



Concentration-compactness principle for Trudinger–Moser inequalities on Heisenberg groups and existence of ground state solutions

Jungang Li¹ · Guozhen Lu¹ · Maochun Zhu²

Received: 7 March 2017 / Accepted: 8 April 2018 / Published online: 2 May 2018
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the n -dimensional Heisenberg group, $Q = 2n + 2$ be the homogeneous dimension of \mathbb{H}^n . We extend the well-known concentration-compactness principle on finite domains in the Euclidean spaces of Lions (Rev Mat Iberoam 1:145–201, 1985) to the setting of the Heisenberg group \mathbb{H}^n . Furthermore, we also obtain the corresponding concentration-compactness principle for the Sobolev space $HW^{1,Q}(\mathbb{H}^n)$ on the entire Heisenberg group \mathbb{H}^n . Our results improve the sharp Trudinger–Moser inequality on domains of finite measure in \mathbb{H}^n by Cohn and Lu (Indiana Univ Math J 50(4):1567–1591, 2001) and the corresponding one on the whole space \mathbb{H}^n by Lam and Lu (Adv Math 231:3259–3287, 2012). All the proofs of the concentration-compactness principles for the Trudinger–Moser inequalities in the literature even in the Euclidean spaces use the rearrangement argument and the Polyá–Szegő inequality. Due to the absence of the Polyá–Szegő inequality on the Heisenberg group, we will develop a different argument. Our approach is surprisingly simple

The results of this paper were presented by Zhu at the Workshop in Fourier Analysis in Sanya Mathematical Forum in August, 2016 and by Lu at the AMS special session on Geometric Aspects of Harmonic Analysis in Maine in September, 2016.

Communicated by A. Malchiodi.

M. Zhu was partly supported by Natural Science Foundation of China (11601190), Natural Science Foundation of Jiangsu Province (BK20160483) and Jiangsu University Foundation Grant (16JDG043). J. Li and G. Lu was partly supported by a US National Science Foundation Grant DMS No. 1700918, a Collaboration Grant No. 519099 and a Simons Fellowship from the Simons Foundation.

✉ Maochun Zhu
zhumaochun2006@126.com

Jungang Li
jungang.li@uconn.edu

Guozhen Lu
guozhen.lu@uconn.edu

¹ Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA

² Faculty of Science, Jiangsu University, Zhenjiang 212013, China

and general and can be easily applied to other settings where symmetrization argument does not work. As an application of the concentration-compactness principle, we establish the existence of ground state solutions for a class of Q -Laplacian subelliptic equations on \mathbb{H}^n :

$$-\operatorname{div} \left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u \right) + V(\xi) |u|^{Q-2} u = \frac{f(u)}{\rho(\xi)^\beta}$$

with nonlinear terms f of maximal exponential growth $\exp(\alpha t^{\frac{Q}{Q-1}})$ as $t \rightarrow +\infty$. All the results proved in this paper hold on stratified groups with the same proofs. Our method in this paper also provide a new proof of the classical concentration-compactness principle for Trudinger-Moser inequalities in the Euclidean spaces without using the symmetrization argument.

Mathematics Subject Classification 46E35 · 35J92 · 35H20

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ and $W_0^{1,q}(\Omega)$ be the usual Sobolev space, that it, the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{W^{1,q}(\Omega)} = \left(\int_{\Omega} (|u|^q + |\nabla u|^q) dx \right)^{\frac{1}{q}}.$$

If $1 \leq q < n$, the classical Sobolev embedding says that $W_0^{1,q}(\Omega) \hookrightarrow L^s(\Omega)$ for $1 \leq s \leq q^*$, where $q^* := \frac{nq}{n-q}$. When $q = n$, it is known that

$$W_0^{1,n}(\Omega) \hookrightarrow L^s(\Omega) \quad \text{for any } n \leq s < +\infty,$$

but $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$. When Ω is of finite measure, the analogue of the Sobolev embedding is the well-known Trudinger’s inequality, which was established independently by Yudovič [58], Pohožaev [52], and Trudinger [56]. In 1971, Moser sharpened in [51] Trudinger’s inequality, and proved the following inequality:

$$\sup_{\substack{u \in W_0^{1,n}(\Omega) \\ \|\nabla u\|_{L^n(\Omega)} \leq 1}} \int_{\Omega} e^{\alpha|u|^{\frac{n}{n-1}}} dx < \infty \quad \text{iff } \alpha \leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}, \tag{1.1}$$

where ω_{n-1} is the $n - 1$ dimensional surface measure of the unit ball in \mathbb{R}^n and $|\Omega| < \infty$. Inequality (1.1) is known as the *Trudinger–Moser inequality*. In 1985, Lions [42] established the concentration-compactness principle associated with (1.1), which tells us that, if $\{u_k\}$ is a sequence of functions in $W_0^{1,n}(\Omega)$ with $\|\nabla u_k\|_n = 1$ such that $u_k \rightharpoonup u$ weakly in $W^{1,n}(\Omega)$, then for any $0 < p < M_{n,u} := (1 - \|\nabla u\|_n^n)^{-1/(n-1)}$, one has

$$\sup_k \int_{\Omega} e^{\alpha_n p |u_k|^{\frac{n}{n-1}}} dx < \infty. \tag{1.2}$$

This conclusion gives more precise information and is stronger than (1.1) when $u_k \rightharpoonup u \neq 0$ weakly in $W_0^{1,n}(\Omega)$.

When $|\Omega| = +\infty$, the inequality (1.1) is meaningless. In this case, the first related inequalities have been considered by Cao [6] in the case $N = 2$ and for any dimension by do Ó [16] and Adachi-Tanaka [1]. For two-weighted subcritical Trudinger–Moser inequalities,

see [21,23]. Note that, unlike (1.1), all these results have been proved in the subcritical growth case, that is $\alpha < \alpha_n$. In [54], Ruf showed that in the case $N = 2$, the exponent $\alpha_2 = 4\pi$ becomes admissible if the Dirichlet norm $\int_{\Omega} |\nabla u|^2 dx$ is replaced by $W^{1,2}$ norm $\int_{\Omega} (|u|^2 + |\nabla u|^2) dx$. Later, Li and Ruf [40] established the same critical inequality as in [54] in arbitrary dimensions. These critical and subcritical inequalities have been proved to be equivalent by Lam et al. in [36]. The asymptotic estimates for the supremums for subcritical and critical Trudinger–Moser inequalities and their relationship established in [36] have been found useful in proving the existence and nonexistence of extremal functions for the Trudinger–Moser inequalities (see [21,25–27]).

While there has been much progress for Trudinger–Moser type inequalities and the concentration-compactness phenomenon on the Euclidean spaces, much less is known on the Heisenberg group. We recall that most of the proofs for Trudinger–Moser inequalities in the Euclidean space are based on the rearrangement argument. When one considers the Trudinger–Moser inequalities in the subelliptic setting, one often attempts to use the radial non-increasing rearrangement u^* of functions u . Unfortunately, it is not known to be true that the L^p norm of the subelliptic gradient of the rearrangement of a function is dominated by the L^p norm of the subelliptic gradient of the function. In other words, the Pólya-Szegő type inequality in the subelliptic setting like

$$\|\nabla_{\mathbb{H}} u^*\|_{L^p} \leq \|\nabla_{\mathbb{H}} u\|_{L^p} \tag{1.3}$$

is not available. Actually, from the work of Jerison and Lee [24] on sharp L^2 to $L^{\frac{2Q}{Q-2}}$ inequality on the Heisenberg group with applications to the solution to the CR Yamabe problem, we know that this inequality fails to hold for the case $p = 2$ in Heisenberg groups.

The sharp Trudinger–Moser inequality on Heisenberg groups was due to Cohn and Lu [10] and has been extended to the Heisenberg type groups and Carnot groups in [45] and [5] and with singular weights in [33]. Furthermore, Lam and Lu developed in [29,30] a rearrangement-free argument by considering the level sets of the functions under consideration, this argument enables them to deduce the global critical Trudinger–Moser inequalities on the entire space from the local ones on the level sets (see also work by Lam et al. [34] for subcritical inequalities and [43] for adaptation of such an argument). Therefore, both sharp critical and subcritical Trudinger–Moser inequalities are established on the entire Heisenberg group in [29,34].

More recently, Černý et al. in [8] discover a new approach to obtain and sharpen Lions’s concentration compactness principles (1.2) as well as fill in a gap in [42]. This approach was further extended to study the concentration-compactness principle for the whole space \mathbb{R}^n by do Ó et al. in [18]. Their results can be stated as follows: let $\{u_k\}$ be a sequence of functions in $W_0^{1,n}(\mathbb{R}^n)$ with $\|u_k\|_{W^{1,n}(\mathbb{R}^n)} = 1$ such that $u_k \rightharpoonup u$ weakly in $W^{1,n}(\mathbb{R}^n)$, then for any $0 < p < \tilde{M}_{n,u} := \left(1 - \|u\|_{W^{1,n}(\mathbb{R}^n)}^n\right)^{-1/(n-1)}$,

$$\sup_k \int_{\mathbb{R}^n} e^{\alpha_n p |u_k|^{\frac{n}{n-1}}} dx < \infty. \tag{1.4}$$

Furthermore, $\tilde{M}_{n,u}$ is sharp in the sense that there exists a sequence $\{u_k\}$ satisfying $\|u_k\|_{W^{1,n}(\mathbb{R}^n)} = 1$ and $u_k \rightharpoonup u$ weakly in $W^{1,n}(\mathbb{R}^n)$ such that the supremum (1.4) is infinite for $p > \tilde{M}_{n,u}$.¹ We also quote a recent work on sharp Trudinger–Moser type inequalities in the spirit of Lions’ work on the whole spaces [35].

¹ The sequence $\{u_k\}$ constructed in [17] cannot show that the supremum (1.4) is infinite for $p = \tilde{M}_{n,u}$ (see Remark 1).

Nevertheless, we mention that arguments of [8, 17] still rely on the Polyá–Szegő inequality in the Euclidean spaces and such an inequality is not available in the subelliptic setting.

Now, it is fairly natural to ask whether the concentration–compactness principles (1.2) and (1.4) still holds for the subelliptic setting in spite of its absence of the Polyá–Szegő inequality in such a setting. In this paper, we will give an affirmative answer to this question. More precisely, we first prove a concentration–compactness principle for domains with finite measure on Heisenberg groups (Theorem 2.1), and then prove the concentration–compactness principle for the horizontal Sobolev space $HW^{1,Q}(\mathbb{H}^n)$ (Theorem 2.2, for definition of $HW^{1,Q}(\mathbb{H}^n)$ see Sect. 2). Theorem 2.1 sharpens the Trudinger–Moser inequality by Cohn and Lu [10] and recent one of Lam et al. [33], Theorem 2.2 improves the sharp Trudinger–Moser inequality by Lam and Lu [29].

