SHARP SINGULAR ADAMS INEQUALITIES IN HIGH ORDER SOBOLEV SPACES

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Abstract. In this paper, we prove a version of weighted inequalities of exponential type for fractional integrals with sharp constants in any domain of finite measure in $\mathbb{R}^n$. Using this we prove a sharp singular Adams inequality in high order Sobolev spaces in bounded domain at critical case. Then we prove sharp singular Adams inequalities for high order derivatives on unbounded domains. Our results extend the singular Moser-Trudinger inequalities of first order in [4, 29, 24, 8] to the higher order Sobolev spaces $W^{m,\frac{n}{m}}$ and the results of [30] on Adams type inequalities in unbounded domains to singular case. Our singular Adams inequality on $W^{2,2}\left(\mathbb{R}^4\right)$ with standard Sobolev norm at the critical case settles a unsolved question remained in [37].

1. Introduction

Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$ be a smooth bounded domain, and $W_0^{1,n}(\Omega)$ be the completion of $C_0^{\infty}(\Omega)$ under the norm $\|u\|_{W_0^{1,n}(\Omega)}=\left[\int_\Omega\left(|u|^2+|\nabla u|^2\right)dx\right]^{1/2}$. The classical Moser-Trudinger inequality [27, 28, 34, 36] which plays an important role in analysis says that

$$\sup_{u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_n\leq 1}\frac{1}{|\Omega|}\int_\Omega\exp\left(\beta|u|^{\frac{n}{n-1}}\right)dx<+\infty$$

for any $\beta\leq\beta_n=n\omega_{n-1}^{\frac{1}{n-1}}$, where $\omega_{n-1}=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the area of the surface of the unit $n-$ball. Moreover, this constant β_n is sharp in the sense that if $\beta>\beta_n$, then supremum is infinity. Here and in the sequel, for any real number $p>1$, $\|\cdot\|_p$ denotes the L^p-norm with respect to the Lebesgue measure.

There is also another famous inequality in analysis: the Hardy inequality. Thus it is very natural to establish an interpolation of Hardy inequality and Moser-Trudinger inequality. Inspired by the following Hardy inequality [**?**]:

$$\left(\frac{n-1}{n}\right)^n\int_\Omega\frac{|u|^n}{|x|^n\left(\log\frac{R}{|x|}\right)^n}dx\leq\int_\Omega|\nabla u|^n$$

where $R\geq es\sup_\Omega|x|$, Adimurthi and Sandeep proved in [4] a singular Moser-Trudinger inequality with the sharp constant:

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Theorem A. *Let Ω be an open and bounded set in $\mathbb{R}^n$. There exists a constant $C_0 = C_0(n, |\Omega|) > 0$ such that*

$$\int_{\Omega} \frac{\exp\left(\beta |u|^{\frac{n}{n-1}}\right)}{|x|^{\alpha}} dx \leq C_0$$

for any $\alpha \in [0, n)$, $0 \leq \beta \leq \left(1 - \frac{\alpha}{n}\right) \beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_{\Omega} |\nabla u|^n dx \leq 1$. Moreover, this constant $\left(1 - \frac{\alpha}{n}\right) \beta_n$ is sharp in the sense that if $\beta > \left(1 - \frac{\alpha}{n}\right) \beta_n$, then the above inequality can no longer hold with some C_0 independent of u.

There is another improved Moser-Trudinger inequality on the disk in $\mathbb{R}^2$, which was recently proved and studied in [7, 26]:

$$\sup_{u \in W_0^{1,2}(B),\ \|\nabla u\|_2 \leq 1} \int_B \frac{\exp\left(4\pi |u|^2\right) - 1}{\left(1 - |x|^2\right)^2} dx < +\infty.$$

Very recently, Wang and Ye [35] proved an interesting Hardy-Moser-Trudinger inequality on the unit disk in $\mathbb{R}^2$, which improves the classical Moser-Trudinger inequality and the classical Hardy inequality at the same time. Namely, there exists a constant $C_0 > 0$ such that

$$\int_B e^{\frac{4\pi u^2}{H(u)}} dx \leq C_0 < \infty,\ \forall u \in C_0^{\infty}(B) \setminus \{0\},$$

where

$$H(u) = \int_B |\nabla u|^2 dx - \int_B \frac{u^2}{\left(1 - |x|^2\right)^2} dx.$$

We notice that when Ω has infinite volume, the usual Moser-Trudinger inequalities become meaningless. In the case $|\Omega| = +\infty$, the following modified Moser-Trudinger type inequality can be established:

Theorem B. *For all $\beta > 0$, $0 \leq \alpha < n$ and $u \in W^{1,n}(\mathbb{R}^n)$ $(n \geq 2)$, there holds*

$$\int_{\mathbb{R}^n} \frac{\phi\left(\beta |u|^{\frac{n}{n-1}}\right)}{|x|^{\alpha}} dx < \infty.$$

Furthermore, we have for all $\beta \leq \left(1 - \frac{\alpha}{n}\right) \beta_n$ and $\tau > 0$,

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^n} \frac{\phi\left(\beta |u|^{\frac{n}{n-1}}\right)}{|x|^{\alpha}} dx < \infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$$

$$\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^n} \left(|\nabla u|^n + \tau |u|^n\right) dx\right)^{1/n}.$$

Moreover, this constant $\left(1 - \frac{\alpha}{n}\right) \beta_n$ is sharp in the sense that if $\beta > \left(1 - \frac{\alpha}{n}\right) \beta_n$, then the supremum is infinity.

The above modified Moser-Trudinger type inequality when $\alpha = 0$ was established by B. Ruf [29] in dimension two and Y.X. Li and Ruf [24] in general dimension. It was then extended to the singular case $0 \le \alpha < n$ by Adimurthi and Yang [8]. Indeed, such type of inequality on unbounded domains in the subcritical case $\beta < \beta_n$ ($\alpha = 0$) was first established by D. Cao [12] in dimension two and by Adachi and Tanaka [1] in high dimension.

In the case of compactly supported functions, D. Adams [2] extended the original Moser-Trudinger inequality to the higher order space $W_0^{m,\frac{n}{m}}(\Omega)$. In fact, Adams proved the following inequality:

Theorem C. *There exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \le 1$, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|u(x)|^{\frac{n}{n-m}})dx \le C_0$$

for all $\beta \le \beta(n,m)$ where

$$\beta(n,\ m)\ = \begin{cases} \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\ \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, for any $\beta > \beta(n,m)$, the integral can be made as large as possible.

Note that $\beta(n,1)$ coincides with Moser's value of β_n and $\beta(2m,m) = 2^{2m}\pi^m\Gamma(m+1)$ for both odd and even m. Here, we use the symbol $\nabla^m u$, where m is a positive integer, to denote the $m-$th order gradient for $u \in C^m$, the class of $m-$th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}}u & \text{for } m \text{ even} \\ \nabla\triangle^{\frac{m-1}{2}}u & \text{for } m \text{ odd} \end{cases}.$$

where ∇ is the usual gradient operator and $\triangle$ is the Laplacian. We use $||\nabla^m u||_p$ to denote the L^p norm ($1 \le p \le \infty$) of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $\left(\sum_{j=0}^{k}||\nabla^j u||^p_{L^p(\Omega)}\right)^{1/p}$.

