

N-Laplacian equations in \mathbb{R}^N with subcritical and critical growth without the Ambrosetti-Rabinowitz condition.

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ABSTRACT. Let Ω be a bounded domain in \mathbb{R}^N . In this paper, we consider the following nonlinear elliptic equation of N -Laplacian type:

$$(0.1) \quad \begin{cases} -\Delta_N u = f(x, u) \\ u \in W_0^{1,2}(\Omega) \setminus \{0\} \end{cases}$$

when f is of subcritical or critical exponential growth. This nonlinearity is motivated by the Moser-Trudinger inequality. In fact, we will prove the existence of a nontrivial non-negative solution to (0.1) without the Ambrosetti-Rabinowitz (AR) condition. Earlier works in the literature on the existence of nontrivial solutions to N -Laplacian in \mathbb{R}^N when the nonlinear term f has the exponential growth only deal with the case when f satisfies the (AR) condition. Our approach is based on a suitable version of the Mountain Pass Theorem introduced by G. Cerami [11, 12]. This approach can also be used to yield an existence result for the p -Laplacian equation ($1 < p < N$) in the subcritical polynomial growth case.

1. Introduction

Let Ω be a bounded smooth domain in \mathbb{R}^N and we consider the following class of nonlinear elliptic equations

$$(1.1) \quad \begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \setminus \{0\} \end{cases}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian. It is well known that problems involving the p -Laplacian appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, nonlinear elasticity and reaction-diffusions. The main purpose of this paper is to establish existence results of nontrivial nonnegative solutions to the above problem of N -Laplacian when the nonlinear term f has the exponential growth but without satisfying the Ambrosetti-Rabinowitz condition. In these cases, the

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original version of the Mountain Pass Theorem of Ambrosetti-Rabinowitz [7, 32] is not sufficient for our purpose. Therefore, we will adapt a suitable version of Mountain Pass Theorem introduced by Cerami [11, 12] to accomplish our goal. Our approach also yields an existence result of nontrivial nonnegative solutions when $1 < p < N$ and f satisfies a certain subcritical polynomial growth condition weaker than those in the literature.

In the case $p = N$, motivated by the Trudinger-Moser inequality (see Lemma 3), existence of nontrivial solutions to N -Laplacian when f has the exponential growth have been studied by many authors. See for example, Carleson-Chang [10], Atkinson-Peletier [8], Adimurthi et al [1, 2, 3, 4, 5, 6], Marcos Do O et al [27, 28, 29, 30], de Figueiredo et al [15, 16], etc. using the classical Critical Point Theory first developed by Ambrosetti-Rabinowitz in their celebrated work [7], see also [32]. The key issue in using such a theory is the verification of conditions which allow the use of the Palais-Smale condition.

When $1 < p < N$, there have been substantial amount of works to study the existence of the nontrivial solution for (1.1). Nevertheless, almost all of the works involve the nonlinear term $f(x, u)$ of a subcritical (polynomial) growth, say,

(SCP) : There exist positive constants c_1 and c_2 and $q_0 \in (p - 1, p^* - 1)$ such that

$$0 \leq f(x, t) \leq c_1 + c_2 t^{q_0} \text{ for all } t \geq 0 \text{ and } x \in \bar{\Omega}$$

where $p^* = Np/(N - p)$ denotes the critical Sobolev exponent. In this case, we can treat the problem (1.1) variationally in the Sobolev space $W_0^{1,p}(\Omega)$ thanks to the standard Mountain Pass Theorem. Since Ambrosetti and Rabinowitz proposed the Mountain-pass Theorem in their celebrated paper [7], critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. Indeed, if we define the Euler-Lagrange function associated to problem (1.1):

$$J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$$

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} F(x, u) dx$$

where

$$F(x, u) = \int_0^u f(x, s) ds$$

then the critical point of J are precisely the weak solutions of problem (1.1). One of the main conditions that appeared in many works is the so-called Ambrosetti-Rabinowitz condition:

(AR) : There are constants $\theta > p$ and $s_0 > 0$ such that

$$0 < \theta F(x, s) \leq s f(x, s), \quad |s| \geq s_0, \quad \forall x \in \Omega$$

In fact, the (AR) condition is quite natural and plays an important role in studying problem (1.1), for example, it ensures the boundedness of the Palais-Smale sequence. On the other hand, this condition is very restrictive and eliminates many interesting and important nonlinearities. We recall that (AR) condition implies another weaker condition

$$f \text{ is } p\text{-superlinear at infinity, i.e., } \lim_{t \rightarrow \infty} \frac{f(x, t)}{|t|^{p-1}} = +\infty, \text{ uniformly in } x \in \Omega.$$

However, there are many functions which satisfy the p -superlinearity at infinity, but do not satisfy the (AR) condition. An example of such functions is

$$f(x, t) = |t|^{p-2} t \log(1 + |t|).$$

Over the years, many researchers studied problem (1.1) by trying to drop the (AR) condition, see for instance [17, 18, 20, 21, 22, 23, 24, 25, 31, 34, 35, 37, 39]. For example, the following assumption has been studied by many authors:

$$\frac{f(x, t)}{|t|^{p-1}} \text{ is non-decreasing with respect to } |t|$$

(see [24, 25, 35] and references therein). Recently, the authors of [14] have used the following condition:

There exists $\theta \geq 1$ such that $\theta G(x, t) \geq G(x, st)$ for all $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$

where $G(x, t) = f(x, t)t - pF(x, t)$, to compute the critical groups of the functional J at infinity, and obtain one nontrivial solution of (1.1). This condition was first introduced by Jeanjean [18], and then was used by numerous authors, for example, [21, 23, 25, 34, 37].

We note that except in [21], the other authors assumed the condition (SCP) in their works in order to get the existence results. One of the main reasons to assume this condition (SCP) is that they can use the Sobolev compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q < p^*$.

