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Poincaré and Sobolev Inequalities for Vector Fields Satisfying Hörmander's Condition in Variable Exponent Sobolev Spaces

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Abstract In this paper, we will establish Poincaré inequalities in variable exponent non-isotropic Sobolev spaces. The crucial part is that we prove the boundedness of the fractional integral operator on variable exponent Lebesgue spaces on spaces of homogeneous type. We obtain the first order Poincaré inequalities for vector fields satisfying Hörmander's condition in variable non-isotropic Sobolev spaces. We also set up the higher order Poincaré inequalities with variable exponents on stratified Lie groups. Moreover, we get the Sobolev inequalities in variable exponent Sobolev spaces on whole stratified Lie groups. These inequalities are important and basic tools in studying nonlinear subelliptic PDEs with variable exponents such as the $p(x)$-subLaplacian. Our results are only stated and proved for vector fields satisfying Hörmander's condition, but they also hold for Grushin vector fields as well with obvious modifications.

Keywords Poincaré inequalities, the representation formula, fractional integrals on homogeneous spaces, vector fields satisfying Hörmander's condition, stratified groups, high order non-isotropic Sobolev spaces with variable exponents, Sobolev inequalities with variable exponents

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1 Introduction

Poincaré type inequalities for vector fields satisfying Hörmander's condition have been extensively studied for the past two decades (see [2, 12, 17–19, 25] etc.). Such a Poincaré inequality

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is of the following form:

$$\left(\frac{1}{|B|}\int_B |f(x)-f_B|^q dx\right)^{\frac{1}{q}} \le cr\left(\frac{1}{|B|}\int_B \left(\sum_j |\langle X_j, \nabla f(x)\rangle|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \tag{1.1}$$

in Euclidean space $\mathbb{R}^N$ for $1 \le p < \infty$ and certain values $q > p$, where $\{X_j\}$ is a collection of smooth vector fields which satisfy the Hörmander condition (see [16]). Here, B denotes any suitably restricted ball of radius r relative to a metric ρ which is naturally associated with the vector fields $\{X_j\}$ as, e.g., in [7, 26], $f_B = |B|^{-1}\int_B f(x)dx$, ∇f is the usual gradient of f and c is a constant independent of f and B. One can show similar results also hold for more general domains as well, such as Bomain chain domains.

Inequality (1.1) was first established by Jerison in [17] for $q = p$ and $1 \le p < \infty$, and this result was improved by Lu in the case $p > 1$ in [18, 19] by showing that the estimate holds for $1 < p < Q$ and $q = \frac{pQ}{Q-p}$, where Q $(\ge N)$ denotes the homogeneous dimension of $\mathbb{R}^N$ associated with the vector fields $\{X_j\}$ (see [12, §2] for the definition). Sharpened results hold for all $1 \le p < Q$ with $q = \frac{pQ}{Q-p}$, and this was proved by Franchi et al. [12] through establishing a stronger representation formula than that in [18] and using a truncation argument (see also the work by Maheux and Saloff-Coste through a self-improving argument [25]). Two-weighted Poincaré inequalities were also established in [12] when two weights satisfy a certain balance condition as introduced in Chanillo and Wheeden [3]. It was further proved in [13] that such a representation formula proved in [12] is actually equivalent to $L^1 \to L^1$ Poincaré's inequality in spaces of homogeneous type.

The results of Poincaré type in [12] are based on a new representation formula for a function in terms of the vector fields $\{X_j\}$. One form of the representation formula states that if ρ denotes the metric corresponding to the vector fields $\{X_j\}$, then

$$|f(x)-f_B| \le C\int_{cB} |Xf(y)|\frac{\rho(x,y)}{|B(x,\rho(x,y))|}dy, \quad x \in B, \tag{1.2}$$

where B is any suitably small ρ-ball. Here, $C > 0$ and $c \ge 1$ are appropriate constants, $|Xf|^2 = \sum_j |\langle X_j,\, \nabla f\rangle|^2$, f_B is the Lebesgue average $|B|^{-1}\int_B f dy$, $B(x,r)$ is the metric ball with center x and radius r, and cB denotes $B(x, cr)$ if $B = B(x,r)$.

Inequality (1.2) was shown to be true on graded nilpotent Lie groups for the left invariant vector fields by Lu in [18] (see Lemma 3.1 there). For general Hörmander vector fields, (1.2) improves significantly an analogous (but weaker) fractional integral estimate in [18] (see Lemma 3.2 there) for their "lifted" versions $\{\tilde{X}_j\}$ and $\tilde{\rho}$ as defined in Rothschild and Stein [27]. A subsequent point wise estimate was also given in [2]. It was further proved in [22] that the constant c in the integral domain in (1.2) can be taken as 1.

Though extensive investigation for Poincaré inequalities for vector fields satisfying Hörmander's condition has been made in L^p Lebesgue spaces, none has been done in the variable exponent $L^{p(x)}$ spaces. This is the main motivation of our paper.

In order to state our results more precisely, we now introduce some additional notation. Let Ω be an open, connected set in $\mathbb{R}^N$. Let $X_1, \ldots, X_m$ be real C^∞ vector fields which satisfy Hörmander's condition, i.e., the rank of the Lie algebra generated by $X_1, \ldots, X_m$ equals N at each point of a neighborhood Ω_0 of $\bar{\Omega}$. As is well known, it is possible to naturally associate with $\{X_j\}$ a metric $\rho(x,y)$ for $x,y \in \Omega$. The geometry of the metric space (Ω,ρ) is described in Fefferman, Phong [7] and Nagel et al. [26], Sanchez-Calle [28]. In particular, the ρ-topology and the Euclidean topology are equivalent in Ω, each metric ball

$$B(x,r) = \{y \in \Omega : \rho(x,y) < r\}, \quad x \in \Omega,\ r > 0,$$

contains some Euclidean ball with center x, and if K is a compact subset of Ω and $r_0 > 0$, there is a constant C_D such that

$$|B(x,2r)| \le C_D |B(x,r)|, \quad x \in K,\ 0 < r < r_0, \tag{1.3}$$

where $|E|$ denotes the Lebesgue measure of a measurable set E. This doubling property of Lebesgue measure is crucial for our results. With this doubling property, the space $(\Omega, \rho, |\cdot|)$ becomes a homogenous space in the sense of Coifman and Weiss [5] (see Section 2 for more details).

If $B = B(x,r)$, we will use the notation $r(B)$ for the radius r of B.

By [26], given a ball $B = B(x,r)$, $x \in K$, $r < r_0$, there exist positive constants γ and c, depending on B, such that

$$|J| \le c \left(\frac{r(J)}{r(I)}\right)^{N\gamma} |I| \tag{1.4}$$

for all balls I, J with $I \subset J \subset B$. We will call γ the (local) doubling order of Lebesgue measure for B. In fact, by [26], $N\gamma$ lies somewhere in the range $N \le N\gamma \le Q$, where $Q = \log_2 C_D$ is the homogeneous dimension (C_D is the constant in (1.3)). We can always choose $N\gamma = Q$, but smaller values may arise for particular vector fields, and these values may vary with $B(x,r)$ and will give sharper results for the Poincaré inequalities.

Given any real-valued function $f \in \mathrm{Lip}(\Omega)$, we denote

$$X_j f(x) = \langle X_j(x), \nabla f(x)\rangle, \quad j = 1, \dots, m,$$

and

$$|Xf(x)|^2 = \sum_{j=1}^m |X_j f(x)|^2,$$

where ∇f is the usual gradient of f and $\langle \cdot,\cdot \rangle$ is the usual inner product on $\mathbb{R}^N$.