Recently, in the setting of the Sobolev space with homogeneous Navier boundary conditions $W_N^{m,\frac{n}{m}}(\Omega)$:

$$W_N^{m,\frac{n}{m}}(\Omega) := \left\{u \in W^{m,\frac{n}{m}} : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \le j \le \left[\frac{m-1}{2}\right]\right\},$$

the Adams inequality was extended by Tarsi [32]. Note that $W_N^{m,\frac{n}{m}}(\Omega)$ contains the Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$ as a closed subspace.

The Adams type inequality on Sobolev spaces $W_0^{m,\frac{n}{m}}(\Omega)$ when Ω has infinite volume and m is an even integer was studied recently by Ruf and Sani [30]. In fact, they proved the following

Theorem D. *If m is an even integer less than n, then there exists a constant $C_{m,n}>0$ such that for any domain $\Omega\subseteq\mathbb{R}^n$*

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\|u\|_{m,n}\leq 1}\int_{\Omega}\phi\left(\beta_0(n,m)|u|^{\frac{n}{n-m}}\right)dx\leq C_{m,n}$$

where

$$\beta_0(n,m)=\frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}}2^m\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\right]^{\frac{n}{n-m}},$$

$$\phi(t)=e^t-\sum_{j=0}^{j_{\frac{n}{m}}-2}\frac{t^j}{j!}$$

$$j_{\frac{n}{m}}=\min\left\{j\in\mathbb{N}:j\geq\frac{n}{m}\right\}\geq\frac{n}{m}.$$

Moreover, this inequality is sharp in the sense that if we replace $\beta_0(n,m)$ by any larger β, then the above supremum will be infinity.

In the above result, Ruf and Sani used the norm

$$\|u\|_{m,n}=\left\|(-\Delta+I)^{\frac{m}{2}}u\right\|_{\frac{n}{m}}$$

which is equivalent to the standard Sobolev norm

$$\|u\|_{W^{m,\frac{n}{m}}}=\left(\|u\|_{\frac{n}{m}}^{\frac{n}{m}}+\sum_{j=1}^{m}\left\|\nabla^j u\right\|_{\frac{n}{m}}^{\frac{n}{m}}\right)^{\frac{m}{n}}.$$

In particular, if $u\in W_0^{m,\frac{n}{m}}(\Omega)$ or $u\in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, then $\|u\|_{W^{m,\frac{n}{m}}}\leq\|u\|_{m,n}$.

Because the result of Ruf and Sani [30] only treats the case when m is even, thus it leaves an open question if Ruf and Sani's theorem still holds when m is odd. Recently, the authors of [23] have established the results of Adams type inequalities on unbounded domains when m is odd. More precisely, the first result of [23] is as follows:

Theorem E. *Let m be an odd integer less than n: $m=2k+1$, $k\in\mathbb{N}$. There holds*

$$\sup_{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\left\|\nabla(-\Delta+I)^k u\right\|_{\frac{n}{m}}^{\frac{n}{m}}+\left\|(-\Delta+I)^k u\right\|_{\frac{n}{m}}^{\frac{n}{m}}\leq 1}\int_{\mathbb{R}^n}\phi\left(\beta(n,m)|u|^{\frac{n}{n-m}}\right)dx<\infty.$$

Moreover, the constant $\beta(n,m)$ is sharp.

In the special case $n=2m$, we have the following stronger results in [23]:

Theorem F. *Let $m=2k+1$, $k\in\mathbb{N}$. For all $\tau>0$, there holds*

$$\sup_{u\in W^{m,2}(\mathbb{R}^{2m}),\left\|\nabla(-\Delta+\tau I)^k u\right\|_2^2+\tau\left\|(-\Delta+\tau I)^k u\right\|_2^2\leq 1}\int_{\mathbb{R}^{2m}}\left(e^{\beta(2m,m)u^2}-1\right)dx<\infty.$$

Moreover, the constant $\beta(2m,m)$ is sharp in the sense that if we replace $\beta(2m,m)$ by any $\beta>\beta(2m,m)$, then the supremum is infinity.

The result of [30] (stated as Theorem D above) for m being even were also extended recently using the standard Sobolev norm by Yang in the special case $n=4$ and $m=2$

[37] and by the authors [23] to the case $n = 2m$ for all m being both odd and even. More precisely, the following has been established by the authors in [23]:

Theorem G. *Let $m \geq 2$ be an integer. For all constants $a_0 = 1, a_1, ..., a_m > 0$, there holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \int_{\mathbb{R}^{2m}} \left(\sum_{j=0}^{m} a_{m-j} |\nabla^j u|^2 \right) dx \leq 1} \int_{\mathbb{R}^{2m}} \left[\exp\left(\beta\left(2m, m\right) |u|^2 \right) - 1 \right] dx < \infty.$$

Furthermore this inequality is sharp, i.e., if $\beta(2m, m)$ is replaced by any $\beta > \beta(2m, m)$, then the supremum is infinite.

As a corollary of the above theorem, we have the following Adams type inequality with the standard Sobolev norm:

Theorem H. *Let $m \geq 1$ be an integer. There holds*

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|u\|_{W^{m,2}} \leq 1} \int_{\mathbb{R}^{2m}} \left[\exp\left(\beta\left(2m, m\right) |u|^2 \right) - 1 \right] dx < \infty.$$

Furthermore this inequality is sharp, i.e., if $\beta(2m, m)$ is replaced by any $\alpha > \beta(2m, m)$, then the supremum is infinite.

Moser-Trudinger type inequalities and Adams type inequalities have important applications in geometric analysis and partial differential equations, especially in the study of the exponential growth partial differential equations where the nonlinear term behaves like $e^{\alpha |u|^{\frac{n}{n-m}}}$ as $|u| \to \infty$. There has been a vast literature in this direction. We refer the interested reader to [10], [13], [3], [4], [6], [8], [5], [15], [16], [14], [21, 22] and the references therein.

In this paper, we will first establish a sharp inequality of exponential type with weights $\frac{1}{|x|^\alpha}$ for the fractional integrals.

Theorem 1.1. *Let $1 < p < \infty$, $0 \leq \alpha < n$ and $\Omega \subset \mathbb{R}^n$ be an open set with $|\Omega| < \infty$. Then there is a constant $c_0 = c_0(p, \Omega)$ such that for all $f \in L^p(\mathbb{R}^n)$ with support contained in Ω,*

$$\int_{\Omega} \frac{\exp\left(\left(1 - \frac{\alpha}{n}\right) \frac{n}{\omega_{n-1}} \left| \frac{I_\gamma * f(x)}{\|f\|_p} \right|^{p'} \right)}{|x|^\alpha} dx \leq c_0,$$

where $\gamma = n/p$ and $I_\gamma * f(x) = \int |x - y|^{\gamma - n} f(y) dy$ is the Riesz potential of order γ.