In the proof of the concentration–compactness principles on Heisenberg groups, though our proof is an argument by contradiction as done in the Euclidean spaces, our method is substantially different from those in [8, 17]. Using the Polyá–Szegő inequality, in the Euclidean space one can reduce the problem to radial functions and then the radial lemmas play an important role. In our setting of the Heisenberg group, for the proof on the bounded domain, instead of dealing with the upper and lower bound of the radial symmetric rearrangement [8], we directly consider the Dirichlet norms of each part under the truncation (the way of truncation will be defined in the proof of the main theorems) while taking weak convergence. This is based on the fact that the constant $M_{n,u}$ only depends on $\|\nabla u\|_n$. For the proof on the whole space, on the one hand, in order to localize the problem as well as to get rid of the rearrangement argument, we in the spirit apply the technique of level set argument developed in [29] (see also [30]). On the other hand, due to the different type of inequalities considered here from those in [29,30], the level sets used here are different from those in [29]. It is worthwhile to note that our approach can be easily applied to the other subelliptic setting such as Carnot groups with virtually no modifications.

As an application of concentration–compactness principles on Heisenberg groups, we study the existence of positive ground state solution to a class of partial differential equations with exponential growth on \mathbb{H}^n of the form:

$$-\operatorname{div}\left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u\right)+V(\xi)|u|^{Q-2} u=\frac{f(u)}{\rho(\xi)^{\beta}} \tag{1.5}$$

for any $0 \leq \beta < Q$, where $V: \mathbb{H}^n \rightarrow \mathbb{R}$ is a continuous potential, and $f: \mathbb{R} \rightarrow \mathbb{R}$ behaves like $\exp\left(\alpha t^{\frac{Q}{Q-1}}\right)$ when $t \rightarrow \infty$ (for the meaning of $\nabla_{\mathbb{H}}$ and $\rho(\xi)$ see Sect. 2).

We remark that the Trudinger–Moser type inequalities play an important role in the study of the existence of solutions to nonlinear partial differential equations of exponential growth in Euclidean spaces. A good deal of works have been done and we just quote some of them on this subject, which are a good starting point for further bibliographic references: [3, 4, 7, 14–19, 22, 28, 31, 38, 39, 41, 46–49, 55, 57, 60], etc.

Existence and multiplicity of nontrivial nonnegative solutions to the equations (1.5) on the Heisenberg groups have been proved in a series of papers [12, 29, 33, 34]. In their argument, they apply the Trudinger–Moser inequality in the whole space \mathbb{H}^n (Lemma 2.3 in Sect. 2) combined with mountain–pass theorem, minimization and Ekelands variational principle. Nevertheless, the existence of *ground state* solutions to the sub-elliptic equation (1.5) on the Heisenberg groups has not been established yet so far. The concentration–compactness principles on Heisenberg groups proved in this paper makes it possible to establish such an existence result.

We end this introduction by mentioning recent works on concentration-compactness principle for singular Trudinger–Moser inequalities in \mathbb{R}^n [59] and singular Adams inequality [2] for bi-Laplacians in \mathbb{R}^4 [9]. In particular, a completely symmetrization-free argument has been developed in a more recent work [37] on concentration-compactness principle for Trudinger–Moser’s inequalities on Riemannian manifolds and stratifies groups. We note that in [37] we avoid completely even a weak form of Polyá–Szegő inequality which is not known to be true on Riemannian manifolds. We also note that sharp weighted Trudinger–Moser inequalities on the gradient terms have also been proved using a quasi-conformal type mapping and a symmetrization lemma associated with power weights to reduce the weighted inequalities to the non-weighted ones (see [20,21,32]). These are cases where the Polyá–Szegő inequality fails to hold with weights.

This paper is organized as follows: in Sect. 2 we recall some basic facts about Heisenberg Groups and state precisely our main results; in Sect. 3 we first prove the concentration compactness principles for Trudinger–Moser inequalities on domains with finite measure—Theorem 2.1, and then we give the proof for the concentration compactness principles for horizontal Sobolev space $HW^{1,Q}(\mathbb{H}^n)$ (Theorem 2.2). As an application, in Sect. 4, we consider the equations (1.5) and establish the existence of the *ground state solutions* and prove Theorem 2.3 by using the minimax argument and Theorem 2.2.

2 Preliminaries and statement of the results

2.1 Background on Heisenberg groups

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the n -dimensional Heisenberg group, whose group structure is given by

$$(x, t) \circ (x', t') = (x + x', t + t' + 2\text{im}(x \cdot \bar{x}')).$$

The Lie algebra of \mathbb{H}^n is generated by the left invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

for $i = 1, \dots, n$. These generators satisfy the non-commutative relationship $[X_i, Y_i] = -4\delta_{ij}T$. Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r > 0$, there is a dilation naturally associated with the Heisenberg group structure which is usually denoted as $\delta_r(x, t) = (rx, r^2t)$. The Jacobian determinant of δ_r is r^Q , where $Q = 2n + 2$ is the homogeneous dimension of \mathbb{H}^n .

We will use $\xi = (x, t)$ to denote any point $(x, t) \in \mathbb{H}^n$, then the anisotropic dilation structure on \mathbb{H}^n introduces a homogeneous norm $|\xi| = (|x|^4 + t^2)^{1/4}$. Let

$$B_r = \{\xi : |\xi| < r\}$$

be the metric ball of center 0 and radius r in \mathbb{H}^n . Since the Lebesgue measure in \mathbb{R}^{2n+1} is the Haar measure on \mathbb{H}^n , one has (writing $|A|$ for the measure of A)

$$|B_r| = \omega_Q r^Q,$$

where ω_Q is a positive constant only depending on Q (see [10]).

We write $|\nabla_{\mathbb{H}^n} u|$ to express the norm of the subelliptic gradient of the function $u : \mathbb{H}^n \rightarrow \mathbb{R}$:

$$|\nabla_{\mathbb{H}^n} u| = \sqrt{\sum (X_i u)^2 + (Y_i u)^2}.$$

Let Ω be an open set in \mathbb{H}^n and $p > 1$. We define the horizontal Sobolev spaces

$$HW^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \|u\|_{HW^{1,p}(\Omega)} < \infty \right\}$$

with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \left(\int_{\Omega} (|\nabla_{\mathbb{H}^n} u(z, t)|^p + |u(z, t)|^p) dxdt \right)^{1/p}.$$

Also, we define the space $HW_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the norm of $HW^{1,p}(\Omega)$. For Sobolev embedding and compact embedding theorems for vector fields satisfying Hörmander’s condition, which includes the Heisenberg groups as a special case, we refer the reader to e.g., [11, 44]. These embedding theorems are needed in proving the existence of the ground state solutions to the quasi-linear subelliptic equations with nonlinearity of exponential growth.

2.2 Some useful known results on Heisenberg groups

In this subsection, we collect some known results which will be used in the following.

Lemma 2.1 [10] *Let $\rho = |\xi|$ be the homogeneous norm of the element $\xi = (x, t) \in \mathbb{H}^n$, and $g(\xi) = g(\rho)$ be a C^1 radial function on \mathbb{H}^n . Then*

$$|\nabla_{\mathbb{H}^n} g(\xi)| = \frac{g'(\rho)}{\rho} |x|.$$

Lemma 2.2 [33] *Let $\alpha_Q = Q \left(2\pi^n \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{Q-1}{2}\right) \Gamma\left(\frac{Q}{2}\right)^{-1} \Gamma(n-1) \right)^{\frac{1}{Q-1}}$, $0 \leq \beta < Q$. There exists a uniform constant c depending only on Q, β such that for all $\Omega \subset \mathbb{H}^n$ with $|\Omega| < \infty$ and $\alpha \leq \alpha_{Q,\beta} = \alpha_Q \left(1 - \frac{\beta}{Q} \right)$, one has*

$$\sup_{\substack{u \in HW_0^{1,Q}(\Omega) \\ \|\nabla_{\mathbb{H}^n} u\|_{L^Q} \leq 1}} \int_{\Omega} \frac{\exp\left(\alpha u(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < c. \tag{2.1}$$

The constant $\alpha_{Q,\beta}$ is the best possible in the sense that if $\alpha > \alpha_{Q,\beta}$, then the supremum above is infinite.

Lemma 2.3 [29] *Let $0 \leq \beta < Q$. There exists a uniform constant c depending only on Q, β such that for all $\alpha \leq \alpha_{Q,\beta}$, one has*

$$\sup_{\substack{f \in HW^{1,Q}(\mathbb{H}^n) \\ \|f\|_{HW^{1,Q}(\mathbb{H}^n)} \leq 1}} \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha f(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < c. \tag{2.2}$$

where $\Phi(t) = e^t - \sum_{j=0}^{Q-2} \frac{t^j}{j!}$. The constant $\alpha_{Q,\beta}$ is the best possible in the sense that if $\alpha > \alpha_{Q,\beta}$, then the supremum in the inequality (2.2) is infinite.

2.3 Statement of the main results

Now, we are ready to state precisely the main results of this paper.

Theorem 2.1 (Concentration compactness for domains with finite measure) *Let $0 \leq \beta < Q$. Assume that $\{u_k\}$ is a sequence in $HW_0^{1,Q}(\Omega)$ with $|\Omega| < \infty$, such that $\|\nabla_{\mathbb{H}^n} u_k\|_Q = 1$ and $u_k \rightharpoonup u \neq 0$ in $HW_0^{1,Q}(\Omega)$. If*

$$0 < p < M_{Q,u} := \frac{1}{\left(1 - \|\nabla_{\mathbb{H}^n} u\|_Q^Q\right)^{1/(Q-1)}}$$

then

$$\sup_k \int_{\Omega} \frac{e^{\alpha_{Q,\beta} p u_k^{\frac{Q}{Q-1}}}}{\rho(\xi)^\beta} d\xi < \infty.$$

Moreover, $M_{Q,u}$ is sharp in the sense that there exists a sequence $\{u_k\}$ satisfying $\|\nabla_{\mathbb{H}^n} u\|_Q^Q = 1$ and $u_k \rightharpoonup u \neq 0$ in $HW_0^{1,Q}(\Omega)$ such that the supremum is infinite for $p \geq M_{Q,u}$.

Theorem 2.2 (Concentration compactness for $HW^{1,Q}(\mathbb{H}^n)$) *Let $0 \leq \beta < Q$. Assume that $\{u_k\}$ is a sequence in $HW^{1,Q}(\mathbb{H}^n)$ such that $\|u_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q = 1$ and $u_k \rightharpoonup u \neq 0$ in $HW^{1,Q}(\mathbb{H}^n)$. If*

$$0 < p < \tilde{M}_{Q,u} := \frac{1}{\left(1 - \|u\|_{HW^{1,Q}(\mathbb{H}^n)}^Q\right)^{1/(Q-1)}},$$

then

$$\sup_k \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty, \tag{2.3}$$

where $\Phi(t) = e^t - \sum_{j=0}^{Q-2} \frac{t^j}{j!}$. Furthermore, $\tilde{M}_{Q,u}$ is sharp in the sense that there exists a sequence $\{u_k\}$ satisfying $\|u_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q = 1$ and $u_k \rightharpoonup u \neq 0$ in $HW^{1,Q}(\mathbb{H}^n)$ such that the supremum is infinite for $p > \tilde{M}_{Q,u}$.