We now recall the Poincaré estimate proved in the unweighted case in [12].

Theorem 1.1 *Let K be a compact subset of Ω. There exists r_0 depending on K, Ω and $\{X_j\}$ such that if $B = B(x,r)$ is a ball with $x \in K$ and $0 < r < r_0$, and if $1 \le p < N\gamma$ and $1/q = 1/p - 1/(N\gamma)$, where γ is defined by (1.4) for B, then*

$$\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q dx\right)^{\frac{1}{q}} \le cr \left(\frac{1}{|B|}\int_B |Xf(x)|^p dx\right)^{\frac{1}{p}}$$

for any $f \in \mathrm{Lip}(\bar{B})$. The constant c depends on K, Ω, $\{X_j\}$, and the constants c and γ in (1.4). Also, f_B may be taken to be the Lebesgue average of f, i.e., $f_B = |B|^{-1}\int_B f(x)dx$.

As mentioned earlier, we may always choose $N\gamma = Q$, and then with $p > 1$, we obtain the principal result of [19].

As the proof of Theorem 1.1 shows (see [12]), if the conclusion is weakened by replacing the integration over B on the right by integration over an appropriate larger ball cB for some $c > 1$, then (1.4) may be replaced by the condition

$$|B| \leq c\left(\frac{r(B)}{r(I)}\right)^{N\gamma} |I|$$

for all balls I with center in cB and $r(I) \leq r(B)$.

Some weighted versions of Poincaré inequality for Hörmander vector fields are proved in [12, 18].

A weight function $w(x)$ on Ω is a nonnegative function on Ω which is locally integrable with respect to Lebesgue measure. We say that a weight $w \in A_p$ $(= A_p(\Omega, \rho, dx))$, $1 \leq p < \infty$, if

$$\left(\frac{1}{|B|}\int_B w\ dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}dx\right)^{p-1} \leq C \quad \text{when } 1 < p < \infty,$$
$$\frac{1}{|B|}\int_B w\ dx \leq C \underset{B}{\rightarrow} \text{ess inf}\, w \quad \text{when } p = 1$$

for all metric balls $B \subset \Omega$. The fact that Lebesgue measure satisfies the doubling condition (1.3) allows us to develop the usual theory of such weight classes as in [1], at least for balls $B = B(x, r)$ with $0 < r < r_0$ and x belonging to a compact subset of Ω. It follows easily from the definition and (1.3) that if $w \in A_p$, then

$$w(B(x, 2r)) \leq Cw(B(x, r))$$

if $0 < r < r_0$ and $x \in K \subset \Omega$, K compact, with $C = C(r_0, K)$, where we use the standard notation $w(E) = \int_E wdx$. We say that any such weight is doubling. All the weights we shall consider will be doubling weights.

Given two weight functions w_1, w_2 on Ω and $1 \leq p < q < \infty$, we will assume that the following local balance condition holds for w_1, w_2 and a ball B with center in K and $r(B) < r_0$:

$$\frac{r(I)}{r(J)}\left(\frac{w_2(I)}{w_2(J)}\right)^{\frac{1}{q}} \leq c\left(\frac{w_1(I)}{w_1(J)}\right)^{\frac{1}{p}} \tag{1.5}$$

for all metric balls I, J with $I \subset J \subset B$. Note that in the case of Lebesgue measure ($w_1 = w_2 = 1$), (1.5) reduces to (1.4) when $1/q = 1/p - 1/(N\gamma)$.

Then the weighted Poincaré inequalities proved in [12] read as follows:

Theorem 1.2 *Let K be a compact subset of Ω. Then there exists r_0 depending on K, Ω and $\{X_j\}$ such that if $B = B(x, r)$ is a ball with $x \in K$ and $0 < r < r_0$, and if $1 \leq p < q < \infty$ and w_1, w_2 are weights satisfying the balance condition* (1.5) *for B, with $w_1 \in A_p(\Omega, \rho, dx)$ and w_2 doubling, then*

$$\left(\frac{1}{w_2(B)}\int_B |f(x) - f_B|^q w_2(x)dx\right)^{\frac{1}{q}} \leq cr\left(\frac{1}{w_1(B)}\int_B |Xf(x)|^p w_1(x)dx\right)^{\frac{1}{p}}$$

for any $f \in \mathrm{Lip}\ (\bar{B})$, with $f_B = w_2(B)^{-1}\int_B f(x)w_2(x)dx$. The constant c depends only on K, Ω, $\{X_j\}$ and the constants in the conditions imposed on w_1 and w_2.

This result includes Theorem 1.1 and the weighted results in [18].

Remark 1.3 As pointed out in [12], Theorem 1.2 has an analogue in case $q = p$ and $1 \le p < \infty$. In fact, the theorem remains true as stated if $1 < p < \infty$ and $q = p$ provided $w_1 \in A_p$ and there exists $s > 1$ such that w_2^s is a doubling weight and the balance condition (1.5) is replaced by the condition

$$\left(\frac{r(I)}{r(J)}\right)^p \frac{\mathcal{A}_s(I, w_2)}{w_2(J)} \le c\frac{w_1(I)}{w_1(J)}$$

for all balls I, J with $I \subset J \subset B$, where

$$\mathcal{A}_s(I, w_2) = |I| \left(\frac{1}{|I|} \int_I w_2^s dx\right)^{\frac{1}{s}}.$$

Note that $w_2(I) \le \mathcal{A}_s(I, w_2)$ for $s > 1$ by Hölder's inequality, and, as is well known, $w_2(I)$ and $\mathcal{A}_s(I, w_2)$ are equivalent if w_2 belongs to some A_{p_0} class and s is sufficiently close to 1.

Poincaré estimates on domains other than balls were also obtained in [12, 19]. In particular, this can be done for domains which satisfy the Boman chain condition, see [12, 19, 23].

The validity of global Poincaré inequalities of higher order on the nilpotent stratified Lie groups has also been discussed (see [20, 21, 23, 24], etc.). Now we recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups) (see [8]). Assume

$$\mathfrak{g} = \bigoplus_{i=1}^{s} V_i,$$

with $[V_i, V_j] \subset V_{i+j}$ for $i + j \le s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \ldots, X_l$ be the basis for V_1 and suppose that $X_1, \ldots, X_l$ generate $\mathfrak{g}$ as a Lie algebra. Then for $2 \le j \le s$, we can choose a basis $\{X_{ij}\}$, $1 \le i \le k_j$, for V_j consisting of commutators of length j. We set $k_1 = l$ and $X_{i1} = X_i$, $i = 1, \ldots, l$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathfrak{g}$, then the exponential mapping is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \le i \le k_j$, $1 \le j \le s$, $N = \sum_{j=1}^{s} k_j$ such that

$$g = \exp\Big(\sum x_{ij} X_{ij}\Big).$$

The higher order Poincaré inequalities on the nilpotent stratified Lie groups were established by Lu [20, 21], Lu and Wheeden [23, 24] and Cohn et al. [4]. We state a theorem from [4].

Theorem 1.4 *Let m be a positive integer, $p \ge 1$, $f \in W^{m,p}_{\rm loc}(\mathbb{G})$ and $X^m f \in L^p(\mathbb{G})$. Then there exists a unique polynomial $P \in \mathcal{P}_m$ such that, for any integer j with $0 \le j < m$,*

$$\left(\int_{\mathbb{G}} |X^j(f - P)|^{q_{mj}} dx\right)^{\frac{1}{q_{mj}}} \le C\left(\int_{\mathbb{G}} |X^m f(x)|^p dx\right)^{\frac{1}{p}} \tag{1.6}$$

for all $1 \le p < \frac{Q}{m-j}$ and $q_{mj} = \frac{pQ}{Q-(m-j)p}$, where C is independent of f, $\mathcal{P}_m$ denotes the polynomials of homogeneous degree less than m for each positive integer m and

$$|X^m f| = \Big(\sum_{I:d(I)=m} |X^I f|^2\Big)^{\frac{1}{2}}.$$

More details, see [4] or Section 5 in this paper.