Next, we will establish a version of singular Adams inequality on bounded domains. More precisely, we will prove that:

Theorem 1.2. *Let $0 \leq \alpha < n$ and Ω be a bounded domain in $\mathbb{R}^n$. Then for all $0 \leq \beta \leq \beta_{\alpha,n,m} = \left(1 - \frac{\alpha}{n}\right) \beta(n, m)$, we have*

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}} \leq 1} \int_{\Omega} \frac{e^{\beta |u|^{\frac{n}{n-m}}}}{|x|^\alpha} dx < \infty. \tag{1.1}$$

When $\beta > \beta_{\alpha,n,m}$, the supremum is infinite. Moreover, when m is an even number, the Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$ in the above supremum can be replaced by a larger Sobolev space $W_N^{m,\frac{n}{m}}(\Omega)$.

Using the above Theorem 1.2, we will then set up the singular Adams inequality for the space $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ when m is an even integer number:

Theorem 1.3. *Let $0 \leq \alpha < n$, $m > 0$ be an even integer less than n. Then for all $0 \leq \beta \leq \beta_{\alpha,n,m} = \left(1 - \frac{\alpha}{n}\right)\beta_0(n,m)$, we have*

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \left\|(-\Delta+I)^{\frac{m}{2}}u\right\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \frac{\phi\left(\beta\left|u\right|^{\frac{n}{n-m}}\right)dx}{\left|x\right|^{\alpha}}dx < \infty \tag{1.2}$$

where $\phi(t) = e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2}\frac{t^j}{j!}$. Moreover, when $\beta > \beta_{\alpha,n,m}$, the supremum is infinite.

Finally, in the special case $n = 2m = 4$, we will prove a singular Adams inequality in the spirit of Theorem G above.

Theorem 1.4. *Let $0 \leq \alpha < 4$. Assume that $\tau > 0$ and $\sigma > 0$ are any two positive constants. Then for all $0 \leq \beta \leq \beta_\alpha = \left(1 - \frac{\alpha}{4}\right)32\pi^2$, we have*

$$\sup_{u \in W^{2,2}(\mathbb{R}^4), \int_{\mathbb{R}^4}\left(|\Delta u|^2 + \tau|\nabla u|^2 + \sigma|u|^2\right) \leq 1} \int_{\mathbb{R}^4} \frac{\left(e^{\beta u^2} - 1\right)}{\left|x\right|^{\alpha}}dx < \infty. \tag{1.3}$$

Moreover, when $\beta > \beta_\alpha$, the supremum is infinite.

As we can see, when $\alpha = 0$, this theorem is already included in Theorem G. When $0 < \alpha < 4$, we note that the above inequality (1.3) for the subcritical case $\beta < \beta_\alpha = \left(1 - \frac{\alpha}{4}\right)32\pi^2$ was proved in [37]. However, the critical case $\beta = \left(1 - \frac{\alpha}{4}\right)32\pi^2$ is much harder to prove. Thus, our Theorem 1.4 in the critical case settles a unsolved question remained in [37].

Our paper is organized as follows: In Section 2, we give some preliminaries. Section 3 deals with the sharp weighted inequality of exponential type for fractional integrals (Theorem 1.1). The singular Adams inequality for the bounded domains (Theorem 1.2) will be proved in Section 4. Theorem 1.2 will be used to prove Theorem 1.3 and Theorem 1.4 in Section 5.

2. Some preliminaries

In this section, we provide some preliminaries. For $u \in W^{m,p}(\Omega)$ with $1 \leq p < \infty$, we will denote by $\nabla^j u$, $j \in \{1, 2, ..., m\}$, the $j-th$ order gradient of u, namely

$$\nabla^j u = \begin{cases} \triangle^{\frac{j}{2}}u & \text{for } j \text{ even} \\ \nabla \triangle^{\frac{j-1}{2}}u & \text{for } j \text{ odd} \end{cases}.$$

We now introduce the Sobolev space of functions with homogeneous Navier boundary conditions:

$$W_N^{m,\frac{n}{m}}(\Omega) := \left\{ u \in W^{m,\frac{n}{m}}(\Omega) : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \le j \le \left[\left[\frac{m-1}{2}\right]\right] \right\}.$$

It is easy to see that $W_N^{m,\frac{n}{m}}(\Omega)$ contains $W_0^{m,\frac{n}{m}}(\Omega)$ as a closed subspace. We also define

$$W_{rad}^{m,\frac{n}{m}}(B_R) := \left\{ u \in W^{m,\frac{n}{m}}(B_R) : u(x) = u(|x|) \text{ a.e. in } B_R \right\},$$
$$W_{N,rad}^{m,\frac{n}{m}}(B_R) = W_N^{m,\frac{n}{m}}(B_R) \cap W_{rad}^{m,\frac{n}{m}}(B_R)$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ is a ball in $\mathbb{R}^n$.

Next, we will discuss the iterated comparison principle. Let Ω be a bounded domain in $\mathbb{R}^n$ and B_R be an open ball with radius $R > 0$ centered at 0 such that $|\Omega| = |B_R|$. Let $u : \Omega \to \mathbb{R}$ be a measurable function. The distribution function of u is defined by

$$\mu_u(t) = |\{x \in \Omega |\, |u(x)| > t\}| \;\; \forall t \ge 0.$$

The decreasing rearrangement of u is defined by

$$u^*(s) = \inf\{t \ge 0 : \mu_u(t) < s\} \;\; \forall s \in [0, |B_R|],$$

and the spherically symmetric decreasing rearrangement of u by

$$u^\#(x) = u^*(\sigma_n |x|^n) \;\; \forall x \in B_R.$$

We have that $u^\#$ is the unique nonnegative integrable function which is radially symmetric, nonincreasing and has the same distribution function as $|u|$.

Now, we introduce the Trombetti and Vazquez iterated comparision principle [33]: let $c > 0$ and u be a weak solution of

$$\begin{cases} -\Delta u + cu = f \text{ in } B_R \\ u \in W_0^{1,2}(B_R) \end{cases} \tag{2.1}$$

where $f \in L^{\frac{2n}{n+2}}(B_R)$. We have the following result that can be found in [33] (Inequality (2.20)):

Proposition 2.1. *If u is a nonnegative weak solution of (2.1) then*

$$-\frac{du^*}{ds}(s) \le \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s (f^* - cu^*)\, dt, \;\; \forall s \in (0, |B_R|). \tag{2.2}$$

Now, we consider the problem

$$\begin{cases} -\Delta v + cv = f^\# \text{ in } B_R \\ v \in W_0^{1,2}(B_R) \end{cases} \tag{2.3}$$

Due to the radial symmetry of the equation, the unique solution v of (2.3) is radially symmetric and we have

$$-\frac{d\widehat{v}}{ds}(s) = \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s (f^* - c\widehat{v})\, dt, \;\; \forall s \in (0, |B_R|) \tag{2.4}$$

where $\widehat{v}(\sigma_n |x|^n) := v(x)$. We have the following comparison of integrals in balls that again can be found in [33]:

