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Elliptic Equations and Systems with Subcritical and Critical Exponential Growth Without the Ambrosetti–Rabinowitz Condition

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Abstract In this paper, we prove the existence of nontrivial nonnegative solutions to a class of elliptic equations and systems which do not satisfy the Ambrosetti–Rabinowitz (AR) condition where the nonlinear terms are superlinear at 0 and of subcritical or critical exponential growth at ∞. The known results without the AR condition in the literature only involve nonlinear terms of polynomial growth. We will use suitable versions of the Mountain Pass Theorem and Linking Theorem introduced by Cerami (Istit. Lombardo Accad. Sci. Lett. Rend. A, 112(2):332–336, 1978 Ann. Mat. Pura Appl., 124:161–179, 1980). The Moser–Trudinger inequality plays an important role in establishing our results. Our theorems extend the results of de Figueiredo, Miyagaki, and Ruf (Calc. Var. Partial Differ. Equ., 3(2):139–153, 1995) and of de Figueiredo, do Ó, and Ruf (Indiana Univ. Math. J., 53(4):1037–1054, 2004) to the case where the nonlinear term does not satisfy the AR condition. Examples of such nonlinear terms are given in Appendix A. Thus, we have established the existence of nontrivial nonnegative solutions for a wider class of nonlinear terms.

Keywords Mountain pass theorem · Critical point theory · Ambrosetti–Rabinowitz condition · Moser–Trudinger inequality · Subcritical and critical exponential growth

Mathematics Subject Classification 35B38 · 35J92 · 35B33 · 35J62

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1 Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. In this paper, we consider the following class of semilinear elliptic problems.

$$\begin{cases} -\Delta u = f(x,u) \\ u \in W_0^{1,2}(\Omega) \setminus \{0\} \end{cases} \tag{1.1}$$

This kind of equation arises naturally in various contexts of physics; for instance, in the study of propagation phenomena of solitary waves, Newtonian fluids, non-Newtonian fluids and nonlinear elasticity problems. It also appears in the search for solitons of certain Lorentz-invariant nonlinear field equations.

The main goal of this paper is to establish the existence of nontrivial nonnegative solutions to the above equation in $\mathbb{R}^2$ when the nonlinear term f has subcritical or critical exponential growth and does not satisfy the AR condition. In contrast to the case when f has polynomial growth but without the AR condition, the usual Mountain Pass Theorem of Ambrosetti–Rabinowitz [5, 37] does not seem to be applicable in dealing with the existence of nontrivial nonnegative solutions in our case. As a result, we will apply a suitable version of the Mountain Pass Theorem which was introduced by G. Cerami [12, 13] in a different context. Thus, our results extend the work of de Figueiredo, Miyagaki, and Ruf [17] to the case when the nonlinear term does not satisfy the AR condition.

In the case $N = 2$, motivated by the Moser–Trudinger inequality (see Lemma 3), existence of nontrivial solutions to problems of the above type when f has exponential growth has been studied by many authors; see, for example, Carleson–Chang [9], Atkinson–Peletier [6], Shaw [40], Adimurthi et al. [1–4], Cao [8], do Ó et al. [34, 35], de Figueiredo et al. [16, 17], Y.X. Li et al. [25–27], Ruf [38], Lu and Yang [30, 31], etc. using the classical Critical Point Theory as first developed by Ambrosetti–Rabinowitz in their celebrated work [5]. The key issue in using such a theory is the verification of conditions which allow the use of the Palais–Smale condition by using the crucial AR condition. In [17], de Figueiredo, Miyagaki, and Ruf studied problem (1.1) in $\mathbb{R}^2$ where the nonlinearity $f(x,u)$ has the maximal growth on u and satisfies the AR condition which allows them to treat problem (1.1) variationally in the Sobolev space $W_0^{1,2}(\Omega)$. More precisely, they treat the so-called subcritical case and also the critical case. Let us now list some of the assumptions used in [17].