As for the variable exponent version, the Poincaré inequalities on bounded John domain were proved in variable Sobolev spaces (the definition of variable Sobolev spaces can be found in [6]) by Harjulehto and Hästö [14] with some assumptions on the exponents, i.e., $p(\cdot)$ is continuous on $\overline{\Omega}$. They also gave an example that $p(\cdot)$ ordinarily needs some regularity and proved a version when $p(\cdot)$ has no regularity but has limited oscillation, i.e., $(p_-)^* \geq p_+$.

Theorem 1.5 ([14]) *Let $D \subset \mathbb{R}^n$ be a bounded John domain, with constant λ. If $p_D^+ \leq (p_D^-)^*$ or $p_D^- \geq n$ and $p_D^+ < \infty$, then there exists a constant $C = C(n,\ p_D^-,\ p_D^+,\ \lambda)$ such that, for every $f \in W^{1,p(\cdot)}(D)$,*

$$\|f - f_D\|_{L^{p(\cdot)}(D)} \leq C(1 + |D|)^2 |D|^{\frac{1}{2} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla f\|_{L^{p(\cdot)}(D)}, \tag{1.7}$$

here, $p_D^+ := \operatorname{ess\,sup}_{x\in D} p(x),\ p_D^- := \operatorname{ess\,inf}_{x\in D} p(x)$.

The result also holds for stronger hypotheses on the exponent $p(\cdot)$, a suitable boundedness for the Hardy–Littlewood maximal function on the variable exponent Lebesgue spaces.

Theorem 1.6 ([6]) *Given a bounded convex set $\Omega \subset \mathbb{R}^n$ with diameter D, let $p(\cdot) : \Omega \to [1, +\infty]$ be such that $p_+ < \infty$ and the maximal operator M is bounded on $L^{p'(\cdot)}(\Omega)$. Then for all $f \in W^{1,p(\cdot)}(\Omega)$,*

$$\|f - f_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla f\|_{L^{p(\cdot)}(\Omega)}, \tag{1.8}$$

here, C is only dependent on n, $p(\cdot)$ and D, $\|M\|_{L^{p'(\cdot)}(\Omega)}$ and the measure of Ω.

In this paper, we consider the Poincaré inequalities in variable exponent non-isotropic Sobolev spaces. First, we set up the first order Poincaré inequalities for vector fields satisfying Hörmander's condition. To this end, we introduce the notion of Boman chair domains in a metric space of homogeneous type (X, ρ, μ) (see Section 2 for more details).

Definition 1.7 *A domain (i.e., an open connected set) Ω in X is said to satisfy the weak Boman chain condition of type σ, Λ, or to be a member of $\mathcal{F}(\sigma, \Lambda)$, if there exist constants $\sigma = 1$, $\Lambda > 0$, and a family $\mathcal{F}$ of metric balls $B \subset \Omega$ such that*

(i) $\Omega = \bigcup_{B\in\mathcal{F}} B$;

(ii) $\sum_{B\in\mathcal{F}} \chi_{\sigma B}(x) \leq \Lambda \chi_\Omega(x)$ *for all* $x \in \mathbb{G}$;

(iii) *there is a "central ball" $B_0 \in \mathcal{F}$ such that, for each ball of $B \in \mathcal{F}$, there is a positive integer $k = k(B)$ and a chain $\{B_j\}_{j=0}^k$ for balls for which $B_k = B$ and each $B_j \cap B_{j+1}$ contains a ball D_j with $B_j \cup B_{j+1} \subset \Lambda D_j$;*

(iv) $B \subset \Lambda B_j$ *for all* $j = 0, \ldots, k(B)$.

Our first theorem is concerning the first order Poincaré inequality on variable exponent non-isotropic Sobolev spaces associated with the vector fields satisfying Hörmander's condition on domains Ω satisfying the Boman chain condition (with respect to the metric associated with the vector fields satisfying Hörmander's condition).

Theorem 1.8 *Given a weak Boman chain domain Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $1 \leq p_- \leq p_+ < Q$, suppose that the maximal operator M is bounded on $L^{(p^*(\cdot)/Q')'}(\Omega)$. Then, for every $f \in \mathrm{Lip}(\overline{\Omega})$,*

$$\|f - f_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq C\|Xf\|_{L^{p(\cdot)}(\Omega)}.$$

Here, $p^*(x) = \frac{Qp(x)}{Q-p(x)}$ and Q is the homogeneous dimension.

As is well known, the metric balls associated with the vector fields satisfying Hörmander's condition are Boman chain domains, the above theorem holds for balls as well.

Next, we set up the higher order Poincaré inequalities on variable exponent Sobolev space on stratified Lie groups $\mathbb{G}$. $\mathrm{LH}(\Omega)$ means the class of log-Hölder continuous functions (see Section 2 for definitions).

Theorem 1.9 *Let m be an integer and let $p(\cdot)$ satisfy $1 \le p_- \le p_+ < Q/m$ and $p(\cdot) \in \mathrm{LH}(\Omega)$. Assume Ω is a weak Boman chain domain in a stratified group $\mathbb{G}$ with a central ball B, and $f \in W^{m,p(\cdot)}(\Omega)$. Then there exists a unique polynomial $p \in \mathcal{P}_m$ such that*

$$\|f - P\|_{L^{p^*_m(\cdot)}(\Omega)} \le C\|X^m f\|_{L^{p(\cdot)}(\Omega)}, \tag{1.9}$$

with $p^*_m(x) = \frac{Qp(x)}{Q-mp(x)}$. Here, $\mathcal{P}_m$ is the polynomials of homogeneous degree less than m for each positive integer m.

Finally, we establish the Sobolev inequalities in variable Sobolev spaces on whole stratified Lie groups.

Theorem 1.10 *Suppose that m is a positive integer and let $p(\cdot)$ satisfy $1 \le p_- \le p_+ < Q/m$ and $p(\cdot) \in \mathrm{LH}(\mathbb{G})$, $f \in W^{m,p(\cdot)}(\mathbb{G})$. Then*

$$\|f\|_{L^{p^*_m(\cdot)}(\mathbb{G})} \le C\|X^m f\|_{L^{p(\cdot)}(\mathbb{G})}, \tag{1.10}$$

with $p^*_m(x) = \frac{Qp(x)}{Q-mp(x)}$.

We end this introduction with the following remark. The Poincaré inequalities for Grushin type vector fields on variable exponent spaces can be established using the same method employed in this paper. The representation formula for Grushin vector fields were established by Franchi et al. (see [9–11]).

The organization of this paper is as follows. In Section 2, we will give some preliminaries. Then, we will establish the boundedness of the fractional integral operator in variable Lebesgue spaces in Section 3. In Section 4, we will prove Theorem 1.8. Then we will prove the high order Poincaré inequalities on stratified Lie groups (Theorem 1.9) in Section 5. Finally, we obtain the Sobolev inequalities in variable exponent Lebesgue spaces on whole stratified Lie groups (Theorem 1.10) in Section 6.

2 Preliminaries

2.1 Variable Exponent Non-isotropic Sobolev Spaces

Here, we first give the definition of homogeneous spaces. We say X is the quasi-metric space with the quasi-metric d, if the function $d: X \times X \to [0, \infty)$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \le K[d(x, z) + d(z, y)]$ for some constant $K \ge 1$.