Proposition 2.2. *Let u, v be weak solutions of (2.1) and (2.3) respectively. For every $r \in (0, R)$ we have*

$$\int_{B_r} u^{\#} dx \leq \int_{B_r} v dx.$$

and for every convex nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ we have

$$\int_{B_r} \phi(|u|)\, dx \leq \int_{B_r} \phi(|v|)\, dx.$$

Next, we adapt the comparison principle to the polyharmonic operator. Let $u \in W^{m,2}(B_R)$ be a weak solution of

$$\begin{cases} (-\Delta + cI)^k u = f \text{ in } B_R \\ u \in W_N^{2k,2}(B_R) \end{cases} \tag{2.5}$$

where $m = 2k$ and $f \in L^{\frac{2n}{n+2}}(B_R)$. If we consider the problem

$$\begin{cases} (-\Delta + cI)^k v = f^{\#} \text{ in } B_R \\ v \in W_N^{2k,2}(B_R) \end{cases} \tag{2.6}$$

then we have the following comparison of integrals in balls:

Proposition 2.3. *Let u, v be weak solutions of the polyharmonic problems (2.5) and (2.6) respectively. Then for every $r \in (0, R)$ we have*

$$\int_{B_r} u^{\#} dx \leq \int_{B_r} v dx.$$

Proof. The proof adapts the comparison principle as in [33] and [30]. We include a proof for its completeness. Since equations in (2.5) and (2.6) are considered with homogeneous Navier boundary conditions, they may be rewritten as second order systems:

$$(P1)\begin{cases} -\Delta u_1 + cu_1 = f \text{ in } B_R \\ u_1 \in W_0^{1,2}(B_R) \end{cases} \quad (Pi)\begin{cases} -\Delta u_i + cu_i = u_{i-1} \text{ in } B_R \\ u_i \in W_0^{1,2}(B_R) \end{cases} \quad i \in \{2, 3, ..., k\}$$

$$(Q1)\begin{cases} -\Delta v_1 + cv_1 = f^{\#} \text{ in } B_R \\ v_1 \in W_0^{1,2}(B_R) \end{cases} \quad (Qi)\begin{cases} -\Delta v_i + cv_i = v_{i-1} \text{ in } B_R \\ v_i \in W_0^{1,2}(B_R) \end{cases} \quad i \in \{2, 3, ..., k\}$$

where $u_k = u$ and $v_k = v$. Thus we have to prove that for every $r \in (0, R)$

$$\int_{B_r} u_k^{\#} dx \leq \int_{B_r} v_k dx. \tag{2.7}$$

By the above proposition (Proposition 2.2), we have

$$\int_{B_r} u_1^{\#} dx \leq \int_{B_r} v_1 dx.$$

Now, if we assume that

$$\int_{B_r} u_j^{\#} dx \leq \int_{B_r} v_j dx \text{ for all } j = 1, ..., i,$$

we will prove that

$$\int_{B_r} u_{i+1}^{\#} dx \leq \int_{B_r} v_{i+1} dx.$$

With no loss of generality, we may assume that $u_{i+1} \geq 0$. In fact, let $\overline{u}_{i+1}$ be a weak solution of

$$\begin{cases} -\Delta \overline{u}_{i+1} + c\overline{u}_{i+1} = |u_i| \text{ in } B_R \\ \overline{u}_{i+1} \in W_0^{1,2}(B_R) \end{cases}$$

then the maximum principle implies that $\overline{u}_{i+1} \geq 0$ and $\overline{u}_{i+1} \geq |u_{i+1}|$ in B_R.

Since u_{i+1} is a nonnegative weak solution of $(P(i+1))$ and v_{i+1} is a nonnegative weak solution of $(Q(i+1))$, then by Proposition 2.1 we have

$$-\frac{du_{i+1}^*}{ds}(s) \leq \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s \left(u_i^* - cu_{i+1}^*\right) dt, \ \forall s \in (0, |B_R|),$$

$$-\frac{d\widehat{v}_{i+1}}{ds}(s) = \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s \left(\widehat{v}_i - c\widehat{v}_{i+1}\right) dt, \ \forall s \in (0, |B_R|)$$

Thus for all $s \in (0, |B_R|)$, we have

$$\frac{d\widehat{v}_{i+1}}{ds}(s) - \frac{du_{i+1}^*}{ds}(s) - \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s \left(c\widehat{v}_{i+1} - cu_{i+1}^*\right) dt \leq \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s \left(u_i^* - \widehat{v}_i\right) dt.$$

Thanks to the induction hypotheses, we get that

$$\int_0^s \left(u_i^* - \widehat{v}_i\right) dt \leq 0 \ , \forall s \in (0, |B_R|)$$

and then

$$\frac{d\widehat{v}_{i+1}}{ds}(s) - \frac{du_{i+1}^*}{ds}(s) - \frac{s^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} \int_0^s \left(c\widehat{v}_{i+1} - cu_{i+1}^*\right) dt \leq 0.$$

Setting

$$y(s) = \int_0^s \left(\widehat{v}_{i+1} - u_{i+1}^*\right) dt \ \forall s \in (0, |B_R|)$$

we get

$$\begin{cases} y'' - \frac{cs^{\frac{2}{n}-2}}{n^2\sigma_n^{2/n}} y \leq 0, \ \forall s \in (0, |B_R|) \\ y(0) = y'(|B_R|) = 0 \end{cases}.$$

By maximum principle, we have that $y \geq 0$ which is what we need. □

From the above proposition, we have the following corollary:

Corollary 2.1. *Let u, v be weak solutions of the polyharmonic problems (2.5) and (2.6) respectively. Then for every convex nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ we have*

$$\int_{B_r} \phi(|u|)\, dx \leq \int_{B_r} \phi(|v|)\, dx.$$

Now, we state the following known result from [9, 19]:

Lemma 2.1. *Let $f(s),\ g(s)$ be measurable, positive functions such that*

$$\int_{[0,r]} f(s)ds \le \int_{[0,r]} g(s)ds,\ r \in [0,R]\,;$$

if $h(s) \ge 0$ is a decreasing function then

$$\int_{[0,r]} f(s)h(s)ds \le \int_{[0,r]} g(s)h(s)ds,\ r \in [0,R]\,.$$

Then we have the following:

Proposition 2.4. *Let $u,\ v$ be weak solutions of (2.5) and (2.7) respectively. For every convex nondecreasing function $\phi : [0,+\infty) \to [0,+\infty)$ we have*

$$\int_{B_R} \frac{\phi\left(|u|\right)}{|x|^{\alpha}} dx \le \int_{B_R} \frac{\phi\left(|v|\right)}{|x|^{\alpha}} dx,\ 0 \le \alpha < n.$$

Next, we provide some Radial Lemmas which will be used in the proof of Theorem 1.2. See [11, 18, 23, 30, 32]:

Lemma 2.2. *If $u \in W^{1,\frac{n}{m}}\left(\mathbb{R}^n\right)$ then*

$$|u(x)| \le \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \left\|u\right\|_{W^{1,\frac{n}{m}}}$$

for a.e. $x \in \mathbb{R}^n$.

Lemma 2.3. *If $u \in L^p\left(\mathbb{R}^n\right),\ 1 \le p < \infty$, is a radial nonincreasing function, then*

$$|u(x)| \le \left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{p}} \frac{1}{|x|^{\frac{n}{p}}} \left\|u\right\|_{L^p(\mathbb{R}^n)}$$

for a.e. $x \in \mathbb{R}^n$.