In this paper, our first main result will be to study problem (1.1) in the improved subcritical polynomial growth

$$(SCPI) : \lim_{s \rightarrow +\infty} \frac{f(x, s)}{|s|^{p^*-1}} = 0$$

which is much weaker than (SCP) . Note that in this case, we don't have the Sobolev compact embedding anymore. Our work again is without the (AR) -condition. In fact, this condition was studied by Liu and Wang in [21] in the case of Laplacian (i.e., $p = 2$) by the Nehari manifold approach. However, we will show that we can use a suitable version of the Mountain Pass Theorem to get the nontrivial solution to (1.1) in the general case $1 < p < N$. This result is stronger than those in [17, 23, 25, 34].

Let us now state our result: Consider the problem:

$$(P) \quad \begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \setminus \{0\} \\ u \geq 0 \end{cases}$$

Suppose that

(L1) : $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(x, u) \geq 0$, $\forall (x, u) \in \Omega \times [0, \infty)$ and $f(x, u) = 0$, $\forall (x, u) \in \Omega \times (-\infty, 0]$.

(L2) : $\lim_{u \rightarrow +\infty} \frac{F(x, u)}{u^p} = +\infty$ uniformly on $x \in \Omega$ where $F(x, u) = \int_0^u f(x, t) dt$.

(L3) : There is $C_* \geq 0$, $\theta \geq 1$ such that $H(x, t) \leq \theta H(x, s) + C_*$ for all $0 < t < s$, $\forall x \in \Omega$ where $H(x, u) = uf(x, u) - pF(x, u)$.

(L4) : $\limsup_{u \rightarrow 0^+} \frac{pF(x, u)}{|u|^p} < \lambda_1(\Omega)$, uniformly on $x \in \Omega$.

where

$$\lambda_1(\Omega) = \inf \left\{ \frac{\int |\nabla u|^p dx}{\int |u|^p dx} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}$$

then

Theorem 1. *Let $1 < p < N$ and assume that f has the improved subcritical polynomial growth on Ω (condition (SCPI)) and satisfies (L1), (L2), (L3) and (L4). Then, problem (P) has a nontrivial solution.*

Since we are only concerned with the nonnegative solution, the condition (L1) is natural. Moreover, condition (L2) is just a consequence of the p -superlinear at infinity of f . The type of condition (L3) was first introduced by Jeanjean [18] and was used in subsequent works, see [17, 23, 25, 34]. Finally, in earlier works (see e.g., [17, 23, 25]), they also often assumed that

$$\lim_{u \rightarrow 0^+} \frac{f(x, u)}{u^{p-1}} = 0 \text{ uniformly on } x \in \Omega$$

which is stronger than our condition (L4).

In case of $p = N$, we have $p^* = +\infty$. In this case, every polynomial growth is admitted, but one knows by easy examples that $W_0^{1,N}(\Omega) \not\subseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \rightarrow \mathbb{R}^+$ with maximal growth such that

$$\sup_{u \in W_0^{1,N}(\Omega), \|u\| \leq 1} \int_{\Omega} g(u) dx < \infty$$

It was shown by Trudinger [36] and Moser [26] that the maximal growth is of exponential type. So, we must redefine the subcritical (exponential) growth and the critical (exponential) growth in this case as follows:

(SCE) : f has subcritical (exponential) growth on Ω , i.e., $\lim_{u \rightarrow +\infty} \frac{|f(x, u)|}{\exp(\alpha |u|^{N/(N-1)})} = 0$, uniformly on $x \in \Omega$ for all $\alpha > 0$.

(CG) : f has critical growth on Ω , i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{u \rightarrow +\infty} \frac{|f(x, u)|}{\exp(\alpha |u|^{N/(N-1)})} = 0, \text{ uniformly on } x \in \Omega, \forall \alpha > \alpha_0$$

and

$$\lim_{u \rightarrow +\infty} \frac{|f(x, u)|}{\exp(\alpha |u|^{N/(N-1)})} = +\infty, \text{ uniformly on } x \in \Omega, \forall \alpha < \alpha_0$$

When $p = N$ and f has the subcritical exponential growth (SCE), again we can use the Mountain Pass theorem together with the (AR) condition to get the nontrivial solution to (1.1). Nevertheless, it seems that there are no works when the nonlinear term f does not satisfy the (AR) condition in this case. Thus, the second main result of this paper is to establish the existence of nontrivial nonnegative solutions to (P) when f has the subcritical exponential growth (SCE). More precisely, we will study the existence of the nonnegative nontrivial solution to problem (P) where we don't need to use the (AR) condition. Our result is as follows:

Theorem 2. *Let $p = N$ and assume that f has the subcritical exponential growth on Ω (condition (SCE)) and satisfies (L1), (L2), (L3) and (L4). Then, problem (P) has a nontrivial solution.*

When $p = N$ and f has the critical exponential growth (CG), the study of the problem (1.1) becomes much more difficult than in the case of subcritical exponential growth. Similar to the case of the critical polynomial growth in R^N ($N \geq 3$) for the Laplacian studied by Brezis and Nirenberg in their pioneering work [9]), our Euler-Lagrange functional does not satisfy the Palais-Smale condition at all level anymore. Instead, the authors in [1, 29, 30] used the extremal function sequences related to Moser-Trudinger inequality to prove that J satisfies the Palais-Smale at a certain level. Moreover, this Palais-Smale sequence was shown to be bounded and then derived a nontrivial solution. The idea of choosing the testing functions which are extremal to the Moser-Trudinger inequality is inspired by the work of Brezis and Nirenberg where the testing functions are extremal to the Sobolev embedding inequality.

However, in the works [1, 29, 30], they need to assume a much more restrictive condition

$$(ARR) : \exists t_0 > 0, \exists M > 0 \text{ such that } \forall |u| \geq t_0, \forall x \in \Omega, 0 < F(x, u) \leq M |f(x, u)|$$

It's clear that the condition (ARR) implies the (AR) condition.