The following natural question still remains open at this time.

Problem 1 *Does (2.3) still hold when $p = \tilde{M}_{Q,u}$?*

Now, Let us give the definition of the ground state solution of (1.5):

Definition 1 (Ground state solution) *A function u is said to be the ground state solution of (1.5), if u is positive and minimizes the energy functional associated to the Eq. (4.1) defined by*

$$J(u) = \frac{1}{Q} \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}^n} u|^Q + V(\xi) |u|^Q\right) d\xi - \int_{\mathbb{H}^n} \frac{F(u)}{\rho(\xi)^\beta} d\xi$$

within the set of nontrivial solutions of (1.5).

For the Eq. (4.1), we obtain the following

Theorem 2.3 *Under the hypotheses of (H1) and (H2) in Sect. 4, the Q -sub-Laplacian equations (1.5) has a positive ground state solution.*

Throughout this paper, denote by the letter c some positive constant which may vary from line to line.

3 Concentration-compactness principles on Heisenberg groups

3.1 Concentration-compactness principle for domains with finite measure

In this subsection, we give the

Proof of Theorem 2.1 Since $\|\nabla_{\mathbb{H}}u\|_Q \leq \lim_k \|\nabla_{\mathbb{H}}u_k\|_Q = 1$, we split the proof into two cases.

Case 1: $\|\nabla_{\mathbb{H}}u\|_Q < 1$. We assume by contradiction for some $p_1 < M_{Q,u}$, we have

$$\sup_k \int_{\Omega} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = +\infty.$$

Set

$$\Omega_L^k = \{\xi \in \Omega : u_k(\xi) \geq L\},$$

where L is some constant. Let $v_k = u_k - L$. Then for any $\varepsilon > 0$, one has

$$u_k^{\frac{Q}{Q-1}} \leq (1 + \varepsilon) v_k^{\frac{Q}{Q-1}} + C(\varepsilon, Q) L^{\frac{Q}{Q-1}}. \tag{3.1}$$

Since $0 \leq \beta < Q$, we have

$$\begin{aligned} \int_{\Omega} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi &= \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi + \int_{\Omega \setminus \Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\ &\leq \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\ &\quad + c \exp\left(\alpha_{Q,\beta} p_1 L^{\frac{Q}{Q-1}}\right) \int_{\Omega} \frac{1}{\rho(\xi)^\beta} d\xi \\ &\leq \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi + c(L, Q, |\Omega|, \beta), \end{aligned}$$

and then

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \infty.$$

By (3.1) we have

$$\int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \leq \exp\left(\alpha_{Q,\beta} p_1 C(\varepsilon, Q) L^{\frac{Q}{Q-1}}\right) \times \int_{\Omega_L^k} \frac{\exp\left((1+\varepsilon)\alpha_{Q,\beta} p_1 v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi.$$

Thus

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\bar{p}_1 \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \infty,$$

where $\bar{p}_1 = (1 + \varepsilon) p_1 < M_{Q,u}$.

Now, we define

$$T^L(u) = \min\{L, u\} \text{ and } T_L(u) = u - T^L(u)$$

and choose L such that

$$\frac{1 - \|\nabla_{\mathbb{H}} u\|_Q^Q}{1 - \|\nabla_{\mathbb{H}} T^L u\|_Q^Q} > \left(\frac{\bar{p}_1}{M_{Q,u}}\right)^{Q-1}. \tag{3.2}$$

We claim that

$$\limsup_k \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi < \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

If not, then up to a subsequence, one has

$$\int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi = \int_{\Omega} |\nabla_{\mathbb{H}} T_L u_k|^Q d\xi \geq \left(\frac{1}{\bar{p}_1}\right)^{Q-1} + o_k(1). \tag{3.3}$$

Thus,

$$\begin{aligned} \left(\frac{1}{\bar{p}_1}\right)^{Q-1} + \int_{\Omega} |\nabla_{\mathbb{H}} T^L u_k|^Q d\xi + o_k(1) &\leq \int_{\Omega} |\nabla_{\mathbb{H}} T_L u_k|^Q d\xi + \int_{\Omega \setminus \Omega_L^k} |\nabla_{\mathbb{H}} u_k|^Q d\xi \\ &= \int_{\Omega_L^k} |\nabla_{\mathbb{H}} u_k|^Q d\xi + \int_{\Omega \setminus \Omega_L^k} |\nabla_{\mathbb{H}} u_k|^Q d\xi = 1. \end{aligned}$$

For $L > 0$ fixed, $T^L u_k$ is also bounded in $HW^{1,Q}(\Omega)$. Hence, up to a subsequence, $T^L u_k \rightharpoonup T^L u$ in $HW^{1,Q}(\Omega)$ and $T^L u_k \rightarrow T^L u$ almost everywhere in Ω . By the lower semicontinuity of the norm in $HW^{1,Q}(\Omega)$ and the above inequality, we have

$$\bar{p}_1 \geq \frac{1}{\left(1 - \liminf_{k \rightarrow \infty} \|\nabla_{\mathbb{H}} T^L u_k\|_Q^Q\right)^{\frac{1}{Q-1}}} \geq \frac{1}{\left(1 - \|\nabla_{\mathbb{H}} T^L u\|_Q^Q\right)^{\frac{1}{Q-1}}},$$

combining with (3.2), we derive

$$\bar{p}_1 \geq \frac{1}{\left(1 - \|\nabla_{\mathbb{H}} T^L u\|_Q^Q\right)^{\frac{1}{Q-1}}} > \frac{\bar{p}_1}{M_{Q,u}} \frac{1}{\left(1 - \|\nabla_{\mathbb{H}} u\|_Q^Q\right)^{\frac{1}{Q-1}}} = \bar{p}_1,$$

which is a contradiction. Therefore

$$\limsup_k \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi < \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

By the Trudinger–Moser inequality (2.1), we derive

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\bar{p}_1 \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty,$$

which is also a contradiction. The proof is finished in this case.

Case 2: $\|\nabla_{\mathbb{H}} u\|_Q = 1$. We can iterate the procedure as in Case 1 and get

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\bar{p}_1 \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} = \infty,$$

where $\bar{p}_1 = (1 + \varepsilon) p_1$. Then we have

$$\limsup_k \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi = \limsup_k \int_{\Omega} |\nabla_{\mathbb{H}} T_L u_k|^Q d\xi \geq \left(\frac{1}{\bar{p}_1}\right)^{Q-1},$$

thus,

$$\begin{aligned} \|\nabla_{\mathbb{H}} T^L u\|_Q^Q &\leq \liminf_k \int_{\Omega} |\nabla_{\mathbb{H}} T^L u_k|^Q d\xi = 1 \\ &\quad - \limsup_k \int_{\Omega} |\nabla_{\mathbb{H}} T_L u_k|^Q d\xi \leq 1 - \left(\frac{1}{\bar{p}_1}\right)^{Q-1}. \end{aligned}$$

On the other hand, since $\|\nabla_{\mathbb{H}} u\|_Q = 1$, we can take L large enough such that

$$\|\nabla_{\mathbb{H}} T^L u\|_Q^Q > 1 - \frac{1}{2} \left(\frac{1}{\bar{p}_1}\right)^{Q-1},$$

which is contradiction, and the proof is finished in this case.

Now, we prove the sharpness of $M_{Q,u}$. For some $r > 0$, we defined $\omega_k(\xi)$ by

$$\omega_k(\xi) = \begin{cases} Q^{\frac{1-Q}{Q}} (c_Q)^{-\frac{1}{Q}} k^{\frac{Q-1}{Q}} & \text{if } |\xi| \in \left[0, r e^{-\frac{k}{Q}}\right] \\ Q^{\frac{1}{Q}} (c_Q)^{-\frac{1}{Q}} \log\left(\frac{r}{|\xi|}\right) k^{-\frac{1}{Q}} & \text{if } |\xi| \in \left[r e^{-\frac{k}{Q}}, r\right] \\ 0 & \text{if } |\xi| \in [r, \infty], \end{cases} \tag{3.4}$$

where $c_Q = \int_{\Sigma} |x^*|^Q d\xi x^* = \frac{x}{|\xi|}$ and Σ is the unit sphere on \mathbb{H}^n .

We can verify that $\omega_k(\xi) \in HW_0^{1,Q}(\Omega)$. Actually, from Lemma 2.1 we have

$$\begin{aligned} \int_{\Omega} |\nabla_{\mathbb{H}} \omega_k(\xi)|^Q d\xi &= \int_{\Sigma} \int_{r e^{-\frac{k}{Q}}}^r \left| Q^{\frac{1}{Q}} (c_Q)^{-\frac{1}{Q}} k^{-\frac{1}{Q}} \frac{|x^*|}{\rho(\xi)} \right|^Q \rho(\xi)^{Q-1} d\rho(\xi) d\mu(x^*) \\ &= \frac{Q}{c_Q} \frac{1}{k} c_Q \int_{r e^{-\frac{k}{Q}}}^r \rho^{-1} d\rho = 1 \end{aligned}$$

and $\omega_k(\xi) \rightarrow 0$ in $HW_0^{1,Q}(\Omega)$.

Now, for $R = 3r$, we define

$$u = \begin{cases} A & \text{if } |\xi| \in [0, \frac{2R}{3}] \\ 3A - \frac{3A}{R} |\xi| & \text{if } |\xi| \in [\frac{2R}{3}, R] \\ 0 & \text{if } |\xi| \in [R + \infty], \end{cases} \tag{3.5}$$

where $A > 0$ is chosen in such a way that $\|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)} = \delta < 1$. Defining

$$u_k = u + (1 - \delta^Q)^{1/Q} \omega_k. \tag{3.6}$$

Observing that $\nabla_{\mathbb{H}^n} u$ and $\nabla_{\mathbb{H}^n} \omega_k$ have disjoint supports, we have

$$\begin{aligned} \|\nabla_{\mathbb{H}^n} u_k\|_{L^Q(\Omega)}^Q &= \|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)}^Q + (1 - \delta^Q) \\ &= 1, \end{aligned}$$

moreover, $u_k \rightharpoonup u$ in $HW_0^{1,Q}(\Omega)$.