(H1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,0) = 0$.
(H2) $\exists t_0 > 0, \exists M > 0$ such that $\forall |u| \geq t_0, \forall x \in \Omega$,

$$0 < F(x,u) = \int_0^u f(x,t)dt \leq M\left|f(x,u)\right|.$$

(H3) $0 < F(x,u) \leq \frac{1}{2} f(x,u)u, \forall u \in \mathbb{R} \setminus \{0\}, \forall x \in \Omega$.
(SG) f has subcritical growth on Ω at $+\infty(-\infty)$, i.e.,

$$\lim_{u\to+\infty(-\infty)} \frac{|f(x,u)|}{\exp(\alpha|u|^2)} = 0, \quad \text{uniformly on } x \in \Omega \text{ for all } \alpha > 0.$$

(CG) f has critical growth on Ω at $+\infty(-\infty)$, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{u\to+\infty(-\infty)} \frac{|f(x,u)|}{\exp(\alpha|u|^2)} = 0, \quad \text{uniformly on } x \in \Omega,\ \forall \alpha > \alpha_0$$

and

$$\lim_{u\to+\infty(-\infty)} \frac{|f(x,u)|}{\exp(\alpha|u|^2)} = +\infty, \quad \text{uniformly on } x \in \Omega,\ \forall \alpha < \alpha_0.$$

It is well known that there exists the smallest eigenvalue $\lambda_1(\Omega) > 0$ to the Dirichlet problem on Ω, and in fact $\lambda_1(\Omega)$ can be variationally characterized as

$$\lambda_1(\Omega) = \inf\left\{ \frac{\int |\nabla u|^2 dx}{\int |u|^2 dx} : u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\}.$$

Then the authors of [17] have established the following result.

Theorem 1 *Assume* (H1), (H2), (H3), *and that f has subcritical growth* (SG) *at both $+\infty$ and $-\infty$. Furthermore, suppose that*

$$\text{(H4)} \qquad \limsup_{t\to 0} \frac{2F(x,t)}{t^2} < \lambda_1, \quad \textit{uniformly in } (x,t).$$

Then problem (1.1) *has a nontrivial solution.*

Theorem 2 *Assume* (H1), (H2), (H3), *and that f has critical growth* (CG) *at both $+\infty$ and $-\infty$. Furthermore, assume* (H4) *and*

$$\text{(H5)} \qquad \lim_{t\to+\infty} f(x,t)\exp\bigl(-\alpha_0|t|^2\bigr)t \geq \beta > \frac{4}{3\alpha_0 d^2}, \quad \textit{uniformly in } (x,t),$$

where d is the inner radius of Ω, i.e., $d :=$ radius of the largest open ball $\subset \Omega$. Then problem (1.1) *has a nontrivial solution.*

Note that as a consequence of the conditions (H1) and (H2), we have the following well-known Ambrosetti–Rabinowitz condition.

(AR) $\exists R_0 > 0, \exists \theta > 2$ such that $\forall |u| \geq R_0, \forall x \in \Omega$,

$$0 < \theta F(x,u) \leq u f(x,u).$$

The AR condition has appeared in most of the studies for superlinear problems and plays an important role in studying the existence of nontrivial solutions of many nonlinear elliptic boundary value problems of Laplacian and p-Laplacian type. Since Ambrosetti and Rabinowitz proposed the Mountain Pass Theorem in their celebrated paper [5], critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. In the subcritical (polynomial growth) case, the AR condition ensures that the Euler–Lagrange functional associated with a (1.1)-type problem has a mountain pass geometry and also guarantees the boundedness of the Palais–Smale sequence, so we can get the nontrivial solution by using suitable versions of the Mountain Pass Theorem.

On the other hand, there are many cases where the nonlinear term $f(x,u)$ does not satisfy the AR condition; see Appendix A. Thus it becomes interesting to know if a nontrivial solution exists in such situations.