Our third main purpose of this paper is to study problem (P) without using the (ARR) condition or (AR) condition. Indeed, we get the following result:

Theorem 3. *Let $p = N$ and assume (L1), (L2), (L3) with $\theta = 1$ and $C^* = 0$, (L4) and that f has critical growth on Ω (CG), say, at α_0 . Furthermore assume that*

(L5): $\lim_{t \rightarrow +\infty} f(x, t) \exp\left(-\alpha_0 |t|^{N/(N-1)}\right) t \geq \beta > \left(\frac{N}{d}\right)^N \frac{1}{M\alpha_0^{N-1}}$, uniformly in (x, t) where d is the inner radius of Ω , i.e. $d := \text{radius of the largest open ball } \subset \Omega$;

$$\mathcal{M} = \lim_{n \rightarrow \infty} n \int_0^1 \exp n (t^{N/(N-1)} - t) dt (\geq 2)$$

and

(L6): f is in the class (L_0) , i.e., for any $\{u_n\}$ in $W_0^{1,N}(\Omega)$, if $\begin{cases} u_n \rightarrow 0 \text{ in } W_0^{1,N}(\Omega) \\ f(x, u_n) \rightarrow 0 \text{ in } L^1(\Omega) \end{cases}$, then $F(x, u_n) \rightarrow 0$ in $L^1(\Omega)$ (up to a subsequence).

Then, problem (P) has a nontrivial solution.

It is easy to see that condition (L2) in Theorem 3 is just a consequence of the critical exponential growth condition (CG) and therefore it is automatically satisfied.

The following remarks are in order. First of all, in dimension two we have recently established in [19] the existence of nontrivial nonnegative solutions to the Laplacian equation (i.e., $p = 2$) when the nonlinear term f has the subcritical or critical exponential growth of order $\exp(\alpha u^2)$ but without satisfying the Ambrosetti-Rabinowitz condition. These results in dimension two in [19] extend those of [16] to the case when f does not have the (AR) condition. Second, there have been many works in the literature in which the (AR) condition was replaced by other alternative conditions when f has the polynomial growth. Our results in this paper appear to be the first time in high dimension

for N -Laplacian when f has the subcritical or critical exponential growth and without (AR) condition.

As far as the case when the nonlinear term f has the polynomial growth is concerned, we recall that, in [38], Willem and Zou used

$$\begin{aligned} H(x, s) \text{ is increasing in } s, \forall x \in \Omega; \quad sf(x, s) \geq 0 \quad \forall s \in \mathbb{R}, \\ sf(x, s) \geq C_0 |s|^\mu, \quad \forall |s| \geq s_0 > 0, \quad \forall x \in \Omega \end{aligned}$$

where $\mu > 2$ and $C_0 > 0$, instead of (AR) . It's clear that this condition is much stronger than our conditions. Also, in [13], the authors replaced (AR) condition by

$$\liminf_{s \rightarrow \infty} \frac{H(x, s)}{|s|^\mu} \geq k > 0, \quad \text{uniformly a.e. } x \in \Omega,$$

where $\mu \geq \mu_0 > 0$. In [33], Schechter and Zou assumed that

$$H(x, s) \text{ is convex in } s, \quad \forall x \in \Omega$$

or there are constants $C > 0$, $\mu > 2$ and $r \geq 0$, such that

$$\mu F(x, t) - tf(x, t) \leq C(1 + t^2), \quad |t| \geq r.$$

As remarked in [25], the later condition is in fact equivalent to (AR) and it's easy to see that the convexity on H is much stronger than our condition. Indeed, observe that function $H(x, s)$ is a "quasi-monotonic" function, and also if H is monotonic function in $s < 0$ and $s > 0$, or a convex function in \mathbb{R} , then it satisfies $(L3)$ with $\theta = 1$.

The organization of the paper is as follows. In section 2, we collect some known results of Mountain Pass Theorem in critical point theory ([7], [32], [11], [12]). In particular, it is necessary to adapt the appropriate version of the Mountain Pass Theorem due to Cerami [11, 12] to remove the Ambrosetti-Rabinowitz condition. Section 3 provides the proof of Theorem 1, i.e., the existence of nontrivial nonnegative solutions to Problem (P) when the nonlinear term f has the improved subcritical polynomial growth (SCPI). Section 4 deals with the case when the nonlinear term f has the subcritical exponential growth and gives the proof of Theorem 2. Section 5 contains the proof of Theorem 3 and establishes the existence of nontrivial solutions when f has the critical exponential growth.

2. Preliminaries and Mountain Pass Theorems

Let Ω be a bounded domain in \mathbb{R}^N . We denote

$$\begin{aligned} \|u\| &= \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p} \\ \|u\|_p &= \left(\int_{\Omega} |u|^p dx \right)^{1/p} \\ \lambda_1(\Omega) &= \inf \left\{ \frac{\|u\|^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} \\ d &= \text{radius of the largest open ball } \subset \Omega \end{aligned}$$

Define the Euler-Lagrange functional associated to problem (P):

$$J(u) = \frac{1}{p} \|u\|^p - \int_{\Omega} F(x, u) dx, \quad u \in W_0^{1,p}(\Omega)$$

From the hypotheses on f , by the standard arguments and the Moser-Trudinger inequality (see Lemma 3), we can easily see that J is well-defined. Also, it's standard to check that J is $C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and

$$DJ(u)v = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \int_{\Omega} f(x, u) v dx, \quad v \in W_0^{1,p}(\Omega)$$

Thus, the critical point of J are precisely the weak solutions of problem (P). We will prove the existence of such critical points by the Mountain Pass Theorem.

Definition 1. Let $(X, \|\cdot\|_X)$ be a real Banach space with its dual space $(X^*, \|\cdot\|_{X^*})$ and $I \in C^1(X, \mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the $(PS)_c$ condition if for any sequence $\{x_n\} \subset X$ with

$$I(x_n) \rightarrow c, \quad DI(x_n) \rightarrow 0 \text{ in } X^*$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X . Also, we say that I satisfies the $(C)_c$ condition if for any sequence $\{x_n\} \subset X$ with

$$I(x_n) \rightarrow c, \quad \|DI(x_n)\|_{X^*} (1 + \|x_n\|_X) \rightarrow 0$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X .