Consequently,

$$\begin{aligned} \int_{\Omega} \frac{\exp\left(\alpha_{Q,\beta} M_{Q,u} u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi &= \int_{\Omega} \frac{\exp\left(\frac{\alpha_{Q,\beta} u_k^{\frac{Q}{Q-1}}}{(1-\delta^Q)^{1/(Q-1)}}\right)}{\rho(\xi)^\beta} d\xi \\ &\geq \int_{B_r \exp(-\frac{k}{Q})} \frac{\exp\left(\frac{\alpha_{Q,\beta} (A+(1-\delta^Q)^{1/Q} \omega_k)^{\frac{Q}{Q-1}}}{(1-\delta^Q)^{1/(Q-1)}}\right)}{\rho(\xi)^\beta} d\xi \\ &= \int_{B_r \exp(-\frac{k}{Q})} \frac{\exp\left(\alpha_{Q,\beta} (C + \omega_k)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\ &\geq \exp\left(\left(C' + \left(1 - \frac{\beta}{Q}\right)k\right)^{\frac{Q-1}{Q}}\right) \int_{B_r \exp(-\frac{k}{Q})} \frac{1}{\rho(\xi)^\beta} d\xi \\ &\geq C'' \exp\left(\left(C' + \left(1 - \frac{\beta}{Q}\right)k\right)^{\frac{Q-1}{Q}} - \left(1 - \frac{\beta}{Q}\right)k\right) \rightarrow \infty, \end{aligned}$$

for some positive constant C, C', C'' , and the theorem is finished. □

3.2 Concentration-compactness principle for the whole space \mathbb{H}^n

In order to prove Theorem 2.2, we need the following

Lemma 3.1 *Let $\{u_k\}$ be a sequence in $HW^{1,Q}(\mathbb{H}^n)$ strongly convergent. Then there exist a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ and $\omega(\xi) \in HW^{1,Q}(\mathbb{H}^n)$ such that $|u_{k_i}| \leq \omega(\xi)$ almost everywhere on \mathbb{H}^n .*

Proof The proof is similar as [17, Proposition 1], and we omit it. □

Now, we give the

Proof of Theorem 2.2 As in the proof of Theorem 2.1, we split the proof into two cases.

Case 1: $\|u\|_{HW^{1,Q}(\mathbb{H}^n)} < 1$. We assume by contradiction for some $p_1 < \tilde{M}_{Q,u}$, we have

$$\sup_k \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = +\infty.$$

Set

$$A(u_k) = 2^{-\frac{1}{Q(Q-1)}} \|u_k\|_{L^Q(\mathbb{H}^n)}, \Omega(u_k) = \{\xi \in \mathbb{H}^n : u_k(\xi) > A(u_k)\}$$

and

$$\Omega_L^k = \{\xi \in \mathbb{H}^n, u_k(\xi) \geq L\},$$

where L is some constant which will be determined later. We can easily verify that

$$A(u_k) < 1 \text{ and } |\Omega(u_k)| \leq 2^{\frac{1}{Q-1}}.$$

Now, we write

$$\int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \left(\int_{\Omega(u_k)} + \int_{\mathbb{H}^n \setminus \Omega(u_k)}\right) \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi,$$

Similar to the proof of [29, Theorem 1.6], we can show that

$$\int_{\mathbb{H}^n \setminus \Omega(u_k)} \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \leq c(p_1, Q, \beta).$$

Therefore, we have

$$\sup_k \int_{\Omega(u_k)} \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \infty$$

Let $v_k = u_k - L$. Then for any $\varepsilon > 0$, one has

$$u_k^{\frac{Q}{Q-1}} \leq (1 + \varepsilon) v_k^{\frac{Q}{Q-1}} + c(\varepsilon, Q) L^{\frac{Q}{Q-1}}. \tag{3.7}$$

By (3.7), we have

$$\begin{aligned} \int_{\Omega(u_k)} \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi &\leq \exp\left(\alpha_{Q,\beta} p_1 c(\varepsilon, Q) L^{\frac{Q}{Q-1}}\right) \\ &\int_{\Omega(u_k)} \frac{\exp\left((1 + \varepsilon) \alpha_{Q,\beta} p_1 v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi. \end{aligned}$$

Thus

$$\sup_k \int_{\Omega(u_k)} \frac{\exp\left(\alpha_Q \bar{p}_1 v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \infty,$$

where $\bar{p}_1 = (1 + \varepsilon) p_1 < \tilde{M}_{Q,u}$.

Since $|\Omega(u_k)| \leq 2^{\frac{1}{Q-1}}$, we have

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} \bar{p}_1 v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \infty.$$

Now, we define

$$T^L(u) = \min\{L, u\} \text{ and } T_L(u) = u - T^L(u).$$

and choose L such that

$$\frac{1 - \|u\|_{HW^{1,Q}(\mathbb{H}^n)}^Q}{1 - \|T^L u\|_{HW^{1,Q}(\mathbb{H}^n)}^Q} > \left(\frac{\bar{p}_1}{\tilde{M}_{Q,u}}\right)^{Q-1}. \tag{3.8}$$

We claim that

$$\limsup_k \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi < \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

If not, up to a subsequence, one has

$$\int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi = \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T_L u_k|^Q d\xi \geq \left(\frac{1}{\bar{p}_1}\right)^{Q-1} + o_k(1). \tag{3.9}$$

Thus,

$$\begin{aligned} & \left(\frac{1}{\bar{p}_1}\right)^{Q-1} + \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T^L u_k|^Q d\xi + \int_{\mathbb{H}^n} |T^L u_k|^Q d\xi + o_k(1) \\ & \leq \left(\frac{1}{\bar{p}_1}\right)^{Q-1} + \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T^L u_k|^Q d\xi + \int_{\mathbb{H}^n} |u_k|^Q d\xi + o_k(1) \\ & \leq \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} T_L u_k|^Q d\xi + \int_{\mathbb{H}^n \setminus \Omega_L^k} |\nabla_{\mathbb{H}} u_k|^Q d\xi + \int_{\mathbb{H}^n} |u_k|^Q d\xi \\ & = \int_{\Omega_L^k} |\nabla_{\mathbb{H}} u_k|^Q d\xi + \int_{\Omega \setminus \Omega_L^k} |\nabla_{\mathbb{H}} u_k|^Q d\xi + \int_{\mathbb{H}^n} |u_k|^Q d\xi = 1. \end{aligned}$$

For $L > 0$ fixed, $T^L u_k$ is also bounded in $HW^{1,Q}(\mathbb{H}^n)$. Hence, up to a subsequence, $T^L u_k \rightharpoonup T^L u$ in $HW^{1,Q}(\mathbb{H}^n)$ and $T^L u_k \rightarrow T^L u$ almost everywhere on \mathbb{H}^n . By the lower semicontinuity of the norm in $HW^{1,Q}(\mathbb{H}^n)$ and the above inequality, we have

$$\bar{p}_1 \geq \frac{1}{\left(1 - \liminf_{k \rightarrow \infty} \|T^L u_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q\right)^{\frac{1}{Q-1}}} \geq \frac{1}{\left(1 - \|T^L u\|_{HW^{1,Q}(\mathbb{H}^n)}^Q\right)^{\frac{1}{Q-1}}},$$

combining with (3.8), we have

$$\bar{p}_1 \geq \frac{1}{\left(1 - \|T^L u\|_{HW^{1,Q}(\mathbb{H}^n)}^Q\right)^{\frac{1}{Q-1}}} > \frac{\bar{p}_1}{\tilde{M}_{Q,u}} \frac{1}{\left(1 - \|T^L u\|_{HW^{1,Q}(\mathbb{H}^n)}^Q\right)^{\frac{1}{Q-1}}} = \bar{p}_1,$$

which is a contradiction. Therefore

$$\limsup_k \int_{\Omega_L^k} |\nabla_{\mathbb{H}} v_k|^Q d\xi < \left(\frac{1}{\bar{p}_1}\right)^{Q-1}.$$

By the Trudinger–Moser inequality (2.1), we have

$$\sup_k \int_{\Omega_L^k} \frac{\exp\left(\alpha_{Q,\beta} \bar{p}_1 v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty,$$

which is also a contradiction. The proof is finished in this case.

Case 2: $\|u\|_{HW^{1,Q}(\mathbb{H}^n)} = 1$. Since $HW^{1,Q}(\mathbb{H}^n)$ is uniformly convex Banach space and $u_k \rightharpoonup u$ weakly in $HW^{1,Q}(\mathbb{H}^n)$, by Radon’s Theorem, we have $u_k \rightarrow u$ strongly in $HW^{1,Q}(\mathbb{H}^n)$. Using Lemma 3.1, there exists some $\omega(\xi) \in HW^{1,Q}(\mathbb{H}^n)$, such that up to a subsequence, $|u_k| \leq \omega(\xi)$ a.e. in \mathbb{H}^n . Therefore

$$\int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p_1 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \leq \int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi.$$

Now, we show

$$\int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty. \tag{3.10}$$

Set $\Omega(\omega) = \{\xi \in \mathbb{H}^n : \omega > 1\}$, we have

$$\begin{aligned} \int_{\mathbb{H}^n} |\omega(\xi)|^Q d\xi &\geq \int_{\Omega(\omega)} |\omega(\xi)|^Q d\xi \\ &\geq |\Omega(\omega)|. \end{aligned}$$

Similar as [29], we can derive

$$\int_{\mathbb{H}^n \setminus \Omega(\omega)} \frac{\Phi\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < C(p_1, Q, \beta).$$

Now, we only need to show

$$\int_{\Omega(\omega)} \frac{\exp\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty.$$

Let $\omega^*(\xi)$ be the non-increasing rearrangement of $\omega(\xi)$ in $\Omega(\omega)$. Then

$$\int_{\Omega(\omega)} \frac{\exp\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \int_{B_R} \frac{\exp\left(\alpha_{Q,\beta} p_1 \omega^*(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi,$$

where $|B_R| = |\Omega(\omega)|$. We introduce the variable t by $\rho(\xi)^Q = R^Q e^{-t}$, and set

$$\varphi(t) = Q^{1-\frac{1}{Q}} c^{\frac{1}{Q}} \omega^*(\xi).$$

Then by Lemma 2.1 and the result of Manfredi and Vera De Serio [50] that there exists a constant $c \geq 1$ depending only on Q such that,

$$\int_0^\infty |\varphi'(t)|^Q dt = \int_{B_R} |\nabla_{\mathbb{H}} \omega^*(\xi)|^Q d\xi \leq c \int_{\Omega(\omega)} |\nabla_{\mathbb{H}} \omega(\xi)|^Q d\xi < \infty.$$

Moreover, we have

$$\begin{aligned} \int_{\Omega(\omega)} \frac{\exp\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi &\leq \int_{\Omega(\omega)} \frac{\exp\left(\alpha_{Q,\beta} p_1 \omega^*(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\ &= |\Omega(\omega)| \cdot R^{-\beta} \int_0^\infty \exp\left(\left(1 - \frac{\beta}{Q}\right) \left(p_1 \varphi(t)^{\frac{Q}{Q-1}} - t\right)\right) dt. \end{aligned}$$

This follows from the Hardy–Littlewood inequality by noticing that the rearrangement of $\rho(\xi)^{-\beta}$ is just itself.