A positive measure μ is a doubling measure on X if for some positive C,

$$\mu(B(x, 2r)) \le C\mu(B(x, r)), \quad x \in X,\, r > 0,$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ is the ball of radius r and centered in x. Then, the quasi-metric space X matched with a doubling measure μ on it is extended to general homogeneous spaces, and we denote it by (X, μ).

The Hardy–Littlewood maximal operator and the fractional integral operator of order α on homogeneous spaces are defined respectively by

$$Mf(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B f(y)d\mu(y),$$
$$I_\alpha f(x) = \int_X f(y) \frac{d(x,y)^\alpha}{\mu(B(x,d(x,y)))} d\mu(y).$$

Next, we define the A_p weight on homogeneous spaces (X,μ).

We say $\omega \in A_p$, $1 < p < \infty$, if

$$\left[\int_B \omega(x)d\mu(x)\right]\left[\int_B \omega(x)^{-1/(p-1)}d\mu(x)\right]^{p-1} \le C_\omega \mu(B)^p$$

for all balls $B \subset \Omega$. And we say $\omega \in A_1$, if

$$\int_B \omega(x)d\mu(x) \le C_\omega \mu(B) \operatorname*{ess\,inf}_{x\in B} \omega(x)$$

for all balls B.

The following basic properties for $\omega \in A_p$, $1 < p < \infty$, are well known and can be verified easily.

Lemma 2.1 *Suppose $\omega \in A_p$, $1 < p < \infty$. Then*

(1) *if $p_1 \le p_2$, $1 \le p_1 < p_2$, then $A_{p_1} \subsetneqq A_{p_2}$;*

(2) *if $\omega \in A_p$ $(1 \le p < \infty)$, $0 < \alpha < 1$, then,*

$$\omega^{-\frac{1}{p-1}} \in A_{p'}, \quad \omega^\alpha \in A_{\alpha p+1-\alpha};$$

(3) *$\omega \in A_p$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}$;*

(4) *if $\omega \in A_p$, then $\omega d\mu$ satisfies the doubling measure condition.*

We are ready to give the definition of the variable exponent non-isotropic Sobolev spaces associated with the vector fields satisfying the Höremainder condition. First, we define the variable Lebesgue space on homogeneous spaces.

Definition 2.2 *Given a homogeneous space (X, μ), an open set $\Omega \subset X$ and a μ-measurable function $p(\cdot) : \Omega \to [1,\infty]$, let $L^{p(\cdot)}(\Omega)$ denote the Banach function space of μ-measurable f on Ω such that*

$$\int_\Omega |f(x)|^{p(x)} d\mu(x) < \infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} d\mu(x) \le 1\right\}.$$

If there is no ambiguity over the domain Ω, we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$.

For brevity, hereafter let

$$\mathcal{P}(\Omega) = \{p(\cdot) : \Omega \to [1,\infty] \text{ is a } \mu\text{-measurable function}\}.$$
$$p_-(E) := \operatorname*{ess\,inf}_{x\in E} p(x) \quad \text{and} \quad p_+(E) := \operatorname*{ess\,sup}_{x\in E} p(x).$$

If the domain is clear, we will simply write $p_- = p_-(E)$, $p_+ = p_+(E)$. Given $p(\cdot)$, we define the conjugate exponent function $p'(\cdot)$ by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega,$$

with the convention that $1/\infty = 0$.

In classical L^p space, the Hardy–Littlewood maximal operator M is bounded in L^p $(p > 1)$. However, the boundedness of M on variable Lebesgue space $L^{p(\cdot)}$ is not trivial. In particular, in spaces of homogeneous type, there are many conditions that guarantee M is bounded on variable Lebesgue space $L^{p(\cdot)}$, see [15]. Here we give the common condition known as log-Hölder continuity condition as follows:

Definition 2.3 *Given a set $\Omega \subset X$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$, we say that $p(\cdot) \in \mathrm{LH}(\Omega)$, if there exists a constant C_0 such that for all x, $y \in \Omega$, $|d(x,y)| < 1/2$,*

$$|p(x) - p(y)| \le \frac{C_0}{-\log(d(x,y))}.$$

Now, we can define variable non-isotropic Sobolev spaces for vector fields satisfying Hörmander's condition.

Definition 2.4 *Let Ω be open, connected set in $\mathbb{R}^N$, and $X_1, X_2, \ldots, X_m$ be real C^∞ vector fields which satisfy Hörmander's condition. Given $p(\cdot) \in \mathcal{P}(\Omega)$, we define the Sobolev space $W^{1,p(\cdot)}(\Omega)$ for $p_+(\Omega) < \infty$ as follows:*

$$W^{1,p(\cdot)}(\Omega) = \Big\{ f \in L^{p(\cdot)}(\Omega), |Xf| \in L^{p(\cdot)}(\Omega) : \int_\Omega |f(x)|^{p(x)} + |Xf(x)|^{p(x)} dx < \infty \Big\},$$

equipped with the norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|Xf\|_{L^{p(\cdot)}(\Omega)}.$$

If there is no ambiguity about the domain, we often write $\|f\|_{1,p(\cdot)}$ instead of $\|f\|_{W^{1,p(\cdot)}(\Omega)}$.

2.2 Some Important Lemmas for the Proof of Our Main Theorem

In this subsection, we state some lemmas which will be used in the proof of our main theorem. First, we state an extrapolation lemma which will be very useful. The following lemma in Euclidean spaces was proved in [6, Theorem 5.28]. Our proof here is an adaptation to the homogeneous spaces for the sake of completeness.

Lemma 2.5 *Given $\Omega \subset X$, suppose that for some p_0, q_0, $1 \le p_0 \le q_0$, the family $\mathcal{F}$ is such that, for all $\omega \in A_1$,*

$$\Big(\int_\Omega F(x)^{q_0} \omega(x) d\mu(x) \Big)^{1/q_0} \le C_0 \Big(\int_\Omega G(x)^{p_0} \omega(x)^{p_0/q_0} d\mu(x) \Big)^{1/p_0}, \quad (F, G) \in \mathcal{F}. \tag{2.1}$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_0 \le p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$, define $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}. \tag{2.2}$$

If the Hardy–Littlewood maximal operator M is bounded on $L^{(q(\cdot)/q_0)'}(\Omega)$, then

$$\|F\|_{q(\cdot)} \le C_{p(\cdot)} \|G\|_{p(\cdot)} \quad \text{for } (F, G) \in \mathcal{F}. \tag{2.3}$$

Proof Fix $p(\cdot), q(\cdot) : \Omega \to [1, \infty)$ as in the hypotheses, and let

$$\overline{p}(x) = p(x)/p_0, \quad \overline{q}(x) = q(x)/q_0.$$

By the assumption, the Hardy–Littlewood maximal operator is bounded on $L^{\overline{q}'(\cdot)}(\Omega)$. Define an iteration algorithm $\mathcal{R}$ on $L^{\overline{q}'(\cdot)}(\Omega)$ by

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{\overline{q}'(\cdot)}(\Omega)}},$$

where, for $k \geq 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes k iterations of the Hardy–Littlewood maximal operator M and $M^0 h = |h|$. Then, we have

(a) for all $x \in \Omega$, $|h(x)| \leq \mathcal{R}h(x)$;

(b) $\mathcal{R}$ is bounded on $L^{p(\cdot)}(\Omega)$ and $\|\mathcal{R}h\|_{p(\cdot)} \leq 2\|h\|_{\overline{q}'(\cdot)}$;

(c) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{\overline{q}'(\cdot)}(\Omega)}$.