We have the following versions of the Mountain Pass Theorem (see [7, 11, 12, 23]):

Lemma 1. Let $(X, \|\cdot\|_X)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, $I(0) = 0$ and

(i) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.

(ii) There is an $e \in X \setminus B_\rho$ such that $I(e) \leq 0$.

Then $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \geq \alpha$ is a critical value of I where

$$\Gamma = \{ \gamma \in C^0([0, 1], X), \gamma(0) = 0, \gamma(1) = e \}.$$

Lemma 2. Let $(X, \|\cdot\|_X)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies $I(0) = 0$ and

(i) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.

(ii) There is an $e \in X \setminus B_\rho$ such that $I(e) \leq 0$.

Let C_M be characterized by

$$C_M = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))$$

where

$$\Gamma = \{ \gamma \in C^0([0, 1], X), \gamma(0) = 0, \gamma(1) = e \}.$$

Then I possesses a $(C)_{C_M}$ sequence.

As we remarked earlier, our results are motivated by the so-called Moser-Trudinger inequality which can be found in [26]. As we know, if $\Omega \subset \mathbb{R}^N$ ($N > p$) is a bounded

domain, then the Sobolev imbedding theorem states that $W_0^{1,p}(\Omega) \subset L^q(\Omega)$, for $1 \leq q \leq p^* = \frac{pN}{N-p}$, or equivalently,

$$\sup_{u \in W_0^{1,p}(\Omega), \|u\| \leq 1} \int_U |u|^q dx \leq C(\Omega), \text{ for } 1 \leq q \leq p^*,$$

while the supremum is infinite for $q > p^*$. In the case $p = N$, it was shown by Trudinger [36] and Moser [26] that the maximal growth is of exponential type. More precisely, we have the following lemma:

Lemma 3. *Let $u \in W_0^{1,N}(\Omega)$, then $\exp(|u|^{N/(N-1)}) \in L^q(\Omega)$ for all $1 \leq q < \infty$. Moreover,*

$$\sup_{u \in W_0^{1,N}(\Omega), \|u\| \leq 1} \int_{\Omega} \exp(\alpha |u|^{N/(N-1)}) dx \leq C(\Omega) \text{ for } \alpha \leq \alpha_N.$$

The inequality is optimal: for any growth $\exp(\alpha |u|^{N/(N-1)})$ with $\alpha > \alpha_N$ the corresponding supremum is $+\infty$.

3. The improved subcritical polynomial growth (SCPI)-Proof of Theorem 1

In this section, we study the problem (P) in the case $1 < p < N$. As we mentioned earlier, there have been a lot of papers about the existence of nontrivial nonnegative solutions without the the (AR)-condition in the case of subcritical polynomial growth. Nevertheless, almost all of them consider the problem (P) under the nonlinear term f satisfies the condition (SCP) which is stronger than our condition (SCPI). In [21], the authors had a similar result to ours by using the Nehari condition type to replace for the (AR) condition. Here, we will show that we can use a suitable Mountain Pass Theorem to get our desired result.

Lemma 4. *Let f satisfy (L1), (L2), (L4), (SCPI). Then J satisfies the conditions (i) and (ii) of Lemma 1.*

PROOF. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, $u \geq 0$. By (L2), for all M , there exists d such that for all $(x, s) \in \Omega \times \mathbb{R}^+$

$$(3.1) \quad F(x, s) \geq Ms^p - d.$$

Then

$$\begin{aligned} J(tu) &\leq \frac{t^p}{p} \|u\|^p - Mt^p \int_{\Omega} |u|^p dx + O(1) \\ &= t^p \left(\frac{\|u\|^p}{p} - M \int_{\Omega} |u|^p dx \right) + O(1) \end{aligned}$$

Now, choose $M > \frac{\|u\|^p}{p \int_{\Omega} |u|^p dx}$, we have $J(tu) \rightarrow -\infty$ as $t \rightarrow \infty$, so J satisfies (ii) of Lemma 1.

Next, by (L4) and (SCPI), there exist $C, \tau > 0$ such that

$$(3.2) \quad F(x, s) \leq \frac{1}{p} (\lambda_1 - \tau) |s|^p + C |s|^{p^*}, \quad \forall (x, s) \in \Omega \times \mathbb{R}$$

Thus by the definition of $\lambda_1(\Omega)$ and the Sobolev embedding:

$$J(u) \geq \frac{1}{p} \left(1 - \frac{(\lambda_1 - \tau)}{\lambda_1} \right) \|u\|^p - C \|u\|^{p^*}$$

Since $\tau > 0$ and $p^* > p$, we may choose $\rho, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\| = \rho$ and so, J satisfies (i) of the Lemma 1. \square

Next, we will check that J satisfies the $(C)_c$ for all real numbers c .

Lemma 5. *Assume (L1), (L2), (L3) and (L4) hold. If f has the improved subcritical polynomial growth on Ω (SCPI), then J satisfies $(C)_c$ for all $c \in \mathbb{R}$.*

PROOF. Let $\{u_n\}$ be a Cerami sequence in $W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} (1 + \|u_n\|) \|DJ(u_n)\| &\rightarrow 0 \\ J(u_n) &\rightarrow c \end{aligned}$$

i.e.

$$(3.3) \quad (1 + \|u_n\|) \left| \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla v dx - \int_{\Omega} f(x, u_n) v dx \right| \leq \varepsilon_n \|v\|$$

$$\frac{1}{p} \|u_n\|^p - \int_{\Omega} F(x, u_n) dx \rightarrow c$$

where $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$. We first show that $\{u_n\}$ is bounded which is our main purpose in this paper. Indeed, suppose that

$$(3.4) \quad \|u_n\| \rightarrow \infty$$

Setting

$$v_n = \frac{u_n}{\|u_n\|}$$

then $\|v_n\| = 1$ so we can suppose that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. We may similarly show that $v_n^+ \rightharpoonup v^+$ in $W_0^{1,p}(\Omega)$, where $w^+ = \max\{w, 0\}$. Since Ω is bounded, Sobolev's imbedding

theorem implies that $\begin{cases} v_n^+(x) \rightarrow v^+(x) \text{ a.e. in } \Omega \\ v_n^+ \rightarrow v^+ \text{ in } L^q(\Omega), \forall 1 \leq q < p^* \end{cases}$. We wish to show that $v^+ = 0$ a.e. Ω . Indeed, if $\Omega^+ = \{x \in \Omega : v^+(x) > 0\}$ has a positive measure, then in Ω^+ , we have

$$\lim_{n \rightarrow \infty} u_n^+(x) = \lim_{n \rightarrow \infty} v_n^+(x) \|u_n\| = +\infty$$

and thus by (L2):

$$\lim_{n \rightarrow \infty} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^p} = +\infty \text{ a.e. in } \Omega^+$$