Since $\int_0^\infty |\varphi'(t)|^Q dt < \infty$, then for all $\varepsilon > 0$, there exists $T = T(\varepsilon)$ such that

$$\int_T^\infty |\varphi'(t)|^Q dt < \varepsilon^Q.$$

Hence, by Hölder’s inequality

$$\begin{aligned} \varphi(t) &= \varphi(T) + \int_T^t \varphi'(t) dt \\ &\leq \varphi(T) + \left(\int_T^t |\varphi'(t)|^Q dt\right)^{\frac{1}{Q}} \cdot |t - T|^{\frac{Q-1}{Q}} \\ &\leq \varphi(T) + \varepsilon |t - T|^{\frac{Q-1}{Q}}. \end{aligned}$$

There exists \tilde{T} such that

$$p_1 \varphi(t)^{\frac{Q}{Q-1}} \leq \frac{t}{2} \text{ for all } t > \tilde{T}.$$

Therefore $\int_{\Omega(\omega)} \frac{\exp\left(\alpha_{Q,\beta} p_1 \omega(\xi)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi < \infty$, and the proof is finished in this case.

Now, we prove the sharpness of $\tilde{M}_{Q,u}$. For some $r > 0$ and $R = 3r$, we define $\omega_k(\xi), u \in HW^{1,Q}(\mathbb{H}^n)$ as (3.4),(3.5), respectively. The constant A is chosen in such a way that $\|u\|_{HW^{1,Q}(\mathbb{H}^n)} = \delta < 1$. Defining

$$u_k = u + \left(1 - \delta^Q\right)^{1/Q} \omega_k.$$

We can easily verify that

$$\begin{aligned} \| \omega_k \|_{L^p(\mathbb{H}^n)} &\rightarrow 0, \text{ for any } p \geq 1, \\ \| \nabla_{\mathbb{H}} u_k \|_{L^Q(\mathbb{H}^n)}^Q &= \| \nabla_{\mathbb{H}} u \|_{L^Q(\mathbb{H}^n)}^Q + \left(1 - \delta^Q\right), \end{aligned} \tag{3.11}$$

and

$$u_k \rightharpoonup u \text{ weakly in } HW^{1,Q}(\mathbb{H}^n).$$

Moreover, from (3.11) we have

$$\begin{aligned} \int_{\mathbb{H}^n} |u_k|^Q d\xi &= \int_{\mathbb{H}^n} \left| u + (1 - \delta Q)^{1/Q} \omega_k \right|^Q d\xi \\ &= \int_{\mathbb{H}^n} |u|^Q d\xi + \xi_k, \end{aligned}$$

where $\xi_k \rightarrow O\left(\left(\frac{1}{k}\right)^{1/Q}\right)$, and then we have $\|u_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q = 1 + \xi_k$. Set $v_k = \frac{u_k}{(1+\xi_k)^{1/Q}}$, we have

$$v_k \rightharpoonup u \text{ weakly in } HW^{1,Q}(\mathbb{H}^n) \text{ with } \|v_k\|_{HW^{1,Q}(\mathbb{H}^n)}^Q = 1.$$

Consequently, for any $\varepsilon_0 > 0$ and $p = (1 + \varepsilon_0) \tilde{M}_{Q,u}$, one has

$$\int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} \tilde{M}_{Q,u} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \geq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\frac{(1 + \varepsilon_0) \alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}}{(1 - \delta Q)^{1/(Q-1)}}\right) d\xi$$

(using the fact that $\xi_k \rightarrow 0$)

$$\begin{aligned} &\int_{\mathbb{H}^n} \frac{\Phi\left(\alpha_{Q,\beta} \tilde{M}_{Q,u} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\ &\geq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\frac{\alpha_{Q,\beta} \left((1 + \varepsilon'_0) \left(A + (1 - \delta Q)^{1/Q} \omega_k\right)\right)^{\frac{Q}{Q-1}}}{(1 - \delta Q)^{1/(Q-1)}}\right) d\xi \\ &= \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\alpha_{Q,\beta} \left[(1 + \varepsilon'_0) (C + \omega_k)\right]^{\frac{Q}{Q-1}}\right) d\xi \\ &\geq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\left(1 - \frac{\beta}{Q}\right) \left[(1 + \varepsilon'_0) \left(C' + k^{\frac{Q-1}{Q}}\right)\right]^{\frac{Q}{Q-1}}\right) d\xi \\ &\geq C'' \exp\left(\left[\left(1 - \frac{\beta}{Q}\right) (1 + \varepsilon'_0) \left(C' + k^{\frac{Q-1}{Q}}\right)\right]^{\frac{Q}{Q-1}} - \left(1 - \frac{\beta}{Q}\right) k\right) \rightarrow \infty \end{aligned}$$

for some positive constant $\varepsilon'_0, C, C', C''$, and the theorem is finished. □

Remark 1 The sequence $\{v_k\}$ is not enough to show that the supremum (2.3) is infinite when $p = \tilde{M}_{Q,u}$. Actually, we have

$$\begin{aligned} & \int_{B_{r \exp(-\frac{k}{Q})}} \frac{\Phi\left(\alpha_{Q,\beta} \tilde{M}_{Q,u} v_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\frac{\alpha_{Q,\beta} v_k^{\frac{Q}{Q-1}}}{(1-\delta Q)^{1/(Q-1)}}\right) d\xi \\ & \leq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\frac{\alpha_{Q,\beta} \left((1+\xi_k)^{-1/Q}\right) \left(A + (1-\delta Q)^{1/Q} \omega_k\right)^{\frac{Q}{Q-1}}}{(1-\delta Q)^{1/(Q-1)}}\right) d\xi \\ & \leq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\left(1-\frac{\beta}{Q}\right) \left[\left(1+\xi_k\right)^{-1/Q}\right] \left(C' + k^{\frac{Q-1}{Q}}\right)^{\frac{Q}{Q-1}}\right) d\xi \\ & \quad \left(\text{since } 1 - (1 + \xi_k)^{-1/Q} = o\left(\left(\frac{1}{k}\right)^{1/Q}\right)\right) \\ & \leq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\left(1-\frac{\beta}{Q}\right) \left[\left(1-C''\left(\frac{1}{k}\right)^{1/Q}\right) \left(C' + k^{\frac{Q-1}{Q}}\right)^{\frac{Q}{Q-1}}\right]\right) d\xi \\ & \leq \int_{B_{r \exp(-\frac{k}{Q})}} \frac{1}{\rho(\xi)^\beta} \exp\left(\left(1-\frac{\beta}{Q}\right) \left[k^{\frac{Q-1}{Q}} - C''k^{\frac{Q-2}{Q}}\right]^{\frac{Q}{Q-1}}\right) d\xi \\ & \leq C'''' \exp\left(\left(1-\frac{\beta}{Q}\right) k \left[1 - C''''k^{\frac{-1}{Q}}\right] - \left(1-\frac{\beta}{Q}\right) k\right) \\ & \leq C'''' \exp\left(-C'''' \left(1-\frac{\beta}{Q}\right) k^{\frac{Q-1}{Q}}\right) < \infty \end{aligned}$$

for some positive constant C', C'', C''' and C'''' . We remark that this argument is also suitable for the sequence constructed in [18] for the sharpness of $\tilde{M}_{n,u} = (1 - \|u\|_{W^{1,n}(\mathbb{R}^n)})^{-1/(n-1)}$.

4 Q -sub-Laplacian equations of exponential growth on \mathbb{H}^n .

In this section, let's consider the following nonlinear partial differential equations on \mathbb{H}^n :

$$-\operatorname{div}\left(|\nabla_{\mathbb{H}^n} u|^{Q-2} \nabla_{\mathbb{H}^n} u\right) + V(\xi) |u|^{Q-2} u = \frac{f(u)}{\rho(\xi)^\beta}, \tag{4.1}$$

where $0 \leq \beta < Q$.

The main features of this class of equations (4.1) are that it is defined in the whole space \mathbb{H}^n and involves critical growth and the nonlinear operator is Q -sub-Laplacian. In spite of a possible failure of the Palais–Smale (PS) compactness condition, we apply the mini-max argument based on the concentration-compactness principle for $HW^{1,Q}(\mathbb{H}^n)$ —Theorem 2.2 as in [18].

The basic assumptions about f and V are collected in the following:

(H1) Assumptions for potential V

The potential $V : \mathbb{H}^n \rightarrow \mathbb{R}$ is a continuous potential, and satisfies:

- (V1) V is a continuous function such that $V(\xi) \geq 1$ for all $\xi \in \mathbb{H}^n$, and one of the following two conditions:
- (V2) $V(\xi) \rightarrow \infty$ as $\rho(\xi) \rightarrow \infty$; or more generally, for every $M > 0$ $\{ \xi \in \mathbb{H}^n : V(\xi) > M \} < \infty$;
- (V3) the function $V(\xi)^{-1}$ belongs to $L^1(\mathbb{H}^n)$.

(H2) Assumptions for f

The function $f(t) : \mathbb{R} \rightarrow \mathbb{R}$ behaves like $\exp\left(\alpha t^{\frac{Q}{Q-1}}\right)$ when $|t| \rightarrow \infty$. Precisely, we assume that f satisfies the following conditions:

- (f1) there exist constants $\alpha_0, b_0, b_1 > 0$ such that for all $t \in \mathbb{R}$,

$$f(t) \leq b_0 t^{Q-1} + b_1 \Phi\left(\alpha_0 t^{\frac{Q}{Q-1}}\right);$$

- (f2) there exists $\lambda > Q$ such that for all $\xi \in \mathbb{H}^n$ and $t > 0$,

$$0 < \lambda F(t) := \lambda \int_0^t f(s) ds \leq t f(t);$$

- (f3) there exist constant $R_0, M_0 > 0$ such that for all $\xi \in \mathbb{H}^n$ and $t > R_0$,

$$0 \leq F(t) \leq M_0 f(t);$$

- (f4) there exist constant $\mu > Q$ and C_μ such that for all $t \geq 0$,

$$f(t) \geq C_\mu t^{\mu-1} \tag{4.2}$$

with C_μ satisfying

$$C_\mu > \left(\frac{\alpha_{Q,\beta}}{\alpha_0}\right)^{\frac{(Q-\mu)(Q-1)}{Q}} \left(\frac{\mu-Q}{\mu}\right)^{\frac{\mu-Q}{Q}} \lambda_\mu^{\mu/Q},$$

where

$$\lambda_\mu := \inf_{u \in \mathcal{S} \setminus 0} \frac{\|u\|^Q}{\int_{\mathbb{H}^n} \frac{|u|^\mu}{\rho(\xi)^\beta} d\xi};$$

- (f5) $\limsup_{t \rightarrow 0^+} \frac{F(t)}{k_0 |s|^Q} < \lambda_Q := \inf_{u \in \mathcal{S} \setminus 0} \frac{\|u\|^Q}{\int_{\mathbb{H}^n} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi}$.

Since we are interested in nonnegative weak solutions, we also suppose the following

- (f6) $F(t) = 0$ if $t \leq 0$.