For further details, see Lemma 2.7.

Fix a pair $(F, G) \in \mathcal{F}$ such that $F \in L^{q(\cdot)}$ (i.e., so that the left-hand side of (2.3) is finite). Then we have

$$\|F\|^{q_0}_{L^{q(\cdot)}(\Omega)} = \|F^{q_0}\|_{L^{q(\cdot)}(\Omega)} \leq k^{-1}_{p(\cdot)} \sup \int_\Omega F(x)^{q_0} h(x) d\mu(x),$$

where the supremum is taken over all non-negative $h \in L^{\overline{q}'(\cdot)}(\Omega)$ with $\|h\|_{\overline{q}'(\cdot)} = 1$.

For any such function h, we will show that

$$\int_\Omega F(x)^{q_0} h(x) d\mu(x) \leq C\|G\|^{q_0}_{L^{p(\cdot)}(\Omega)},$$

with the constant C independent of h.

First note that by property (a), we have

$$\int_\Omega F(x)^{q_0} h(x) d\mu(x) \leq \int_\Omega F(x)^{q_0} \mathcal{R}h(x) d\mu(x). \tag{2.4}$$

We want to apply our hypothesis (2.1) to the term on the right-hand side of (2.4).

To do so, we have to show that it is finite.

In fact,

$$\begin{aligned}\int_\Omega F(x)^{q_0} \mathcal{R}h(x) d\mu(x) &\leq K_{p(\cdot)} \|F^{q_0}\|_{\overline{q}(\cdot)} \|\mathcal{R}h\|_{\overline{q}'(\cdot)} \\ &\leq 2K_{p(\cdot)} \|F\|^{q_0}_{\overline{q}(\cdot)} \|h\|_{\overline{q}'(\cdot)} \\ &< \infty.\end{aligned}$$

Therefore, by property (c), (2.1) holds with $\omega = \mathcal{R}h$. Furthermore, the constant C_0 is independent of h. Hence,

$$\begin{aligned}\int_\Omega F(x)^{q_0} \mathcal{R}h(x) d\mu(x) &\leq C_0^{q_0} \left(\int_\Omega G(x)^{p_0} \mathcal{R}h(x)^{p_0/q_0} d\mu(x) \right)^{q_0/p_0} \\ &\leq C_0^{q_0} \|G^{p_0}\|^{q_0/p_0}_{\overline{p}(\cdot)} \|\mathcal{R}h^{p_0/q_0}\|^{q_0/p_0}_{\overline{p}'(\cdot)} \\ &= C_0^{q_0} \|G\|^{q_0}_{\overline{p}(\cdot)} \|\mathcal{R}h^{p_0/q_0}\|^{q_0/p_0}_{\overline{p}'(\cdot)}.\end{aligned}$$

To complete the proof, we need to show that $\|\mathcal{R}h^{p_0/q_0}\|^{q_0/p_0}_{\overline{p}'(\cdot)}$ is bounded by a constant independent of h. By the definition of $q(\cdot)$,

$$\overline{p}' = \frac{p(x)}{p(x) - p_0} = \frac{q_0}{p_0} \frac{q(x)}{q(x) - q_0} = \frac{q_0}{p_0} \overline{q}'(x).$$

Therefore,

$$\|\mathcal{R}h^{p_0/q_0}\|_{\overline{p}'(\cdot)}^{q_0/p_0} = \|\mathcal{R}h\|_{\overline{q}'(\cdot)} \le 2\|h\|_{\overline{q}'(\cdot)} = 2.$$

This completes our proof of our Lemma 2.5. □

Secondly, we state a particular extrapolation lemma.

Lemma 2.6 *Given $\Omega \subset X$, suppose that for some $p_0 \ge 1$, the family $\mathcal{F}$ is such that, for all $\omega \in A_1$,*

$$\int_\Omega F(x)^{p_0}\omega(x)d\mu(x) \le C_0 \int_\Omega G(x)^{p_0}\omega(x)d\mu(x), \quad (F, G) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_0 \le p_- \le p_+ < \infty$, and the maximal operator M is bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$, then

$$\|F\|_{p(\cdot)} \le C_{p(\cdot)}\|G\|_{p(\cdot)}, \quad (F, G) \in \mathcal{F}.$$

Then, we will construct an A_1 weight which will be used when we apply Lemma 2.5.

Lemma 2.7 *Given $\Omega \subset X$ and $h \in L^{p(\cdot)}(\Omega)$, define*

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{p(\cdot)}(\Omega)}},$$

where, for $k \ge 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes k iterations of operator and $M^0 h = |h|$. Then this operator has the following properties:

(a) *for all $x \in \Omega$, $|h(x)| \le \mathcal{R}h(x)$;*

(b) *$\mathcal{R}$ is bounded on $L^{p(\cdot)}(\Omega)$ and $\|\mathcal{R}h\|_{p(\cdot)} \le 2\|h\|_{p(\cdot)}$;*

(c) *$\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \le 2\|M\|_{L^{p(\cdot)}(\Omega)}$.*

Proof Property (a) immediately follows from the definition. Since

$$\|\mathcal{R}h\|_{p(\cdot)} \le \sum_{k=0}^{\infty} \frac{\|M^k h\|_{p(\cdot)}}{2^k \|M\|^k_{L^{p(\cdot)}(\Omega)}} \le \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} 2^{-k} = 2\|h\|_{p(\cdot)}.$$

We get property (b).

And property (c) follows by the subadditivity and homogeneity of the maximal operator:

$$\begin{aligned} M(\mathcal{R}h)(x) &\le \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^k \|M\|^k_{L^{p(\cdot)}(\Omega)}} \\ &\le 2\|M\|_{L^{p(\cdot)}} \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^{k+1}\|M\|^{k+1}_{L^{p(\cdot)}(\Omega)}} \\ &\le 2\|M\|_{L^{p(\cdot)}} \mathcal{R}h(x). \end{aligned}$$

□

Finally, we state a result for us to test the condition (2.1) in Lemma 2.5.

Lemma 2.8 ([29]) *Suppose $1 < p \le q < \infty$, (X, d) is a quasi-metric space, μ is a doubling measure on X, and $\omega(x)$ and $\nu(x)$ are nonnegative μ-measurable functions on X. Let $\varphi(B)$ be given by*

$$\varphi(B) = \sup\{K(x, y) : x, y \in B,\, d(x, y) \ge C(K)r(B)\},$$

where $K(x, y)$ is the kernel of I_α, $r(B)$ is the radius of B, and $C(K) = K^{-4}/9$. If $p < q$, $\omega d\mu$ and $\nu^{1-p'}d\mu$ are doubling measures, then the following weighted inequality

$$\left(\int_X [I_\alpha f(x)]^q \omega(x)d\mu(x)\right)^{1/q} \le \left(\int_X f(x)^p \nu(x)d\mu(x)\right)^{1/p} \tag{2.5}$$

holds if the condition

$$\varphi(B)\left(\int_B \omega d\mu\right)^{1/q}\left(\int_B \nu^{1-p'}d\mu\right)^{1/p'} \leq C \quad \textit{for all balls } B \subset X \tag{2.6}$$

holds.

Now, we give a weak type inequality in a metric space of homogeneous type (Ω, ρ, μ). Since our main interest is to prove the Poincaré inequalities for vector fields satisfying Hörmander's condition, we assume the doubling order of the metric space is Q, the homogenous dimension (see (1.4) for its definition).