This means that

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^p} |v_n^+(x)|^p = +\infty \text{ a.e. in } \Omega^+$$

and so

$$(3.6) \quad \int_{\Omega^+} \liminf_{n \rightarrow \infty} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^p} |v_n^+(x)|^p dx = +\infty$$

Also, by (3.3), we see that

$$\|u_n\|^p = pc + p \int_{\Omega} F(x, u_n^+(x)) dx + o(1)$$

which implies that

$$\int_{\Omega} F(x, u_n^+(x)) dx \rightarrow +\infty$$

and

$$\begin{aligned} (3.7) \quad & \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, u_n^+(x))}{\|u_n\|^p} dx \\ &= \liminf_{n \rightarrow \infty} \frac{\int_{\Omega^+} F(x, u_n^+(x)) dx}{pc + p \int_{\Omega} F(x, u_n^+(x)) dx + o(1)} \\ &= \frac{1}{p} \end{aligned}$$

Now, note that $F(x, s) \geq 0$, by Fatou's lemma and (3.6) and (3.7), we get a contradiction. So $v \leq 0$ a.e.

Letting $t_n \in [0, 1]$ such that

$$J(t_n u_n) = \max_{t \in [0, 1]} J(t u_n)$$

For all $R > 0$, by (SCPI), there exists $C > 0$ such that

$$(3.8) \quad F(x, s) \leq C|s| + \frac{1}{R^{p^*}} s^{p^*}, \quad \forall (x, s) \in \Omega \times \mathbb{R}.$$

Also since $\|u_n\| \rightarrow \infty$, we have for n sufficient large:

$$(3.9) \quad J(t_n u_n) \geq J\left(\frac{R}{\|u_n\|} u_n\right) = J(Rv_n)$$

and by (3.8) with note that $\int_{\Omega} F(x, v_n) dx = \int_{\Omega} F(x, v_n^+) dx$:

$$\begin{aligned} (3.10) \quad pJ(Rv_n) &\geq R^p - pC \int_{\Omega} |Rv_n^+(x)| dx - \frac{p}{R^{p^*}} \int_{\Omega} |Rv_n^+|^{p^*} dx \\ &= R^p - pRC \int_{\Omega} |v_n^+(x)| dx - p \int_{\Omega} |v_n^+|^{p^*} dx \end{aligned}$$

Since $v_n^+ \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$, thus $\int_{\Omega} |v_n^+|^{p^*} dx$ is bounded by a universal constant $C(\Omega) > 0$ and also $\int_{\Omega} |v_n^+(x)| dx \rightarrow 0$. Thus if we let $n \rightarrow \infty$ in (3.10), and then let $R \rightarrow \infty$ and using (3.9), we get

$$(3.11) \quad J(t_n u_n) \rightarrow \infty$$

Note that $J(0) = 0$ and $J(u_n) \rightarrow c$, we can suppose that $t_n \in (0, 1)$. Thus $DJ(t_n u_n) t_n u_n = 0$, i.e.,

$$t_n^p \|u_n\|^p = \int_{\Omega} f(x, t_n u_n) t_n u_n dx$$

Also, by (3.3)

$$\begin{aligned} \int_{\Omega} [f(x, u_n) u_n - pF(x, u_n)] dx &= \|u_n\|^p + pc - \|u_n\|^p + o(1) \\ &= pc + o(1) \end{aligned}$$

So by (L3) :

$$\begin{aligned} pJ(t_n u_n) &= t_n^p \|u_n\|^p - p \int_{\Omega} F(x, t_n u_n) dx \\ &= \int_{\Omega} [f(x, t_n u_n) t_n u_n - pF(x, t_n u_n)] dx \\ &\leq \theta \int_{\Omega} [f(x, u_n) u_n - pF(x, u_n)] dx + O(1) \\ &\leq O(1) \end{aligned}$$

which is a contraction to (3.11). This proves that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Without loss of generality, we can suppose that

$$\begin{cases} u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \\ u_n(x) \rightarrow u(x) \text{ a.e. } \Omega \\ u_n \rightarrow u \text{ in } L^q(\Omega), \forall 1 \leq q < p^*. \end{cases}$$

Now, since f has the subcritical growth on Ω , for every $\varepsilon > 0$, we can find a constant $C(\varepsilon) > 0$ such that

$$f(x, s) \leq C(\varepsilon) + \varepsilon |s|^{p^*-1}, \quad \forall (x, s) \in \Omega \times \mathbb{R}$$

then

$$\begin{aligned} &\left| \int_{\Omega} f(x, u_n) (u_n - u) dx \right| \\ &\leq C(\varepsilon) \int_{\Omega} |(u_n - u)| dx + \varepsilon \int_{\Omega} |(u_n - u)| |u_n|^{p^*-1} dx \\ &\leq C(\varepsilon) \int_{\Omega} |(u_n - u)| dx + \varepsilon \left(\int_{\Omega} (|u_n|^{p^*-1})^{p^*/(p^*-1)} dx \right)^{(p^*-1)/p^*} \left(\int_{\Omega} |u_n - u|^{p^*} dx \right)^{1/p^*} \\ &\leq C(\varepsilon) \int_{\Omega} |(u_n - u)| dx + \varepsilon C(\Omega) \end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $\int_{\Omega} |(u_n - u)| dx \rightarrow 0$. Since $\varepsilon > 0$ is arbitrary, we can conclude that $\int_{\Omega} f(x, u_n) (u_n - u) dx \rightarrow 0$. Thus we can conclude that

$$(3.12) \quad \int_{\Omega} (f(x, u_n) - f(x, u)) (u_n - u) dx \xrightarrow{n \rightarrow \infty} 0$$