In recent years, there has been some work in the absence of the Ambrosetti–Rabinowitz condition when the nonlinear terms have polynomial growth. For example, Miyagaki and Souto [32] studied a problem with parameter λ and adapted some monotonicity arguments as in [41, 42]. In [19, 20, 36, 42, 45], the authors used a more suitable version of the Mountain Pass Theorem to overcome this difficulty. Nevertheless, the authors of these papers treated the equations with subcritical polynomial growth of the nonlinearity $f(x,u)$:

(SCP) There exist positive constants a and b such that $\left|f(x,s)\right| \le a + b|s|^p$,

where $0 \le p < 2^* - 1 = \frac{N+2}{N-2}$ $(= \infty$ if $N = 2)$. This condition implies that the growth of F is less than or equal to $p+1$ (which is $< 2^*$), and it is needed in the above works because it allows them to use the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$. Note that from the condition (SCP), we can easily deduce a weaker condition,

$$\text{(SCPI)} \qquad \lim_{s\to+\infty} \frac{f(x,s)}{|s|^{2^*-1}} = 0,$$

where we no longer have the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$. In the case of this subcritical polynomial growth (SCPI), Liu and Wang [28] can replace the AR condition by the Nehari type condition and use the Nehari manifold approach to derive the existence of a nontrivial solution. Such a Nehari type argument is also used in [29] to produce ground states of superlinear Schrödinger equations, whose nonlinearity does not satisfy AR.

Though there has been extensive work on the existence of nontrivial nonnegative solutions to elliptic equations with nonlinear terms of polynomial growth without the AR condition, not much has been done when the nonlinear term of subcritical or critical exponential growth without the AR condition. The primary purpose of this paper is to establish the existence of a nontrivial nonnegative solution for such a class of elliptic equations in $\mathbb{R}^2$ using an appropriate version of the Mountain Pass Theorem introduced by G. Cerami [12, 13]. Moreover, we will also provide a different approach from that of [28] for elliptic equations in $\mathbb{R}^N$ for $N > 2$ in the case of subcritical polynomial growth (SCPI), using this version of the Mountain Pass Theorem instead of the Nehari manifold approach. The main ingredient in this approach of using Cerami's version of the Mountain Pass Theorem is to show that a subsequence of the Palais–Smale sequence is bounded in $W_0^{1,2}(\Omega)$. Then we show that this subsequence converges weakly to a nontrivial solution in $W_0^{1,2}(\Omega)$.

In this paper, we will study the existence of the nontrivial solution to a problem of type (1.1) without the AR conditions in both the subcritical and critical exponential growth cases (SG) and (CG) defined above. More precisely, we consider the following problem in $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} -\Delta u = f(x,u) \\ u \in W_0^{1,2}(\Omega) \setminus \{0\} \\ u \geq 0. \end{cases} \tag{P}$$

We now state our main conditions.

(L1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,u) \geq 0, \forall (x,u) \in \Omega \times [0,\infty)$ and $f(x,u) = 0, \forall (x,u) \in \Omega \times (-\infty, 0]$.
(L2) $\lim_{u\to+\infty} \frac{F(x,u)}{u^2} = +\infty$ uniformly on $x \in \Omega$ where $F(x,u) = \int_0^u f(x,t)dt$.
(L3) There is $C_* \geq 0, \theta \geq 1$ such that $H(x,t) \leq \theta H(x,s) + C_*$ for all $0 < t < s$, $\forall x \in \Omega$ where $H(x,u) = uf(x,u) - 2F(x,u)$.
(SCE) f has subcritical (exponential) growth on Ω, i.e., $\lim_{u\to+\infty} \frac{|f(x,u)|}{\exp(\alpha|u|^2)} = 0$, uniformly on $x \in \Omega$ for all $\alpha > 0$.
(CG) f has critical (exponential) growth on Ω, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{u\to+\infty} \frac{|f(x,u)|}{\exp(\alpha|u|^2)} = 0, \quad \text{uniformly on } x \in \Omega, \forall \alpha > \alpha_0$$

and

$$\lim_{u\to+\infty} \frac{|f(x,u)|}{\exp(\alpha|u|^2)} = +\infty, \quad \text{uniformly on } x \in \Omega, \forall \alpha < \alpha_0.$$

The first main theorem of this paper is the following existence result when the nonlinear term f has subcritical exponential growth without satisfying the AR condition.

Theorem 3 *In the case $N = 2$, assume that f has subcritical exponential growth on Ω (condition* (SCE)*) and satisfies* (L1), (L2), *and* (L3). *Furthermore assume that*

$$\text{(L4)} \quad \limsup_{u\to 0+} \frac{2F(x,u)}{|u|^2} < \lambda_1(\Omega), \quad \textit{uniformly on } x \in \Omega.$$

Then, problem (P) *has a nontrivial solution.*

We notice that using our method as in the proof of Theorem 3, we can prove a similar result when the nonlinear term satisfies the (SCPI) condition. It improves results of Miyagaki and Souto [32] to the (SCPI) growth and also gives a different approach from that of [28, 29].