3. Proof of Theorem 1.1: Sharp inequality of exponential type for fractional integrals

We begin with proving the following result that is a modified version of the key lemma used to prove the Adams inequality in [2]:

Lemma 3.1. *Let $0 < \alpha \le 1$, $1 < p < \infty$ and $a(s,t)$ be a non-negative measurable function on $(-\infty,\infty) \times [0,\infty)$ such that (a.e.)*

$$a(s,t) \le 1,\ \text{when } 0 < s < t, \tag{3.1}$$

$$\sup_{t>0}\left(\int_{-\infty}^{0} + \int_{t}^{\infty} a(s,t)^{p'} ds\right)^{1/p'} = b < \infty. \tag{3.2}$$

Then there is a constant $c_0 = c_0(p,b)$ such that if for $\phi \geq 0$,

$$\int_{-\infty}^{\infty} \phi(s)^p ds \leq 1, \tag{3.3}$$

then

$$\int_{0}^{\infty} e^{-F_\alpha(t)} dt \leq c_0 \tag{3.4}$$

where

$$F_\alpha(t) = \alpha t - \alpha \left(\int_{-\infty}^{\infty} a(s,t)\phi(s) ds \right)^{p'}. \tag{3.5}$$

We sketch a proof here.

Proof. First, we have

$$\int_{0}^{\infty} e^{-F_\alpha(t)} dt = \int_{-\infty}^{\infty} |E_\lambda| \, e^{-\lambda} d\lambda. \tag{3.6}$$

where $E_\lambda = \{t \geq 0 : F_\alpha(t) \leq \lambda\}$.

We will separate the proof into two steps.

Step 1: There is a constant $c = c(p,b,\alpha) > 0$ such that $F_\alpha(t) \geq -c$ for all $t \geq 0$.

Indeed, we will show that if $E_{\alpha\lambda} \neq \emptyset$, then $\lambda \geq -c$, and furthermore that if $t \in E_{\alpha\lambda}$, then

$$\left(b^{p'} + t\right)^{1/p} \left(\int_{t}^{\infty} \phi(s)^p ds \right)^{1/p} \leq A_1 + B_1 |\lambda|^{1/p}. \tag{3.7}$$

In fact, if $E_{\alpha\lambda} \neq \emptyset$ and $t \in E_{\alpha\lambda}$, then

$$\begin{aligned} t - \lambda &\leq t - \frac{F_\alpha(t)}{\alpha} \\ &\leq \left(\int_{-\infty}^{\infty} a(s,t)\phi(s) ds \right)^{p'} \end{aligned}$$

Repeating the argument as in the proof of Lemma 1 in [2], we then have completed Step 1.

Step 2: $|E_\lambda| \leq A|\lambda| + B$, for constants A and B depending only on p, b and α.

The proof of Step 2 is very similar to that in [2]. Thus we finish the proof of the Lemma. □

Using the above lemma, we can provide the

Proof of Theorem 1.1:

Set $u(x) = I_{n/p} * f(x)$, for $f \geq 0$. We use the notations $g(x) = |x|^{\gamma - n}$ and $u^{**}(t) = \frac{1}{t}\int_0^t u^*(s)ds$. Then by O'Neil's lemma, we have that

$$\begin{aligned} u^*(t) \leq u^{**}(t) &\leq t f^{**}(t) g^{**}(t) + \int\limits_0^t f^*(s) g^*(s) ds \\ &= \left(\frac{\omega_{n-1}}{n}\right)^{1/p'} \left(p t^{-1/p'} \int\limits_0^t f^*(s) ds + \int\limits_t^{|\Omega|} f^*(s) s^{-1/p'} ds \right). \end{aligned}$$

Now, we change variables by setting $\phi(s) = |\Omega|^{1/p} f^*(|\Omega| e^{-s}) e^{-s/p}$, so that

$$\begin{aligned} \int_\Omega f(x)^p dx &= \int\limits_0^{|\Omega|} f^*(t)^p dt \\ &= \int\limits_0^\infty \phi(s)^p ds. \end{aligned}$$

By the Hardy-Littlewood inequality, note that with $h(x) = \frac{1}{|x|^\alpha}$, then $h^*(t) = \left(\frac{\sigma_n}{t}\right)^{\frac{\alpha}{n}}$, we have

$$\begin{aligned} &\int_\Omega \frac{\exp\left(\left(1 - \frac{\alpha}{n}\right) \frac{n}{\omega_{n-1}} |u(x)|^{p'}\right)}{|x|^\alpha} dx \\ &\leq \sigma_n^{\frac{\alpha}{n}} \int\limits_0^{|\Omega|} \frac{e^{\left(1-\frac{\alpha}{n}\right)\frac{n}{\omega_{n-1}} u^*(t)^{p'}}}{t^{\frac{\alpha}{n}}} \\ &= \sigma_n^{\frac{\alpha}{n}} |\Omega|^{1-\frac{\alpha}{n}} \int\limits_0^\infty \exp\left[\left(1 - \frac{\alpha}{n}\right) \frac{n}{\omega_{n-1}} u^*\left(|\Omega| e^{-s}\right)^{p'} - \left(1 - \frac{\alpha}{n}\right) s\right] ds \\ &\leq \sigma_n^{\frac{\alpha}{n}} |\Omega|^{1-\frac{\alpha}{n}} \times \\ &\int\limits_0^\infty \exp\left[\left(1 - \frac{\alpha}{n}\right) \left(p \left(|\Omega| e^{-s}\right)^{-\frac{1}{p'}} \int\limits_0^{|\Omega| e^{-s}} f^*(z) dz + \int\limits_{|\Omega| e^{-s}}^{|\Omega|} f^*(z) z^{-\frac{1}{p'}} dz \right)^{p'} - \left(1 - \frac{\alpha}{n}\right) s\right] ds \\ &= \sigma_n^{\frac{\alpha}{n}} |\Omega|^{1-\frac{\alpha}{n}} \int\limits_0^\infty \exp\left[\left(1 - \frac{\alpha}{n}\right) \left(p e^{s/p'} \int\limits_s^\infty \phi(w) e^{-\frac{w}{p'}} dw + \int\limits_0^s \phi(w) \right)^{p'} - \left(1 - \frac{\alpha}{n}\right) s\right] ds \\ &= \sigma_n^{\frac{\alpha}{n}} |\Omega|^{1-\frac{\alpha}{n}} \int\limits_0^\infty \exp\left[-F_{\left(1-\frac{\alpha}{n}\right)}(s)\right] ds. \end{aligned}$$

where $F_{\left(1-\frac{\alpha}{n}\right)}(s)$ is as in Lemma 3.1 with

$$a(s,t)=\begin{cases} 1 \ \text{ for } 0<s<t \\ pe^{(t-s)/p'} \ \text{ for } t<s<\infty \\ 0 \ \text{ for } -\infty<s\leq 0 \end{cases}.$$

Thus it suffices to prove that

$$\int_0^\infty \phi(s)^p ds \leq 1 \text{ implies } \int_0^\infty \exp\left[-F_{\left(1-\frac{\alpha}{n}\right)}(s)\right] ds \leq c_0,$$

but this follows from Lemma 3.1 immediately.