Lemma 2.9 *Given Ω, suppose that for some p_0, q_0, $1 \leq p_0 \leq q_0$, the family $\mathcal{F}$ is such that for all $\omega \in A_1$,*

$$\omega\left(\{x \in \Omega : F(x) > t\}\right) \leq C_0\left(\frac{1}{t^{p_0}}\int_\Omega G(x)^{p_0}\omega(x)^{p_0/q_0}d\mu(x)\right)^{q_0/p_0}, \quad (F, G) \in \mathcal{F}. \tag{2.7}$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_0 < p_- \leq p_+ < \frac{p_0 q_0}{q_0 - p_0}$, define $q(\cdot)$ by

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{Q}$$

and assume that the Hardy–Littlewood maximal operator M is bounded on $L^{(q(\cdot)/q_0)'}(\Omega)$, then for all $t > 0$,

$$\|t\chi_{\{x\in\Omega: F(x)>t\}}\|_{L^{q(\cdot)}(\Omega)} \leq C_{p(\cdot)}\|G|_{L^{p(\cdot)}(\Omega)}, \quad (F, G) \in \mathcal{F}. \tag{2.8}$$

Proof Define a new family $\widetilde{\mathcal{F}}$ consisting of the pairs

$$(F_t, G) = (t\chi_{\{x\in\Omega: F(x)>t\}}, G), \quad (F, G) \in \mathcal{F},\ t > 0.$$

Then we can restate (2.7) as follows: for every $\omega \in A_1$,

$$\|F_t\|_{L^{q_0}(\omega)} = t\omega(\{x \in \Omega : F(x) > t\})^{1/q_0} \leq C_0^{1/q_0}\|G\|_{L^{p_0}\omega^{p_0/q_0}}, \quad (F_t, G) \in \widetilde{\mathcal{F}}.$$

Therefore, we can apply Lemma 2.5 to the family $\widetilde{\mathcal{F}}$ to conclude that

$$\|F_t\|_{L^{p(\cdot)}(\Omega)} \leq C_{p(\cdot)}\|G\|_{L^{p(\cdot)}(\Omega)},$$

which is exactly (2.8). □

3 An Inequality for Fractional Integrals in Variable Lebesgue Spaces on Spaces of Homogeneous Type

In order to prove Theorem 1.8, we need the following important lemma which describes the boundedness for fractional integrals in variable Lebesgue spaces on spaces of homogeneous type. A similar lemma in Euclidean spaces was proved in [6, Theorem 5.46]).

Lemma 3.1 *Fix α, $0 < \alpha < Q$. Given $p(\cdot) \in \mathcal{P}(X)$ such that $1 < p_- \leq p_+ < Q/\alpha$, define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{Q}.$$

If there exists $q_0 > \frac{Q}{Q-\alpha}$ such that M is bounded on $L^{(q(\cdot)/q_0)'}(X)$, then

$$\|I_\alpha f\|_{L^{q(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}. \tag{3.1}$$

If $p_- = 1$ *and if the operator* M *is bounded on* $L^{(q(\cdot)/q_0)'}(X)$ *when* $q_0 = \frac{Q}{Q-\alpha}$, *then for every* $t > 0$,

$$\|t\chi_{\{x\in X:\mu(I_\alpha f(x))>t\}}\|_{L^{q(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}. \tag{3.2}$$

Here, Q *is homogeneous dimension (the doubling order as defined in* (1.4)*),* M *is the Hardy–Littlewood maximal operator on homogeneous spaces, and* I_α *is the fractional integral operator on homogeneous spaces with index* α.

Proof We only need to prove the case $\alpha = 1$. The general case is similar. First, we show that

$$\left(\int_\Omega I_1 f(x)^{q_0}\omega(x)d\mu(x)\right)^{1/q_0} \le C_0\left(\int_\Omega f(x)^{p_0}\omega(x)^{p_0/q_0}d\mu(x)\right)^{1/p_0}. \tag{3.3}$$

In fact, by Lemma 2.8, we only need to prove that (2.6) holds for $\nu = \omega^{p/q}$. Since $\omega \in A_1 \subset A_p$, i.e.,

$$\left[\int_B \omega(x)d\mu(x)\right]\left[\int_B \omega(x)^{-1/(p-1)}d\mu(x)\right]^{p-1} \le C_\omega \mu(B)^p. \tag{3.4}$$

And $k(x,y) = \frac{d(x,y)}{\mu(B(x,d(x,y)))}$, so

$$\varphi(B) = \sup\{K(x,y) : x,y\in B,\ d(x,y) \ge C(K)r(B)\} \le C\mu(B)^{\frac{1}{Q}-1}.$$

Therefore, by Hölder inequality (noticing that $p < q$), the A_p condition, we have

$$\begin{aligned}
&\varphi(B)\left(\int_B \omega d\mu\right)^{1/q}\left(\int_B \nu^{1-p'}d\mu\right)^{1/p'}\\
&= \varphi(B)\left(\int_B \omega d\mu\right)^{1/q}\left(\int_B \omega^{\frac{p}{q}(-\frac{1}{p-1})}d\mu\right)^{\frac{p-1}{p}}\\
&\le \varphi(B)\left(\int_B \omega d\mu\right)^{1/q}\left[\left(\int_B \omega^{\frac{p}{q}(-\frac{1}{p-1})\frac{q}{p}}d\mu\right)^{\frac{p}{q}}\mu(B)^{1-\frac{p}{q}}\right]^{\frac{p-1}{p}}\\
&\le C\mu(B)^{\frac{1}{Q}-1}\left(\int_B \omega d\mu\right)^{1/q}\left(\int_B \omega^{-\frac{1}{p-1}}d\mu\right)^{\frac{p-1}{q}}\mu(B)^{(1-\frac{p}{q})(1-\frac{1}{p})}\\
&\le C\mu(B)^{\frac{1}{Q}-1}\mu(B)^{\frac{p}{q}+(1-\frac{p}{q})(1-\frac{1}{p})}\\
&\le C\mu(B)^{\frac{p}{q}+(1-\frac{p}{q})(1-\frac{1}{p})-\frac{Q-1}{Q}}.
\end{aligned}$$

We can easily calculate that

$$\frac{p}{q} + \left(1-\frac{p}{q}\right)\left(1-\frac{1}{p}\right) - \frac{Q-1}{Q} = 0.$$

It is not hard to show that $\omega d\mu$ is a doubling measure by $\omega \in A_1 \subset A_p$. And, by $\omega^{-\frac{1}{p-1}} \in A_{p'}$, we also have

$$\nu^{1-p'} = \omega^{\frac{p}{q}(1-p')} = \omega^{\frac{p}{q}(-\frac{1}{p-1})} \in A_{\frac{p}{q}p'+1-\frac{p}{q}}.$$

Therefore, $\nu^{1-p'}d\mu$ is also a doubling measure. Thus, by Lemma 2.8, we get (2.6).

So, by Lemma 2.8, we have

$$\left(\int_X [I_\alpha f(x)]^q\omega(x)d\mu(x)\right)^{1/q} \le \left(\int_X f(x)^p\omega(x)^{p/q}d\mu(x)\right)^{1/p}.$$

Finally, we can easily prove Lemma 3.1 by the extrapolation lemma on homogeneous spaces (i.e., Lemma 2.5). And the weak type is very similar, here we omit it. □

4 Proof of Theorem 1.8

First, we recall a representation formula given by Franchi et al. [12] and improved by Lu and Wheeden in [22] which is crucial for the proof of Theorem 1.8.