From condition (f1), we conclude that for all $t \in \mathbb{R}$,

$$F(t) \leq b_2 \Phi\left(\alpha_1 t^{\frac{Q}{Q-1}}\right)$$

for some constant $b_2, \alpha_1 > 0$. From (3.10), we have $\frac{F(u)}{\rho(\xi)^\beta} \in L^1(\mathbb{H}^n)$ for all $u \in \mathcal{S}$. Therefore, the associated functional to the Eq. (4.1) defined by

$$J(u) = \frac{1}{Q} \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u|^Q + V(\xi) |u|^Q \right) d\xi - \int_{\mathbb{H}^n} \frac{F(u)}{\rho(\xi)^\beta} d\xi \tag{4.3}$$

is well-defined. Moreover, J is a C^1 functional on \mathcal{S} with

$$DJ(u)v = \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u \nabla_{\mathbb{H}} v + V(\xi) |u|^{Q-2} uv \right) d\xi - \int_{\mathbb{H}^n} \frac{f(u)v}{\rho(\xi)^\beta} d\xi$$

for all $v \in \mathcal{S}$. Thus, $DJ(u) = 0$ if and only if $u \in \mathcal{S}$ is a weak solution to equation (4.1).

We define the following space associated with the potential V :

$$\mathcal{S} = \left\{ u \in HW^{1,Q}(\mathbb{H}^n) : \|u\| < \infty \right\}$$

with the norm $\|u\| := \left(\int_{\mathbb{H}^n} (|\nabla_{\mathbb{H}} u|^Q + V(\xi)|u|^Q) d\xi \right)^{1/Q}$.

From the hypothesis (H1), we have the following compactness result:

Lemma 4.1 *If $V(\xi)$ satisfy the hypothesis (H1), then for all $Q \leq q < \infty$, the embedding*

$$\mathcal{S} \hookrightarrow L^q(\mathbb{H}^n)$$

is compact.

Proof Though the proof is analogous to the one for the Euclidean case in [13,53], we will need to justify some conclusions on the Heisenberg groups. To this end, we will simply point out the essential components and omit the details.

Let $\{u_k\}$ be a sequence such that $\|u_k\|^Q < C$. In order to prove this result, we only need to show that $u_k \rightarrow 0$ strongly in $L^q(\mathbb{H}^n)$ for any $Q \leq q < \infty$, whenever $u_k \rightarrow 0$ weakly in \mathcal{S} , as $k \rightarrow \infty$.

By the compact embedding theorem on the Heisenberg group proved in [45], we have for any given $p < Q$, the embedding $HW^{1,p}(B_R) \hookrightarrow L^q(B_R)$ is compact for any $q < \frac{pQ}{Q-p}$. So the embedding $HW^{1,Q}(B_R) \hookrightarrow L^q(B_R)$ is compact for any $q < \infty$. The remaining proof then follows from the classical proof in the Euclidean space.

For any $\varepsilon > 0$, from (V2), we can choose some $R > 0$ such that

$$V(\xi) \geq \frac{2C}{\varepsilon} \tag{4.4}$$

for all ξ satisfying $\rho(\xi) \geq R$. Since the embedding $HW^{1,Q}(B_R) \hookrightarrow L^q(B_R)$ is compact, we know $u_k \rightarrow 0$ strongly in $L^q(B_R)$, and then there exists a integer $N > 0$ such that when $k > N$,

$$\int_{B_R} |u_k|^Q d\xi < \frac{\varepsilon}{2}. \tag{4.5}$$

On the other hand, from (4.4) we have

$$\frac{2C}{\varepsilon} \int_{\mathbb{H}^n \setminus B_R} |u_k|^Q d\xi \leq \int_{\mathbb{H}^n \setminus B_R} V(\xi) |u_k|^Q d\xi < C,$$

that is

$$\int_{\mathbb{H}^n \setminus B_R} |u_k|^Q d\xi < \frac{\varepsilon}{2}. \tag{4.6}$$

Combining (4.5) and (4.6) we obtain

$$\int_{\mathbb{H}^n} |u_k|^Q d\xi \leq \varepsilon. \tag{4.7}$$

For any $q < p < \infty$, we define

$$\lambda = \frac{Q(p-q)}{q(p-Q)} \text{ and } \mu = \frac{p(q-Q)}{q(p-Q)}.$$

Then $\lambda > 0$ and $\mu > 0$. By Hölder’s inequality, the Trudinger–Moser inequality (2.2) and (4.7), we have

$$\int_{\mathbb{H}^n} |u_k|^q d\xi \leq c\varepsilon^{\frac{p-q}{p-Q}}.$$

The proof is thus finished. □

4.1 Palais–Smale compactness condition

In this subsection, we analyze the compactness of Palais–Smale sequences of the functional J . This is the crucial step in the study of existence results for Eq. (4.1).

First, we recall the definition of Palais–Smale Condition:

Definition 2 (*Palais–Smale condition*) A sequence $\{u_k\}$ in \mathcal{S} is called a local Palais–Smale sequence at level d for the functional J ($(PS)_d$ sequence), if

$$J(u_k) \rightarrow d \text{ and } \|DJ(u_k)\| \rightarrow 0, \text{ as } k \rightarrow \infty,$$

the functional J is said to satisfy the Palais–Smale condition at level d ($(PS)_d$ condition), if any $(PS)_d$ sequence has a convergent subsequence.

Lemma 4.2 *Under the hypotheses of (H1) and (H2). The functional J satisfies the Palais–Smale condition at level d for any $d < \frac{1}{Q} \left(\frac{\alpha_Q \beta}{\alpha_0}\right)^{Q-1}$.*

Proof Though the general scheme of the proof is a combination of the ideas in [18, Proposition 4.1], the compact embedding theorem and ideas from [29, Lemma 5.5 and (6.7)], there are a number key points that are required to be re-established on the Heisenberg groups and we will sketch those major points and omit the details.

Let $\{u_k\}$ be a $(PS)_d$ sequence for J , that is,

$$J(u_k) \rightarrow d \tag{4.8}$$

and $|DJ(u_k)v| \rightarrow 0$ for all $v \in \mathcal{S}$, as $k \rightarrow \infty$. Then

$$\begin{aligned} & \int_{\mathbb{H}^n} \frac{f(u_k)v}{\rho(\xi)^\beta} d\xi - \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u_k|^{Q-2} \nabla_{\mathbb{H}} u_k \nabla_{\mathbb{H}} v + V(\xi) |u_k|^{Q-2} u_k v \right) d\xi \\ & \leq \varepsilon_k \|v\| \end{aligned} \tag{4.9}$$

for all $v \in \mathcal{S}$, where $\varepsilon_k \rightarrow 0$, as $k \rightarrow \infty$.

Choosing $v = u_k$ in (4.9), by (4.8) we get

$$\begin{aligned} & \int_{\mathbb{H}^n} \frac{f(u_k)u_k}{\rho(\xi)^\beta} - \int_{\mathbb{H}^n} \frac{F(u_k)\lambda}{\rho(\xi)^\beta} + \frac{\lambda}{Q} \|u_k\|^Q - d\lambda - \|u_k\|^Q \\ & \leq \varepsilon_k \|u_k\|, \end{aligned}$$

From (f2), we have

$$\frac{\lambda - Q}{Q} \|u_k\|^Q \leq c + \varepsilon_k \|u_k\|,$$

hence, u_k is bounded in \mathcal{S} . Since for any $q \geq Q$, the embedding $\mathcal{S} \hookrightarrow L^q(\mathbb{H}^n)$ is compact, we can assume that,

$$\begin{aligned} u_k &\rightharpoonup u \text{ weakly in } \mathcal{S}, \\ u_k &\rightarrow u \text{ strongly in } L^q(\mathbb{H}^n) \text{ for any } q \geq Q, \\ u_k &\rightarrow u \text{ for almost all } \xi \in \mathbb{H}^n. \end{aligned} \tag{4.10}$$

From (4.10), we can verify that

$$\int_{\mathbb{H}^n} \frac{|u_k - u|^s}{\rho(\xi)^\beta} d\xi \rightarrow 0 \quad \text{as } k \rightarrow \infty \tag{4.11}$$

for any $s \in [Q, \infty)$ and $\beta \in [0, Q)$.

Thanks to [29, Lemma 5.5 and (6.7)], we have

$$\begin{cases} \frac{f(u_k)}{\rho(\xi)^\beta} \rightarrow \frac{f(u)}{\rho(\xi)^\beta} & \text{in } L^1_{loc}(\mathbb{H}^n) \\ \frac{F(u_k)}{\rho(\xi)^\beta} \rightarrow \frac{F(u)}{\rho(\xi)^\beta} & \text{in } L^1(\mathbb{H}^n) \\ |\nabla_{\mathbb{H}} u_k|^{Q-2} \nabla_{\mathbb{H}} u_k \rightharpoonup |\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u & \text{weakly in } \left(L^{Q/(Q-1)}_{loc}(\mathbb{H}^n) \right)^{2n}. \end{cases} \tag{4.12}$$

From this convergence and passing the limit in (4.9), we get

$$\int_{\mathbb{H}^n} \frac{f(u)v}{\rho(\xi)^\beta} d\xi - \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u \nabla_{\mathbb{H}} v + V(\xi) |u|^{Q-2} uv \right) d\xi = 0$$

for any $v \in C^\infty_0(\mathbb{H}^n)$. By density, taking $v = u$, we have

$$\int_{\mathbb{H}^n} \frac{f(u)u}{\rho(\xi)^\beta} d\xi - \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u|^Q + V(\xi) |u|^Q \right) d\xi = 0,$$

from (f2), we get

$$\int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u|^Q + V(\xi) |u|^Q \right) d\xi \geq Q \int_{\mathbb{H}^n} \frac{F(u)}{\rho(\xi)^\beta} d\xi,$$

thus, $J(u) \geq 0$.

Next, we prove the strong convergence of $\{u_k\}$. For this purpose, we split the proof into two cases:

Case 1: $d = 0$. From (4.12) and (4.8), we have

$$\|u\|^Q \leq \lim_k \|u_k\|^Q = Q \int_{\mathbb{H}^n} \frac{F(u)}{\rho(\xi)^\beta} d\xi,$$

hence $J(u) \leq 0$. Therefore $J(u) = 0$ and $\lim_k \|u_k\|^Q = \|u\|^Q$. Since \mathcal{S} is a uniformly convex Banach space, by Radon’s Theorem, $u_k \rightarrow u$ strongly in \mathcal{S} .

Case 2: $d \neq 0$.

We can first prove that $u \neq 0$ by an argument of contradiction similar to the one given in [18]. We will omit the details here.

Next, set $\tilde{u}_k = \frac{u_k}{\|u_k\|}$ and $\tilde{u} = \frac{u}{\|u\|}$. Then $\|u_k\| = 1$ and $\tilde{u}_k \rightharpoonup \tilde{u}$ weakly in \mathcal{S} . If $\|\tilde{u}\| = 1$, we have $\lim_k \|u_k\| = \|u\|$, and then $u_k \rightarrow u$ strongly in \mathcal{S} .