By (3.3), we have

$$(3.13) \quad \langle DJ(u_n) - DJ(u), (u_n - u) \rangle \xrightarrow{n \rightarrow \infty} 0$$

From (3.12) and (3.13), we get

$$\int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx \rightarrow 0$$

Using an elementary inequality

$$2^{2-p} |b - a|^p \leq \langle |b|^{p-2} b - |a|^{p-2} a, b - a \rangle, \quad \forall a, b \in \mathbb{R}^p$$

we can deduce that

$$\nabla u_n \rightarrow \nabla u \text{ in } L^p(\Omega)$$

So we have $u_n \xrightarrow{n \rightarrow \infty} u$ strongly in $W_0^{1,p}(\Omega)$ which means that J satisfies $(C)_c$. \square

3.1. Proof of Theorem 1. Combing Lemma 5 and Mountain Pass Theorem (Lemma 1), we can easily deduce that the problem (P) has a nontrivial weak solution.

4. The subcritical exponential growth-Proof of Theorem 2

In this section, we will study the problem (P) in the case $p = N \geq 3$ and f satisfies the (SCE). As far as we know, this appears to be the first work with the (AR)-condition free in the subcritical exponential growth.

4.1. The geometry of the functional J. In this subsection, we will check the Mountain Pass properties of the functional J . Similar to Lemma 4, we have the following lemma:

Lemma 6. *Let f satisfy (L2). Then $J(tu) \rightarrow -\infty$ as $t \rightarrow \infty$ for all nonnegative function $u \in W_0^{1,N}(\Omega) \setminus \{0\}$.*

This means that the condition (i) in Lemma 1 is satisfied. Now, we will check the second one:

Lemma 7. *Let f satisfy (L1), (L4), (SCE). Then there exist $\delta, \rho > 0$ such that*

$$J(u) \geq \delta \text{ if } \|u\| = \rho$$

PROOF. By (L4) and (SCE), there exist $\kappa, \tau > 0$ and $q > N$ such that

$$F(x, s) \leq \frac{1}{N} (\lambda_1 - \tau) |s|^N + C \exp\left(\kappa |s|^{N/(N-1)}\right) |s|^q, \quad \forall (x, s) \in \Omega \times \mathbb{R}$$

By Holder's inequality and the Moser-Trudinger embedding, we have:

$$\begin{aligned} \int_{\Omega} \exp\left(\kappa |u|^{N/(N-1)}\right) |u|^q dx &\leq \left(\int_{\Omega} \exp\left(\kappa r \|u\|^{N/(N-1)} \left(\frac{|u|}{\|u\|}\right)^{N/(N-1)}\right) dx \right)^{1/r} \left(\int_{\Omega} |u|^{r'q} dx \right)^{1/r'} \\ &\leq C \left(\int_{\Omega} |u|^{r'q} dx \right)^{1/r'} \end{aligned}$$

if $r > 1$ sufficiently close to 1 and $\|u\| \leq \sigma$, where $\kappa r \sigma^{N/(N-1)} < \alpha_N$. Thus by the definition of λ_1 and the Sobolev embedding:

$$J(u) \geq \frac{1}{N} \left(1 - \frac{(\lambda_1 - \tau)}{\lambda_1}\right) \|u\|^N - C \|u\|^q$$

Since $\tau > 0$ and $q > N$, we may choose $\rho, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\| = \rho$. \square

Again, it's very important to check that J satisfies the $(C)_c$ for all real numbers c . Similar to what we have shown in the previous section, we have the following lemma:

Lemma 8. *Assume (L1), (L2), (L3) and (L4) hold. If f has subcritical exponential growth on Ω (SCE), then J satisfies $(C)_c$ for all $c \in \mathbb{R}$.*

PROOF. Let $\{u_n\}$ be a Cerami sequence in $W_0^{1,N}(\Omega)$ such that

$$\begin{aligned} (1 + \|u_n\|) \|DJ(u_n)\| &\rightarrow 0 \\ J(u_n) &\rightarrow c \end{aligned}$$

i.e.

$$(4.1) \quad (1 + \|u_n\|) \left| \int_{\Omega} |\nabla u_n|^{N-2} \nabla u_n \nabla v dx - \int_{\Omega} f(x, u_n) v dx \right| \leq \varepsilon_n \|v\|$$

$$\frac{1}{N} \|u_n\|^N - \int_{\Omega} F(x, u_n) dx \rightarrow c$$

where $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$. We will show that $\{u_n\}$ is bounded. Again, suppose that

$$(4.2) \quad \|u_n\| \rightarrow \infty$$

Setting

$$v_n = \frac{u_n}{\|u_n\|}$$

then $\|v_n\| = 1$. We can then suppose that $v_n \rightharpoonup v$ in $W_0^{1,N}(\Omega)$ (up to a subsequence). We may similarly show that $v_n^+ \rightharpoonup 0$ in $W_0^{1,N}(\Omega)$, where $w^+ = \max\{w, 0\}$.