Lemma 4.1 *Suppose that μ, ν are doubling measures on a metric space (X, ρ) and give a weak Boman chain domain $\Omega \subset X$. And, there exists a constant $a_1 \geq 1$ such that, for all balls B with $a_1 B \subset \Omega$,*

$$\frac{1}{\nu(B)}\int_B |f - f_{B,\nu}| d\nu \leq C\frac{\rho(B)}{\mu(B)}\int_{a_1 B}|Xf| d\mu,$$

here, B is a ball of radius $\rho(B)$. Then, for ν-a.e. $x \in \Omega$,

$$|f(x) - f_{B_0,\nu}| \leq C\int_\Omega |Xf(y)|\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}d\mu(y), \tag{4.1}$$

where B_0 is the central ball in Ω, $f_{B_0,\nu} = \frac{1}{\nu(B_0)}\int_{B_0} f(y)d\nu(y)$, and C is independent of f and $x \in \Omega$.

To apply extrapolation, we need the corresponding weighted norm inequality.

Lemma 4.2 *Given a Boman chain domain $\Omega \subset X$, and p, $1 \leq p < Q$, $\omega \in A_1$, there is a contant $C = C(\Omega, p, [\omega]_{A_1})$ such that, for all $f \in \mathrm{Lip}(\Omega)$,*

$$\left(\int_\Omega |f(x) - f_\Omega|^{p^*}\omega(x)d\mu(x)\right)^{1/p^*} \leq C\left(\int_\Omega |Xf(x)|^p \omega(x)^{p/p^*}d\mu(x)\right)^{1/p},$$

where $p^* = \frac{Qp}{Q-p}$.

Proof Fix $f \in \mathrm{Lip}(\Omega)$. For each $j \in \mathbb{Z}$, let

$$\Omega_j = \{x \in \Omega : 2^j < |f(x) - f_\Omega| \leq 2^{j+1}\},$$

and define the function f_j by

$$f_j(x) = \begin{cases} |f(x) - f_\Omega| - 2^j, & x \in \Omega_j, \\ 2^j, & x \in \Omega_i,\ i > j, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the function f_j is weakly differentiable and $|Xf_j(x)| = |Xf(x)|\chi_{\Omega_j}$ almost everywhere. Furthermore, if $x \in \Omega_j$, by Lemma 4.1,

$$I_1(Xf_{j-1})(x) \geq C(Q)|f_{j-1}(x) - f_\Omega| = C(Q)2^{j-1}. \tag{4.2}$$

By (4.2) and the weak type inequality

$$\omega\left(\{x \in \Omega : \mu(I_1 h(x)) > t\}\right) \leq C\left(\frac{1}{t^p}\int_\Omega |h(x)|^p\omega(x)^{p/p^*}d\mu(x)\right)^{p^*/p}, \tag{4.3}$$

with $h = |Xf_{j-1}|$, we have

$$\begin{aligned}&\int_\Omega |f(x) - f_\Omega|^{p^*}\omega(x)d\mu(x)\\ &\quad = \sum_j \int_{\Omega_j} |f(x) - f_\Omega|^{p^*}\omega(x)d\mu(x)\end{aligned}$$

$$
\begin{aligned}
&= \sum_j \int_{\Omega_j} 2^{(j+1)p^*} \omega(x) d\mu(x) \\
&= 4^{p^*} C(Q)^{-p^*} \sum_j \int_{\Omega_j} (C(Q)2^{(j-1)})^{p^*} \omega(x) d\mu(x) \\
&\le C \sum_j \int_{\{x\in\Omega: I_1(|Xf_{j-1}|)(x) > C(Q)2^{j-1}\}} (C(Q)2^{(j-1)})^{p^*} \omega(x) d\mu(x) \\
&\le C \sum_j \bigg(\int_\Omega |Xf_{j-1}(x)|^p \omega(x)^{p/p^*} d\mu(x) \bigg)^{p^*/p} \\
&\le C \sum_j \bigg(\int_{\Omega_{j-1}} |Xf(x)|^p \omega(x)^{p/p^*} d\mu(x) \bigg)^{p^*/p} \\
&\le C \bigg(\int_\Omega |Xf(x)|^p \omega(x)^{p/p^*} d\mu(x) \bigg)^{p^*/p}.
\end{aligned}
$$

This completes the proof. □

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8 First, we choose the central ball B of the Boman chain domain Ω. The choice of such a central ball can be as given in, e.g., [12, 22].

We will deal with Theorem 1.8 in two cases.

Case 1 $p_- > 1$. By Lemma 4.1 and the basic property of $L^{q(\cdot)}(\Omega)$, it is not hard to obtain

$$\|f - f_B\|_{L^{q(\cdot)}(\Omega)} \le C\|I_1(Xf)\|_{L^{q(\cdot)}(\Omega)}. \tag{4.4}$$

Next, we will show that

$$\|f - f_\Omega\|_{L^{q(\cdot)}(\Omega)} \lesssim \|f - f_B\|_{L^{q(\cdot)}(\Omega)}.$$

In fact, by the triangle inequality,

$$\|f - f_\Omega\|_{L^{q(\cdot)}(\Omega)} \le \|f - f_B\|_{L^{q(\cdot)}(\Omega)} + \|f_B - f_\Omega\|_{L^{q(\cdot)}(\Omega)}.$$

We estimate the second term by Hölder's inequality:

$$
\begin{aligned}
&\|f_B - f_\Omega\|_{L^{q(\cdot)}(\Omega)} \\
&\quad = |f_B - f_\Omega| \|1\|_{L^{q(\cdot)}(\Omega)} \\
&\quad = \mu(\Omega)^{-1} \|f - f_B\|_{L^1(\Omega)} \|\chi\Omega\|_{L^{q(\cdot)}(\Omega)} \\
&\quad \le c \frac{\|\chi\Omega\|_{L^{q'(\cdot)}(\Omega)} \|\chi\Omega\|_{L^{q(\cdot)}(\Omega)}}{\mu(\Omega)} \|f - f_B\|_{L^{q(\cdot)}(\Omega)}.
\end{aligned}
$$

Finally, fix $\lambda = \mu(\Omega) + 1$. Then

$$\int_\Omega \left(\frac{\chi\Omega}{\lambda}\right)^{q(x)} d\mu(x) = \int_\Omega \lambda^{-q(x)} d\mu(x) \le \lambda^{-q_-}\mu(\Omega) \le \lambda^{-1}(\mu(\Omega)+1) = 1.$$

Therefore, $\|\chi\Omega\|_{L^{q(\cdot)}(\Omega)} \approx \|\chi\Omega\|_{L^{q'(\cdot)}(\Omega)} \le \mu(\Omega) + 1$. Then, by Lemma 3.1, we have

$$\|f - f_\Omega\|_{L^{q(\cdot)}(\Omega)} \le C\|Xf\|_{L^{p(\cdot)}(\Omega)}.$$

Case 2 $p_- = 1$. Define the family $\mathcal{F}$ to be all pairs $(|f - f_\Omega|, |Xf|)$ with $f \in \mathrm{Lip}(\Omega)$. Fix $q(\cdot) = p^*(\cdot)$, and $q_0 = Q'$. By the assumption, the maximal operator M is bounded on

$L^{(q(\cdot)/q_0)'}(\Omega)$. Therefore, by Lemma 4.2 (with $p = 1$) and Lemma 2.5, for all $f \in \mathrm{Lip}(\overline{\Omega})$,

$$\|f - f_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \le C\|Xf\|_{L^{p(\cdot)}(\Omega)},$$

provided the left-hand side is finite, but this is always the case.

