s}(w_j)}^{1/s} \\
&= C\|\vec{b}\|_{\mathrm{BMO}^m}\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)},
\end{aligned}$$

which completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. From the fact that $w^{1/q} \in A(p,q)$ and Lemma 3.9, it follows that $w \in A_{q/p'+1}$. Take a δ such that $0 < \delta < 1/m$. By Lemma 3.5 and Theorem 2.2, we have

$$\begin{aligned}\|T_{\vec{b}}(\vec{f})\|_{L^q(w)} &\le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^q(w)} \le C\|M_\delta^\sharp(T_{\vec{b}}(\vec{f}))\|_{L^q(w)}\\ &\le C\|\vec{b}\|_{\mathrm{Lip}_\beta^m}\sum_{j=1}^m\Big(\|M_{\beta,\delta}(T(\vec{f}))\|_{L^q(w)} + \Big\|M_{\beta,s_0}(f_j)\prod_{i=1,i\neq j}^m M_{s_0}(f_i)\Big\|_{L^q(w)}\Big).\end{aligned}$$

Since $w \in A_{q/p'+1}$, there exists s such that $1 < s < q/p'+1$ and $w \in A_s$. Set $t = \frac{pq}{p(s-1)+q}$. Then $s = \frac{q/t}{(p/t)'}+1$, $1 < t < p < n/\beta$ and $w \in A_{(q/t)/(p/t)'+1}$. Let $\tilde{\beta} = \beta t$, $\tilde{p} = p/t$ and $\tilde{q} = q/t$. Then $0 < \tilde{\beta} < n$, $1 < \tilde{p} < n/\tilde{\beta}$ and $1/\tilde{q} = 1/\tilde{p} - \tilde{\beta}/n$. It follows from Lemma 3.9 that $w^{1/\tilde{q}} \in A(\tilde{p},\tilde{q})$. By Lemmas 3.8 and 3.10, we have

$$\begin{aligned}\|M_{\beta,\delta}(T(\vec{f}))\|_{L^q(w)} &\le \|M_{\beta,t}(T(\vec{f}))\|_{L^q(w)} = \|M_{\tilde{\beta},1}(|T(\vec{f})|^t)^{1/t}\|_{L^q(w)}\\ &= \Big(\int_{\mathbb{R}^n}[M_{\tilde{\beta},1}(|T(\vec{f})|^t)(x)w(x)^{1/\tilde{q}}]^{\tilde{q}}\,dx\Big)^{1/(\tilde{q}t)}\\ &\le C\Big(\int_{\mathbb{R}^n}[|T(\vec{f})(x)|^t w(x)^{1/\tilde{q}}]^{\tilde{p}}\,dx\Big)^{1/(\tilde{p}t)}\\ &= C\|T(\vec{f})\|_{L^p(w^{p/q})} \le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}.\end{aligned}$$

For every $j = 1,\dots,m$, since $1/q_j = 1/p_j - \beta/n$, we have $1/q = 1/q_j + \sum_{i=1,i\neq j}^m 1/p_i$. By Hölder's inequality, we get

$$\Big\|M_{\beta,s_0}(f_j)\prod_{i=1,i\neq j}^m M_{s_0}(f_i)\Big\|_{L^q(w)} \le \|M_{\beta,s_0}(f_j)\|_{L^{q_j}(w_j^{q_j/p_j})}\prod_{i=1,i\neq j}^m\|M_{s_0}(f_i)\|_{L^{p_i}(w_i)}.$$

Set $\tilde{\beta}_j = \beta s_0$, $\tilde{p}_j = p_j/s_0$ and $\tilde{q}_j = q_j/s_0$. Then $0 < \tilde{\beta}_j < n$, $1 < \tilde{p}_j < n/\tilde{\beta}_j$ and $1/\tilde{q}_j = 1/\tilde{p}_j - \tilde{\beta}_j/n$. The fact that $w_j^{s_0/p_j} \in A(p_j/s_0, q_j/s_0)$ means $w_j^{1/\tilde{p}_j} \in A(\tilde{p}_j,\tilde{q}_j)$. By Lemma 3.10, we have

$$\begin{aligned}\|M_{\beta,s_0}(f_j)\|_{L^{q_j}(w_j^{q_j/p_j})} &= \|M_{\tilde{\beta}_j,1}(|f_j|^{s_0})^{1/s_0}\|_{L^{q_j}(w_j^{\tilde{q}_j/\tilde{p}_j})}\\ &= \Big(\int_{\mathbb{R}^n}[M_{\tilde{\beta}_j,1}(|f_j|^{s_0})(x)w_j(x)^{1/\tilde{p}_j}]^{\tilde{q}_j}\,dx\Big)^{1/(\tilde{q}_j s_0)}\\ &\le C\Big(\int_{\mathbb{R}^n}[|f_j(x)|^{s_0}w_j(x)^{1/\tilde{p}_j}]^{\tilde{p}_j}\,dx\Big)^{1/(\tilde{p}_j s_0)}\\ &= C\|f_j\|_{L^{p_j}(w_j)}.\end{aligned}$$

For every $i = 1,\dots,m$ and $i \neq j$, since $w_i \in A_{p_i/s_0}$ and $p_i > s_0$, we have that M is bounded on $L^{p_i/s_0}(w_i)$. Thus,

$$\|M_{s_0}(f_i)\|_{L^{p_i}(w_i)} = \|M(|f_i|^{s_0})\|_{L^{p_i/s_0}(w_i)}^{1/s_0} \le C\||f_i|^{s_0}\|_{L^{p_i/s_0}(w_i)}^{1/s_0} = C\|f_i\|_{L^{p_i}(w_i)}.$$

Therefore, for every $j = 1,\dots,m$,

$$\Big\|M_{\beta,s_0}(f_j)\prod_{i=1,i\neq j}^m M_{s_0}(f_i)\Big\|_{L^q(w)} \le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}.$$

In conclusion,

$$\|T_{\vec{b}}(\vec{f})\|_{L^q(w)} \le C\|\vec{b}\|_{\mathrm{Lip}_\beta^m}\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}.$$

This completes the proof of Theorem 2.4. □

Funding: The first author is partly supported by the National Natural Science Foundation of China (no. 11671397), the Fundamental Research Funds for the Central Universities (no. 2009QS16) and the State Scholarship Fund of China. The second author was partly supported by a US NSF grant and a Simons Collaboration Grant from the Simons Foundation. The third author is partly supported by the National Natural Science Foundation of China (no. 11771195).

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