If $\|\tilde{u}\| < 1$, by (4.8) and (4.12) and the fact that $J(u) \geq 0$, one has

$$d + o_k(1) \geq J(u_k) - J(u) \rightarrow \frac{1}{Q} \left(\|u_k\|^Q - \|u\|^Q \right),$$

thus,

$$\|u_k\|^Q \left(1 - \left\| \frac{u}{\|u_k\|} \right\|^Q\right) \leq Q(d + o_k(1)),$$

that is

$$\|u_k\|^{Q/(Q-1)} < \frac{\frac{\alpha_{Q,\beta}}{\alpha_0} + o_k(1)}{\left(1 - \left\| \frac{u}{\|u_k\|} \right\|^Q\right)^{1/(Q-1)}},$$

therefore when k large, we can choose some $q > 1$ close to 1 sufficiently such that

$$q\alpha_0 \|u_k\|^{Q/(Q-1)} < \frac{\alpha_{Q,q\beta}}{\left(1 - \|\tilde{u}\|^Q\right)^{1/(Q-1)}}. \tag{4.13}$$

By Theorem 2.2, $\frac{\Phi\left(q\alpha_0 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^{q\beta}}$ is bounded in $L^1(\mathbb{H}^n)$.

By Hölder’s inequality, combining (f1), (4.10), (4.11), (4.13), we get

$$\begin{aligned} \left| \int_{\mathbb{H}^n} \frac{f(u_k)(u_k - u)}{\rho(\xi)^\beta} d\xi \right| &\leq c \left(\int_{\mathbb{H}^n} \frac{u_k^Q}{\rho(\xi)^\beta} d\xi \right)^{\frac{Q-1}{Q}} \left(\int_{\mathbb{H}^n} \frac{|u_k - u|^Q}{\rho(\xi)^\beta} d\xi \right)^{\frac{1}{Q}} \\ &\quad + c \left(\int_{\mathbb{H}^n} \frac{\Phi\left(q\alpha_0 u_k^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^{q\beta}} d\xi \right)^{1/q} \left(\int_{\mathbb{H}^n} |u_k - u|^{q'} d\xi \right)^{1/q'} \rightarrow 0. \end{aligned} \tag{4.14}$$

Since $DJ(u_k)(u_k - u) \rightarrow 0$, from (4.14) we derive

$$\int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u_k|^{Q-2} \nabla_{\mathbb{H}} u_k \nabla_{\mathbb{H}}(u_k - u) + V(\xi) |u_k|^{Q-2} u_k (u_k - u) \right) d\xi \rightarrow 0. \tag{4.15}$$

On the other hand, since $u_k \rightharpoonup u$ in \mathcal{S} , we have

$$\int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u \nabla_{\mathbb{H}}(u_k - u) + V(\xi) |u|^{Q-2} u (u_k - u) \right) d\xi \rightarrow 0. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain there is a constant $c > 0$ so that

$$\begin{aligned} \|u_k - u\| &\leq c \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}} u_k|^{Q-2} \nabla_{\mathbb{H}} u_k - |\nabla_{\mathbb{H}} u|^{Q-2} \nabla_{\mathbb{H}} u \right) \nabla_{\mathbb{H}}(u_k - u) \\ &\quad + c \int_{\mathbb{H}^n} V(\xi) \left(|u_k|^{Q-2} u_k - |u|^{Q-2} u \right) (u_k - u) d\xi \rightarrow 0, \end{aligned}$$

where we have used the inequality $(|a|^{Q-2} a - |b|^{Q-2} b)(a - b) \geq 2^{2-Q} |a - b|^Q$, for all $a, b \in \mathbb{R}^{2n}$. The proof is finished. \square

From the proof of [29, Lemmas 5.1 and 5.2.], we have the following geometric conditions of the mountain-pass theorem:

Lemma 4.3 *Suppose that the hypotheses of (H1) and (H2) hold. Then*

- (i) there exists $r, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\| = r$;
- (ii) there exists $e \in \mathcal{S}$ with $\|e\| > r$, such that $J(e) < 0$.

Now, we define the minimax level by

$$d_\infty = \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t)),$$

where $\Gamma = \{g \in C([0, 1], \mathcal{S}) : g(0) = 0 \text{ and } g(1) < 0\}$. From Lemma 4.3, we have $d_\infty > 0$.

Lemma 4.4 *Under the hypotheses of (H1) and (H2). We have $d_\infty < \frac{1}{Q} \left(\frac{\alpha_{Q,\beta}}{\alpha_0}\right)^{Q-1}$.*

Proof Let $\{v_k\}$ be in \mathcal{S} with $\int_{\mathbb{H}^n} \frac{|v_k|^\mu}{\rho(\xi)^\beta} d\xi = 1$ and $\|v_k\|^Q \rightarrow \lambda_\mu$. Then $\{v_k\}$ is bounded in \mathcal{S} . Using the compactness of embedding $\mathcal{S} \hookrightarrow L^q(\mathbb{H}^n)$ for all $q \geq Q$, up to a subsequence, we have

$$\begin{aligned} v_k &\rightharpoonup v_0 \text{ weakly in } \mathcal{S}, \\ v_k &\rightarrow v_0 \text{ strongly in } L^q(\mathbb{H}^n) \text{ for all } q \in [Q, \infty) \\ v_k &\rightarrow v_0 \text{ for almost all } \xi \in \mathbb{H}^n. \end{aligned} \tag{4.17}$$

(4.11) implies that $\int_{\mathbb{H}^n} \frac{|v_0|^\mu}{\rho(\xi)^\beta} d\xi = \lim_k \int_{\mathbb{H}^n} \frac{|v_k|^\mu}{\rho(\xi)^\beta} d\xi = 1$. By the semicontinuity of the norm $\|\cdot\|$, we infer that

$$\|v_0\|^Q \leq \liminf_k \|v_k\|^Q = \lambda_\mu,$$

thus λ_μ is attained by v_0 , we may assume that $v_0 \geq 0$.

From (4.2), we know that $F(t) \geq \frac{C_\mu}{\mu} t^\mu$ for some $\mu > Q$. Hence,

$$\begin{aligned} J(tv_0) &\leq \frac{t^Q}{Q} \int_{\mathbb{H}^n} (|\nabla_{\mathbb{H}} v_0|^Q + V(\xi)|v_0|^Q) d\xi - \frac{C_\mu t^\mu}{\mu} \int_{\mathbb{H}^n} \frac{v_0^\mu}{\rho(\xi)^\beta} d\xi \\ &\rightarrow -\infty, \text{ as } t \rightarrow \infty. \end{aligned}$$

Setting $\tilde{v}_0(t) = tv_0$ with t_0 sufficiently large, then $\tilde{v}_0(t) \in \Gamma$. By (f4), we have

$$\begin{aligned} d_\infty &\leq \max_{t \in [0,1]} J(\tilde{v}_0(t)) \leq \max_{t \in [0,1]} \left(\frac{(t_0 t)^Q}{Q} \|v_0\|^Q - \frac{C_\mu (t_0 t)^\mu}{\mu} \int_{\mathbb{H}^n} \frac{v_0^\mu}{\rho(\xi)^\beta} d\xi \right) \\ &\leq \max_{t>0} \left(\frac{t^Q}{Q} \|v_0\|^Q - \frac{C_\mu}{\mu} t^\mu \int_{\mathbb{H}^n} \frac{v_0^\mu}{\rho(\xi)^\beta} d\xi \right) \\ &\leq \max_{t>0} \left(\frac{t^Q}{Q} \lambda_\mu - \frac{C_\mu}{\mu} t^\mu \right) = \frac{\mu - Q}{Q\mu} \frac{\lambda_\mu^{\mu/(\mu-Q)}}{C_\mu^{Q/(\mu-Q)}} \\ &< \frac{1}{Q} \left(\frac{\alpha_{Q,\beta}}{\alpha_0}\right)^{Q-1}, \end{aligned}$$

and this completes the proof. □

Finally, we come to the

Proof of Theorem 2.3 Let $\{u_k\}$ be a sequence in \mathcal{S} such that

$$J(u_k) \rightarrow d_\infty$$

and $DJ(u_k) \rightarrow 0$. By Lemmas 4.2 and 4.4, the sequence $\{u_k\}$ converges weakly to a weak solution u_0 of (4.1). Now, we show $u_0 > 0$ in \mathbb{H}^n .

Set $u_{0+} := \max\{u_0, 0\}$ and $u_{0-} := \max\{-u_0, 0\}$. Since u_0 satisfies $DJ(u_0) = 0$, we have $DJ(u_0)u_{0-} = 0$, that is,

$$\|u_{0-}\|^Q - \int_{\mathbb{H}^n} \frac{f(u_0)u_{0-}}{\rho(\xi)^\beta} d\xi = 0.$$

On the other hand, from (f6) we have $\int_{\mathbb{H}^n} \frac{f(u_0)u_{0-}}{\rho(\xi)^\beta} d\xi = 0$, and then $\|u_{0-}\|^Q = 0$. Therefore, $u_0 \geq 0$ on \mathbb{H}^n . From $J(u_0) = d_\infty > 0$, we know u_0 is positive on \mathbb{H}^n .

Now, let

$$M_\infty := \inf_{u \in \mathcal{P} \setminus \{0\}} J(u),$$

where $\mathcal{P} := \{u \in \mathcal{S} : DJ(u) = 0\}$.

In order to show that u_0 is a ground state solution of (4.1), we only need to prove $d_\infty \leq M_\infty$. For any $u \in \mathcal{P} \setminus \{0\}$, we define $m(t)$ by $m(t) = J(tu)$. Since $J \in C^1(\mathcal{S}, \mathbb{R})$, we have $m(t)$ is differentiable and

$$m'(t) = DJ(tu)u = t^{Q-1} \|u\|^Q - \int_{\mathbb{H}^n} \frac{f(tu)u}{\rho(\xi)^\beta} d\xi,$$

for any $t > 0$.

From $DJ(u)u = 0$, we derive

$$m'(t) = t^{Q-1} \int_{\mathbb{H}^n} \left(\frac{f(u)}{u^{Q-1}} - \frac{f(tu)}{(tu)^{Q-1}} \right) \frac{u^Q}{\rho(\xi)^\beta} d\xi.$$

By (f2), we know $\frac{f(t)}{t^{Q-1}}$ is increasing for all $s > 0$. From this and the fact $m'(1) = 0$, we know $m'(t) > 0$ if $t \in (0, 1)$, and $m'(t) < 0$ if $t \in (1, \infty)$. Thus, $J(u) = \max_{t \geq 0} J(tu)$.

Setting $\tilde{u}(t) = tt_0u$ with t_0 sufficiently large, we get $\tilde{u}(t) \in \Gamma$, and then

$$d_\infty \leq \max_{t \in [0, 1]} J(\tilde{u}(t)) \leq \max_{t \geq 0} J(tu) = J(u).$$

Therefore, $d_\infty \leq M_\infty$. The proof is completed. □

Acknowledgements The authors would also like to thank the referee for his or her comments which improve its exposition of the paper.