Second, we study the existence of the nontrivial nonnegative solution to problem (P) when f has critical exponential growth (CG) without the condition of (H2) in the case $N = 2$.

Before we state this last main result, we add one more (technical) condition.

(L6) f is in the class (L_0), i.e., for any $\{u_n\}$ in $W_0^{1,2}(\Omega)$, if

$$\begin{cases} u_n \rightharpoonup 0 & \text{in } W_0^{1,2}(\Omega), \\ f(x,u_n) \to 0 & \text{in } L^1(\Omega), \end{cases}$$

then $F(x,u_n) \to 0$ in $L^1(\Omega)$ (up to a subsequence).

Then we have the following result:

Theorem 4 *Assume that* (L1), (L2), (L4), (L6) *hold and that* f *has critical exponential growth* (CG) *on* Ω, *say, at* α_0. *Assume also that* (L3) *holds with* $\theta = 1$ *and* $C^* = 0$. *Furthermore, assume that*

$$\text{(L5)} \qquad \lim_{t\to+\infty} f(x,t)\exp\bigl(-\alpha_0|t|^2\bigr)t \geq \beta > \frac{4}{3\alpha_0 d^2}, \qquad \textit{uniformly in } (x,t),$$

where d *is the inner radius of* Ω, *i.e.*, $d :=$ *radius of the largest open ball* $\subset \Omega$. *Then, problem* (P) *has a nontrivial solution.*

We first note that the assumption (L2) is actually not needed in Theorem 5, since by l'Hôpital's rule (L2) follows from the critical exponential growth assumption on f. We further remark here that in the critical exponential growth case (see the proof of Theorem 5), we will use a different approach from that in the subcritical exponential growth case. In fact, we cannot prove that the Euler–Lagrange functional satisfies the Palais–Smale condition for all $c \in \mathbb{R}$. Instead, we can prove the Euler–Lagrange functional satisfies the Palais–Smale condition for a certain level using the extremal function sequences related to the Moser–Trudinger inequality. See also [1, 34, 35] for more details of such adaptation. Again, the AR condition ensures the boundedness and so the weak convergence of the Palais–Smale sequence to our weak solution. Then the condition of (H2) type guarantees the nontriviality of our weak solution. As far as we know, there has not been any work in the literature without using the condition (H2) (and so without the AR condition) in the case of critical exponential growth.

We should stress that the proof of Theorem 5 in the case of critical exponential growth is motivated by the ideas introduced by H. Brezis and L. Nirenberg [7] in their pioneering work on the solvability of equations with critical growth in dimensions larger than 2. Indeed, our functional under consideration satisfies the Palais–Smale condition only at certain levels. In order to assure that the constructed minimax levels are inside the Palais–Smale region, we use test functions associated with the optimal Moser–Trudinger inequality (while Brezis–Nirenberg used test functions associated with the optimal Sobolev embedding).

Finally, we study the existence of nontrivial solutions for the following Hamiltonian-type systems.

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega \\ -\Delta v = f(u) & \text{in } \Omega \\ u > 0,\ v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{S}$$

Such systems have been widely studied in recent years for bounded domains in $\mathbb{R}^N$, $N \geq 3$; see the recent survey paper [15].

In [18], de Figueiredo, do Ó, and Ruf studied this system when the nonlinearities have the same type of subcritical and critical exponential growth. More precisely, with the following conditions:

(K1) $f, g : [0, \infty) \to [0, \infty)$ are continuous functions;
(K2) $f(s) = o(s)$ and $g(t) = o(t)$ near the origin;
(K3) there exist constants $\theta > 2$ and $t_0 > 0$ such that for all $t \geq t_0$, one has

$$0 < \theta F(t) \leq t f(t) \quad \text{and} \quad 0 < \theta G(t) \leq t g(t)$$

where

$$F(u) = \int_0^u f(t)dt \quad \text{and} \quad G(u) = \int_0^u g(t)dt.$$

(K4) there exist $M > 0$ and $t_1 > 0$ such that for all $t \geq t_1$, one has

$$0 < F(t) \leq M f(t) \quad \text{and} \quad 0 < G(t) \leq M g(t),$$

then [18] has the following result.