In fact, the method we used in Case 2 also can be used to deal with Case 1, since the strong type inequality for the fractional integral operator implies the corresponding weak type inequality. □

5 The Higher Order Poincaré Inequalities on Stratified Groups

First, we recall the polynomials on stratified groups $\mathbb{G}$ by following Folland and Stein [8]. Let $X_1, \dots, X_k$ be the generators of the Lie algebra $\mathfrak{g}$, and $X_1, \dots, X_k, \dots, X_N$ be a basis of $\mathfrak{g}$. We define $d(X_j) = d_j$ as the length of X_j as a commutator, and arrange the order so that $1 \le d_1 \le \cdots \le d_N$. Thus, it is easy to see $d_j = 1$ for $j = 1, \dots, k$. Suppose $\xi_1, \dots, \xi_N$ are the dual basis of $\mathfrak{g}^*$, and let $\eta_i = \xi_i \circ \exp^{-1}$. Thus, $\eta_1, \dots, \eta_N$ are a system of global coordinates on $\mathbb{G}$. A function P on $\mathbb{G}$ is called a polynomial on $\mathbb{G}$ if $P \circ \exp$ is a polynomial on $\mathfrak{g}$. By this definition, $\eta_1, \dots, \eta_N$ are polynomials on $\mathbb{G}$ and generate the algebra of polynomials on $\mathbb{G}$. Therefore, every polynomial on $\mathbb{G}$ can be written uniquely as

$$P = \sum_I a_I \eta^I, \quad \eta^I = \eta_1^{i_1} \cdots \eta_N^{i_N}, \quad a_I \in \mathbb{R},$$

where all but finitely many of the coefficients a_I vanish. Clearly, η^I is homogeneous of degree $d(I) = \sum_{j=1}^N i_j d(i_j)$. If $P = \sum_I a_I \eta^I$, then we define the homogeneous degree (or order) of P to be $\max\{d(I) : a_I \neq 0\}$. If we consider $I = (i_1, \dots, i_k)$, $1 \le i_j \le k$, then $d(I) = |I|$.

Throughout this paper, we use $\mathcal{P}_k$ to denote the polynomials of homogeneous degree less than k for each positive integer k.

Let m be a positive integer, $1 < p_- \le p_+ < \infty$, and Ω be an open set in $\mathbb{G}$. The Sobolev space $W^{m,p(\cdot)}(\Omega)$ associated with the vector fields $X_1, \dots, X_l$ is defined to consist of all functions $f \in L^{p(\cdot)}$ with distributional derivatives $X^I f \in L^{p(\cdot)}(\Omega)$ for every X_I defined by

$$X^I = X_1^{i_1} \cdot X_2^{i_2} \cdot \dots \cdot X_N^{i_N}$$

with $d(I) \le m$. Here, we say that the distributional derivatives $X_I f$ exists and equals a locally integrable function g_I if for every $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega f X_I \phi dx = (-1)^{d(I)} \int_\Omega g_I \phi dx.$$

$W^{m,\,p(\cdot)}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{m,\,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \sum_{1 \le d(I) \le m} \|X^I f\|_{L^{p(\cdot)}(\Omega)}.$$

When $\Omega = \mathbb{G}$, we use $\|f\|_{m,p(\cdot)}$ to denote $\|f\|_{W^{m,p(\cdot)}(\mathbb{G})}$.

To prove Theorem 1.9, we first recall the higher order representation formulas [23], which is also very crucial for us to give the higher order result.

Lemma 5.1 *Let Ω be a weak Boman chain domain in $\mathbb{G}$ with a central ball B_0, and let $f \in C^m(\Omega)$, the class of m-th order differentiable functions. Then for any $m \le Q$, there is a* *polynomial $P_m(B_0, f)$ of order less than m such that for $x \in \Omega$,*

$$|f(x) - P_m(B_0, f)(x)| \le C \int_\Omega |X^m f(y)| \frac{d(x,y)^m}{|B(x, d(x,y))|} dy, \tag{5.1}$$

here, C is independent of f and Q is the homogeneous dimension.

Then, we give the proof of Theorem 1.9.

Proof of Theorem 1.9 By Lemma 5.1, for any $m \le Q$ and every $f \in C^m(\Omega)$, there is a polynomial $P_m(B_0, f)$ of order less than m such that, for $x \in \Omega$,

$$|f(x) - P_m(B_0, f)(x)| \le C \int_\Omega |X^m f(y)| \frac{d(x,y)^m}{|B(x, d(x,y))|} dy.$$

Thus, by Lemma 3.1, we obtain

$$\|f(x) - P_m(B_0, f)(x)\|_{p_m^*(\cdot)} \le C\|X^m f\|_{p(\cdot)}.$$

So, we prove the result of higher order Poincaré inequality on stratified groups for $f \in C^m(\Omega)$.

In fact, the existence of such polynomial $P_m(B_0, f)(x)$ can also be proved for $f \in W^{m,p}$ as done in [20, 21]. Therefore, the above proof goes through for $f \in W^{m,p}$ as well.

Finally, we write $P_m(B_0, f)(x)$ by P. This finishes the proof of Theorem 1.9. □

6 Sobolev Inequality on the Entire Stratified Lie Groups

In this section, we will prove a Sobolev inequality on the entire stratified Lie group $\mathbb{G}$, first, we need to recall the representation formula of Sobolev type [24].

Lemma 6.1 *Suppose that m is any positive integer and $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{G})$. Let Q be the homogeneous dimension of $\mathbb{G}$. Then for a.e. $x \in \mathbb{G}$,*

$$|f(x)| \le C \int_{\mathbb{G}} |X^m f(y)| \frac{d(x,y)^m}{|B(x, d(x,y))|} d\mu(y).$$

In the end, it is easy to prove Theorem 1.10 by Lemma 6.1 and the boundedness of the fractional integral operator on whole stratified Lie groups. Now we sketch the proof of the high-order Sobolev inequality on whole stratified Lie groups.

Proof of Theorem 1.10 By Lemma 6.1, we can immediately obtain

$$\|f\|_{L^{p_m^*(\cdot)}(\mathbb{G})} \le C\|I_m(X^m f)\|_{L^{p_m^*(\cdot)}(\mathbb{G})}. \tag{6.1}$$

Then, we apply Lemma 3.1 with $\alpha = m$ to (6.1),

$$\|I_m(X^m f)\|_{L^{p_m^*(\cdot)}(\mathbb{G})} \le C\|X^m f\|_{L^{p(\cdot)}(\mathbb{G})}.$$

The theorem is completed.

In fact, this theorem can also be proceed by induction. If $m = 1$, then this follows from representation formula and the boundedness of the fractional integral operator.

Now suppose that the result is true for some m. Fix $f \in W^{m+1,p(\cdot)}(\mathbb{G})$. For each I, $d(I) = 1$, $X^I f \in W^{m,p(\cdot)}(\mathbb{G})$, and

$$\begin{aligned}\sum_{d(I)=1} \|X^I f\|_{L^{p_m^*(\cdot)}(\mathbb{G})} &\le C \sum_{d(I)=1} \|X^m X^I f\|_{L^{p(\cdot)}(\mathbb{G})} \\ &\le C \sum_{d(I)=m+1} \|X^I f\|_{L^{p(\cdot)}(\mathbb{G})}.\end{aligned}$$

The Sobolev exponent corresponding to $p^*_m(\cdot)$ is

$$\frac{Qp^*_m(x)}{Q-p^*_m(x)}=\frac{Qp(x)}{Q-(m+1)p(x)}=p^*_{m+1}(x).$$

Since $p(\cdot)\in \mathrm{LH}(\mathbb{G})$, $p^*_m(\cdot)\in \mathrm{LH}(\mathbb{G})$, we can do it as we did when $m=1$ to get

$$\|f\|_{L^{p^*_{m+1}}(\mathbb{G})}\le \sum_{d(I)=1}\|X^I f\|_{L^{p^*_m}(\mathbb{G})}.$$

If we combine these, we get the desired inequality. □

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