4. Proof of Theorem 1.2: A singular Adams inequality on bounded domains

First, we will prove that

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}}\leq 1} \int_\Omega \frac{e^{\beta_{\alpha,n,m}|u|^{\frac{n}{n-m}}}}{|x|^\alpha}dx<\infty. \tag{4.1}$$

To do that, it suffices to dominate an arbitrary C^m function with compact support by a Riesz potential in such a way that the constants are precise. This can be done as in [2] through the following lemma:

Lemma 4.1. *Let $u\in C_0^\infty(\mathbb{R}^n)$. Set $p=\frac{n}{m}$ and $p'=\frac{n}{n-m}$. Then if m is an odd positive integer,*

$$u(x)=(-1)^{\frac{m-1}{2}}\left(\frac{\omega_{n-1}\beta(n,m)}{n}\right)^{-1/p'}\times\int_{\mathbb{R}^n}|x-y|^{m-1-n}(x-y)\cdot\nabla^m u(y)dy$$

and for m an even positive integer

$$u(x)=(-1)^{\frac{m}{2}}\left(\frac{\omega_{n-1}\beta(n,m)}{n}\right)^{-1/p'}\times\int_{\mathbb{R}^n}|x-y|^{m-n}\nabla^m u(y)dy$$

Proof of Theorem 1.2:

It is clear that from Lemma 4.1, we have $\left(\frac{\omega_{n-1}}{n}\right)\beta(n,m)\left|u(x)\right|^{p'}\leq\left[I_m*\left|\nabla^m u\right|(x)\right]^{p'}$ and then we apply Theorem 1.1. This proves the first part of Theorem 1.2.

To show the second part of Theorem 1.2. Now, suppose m is even: $m=2k,\ k\in\mathbb{N}$, we will prove that

$$\sup_{u\in W_N^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}}\leq 1} \int_\Omega \frac{e^{\beta_{\alpha,n,m}|u|^{\frac{n}{n-m}}}}{|x|^\alpha}dx<\infty \tag{4.2}$$

By a density argument, it is enough to prove that

$$\sup_{u\in C_N^\infty(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}}\leq 1} \int_\Omega \frac{e^{\beta_{\alpha,n,m}|u|^{\frac{n}{n-m}}}}{|x|^\alpha}dx<\infty$$

where

$$C_N^{\infty}(\Omega)=\left\{u\in C^{\infty}(\Omega)\cap C^{m-2}\left(\overline{\Omega}\right):u|_{\partial\Omega}=\Delta^{j}u|_{\partial\Omega}=0,\ 1\leq j\leq\left[\frac{m-1}{2}\right]\right\}.$$

Let $u\in C_N^{\infty}(\Omega)$ be such that $\|\nabla^m u\|_{\frac{n}{m}}=\left\|\Delta^k u\right\|_{\frac{n}{m}}\leq 1$ and set $f:=\Delta^k u$ in Ω. Then u is a solution of the Navier boundary value problem

$$\left\{\begin{array}{c}\Delta^k u=f\text{ in }\Omega\\ u=\Delta^j u=0\text{ on }\partial\Omega,\ j\in\{1\leq j<k\}\end{array}\right..$$

Now, we extend f by zero outside Ω

$$\overline{f}(x)=\left\{\begin{array}{l}f(x),\ x\in\Omega\\ 0,\ x\in\mathbb{R}^n\setminus\Omega\end{array}\right..$$

Define

$$\overline{u}=\left(\frac{n}{\omega_{n-1}\beta(n,m)}\right)^{\frac{n-m}{n}}I_m*\left|\overline{f}\right|\text{ in }\mathbb{R}^n,$$

so that we have $(-1)^k\Delta^k u=\left|\overline{f}\right|$ in $\mathbb{R}^n$. It's clear that $\overline{u}\geq 0$ in $\mathbb{R}^n$ and

$$\beta(n,m)\left|\overline{u}\right|^{\frac{n}{n-m}}\leq\frac{n}{\omega_{n-1}}\left(\frac{I_m*\left|\overline{f}\right|}{\|f\|_{\frac{n}{m}}}\right)^{\frac{n}{n-m}}\text{ in }\mathbb{R}^n.$$

It can be proved that $\overline{u}\geq|u|$ (see [30]) and then

$$\begin{aligned}\int_{\Omega}\frac{e^{\beta_{\alpha,n,m}|u|^{\frac{n}{n-m}}}}{|x|^{\alpha}}dx&\leq\int_{\Omega}\frac{e^{\beta_{\alpha,n,m}|\overline{u}|^{\frac{n}{n-m}}}}{|x|^{\alpha}}dx\\&\leq\int_{\Omega}\frac{\exp\left(\left(1-\frac{\alpha}{n}\right)\frac{n}{\omega_{n-1}}\left|\frac{I_\beta*\overline{f}(x)}{\left\|\overline{f}\right\|_{\frac{n}{m}}}\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}}dx\end{aligned}$$

By Theorem 1.1, (4.2) follows.

Moreover, it can be checked that the sequence of test functions which gives the sharpness of Adams' inequality in bounded domains [2] also gives the sharpness of $\beta_{\alpha,n,m}$. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3 and Theorem 1.4

5.1. **Proof of Theorem 1.3.**

Proof. Suppose that $m=2k,\ k\in\mathbb{N}$. Let $u\in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, $\left\|(-\Delta+I)^k u\right\|_{\frac{n}{m}}\leq 1$, by the density of $C_0^{\infty}(\mathbb{R}^n)$ in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$, without loss of generality, we can find a sequence of functions $u_l\in C_0^{\infty}(\mathbb{R}^n)$ such that $u_l\to u$ in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\left|(-\Delta+I)^k u_l\right|^{\frac{n}{m}}dx\leq 1$ and suppose that $\mathrm{supp}u_l\subset B_{R_l}$ for any fixed l. Let $f_l:=(-\Delta+I)^k u_l$, then $\mathrm{supp}f_l\subset B_{R_l}$. Consider the problem

$$\left\{\begin{array}{c}(-\Delta+I)^k v_l=f_l^{\#}\\ v_l\in W_N^{m,2}(B_{R_l})\end{array}\right..$$

By the property of rearrangement, we have

$$\int_{B_{R_l}} \left|(-\Delta+I)^k v_l\right|^{\frac{n}{m}} dx = \int_{B_{R_l}} \left|(-\Delta+I)^k u_l\right|^{\frac{n}{m}} dx \le 1 \tag{5.1}$$

and by the Hardy-Littlewood inequality and Proposition 2.4, we get

$$\int_{B_{R_l}} \frac{\phi\left(\beta_{\alpha,n,m}\left|u_l\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx \le \int_{B_{R_l}} \frac{\phi\left(\beta_{\alpha,n,m}\left|u_l^{\#}\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx \le \int_{B_{R_l}} \frac{\phi\left(\beta_{\alpha,n,m}\left|v_l\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx$$

Now, writing

$$\begin{aligned}\int_{B_{R_l}} \frac{\phi\left(\beta_{\alpha,n,m}\left|v_l\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx &\le \int_{B_{R_0}} \frac{\phi\left(\beta_{\alpha,n,m}\left|v_l\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx + \int_{B_{R_l}\setminus B_{R_0}} \frac{\phi\left(\beta_{\alpha,n,m}\left|v_l\right|^{\frac{n}{n-m}}\right)}{|x|^{\alpha}} dx \\ &= I_1 + I_2\end{aligned}$$

where R_0 is a constant and will be chosen later. Then we will prove that both I_1 and I_2 are bounded uniformly by a constant.