Theorem 5 *Let f, g have subcritical* (SCE) *or critical exponential growth* (SG) *at α_0, and let* (K1)–(K4) *be satisfied. Moreover, assume that*

$$\text{(K5)} \qquad \lim_{t\to\infty} \frac{tf(t)}{\exp(\alpha_0 t^2)} > \frac{4}{\alpha_0 d^2} \quad \textit{and} \quad \lim_{t\to\infty} \frac{tg(t)}{\exp(\alpha_0 t^2)} > \frac{4}{\alpha_0 d^2},$$

where d is the inner radius of the set Ω. Then (*S*) *possesses a nontrivial weak solution.*

Notice that in the above result, the authors required the AR condition for both nonlinear terms f and g. Thus our third goal is that we will prove this result without the AR condition. More precisely, we prove that

Theorem 6 *Let f, g have subcritical* (*SCE*) *or critical exponential growth* (SG) *at α_0, and let* (K1), (K2), *and* (K5) *be satisfied. Moreover, assume that*

(K3′) *there exist constants $\theta > 2$ and $t_0 > 0$ such that for all $t \geq t_0$, one has*

$$0 < 2F(t) \leq t f(t) \quad \textit{and} \quad 0 < \theta G(t) \leq t g(t);$$

(K4′) *f, g are in class* (L_0) (*in the case both f and g are critical*).

Then (S) *possesses a nontrivial weak solution.*

Since our system (S) is a special case of a Hamiltonian system, many basic difficulties appear; for example, the associated functional is strongly indefinite, and the nonlinearities f, g can have critical growth. To overcome these difficulties, we will use the Linking Theorem. Fortunately, the Deformation Lemma still works for the Cerami sequence and thus we can use the Linking Theorem with the Cerami sequence instead of the Palais–Smale condition. This is also our main approach.

Now we make some comments for our conditions. Note that we assume (L1), since we are interested in positive solutions. Also, the AR condition implies a weaker condition

$$F(x,s) \geq c|s|^{\theta} - d \quad \text{with } c, d > 0,\ x \in \Omega,\ s > 0,\ \text{and } \theta > 2$$

and this above condition implies our much weaker condition (L2). Moreover, the condition (L2) is just a consequence of the superlinearity of f at ∞:

$$\lim_{|u| \to +\infty} \frac{f(x,u)}{u} = +\infty.$$

The above condition is often assumed in many works; see [19], for example. Finally, our condition (L3) can be implied by the following stronger condition, which is assumed in many other papers, as in [19, 32]:

$$\text{There exists } s_0 > 0 \text{ such that } \frac{f(x,s)}{s} \text{ is increasing in } s \geq s_0, \quad \forall x \in \Omega.$$

Let us finish this section by comparing our conditions with others in the literature which were used to replace the AR condition in the case of nonlinear terms with polynomial growth. We recall that, in [44], Willem and Zou used

$$H(x,s) \text{ is increasing in } s, \quad \forall x \in \Omega; sf(x,s) \geq 0. \forall s \in \mathbb{R},$$

$$sf(x,s) \geq C_0|s|^{\mu}, \quad \forall |s| \geq s_0 > 0, \forall x \in \Omega,$$

where $\mu > 2$ and $C_0 > 0$, instead of AR. It is easy to see that this condition is stronger than our conditions. In [14], the authors replaced the AR condition by

$$\liminf_{s \to \infty} \frac{H(x,s)}{|s|^{\mu}} \geq k > 0, \quad \text{uniformly a.e. } x \in \Omega,$$

where $\mu \geq \mu_0 > 0$. In [39], Schechter and Zou assumed that

$$H(x,s) \text{ is convex in } s, \quad \forall x \in \Omega,$$

or there are constants $C > 0$, $\mu > 2$, and $r \geq 0$, such that

$$\mu F(x,t) - tf(x,t) \leq C\left(1 + t^2\right), \quad |t| \geq r.$$

As remarked in [32], the latter condition is in fact equivalent to AR, and it's easy to see that the convexity on H is much stronger than our condition. Indeed, observe that function $H(x,s)$ is a "quasi-monotonic" function, and also if H is a monotonic function in $s < 0$ and $s > 0$, or a convex function in $\mathbb{R}$, then it satisfies (L3) with $\theta = 1$.