Again, let $t_n \in [0, 1]$ such that

$$J(t_n u_n) = \max_{t \in [0,1]} J(t u_n)$$

For any given $R > 0$, by (SCE), there exists $C = C(R) > 0$ such that

$$(4.3) \quad F(x, s) \leq C |s| + \exp\left(\frac{\alpha_N}{R^{N/(N-1)}} s^{N/(N-1)}\right), \quad \forall (x, s) \in \Omega \times \mathbb{R}.$$

Also since $\|u_n\| \rightarrow \infty$, we have

$$(4.4) \quad J(t_n u_n) \geq J\left(\frac{R}{\|u_n\|} u_n\right) = J(R v_n)$$

and by (4.3), $\|v_n\| = 1$ and the fact that $\int_{\Omega} F(x, v_n) dx = \int_{\Omega} F(x, v_n^+) dx$, we get

$$(4.5) \quad \begin{aligned} NJ(R v_n) &\geq R^N - NCR \int_{\Omega} |v_n^+(x)| dx - N \int_{\Omega} \exp\left(\alpha_N |v_n^+(x)|^{N/(N-1)}\right) dx \\ &\geq R^N - NCR \int_{\Omega} |v_n^+(x)| dx - N \int_{\Omega} \exp\left(\alpha_N |v_n(x)|^{N/(N-1)}\right) dx \end{aligned}$$

Since $\|v_n\| = 1$, we have that $\int_{\Omega} \exp\left(\alpha_N |v_n(x)|^{N/(N-1)}\right) dx$ is bounded by a universal constant $C(\Omega) > 0$ by the Moser-Trudinger inequality (Lemma 3). Also, since $v_n^+ \rightharpoonup 0$ in $W_0^{1,N}(\Omega)$, we have that $\int_{\Omega} |v_n^+(x)| dx \rightarrow 0$. Thus using (4.4) and letting $n \rightarrow \infty$ in (4.5), and then letting $R \rightarrow \infty$, we get

$$(4.6) \quad J(t_n u_n) \rightarrow \infty$$

Note that $J(0) = 0$ and $J(u_n) \rightarrow c$, we can suppose that $t_n \in (0, 1)$. Thus since $DJ(t_n u_n) t_n u_n = 0$,

$$t_n^N \|u_n\|^N = \int_{\Omega} f(x, t_n u_n) t_n u_n dx$$

So by (L3) :

$$\begin{aligned} NJ(t_n u_n) &= t_n^N \|u_n\|^N - N \int_{\Omega} F(x, t_n u_n) dx \\ &= \int_{\Omega} [f(x, t_n u_n) t_n u_n - NF(x, t_n u_n)] dx \\ &\leq \theta \int_{\Omega} [f(x, u_n) u_n - NF(x, u_n)] dx + O(1) \end{aligned}$$

Also, by (3.3), we have

$$\begin{aligned} \int_{\Omega} [f(x, u_n) u_n - NF(x, u_n)] dx &= \|u_n\|^N + Nc - \|u_n\|^N + o(1) \\ &= Nc + o(1) \end{aligned}$$

which is a contraction to (3.11). This proves that $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. Without loss of generality, suppose that

$$\begin{cases} \|u_n\| \leq K \\ u_n \rightharpoonup u \text{ in } W_0^{1,N}(\Omega) \\ u_n(x) \rightarrow u(x) \text{ a.e. } \Omega \\ u_n \rightarrow u \text{ in } L^p(\Omega), \forall p \geq 1. \end{cases}$$

Now, since f has the subcritical exponential growth (SCE) on Ω , we can find a constant $c_K > 0$ such that

$$f(x, s) \leq c_K \exp\left(\frac{\alpha_N}{2K^{N/(N-1)}} |s|^{N/(N-1)}\right), \forall (x, s) \in \Omega \times \mathbb{R}$$

then by the Moser-Trudinger inequality,

$$\begin{aligned} \left| \int_{\Omega} f(x, u_n) (u_n - u) dx \right| &\leq \int_{\Omega} |f(x, u_n) (u_n - u)| dx \\ &\leq \left(\int_{\Omega} |f(x, u_n)|^2 dx \right)^{1/2} \left(\int_{\Omega} |u_n - u|^2 dx \right)^{1/2} \\ &\leq C \left(\int_{\Omega} \exp\left(\frac{\alpha_N}{K^{N/(N-1)}} |u_n|^{N/(N-1)}\right) dx \right)^{1/2} \|u_n - u\|_2 \\ &\leq C \left(\int_{\Omega} \exp\left(\frac{\alpha_N}{K^{N/(N-1)}} \|u_n\|^{N/(N-1)} \left| \frac{u_n}{\|u_n\|} \right|^{N/(N-1)}\right) dx \right)^{1/2} \|u_n - u\|_2 \\ &\leq C \|u_n - u\|_2 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in $W_0^{1,N}(\Omega)$, $\int_{\Omega} f(x, u) (u_n - u) dx \rightarrow 0$. Thus we can conclude that

$$(4.7) \quad \int_{\Omega} (f(x, u_n) - f(x, u)) (u_n - u) dx \xrightarrow{n \rightarrow \infty} 0$$

Also, by (4.1) we have

$$(4.8) \quad \langle DJ(u_n) - DJ(u), (u_n - u) \rangle \xrightarrow{n \rightarrow \infty} 0$$

From (3.12) and (3.13), we get

$$\int_{\Omega} \left(|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u \right) (\nabla u_n - \nabla u) \rightarrow 0$$

Using an elementary inequality

$$2^{2-N} |b - a|^N \leq \left\langle |b|^{N-2} b - |a|^{N-2} a, b - a \right\rangle, \quad \forall a, b \in \mathbb{R}^N$$

we can deduce that

$$\nabla u_n \rightarrow \nabla u \text{ in } L^N(\Omega)$$

So we have $u_n \xrightarrow{n \rightarrow \infty} u$ strongly in $W_0^{1,N}(\Omega)$ which shows that J satisfies $(C)_c$. \square

4.2. Proof of Theorem 2. Again, by Lemma 8 and Mountain Pass Theorem (Lemma 1), we can easily deduce that the problem (P) has a nontrivial weak solution.

5. The critical exponential growth-Proof of Theorem 3

In this section, we study the problem (P) where Ω is the bounded domain in \mathbb{R}^N and f has the critical growth (CR), say, at $\alpha_0 > 0$. Recall that then we have

$$\lim_{u \rightarrow +\infty} \frac{|f(x, u)|}{\exp\left(\alpha |u|^{N/(N-1)}\right)} = 0, \quad \text{uniformly on } x \in \Omega, \quad \forall \alpha > \alpha_0$$

and

$$\lim_{u \rightarrow +\infty} \frac{|f(x, u)|}{\exp\left(\alpha |u|^{N/(N-1)}\right)} = +\infty, \quad \text{uniformly on } x \in \Omega, \quad \forall \alpha < \alpha_0$$

We now start the proof of Theorem 3.