Using Theorem 1.2, we can estimate I_1. Indeed, we just need to construct an auxiliary radial function $w_l \in W_N^{m,\frac{n}{m}}(B_{R_0})$ with $\|\nabla^m w_l\|_{\frac{n}{m}} \le 1$ which increases the integral we are interested in. Such a function was constructed in [30]. For the sake of completion, we give the detail here. For each $i \in \{1,2,...,m-1\}$ we define

$$g_i(|x|) := |x|^{m-i},\ \forall x \in B_{R_0}$$

so $g_i \in W_{rad}^{m,\frac{n}{m}}(B_{R_0})$. Moreover,

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j |x|^{m-i-2j} \text{ for } j \in \{1,2,...k-i\} \\ 0 \text{ for } j \in \{k-i+1,...,k\}\end{cases} \quad \forall x \in B_{R_0}$$

where

$$c_i^j = \prod_{h=1}^{j} \left[n+m-2(h+i)\right]\left[m-2(i+h-1)\right],\ \forall j \in \{1,2,...k-i\}.$$

Let

$$z_l(|x|) := v_l(|x|) - \sum_{i=1}^{k-1} a_i g_i(|x|) - a_k\ \forall x \in B_{R_0}$$

where

$$a_i := \frac{\Delta^{k-i} v_l(R_0) - \sum_{j=1}^{i-1} a_j \Delta^{k-i} g_j(R_0)}{\Delta^{k-i} g_i(R_0)},\ \forall i \in \{1,2,...k-1\},$$

$$a_k := v_l(R_0) - \sum_{i=1}^{k-1} a_i g_i(R_0).$$

We can check that (see [30])

$$\begin{aligned} z_l &\in W_{N,rad}^{m,\frac{n}{m}}\left(B_{R_0}\right), \\ \nabla^m v_l &= \nabla^m z_l \text{ in } B_{R_0}. \end{aligned}$$

We have the following lemma whose proof can be found in [30].

Lemma 5.1. *For $0<|x|\leq R_0$, there exists some constant $d(m,n,R_0)$ depending only on m,n,R_0 such that*

$$\begin{aligned} \left|v_l\left(|x|\right)\right|^{\frac{n}{n-m}} \leq \left|z_l\left(|x|\right)\right|^{\frac{n}{n-m}} \left(1+c_{m,n}\sum_{j=1}^{k-1}\frac{1}{R_0^{2j\frac{n}{m}-1}}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}+\frac{c_{m,n}}{R_0^{n-1}}\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\right)^{\frac{n}{n-m}} \\ + d(m,n,R_0). \end{aligned}$$

Now, setting

$$w_l\left(|x|\right):=z_l\left(|x|\right)\left(1+c_{m,n}\sum_{j=1}^{k-1}\frac{1}{R_0^{2j\frac{n}{m}-1}}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}+\frac{c_{m,n}}{R_0^{n-1}}\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\right).$$

Since

$$\begin{aligned} z_l &\in W_{N,rad}^{m,\frac{n}{m}}\left(B_{R_0}\right), \\ \nabla^m v_l &= \nabla^m z_l \text{ in } B_{R_0}. \end{aligned}$$

we have

$$w_l \in W_{N,rad}^{m,\frac{n}{m}}\left(B_{R_0}\right)$$

and

$$\left\|\nabla^m w_l\right\|_{\frac{n}{m}} = \left\|\nabla^m z_l\right\|_{\frac{n}{m}}\left(1+c_{m,n}\sum_{j=1}^{k-1}\frac{1}{R_0^{2j\frac{n}{m}-1}}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}+\frac{c_{m,n}}{R_0^{n-1}}\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\right).$$

Note that

$$\begin{aligned} \left\|\nabla^m z_l\right\|_{\frac{n}{m}} &= \left\|\nabla^m v_l\right\|_{\frac{n}{m}} \\ &\leq \left(1-\sum_{j=1}^{k-1}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}-\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\right)^{m/n} \\ &\leq 1-\frac{m}{n}\sum_{j=1}^{k-1}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}-\frac{m}{n}\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}} \end{aligned}$$

we have

$$\begin{aligned} \left\|\nabla^m w_l\right\|_{\frac{n}{m}} \leq& \left(1-\frac{m}{n}\sum_{j=1}^{k-1}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}-\frac{m}{n}\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\right)\times \\ &\times\left(1+c_{m,n}\sum_{j=1}^{k-1}\frac{1}{R_0^{2j\frac{n}{m}-1}}\left\|\Delta^{k-j}v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}+\frac{c_{m,n}}{R_0^{n-1}}\left\|v_l\right\|_{W^{1,\frac{n}{m}}}^{\frac{n}{m}}\right) \\ \leq& 1 \end{aligned}$$

if we choose R_0 sufficiently large.

Finally, note that

$$I_1 \leq e^{\beta_0 d(m,n,R_0)} \int_{B_{R_0}} \frac{e^{\beta_0 w_l^2}}{|x|^\alpha} dx,$$

using Theorem 1.2, we can conclude that I_1 is bounded by a constant since $w_l \in W^{m,\frac{n}{m}}_{N,rad}(B_{R_0})$ and $\|\nabla^m w_l\|_{\frac{n}{m}} \leq 1$.

Now, we will estimate I_2. Note that

$$\begin{aligned} I_2 &= \int_{B_{R_l} \setminus B_{R_0}} \frac{\phi\left(\beta_{\alpha,n,m} |v_l|^{\frac{n}{n-m}}\right)}{|x|^\alpha} dx \\ &\leq \frac{1}{R_0^\alpha} \int_{B_{R_l} \setminus B_{R_0}} \phi\left(\beta_{\alpha,n,m} |v_l|^{\frac{n}{n-m}}\right) dx \end{aligned}$$

By the same argument as that in [30], we can conclude that $I_2 \leq c(m,n,R_0)$.

Combining the above estimates and using the Fatou lemma, we can conclude that

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \left\|(-\Delta+I)^{\frac{m}{2}} u\right\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \frac{\phi\left(\beta_{\alpha,n,m} |u|^{\frac{n}{n-m}}\right) dx}{|x|^\alpha} dx < \infty.$$

When $\beta > \beta_{\alpha,n,m}$, again, it's easy to check that the sequence given by D. Adams [2] will make our supremum blow up. This completes the proof of Theorem 1.3. $\square$

5.2. **Proof of Theorem 1.4.**

Proof. It suffices to prove that

$$\sup_{u \in W^{2,2}(\mathbb{R}^4), \int_{\mathbb{R}^4}\left(|\Delta u|^2 + \tau|\nabla u|^2 + \sigma|u|^2\right) \leq 1} \int_{\mathbb{R}^4} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)u^2} - 1\right)}{|x|^\alpha} dx < \infty.$$

In fact, we will prove a stronger result that

$$\sup_{u \in W^{2,2}(\mathbb{R}^4), \|-\Delta u + cu\|_2 \leq 1} \int_{\mathbb{R}^4} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)u^2} - 1\right)}{|x|^\alpha} dx < \infty \tag{5.2}$$

where $c > 0$ is chosen such that $\|-\Delta u + cu\|_2^2 \leq \int_{\mathbb{R}^4}\left(|\Delta u|^2 + \tau|\nabla u|^2 + \sigma|u|^2\right)$.