References

1. Adachi, S., Tanaka, K.: Trudinger type inequalities in \mathbb{R}^N and their best exponents. Proc. Am. Math. Soc. **128**, 2051–2057 (1999)
2. Adams, D.R.: A sharp inequality of J. Moser for higher order derivatives. Ann. Math. **128**, 385–398 (1988)
3. Adimurthi, M.S.: Global compactness properties of semilinear elliptic equations with critical exponential growth. J. Funct. Anal. **175**, 125–167 (2000)
4. Adimurthi, O.D.: Blow-up analysis in dimension 2 and a sharp form of Trudinger–Moser inequality. Commun. Partial Differ. Equ. **29**, 295–322 (2004)
5. Balogh, Z., Manfredi, J., Tyson, J.: Fundamental solution for the Q -Laplacian and sharp Moser–Trudinger inequality in Carnot groups. J. Funct. Anal. **204**, 35–49 (2003)
6. Cao, D.M.: Nontrivial solution of semilinear elliptic equation with critical exponent in \mathbb{R}^2 . Commun. Partial Differ. Equ. **17**, 407–435 (1992)

7. Carleson, L., Chang, S.Y.A.: On the existence of an extremal function for an inequality of J. Moser. *Bull. Sci. Math.* (2) **110**(2), 113–127 (1986)
8. Černý, R., Cianchi, A., Hencl, S.: Concentration-compactness principles for Moser–Trudinger inequalities: new results and proofs. *Ann. Mat. Pura Appl.* **192**, 225–243 (2013)
9. Chen, L., Li, J., Lu, G., Zhang, C.: Sharpened Adams inequality and ground state solutions to the Bi-Laplacian in \mathbb{R}^4 . *Adv. Nonlinear Stud.* (**to appear**)
10. Cohn, W.S., Lu, G.: Best constants for Moser–Trudinger inequalities on the Heisenberg group. *Indiana Univ. Math. J.* **50**(4), 1567–1591 (2001)
11. Cohn, W.S., Lu, G.: Best constants for Moser–Trudinger inequalities, fundamental solutions and one-parameter representation formulas on groups of Heisenberg type. *Acta Math. Sin. (Engl. Ser.)* **18**, 375–390 (2002)
12. Cohn, W.S., Lam, N., Lu, G., Yang, Y.: The Moser–Trudinger inequality in unbounded domains of Heisenberg group and sub-elliptic equations. *Nonlinear Anal.* **75**, 4483–4495 (2012)
13. Costa, D.G.: On a class of elliptic systems in \mathbb{R}^N . *Electron. J. Differ. Equ.* **07**, 1–14 (1994)
14. de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in \mathbb{R}^2 with nonlinearities in the critical growth range. *Calc. Var. Partial Differ. Equ.* **3**, 139–153 (1995)
15. de Figueiredo, D.G., do Ó, J.M., Ruf, B.: On an inequality by N. Trudinger and J. Moser and related elliptic equations. *Commun. Pure Appl. Math.* **55**, 135–152 (2002)
16. do Ó, J.M.: N -Laplacian equations in \mathbb{R}^N with critical growth. *Abstr. Appl. Anal.* **2**, 301–315 (1997)
17. do Ó, J.M., Medeiros, E., Severo, U.: On a quasilinear nonhomogeneous elliptic equation with critical growth in \mathbb{R}^n . *J. Differ. Equ.* **246**, 1363–1386 (2009)
18. do Ó, J.M., de Souza, M., de Medeiros, E., Severo, U.: An improvement for the Trudinger–Moser inequality and applications. *J. Differ. Equ.* **256**, 1317–1349 (2014)
19. Dong, Y., Yang, Q.: An interpolation of Hardy inequality and Moser–Trudinger inequality on Riemannian manifolds with negative curvature. *Acta Math. Sin. (Engl. Ser.)* **32**(7), 856–866 (2016)
20. Dong, M., Lam, N., Lu, G.: Sharp weighted Trudinger–Moser and Caffarelli–Kohn–Nirenberg inequalities and their extremal functions. *Nonlinear Anal.* **31**, 297–314 (2014)
21. Dong, M., Lu, G.: Best constants and existence of maximizers for weighted Trudinger–Moser inequalities. *Calc. Var. Partial Differ. Equ.* **55**(4), Art. 88 (2016)
22. Flucher, M.: Extremal functions for the Trudinger–Moser inequality in 2 dimensions. *Comment. Math. Helv.* **67**, 471–497 (1992)
23. Ishiwata, M., Nakamura, M., Wadade, H.: On the sharp constant for the weighted Trudinger–Moser type inequality of the scaling invariant form. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31**(2), 297–314 (2014)
24. Jerison, D., Lee, J.M.: Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. *J. Am. Math. Soc.* **1**, 1–13 (1988)
25. Lam, N., Lu, G., Zhang, L.: Existence and nonexistence of extremal functions for sharp Trudinger–Moser inequalities. Preprint
26. Lam, N.: Sharp subcritical and critical Trudinger–Moser inequalities on \mathbb{R}^2 and their extremal functions. *Potential Anal.* **46**(1), 75–103 (2017)
27. Lam, N.: Maximizers for the singular Trudinger–Moser inequalities in the subcritical cases. *Proc. Am. Math. Soc.* **145**(11), 4885–4892 (2017)
28. Lam, N., Lu, G.: Existence and multiplicity of solutions to equations of N -Laplacian type with critical exponential growth in \mathbb{R}^N . *J. Funct. Anal.* **262**(3), 1132–1165 (2012)
29. Lam, N., Lu, G.: Sharp Moser–Trudinger inequality on the Heisenberg group at the critical case and applications. *Adv. Math.* **231**, 3259–3287 (2012)
30. Lam, N., Lu, G.: A new approach to sharp Moser–Trudinger and Adams type inequalities: a rearrangement-free argument. *J. Differ. Equ.* **255**(3), 298–325 (2013)
31. Lam, N., Lu, G.: Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti–Rabinowitz conditions. *J. Geom. Anal.* **24**(1), 118–143 (2014)
32. Lam, N., Lu, G.: Sharp constants and optimizers for a class of Caffarelli–Kohn–Nirenberg inequalities. *Adv. Nonlinear Stud.* **17**(3), 457–480 (2017)
33. Lam, N., Lu, G., Tang, H.: On nonuniformly subelliptic equations of Q -sub-Laplacian type with critical growth in the Heisenberg group. *Adv. Nonlinear Stud.* **12**, 659–681 (2012)
34. Lam, N., Lu, G., Tang, H.: Sharp subcritical Moser–Trudinger inequalities on Heisenberg groups and subelliptic PDEs. *Nonlinear Anal.* **95**, 77–92 (2014)
35. Lam, N., Lu, G., Tang, H.: Sharp affine and improved Moser–Trudinger–Adams type inequalities on unbounded domains in the spirit of lions. *J. Geom. Anal.* **27**(1), 300–334 (2017)
36. Lam, N., Lu, G., Zhang, L.: Equivalence of critical and subcritical sharp Trudinger–Moser–Adams inequalities. *Rev. Mat. Iberoam.* **33**(4), 1219–1246 (2017). [arXiv:1504.04858](https://arxiv.org/abs/1504.04858)

37. Li, J., Lu, G., Zhu, M.: Concentration-compactness for Trudinger–Moser’s inequalities on Riemannian manifolds and stratified groups: a completely symmetrization-free argument. Preprint
38. Li, Y.X.: Moser–Trudinger inequality on compact Riemannian manifolds of dimension two. *J. Partial Differ. Equ.* **14**(2), 163–192 (2001)
39. Li, Y.X.: Remarks on the extremal functions for the Moser–Trudinger inequality. *Acta Math. Sin. (Engl. Ser.)* **22**(2), 545–550 (2006)
40. Li, Y.X., Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in \mathbb{R}^n . *Indiana Univ. Math. J.* **57**, 451–480 (2008)
41. Lin, K.C.: Extremal functions for Moser’s inequality. *Trans. Am. Math. Soc.* **348**, 2663–2671 (1996)
42. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The limit case, part 1. *Rev. Mat. Iberoam.* **1**, 145–201 (1985)
43. Lu, G., Yang, Q.: A sharp Trudinger–Moser inequality on any bounded and convex planar domain. *Calc. Var. Partial Differ. Equ.* **55**(6), Art. 153 (2016)
44. Lu, G.: Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hrmarder’s condition and applications. *Rev. Mat. Iberoam.* **8**(3), 367–439 (1992)
45. Lu, G.: Existence and size estimates for the Green’s functions of differential operators constructed from degenerate vector fields. *Commun. Partial Differ. Equ.* **17**(7–8), 1213–1251 (1992)
46. Lu, G., Tang, H.: Sharp singular Trudinger–Moser inequalities in Lorentz–Sobolev spaces. *Adv. Nonlinear Stud.* **16**(3), 581–601 (2016)
47. Lu, G., Yang, Y.: Sharp constant and extremal function for the improved Moser–Trudinger inequality involving L^p norm in two dimension. *Discrete Contin. Dyn. Syst.* **25**(3), 963–979 (2009)
48. Malchiodi, A., Martinazzi, L.: Critical points of the Moser–Trudinger functional on a disk. *J. Eur. Math. Soc.* **16**, 893–908 (2014)
49. Malchiodi, A., Ruiz, D.: New improved Moser–Trudinger inequalities and singular Liouville equations on compact surfaces. *Geom. Funct. Anal.* **21**, 1196–1217 (2011)
50. Manfredi, J.J., Vera De Serio, V.N.: Rearrangements in Carnot groups. Preprint
51. Moser, J.: A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.* **20**, 1077–1092 (1970–1971)
52. Pohožaev, S.I.: On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR.* **165**, 36–39 (1965). (Russian)
53. Rabinowitz, P.H.: On a class of nonlinear Schrodinger equations. *Z. Angew. Math. Phys.* **43**, 270–291 (1992)
54. Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in \mathbb{R}^2 . *J. Funct. Anal.* **219**, 340–367 (2004)
55. Struwe, M.: Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5**(5), 425–464 (1988)
56. Trudinger, N.S.: On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.* **17**, 473–483 (1967)
57. Yang, Y.: Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space. *J. Funct. Anal.* **262**, 1679–1704 (2012)
58. Yudovič, V.I.: Some estimates connected with integral operators and with solutions of elliptic equations. *Dokl. Akad. Nauk SSSR* **138**, 805–808 (1961). (Russian)
59. Zhang, C., Chen, L.: Concentration-compactness principle of singular Trudinger–Moser inequalities in \mathbb{R}^n and n -Laplace equations. *Adv. Nonlinear Stud.* (2017). <https://doi.org/10.1515/ans-2017-6041>
60. Zhu, J.: Improved Moser–Trudinger inequality involving L^p norm in n dimensions. *Adv. Nonlinear Stud.* **14**, 273–293 (2014)