PROOF. Similar to the previous two sections, by our conditions, we see that our Euler-Lagrange function associated to the problem (P) has the Palais-Smale geometry properties. Now we consider the Moser functions:

$$\widetilde{M}_n(x) = \omega_{N-1}^{-1/N} \begin{cases} (\log n)^{(N-1)/N}, & 0 \leq |x| \leq 1/n \\ \frac{\log(1/|x|)}{(\log n)^{1/N}}, & 1/n \leq |x| \leq 1 \\ 0, & 1 \leq |x| \end{cases}$$

We see that $\widetilde{M}_n \in W_0^{1,N}(B_1(0))$ and $\|\widetilde{M}_n\| = 1$, $\forall n \in \mathbb{N}$. Since d is the inner radius of Ω , we can find $x_0 \in \Omega$ such that $B_d(x_0) \subset \Omega$. Letting $M_n(x) = \widetilde{M}_n\left(\frac{x-x_0}{d}\right)$, which are in $W_0^{1,N}(\Omega)$, $\|M_n\| = 1$ and $\text{supp} M_n = B_d(x_0)$. As in the proof of Theorem 1.3 in [16], we can conclude that

$$\max \{J(tM_n) : t \geq 0\} < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$$

It can be checked easily by a similar argument to that in the previous section that J satisfies the condition (i) and (ii) of Lemma 2 (See Lemmas 6 and 7). So, we can find a Cerami sequence $\{u_n\}$ in $W_0^{1,N}(\Omega)$ such that

$$(5.1) \quad (1 + \|u_n\|) \|DJ(u_n)\| \rightarrow 0$$

$$J(u_n) \rightarrow C_M < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$$

We again want to show that $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. Indeed, if we suppose that $\{u_n\}$ is unbounded, then using the same argument to that used in the previous two sections, we can get that

$$v_n^+ \rightharpoonup 0 \text{ in } W_0^{1,N}(\Omega) \text{ where } v_n = \frac{u_n}{\|u_n\|}.$$

Let $t_n \in [0, 1]$ such that

$$J(t_n u_n) = \max_{t \in [0,1]} J(t u_n)$$

Let $R \in \left(0, \left(\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N}\right)$ and choose $\varepsilon = \frac{\alpha_N}{R^{N/(N-1)}} - \alpha_0 > 0$, by condition (CG), there exists $C > 0$ such that

$$(5.2) \quad F(x, s) \leq C|s| + \left| \frac{\alpha_N}{R^{N/(N-1)}} - \alpha_0 \right| \exp((\alpha_0 + \varepsilon) s^{N/(N-1)}), \quad \forall (x, s) \in \Omega \times \mathbb{R}.$$

Since $\|u_n\| \rightarrow \infty$, we have

$$(5.3) \quad J(t_n u_n) \geq J\left(\frac{R}{\|u_n\|} u_n\right) = J(Rv_n)$$

and by (5.2) and noticing $\|v_n\| = 1$, we have

$$(5.4) \quad NJ(Rv_n) \geq R^N - NCR \int_{\Omega} |v_n^+(x)| dx - N \left| \frac{\alpha_N}{R^{N/(N-1)}} - \alpha_0 \right| \int_{\Omega} \exp((\alpha_0 + \varepsilon) R^{N/(N-1)} v_n^{N/(N-1)}(x)) dx$$

By the Moser-Trudinger inequality (Lemma 3),

$$\int_{\Omega} \exp((\alpha_0 + \varepsilon) R^{N/(N-1)} v_n^{N/(N-1)}(x)) dx = \int_{\Omega} \exp(\alpha_N v_n^{N/(N-1)}(x)) dx$$

is bounded by an universal constant $C(\Omega) > 0$ thanks to the choice of ε . Also, since $v_n^+ \rightharpoonup 0$ in $W_0^{1,N}(\Omega)$, $\int_{\Omega} |v_n^+(x)| dx \rightarrow 0$. Thus if we let $n \rightarrow \infty$ in (5.4), and then let

$R \rightarrow \left[\left(\frac{\alpha_N}{\alpha_0} \right)^{(N-1)/N} \right]^-$ and using (5.3), we get

$$(5.5) \quad \liminf_{n \rightarrow \infty} J(t_n u_n) \geq \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1} > C_M.$$

Note that $J(0) = 0$ and $J(u_n) \rightarrow C_M$, we can suppose that $t_n \in (0, 1)$. Thus since $DJ(t_n u_n) t_n u_n = 0$,

$$t_n^N \|u_n\|^N = \int_{\Omega} f(x, t_n u_n) t_n u_n dx$$

Also, by (5.1)

$$\begin{aligned} \int_{\Omega} [f(x, u_n) u_n - NF(x, u_n)] dx &= \|u_n\|^N + NC_M - \|u_n\|^N + o(1) \\ &= NC_M + o(1) \end{aligned}$$

So by (L3) :

$$\begin{aligned} NJ(t_n u_n) &= t_n^N \|u_n\|^N - N \int_{\Omega} F(x, t_n u_n) dx \\ &= \int_{\Omega} [f(x, t_n u_n) t_n u_n - NF(x, t_n u_n)] dx \\ &\leq \int_{\Omega} [f(x, u_n) u_n - NF(x, u_n)] dx \\ &= NC_M + o(1) \end{aligned}$$

which is a contraction to (5.5). This proves that $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. Now, following the proof of Lemma 4 in [29], we can prove that u is a weak solution of (P). So the last remaining point that we need to show is the nontriviality of u . However, we can get this thanks to our assumption (L6). Indeed, suppose $u = 0$. Arguing as in [29], we get $f(x, u_n) \rightarrow 0$ in $L^1(\Omega)$. Thanks to (L6), $F(x, u_n) \rightarrow 0$ in $L^1(\Omega)$ and we can get

$$\lim_{n \rightarrow \infty} \|u_n\|^N = NC_M < \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$$

and again, follows the proof in [29], we have a contradiction. The proof is now completed. \square

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