Let $u \in W^{2,2}(\mathbb{R}^4)$, $\|-\Delta u + cu\|_2 \leq 1$. By the density of $C_0^\infty(\mathbb{R}^4)$ in $W^{2,2}(\mathbb{R}^4)$, we can find a sequence of functions u_k in $C_0^\infty(\mathbb{R}^4)$ such that $u_k \to u$ in $W^{2,2}(\mathbb{R}^4)$, supp $u \subset B_{R_k}$. Without loss of generality, we assume $\|-\Delta u_k + cu_k\|_2 \leq 1$. By the Fatou lemma, we have

$$\int_{\mathbb{R}^4} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)u^2} - 1\right)}{|x|^\alpha} dx \leq \liminf_{k\to\infty} \int_{B_{R_k}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)u_k^2} - 1\right)}{|x|^\alpha} dx. \tag{5.3}$$

Now, set $f_k := -\Delta u_k + cu_k$ and consider the problem

$$\left\{\begin{array}{c} -\Delta v_k + cv_k = f_k^\# \text{ in } B_{R_k} \\ v_k \in W_0^{1,2}(B_{R_k}) \end{array}\right. .$$

We have that $v_k \in W_N^{2,2}(B_{R_k})$. Moreover, by Proposition 2.4 and the property of rearrangement, we have

$$\left\|-\Delta u_k + cu_k\right\|_2 = \left\|-\Delta v_k + cv_k\right\|_2 \leq 1 \tag{5.4}$$

$$\int_{B_{R_k}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)u_k^2} - 1\right)}{|x|^\alpha} dx \leq \int_{B_{R_k}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2} - 1\right)}{|x|^\alpha} dx.$$

Now, we write

$$\begin{aligned}
&\int_{B_{R_k}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2} - 1\right)}{|x|^\alpha} dx \\
&= \int_{B_{R_0}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2} - 1\right)}{|x|^\alpha} dx + \int_{B_{R_k}\setminus B_{R_0}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2} - 1\right)}{|x|^\alpha} dx \\
&= I_1 + I_2.
\end{aligned}$$

where R_0 only depends on c and will be chosen later.

Choose $R_0 \geq \left(\frac{1}{2\pi^2}\left(\frac{1}{2c} + \frac{1}{c^2}\right)\right)^{1/3}$, then by the Radial Lemma (Lemma 2.2) and (5.4), we have that $|v_k(x)| \leq 1$ when $|x| \geq R_0$. Thus

$$\begin{aligned}
I_2 &= \int_{B_{R_k}\setminus B_{R_0}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2} - 1\right)}{|x|^\alpha} dx \\
&\leq \frac{1}{R_0^\alpha} \int_{B_{R_k}\setminus B_{R_0}} \left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2} - 1\right) dx \\
&\leq \frac{1}{R_0^\alpha} \sum_{j=1}^\infty \frac{\left(32\pi^2\left(1-\frac{\alpha}{4}\right)\right)^j}{j!} \int_{B_{R_k}} v_k^2 \\
&\leq \frac{1}{R_0^\alpha} \frac{1}{c^2} \sum_{j=1}^\infty \frac{\left(32\pi^2\left(1-\frac{\alpha}{4}\right)\right)^j}{j!} \\
&= C(c).
\end{aligned} \tag{5.5}$$

Now, we estimate I_1. Put

$$w_k(|x|) = \left\{ \begin{array}{c} v_k(|x|) - v_k(R_0),\ 0 \leq |x| \leq R_0 \\ 0\ ,\ r \geq R_0 \end{array} \right. .$$

Then it's easy to check that $w_k \in W_N^{2,2}(B_{R_0})$. Moreover, when $0 < |x| \leq R_0$, using Radial Lemmas (Lemma 2.2 and 2.3), we have

$$\begin{aligned}
\left(v_k(|x|)\right)^2 &= \left[w_k(|x|) + v_k(R_0)\right]^2 \\
&= w_k^2(|x|) + 2w_k(|x|)\, v_k(R_0) + \left[v_k(R_0)\right]^2 \\
&\leq w_k^2(|x|) + w_k^2(|x|)\left[v_k(R_0)\right]^2 + 1 + \left[v_k(R_0)\right]^2 \\
&\leq w_k^2(|x|)\left[1 + \frac{C}{R_0^2}\left\|v_k\right\|_{W^{1,2}}^2\right] + d(c, R_0).
\end{aligned}$$

Let

$$z_k(|x|) := w_k\left(|x|\right)\left[1+\frac{C}{R_0^2}\left\|v_k\right\|_{W^{1,2}}^2\right]^{1/2}$$

then $z_k \in W_N^{2,2}\left(B_{R_0}\right)$ since $w_k \in W_N^{2,2}\left(B_{R_0}\right)$. More importantly, we have

$$\begin{aligned}
\left\|\Delta z_k\right\|_2^2 &= \left\|\Delta w_k\right\|_2^2\left[1+\frac{C}{R_0^2}\left\|v_k\right\|_{W^{1,2}}^2\right] \\
&= \left\|\Delta v_k\right\|_2^2\left[1+\frac{C}{R_0^2}\left\|v_k\right\|_{W^{1,2}}^2\right] \\
&\leq (1-2c\left\|\nabla v_k\right\|_2^2 - c^2\left\|v_k\right\|_2^2)(1+\frac{C}{R_0^2}\left\|\nabla v_k\right\|_2^2+\frac{C}{R_0^2}\left\|v_k\right\|_2^2) \\
&\leq 1
\end{aligned}$$

if we choose R_0 sufficiently large.

Then using Theorem 1.2, we have

$$\begin{aligned}
I_1 &= \int_{B_{R_0}} \frac{\left(e^{32\pi^2\left(1-\frac{\alpha}{4}\right)v_k^2}-1\right)}{|x|^\alpha}dx \\
&\leq C\left(c\right)\int_{B_{R_0}} \frac{e^{32\pi^2\left(1-\frac{\alpha}{4}\right)z_k^2}}{|x|^\alpha}dx \\
&\leq C(c).
\end{aligned} \tag{5.6}$$

From (5.5) and (5.6), we get (5.2). Moreover, if we choose $0<c<\min\left\{\frac{\tau}{2},\sqrt{\sigma}\right\}$, then we have

$$\left\|-\Delta u+cu\right\|_2^2 \leq \int_{\mathbb{R}^4}\left(|\Delta u|^2+\tau\left|\nabla u\right|^2+\sigma\left|u\right|^2\right),\ \forall u\in W^{2,2}\left(\mathbb{R}^4\right)$$

and thus the proof of Theorem 1.4 is completed. □

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