Finally, we remark that existence and multiplicity results of nontrivial nonnegative solutions to N-Laplacian equations in $\mathbb{R}^N$ with nonlinear terms of critical exponential growth $\exp(\alpha|u|^{\frac{N}{N-1}})$ and without the AR condition have been recently established by the authors in [21, 22]. Existence of nontrivial and nonnegative solutions to polyharmonic equations of exponential growth and without the AR condition has also been carried out in [23]. We also refer the reader to the article [24] where the authors study a number of classes of nonlinear PDEs of exponential growth in the elliptic and subelliptic setting as applications of the Moser–Trudinger and Adams inequalities in bounded and unbounded domains.

Our paper is organized as follows: In Sect. 2, we give some useful definitions and lemmas. In Sect. 3, we deal with subcritical exponential growth in dimension 2 and give the proof of Theorem 3. In Sect. 4, we deal with the existence of a nontrivial nonnegative solution when the nonlinear term f has critical exponential growth in dimension 2 and provide the proof of Theorem 4. The result about the system is proved in Sect. 5. In Appendix A, we discuss what conditions (H2) in [17] and our condition (L3) mean and provide examples to show that our (L3) is weaker than condition (H2) in [17]. Moreover, we discuss some different situations of critical growth in this section.

2 Preliminary Results

Let Ω be a bounded domain in $\mathbb{R}^2$. We denote

$$\begin{aligned}\|u\| &= \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}\\ \|u\|_2 &= \left(\int_\Omega |u|^2 dx\right)^{1/2}\\ \lambda_1(\Omega) &= \inf\left\{\frac{\|u\|^2}{\|u\|_2^2} : u \in W_0^{1,2}(\Omega)\setminus\{0\}\right\}\\ d &= \text{radius of the largest open ball} \subset \Omega\end{aligned}$$

Define the Euler–Lagrange functional associated with problem (P):

$$J(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx, \quad u \in W_0^{1,2}(\Omega).$$

From the hypotheses on f, by the standard Moser–Trudinger inequality (see Lemma 3), we can easily see that J is well defined. Also, it's standard to check that J is $C^1(W_0^{1,2}(\Omega),\mathbb{R})$ and

$$DJ(u)v = \int_\Omega \nabla u \nabla v dx - \int_\Omega f(x,u)v dx, \quad v \in W_0^{1,2}(\Omega).$$

Thus, the critical points of J are precisely the weak solutions of problem (P). We will prove the existence of such critical points by the Mountain Pass Theorem.

Definition 1 Let $(X, \|\cdot\|_X)$ be a real Banach space with its dual space $(X^*, \|\cdot\|_{X^*})$ and $I \in C^1(X,\mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies the $(PS)_c$ condition if for any sequence $\{x_n\} \subset X$ with

$$I(x_n) \to c, \qquad DI(x_n) \to 0 \quad \text{in } X^*$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X. Also, we say that I satisfies the $(\mathrm{C})_c$ condition if for any sequence $\{x_n\} \subset X$ with

$$I(x_n) \to c, \qquad \left\| DI(x_n) \right\|_{X^*} \left(1 + \|x_n\|_X\right) \to 0$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X.

We have the following versions of the Mountain Pass Theorem (see [5, 10, 12, 13]):

Lemma 1 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $I \in C^1(X, \mathbb{R})$ satisfy the $(\mathrm{C})_c$ condition for any $c \in \mathbb{R}$, $I(0) = 0$ and*

(i) *There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.*
(ii) *There is an $e \in X \setminus B_\rho$ such that $I(e) \leq 0$.*

Then $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \geq \alpha$ is a critical value of I where

$$\Gamma = \left\{ \gamma \in C^0\left([0, 1], X\right), \gamma(0) = 0, \gamma(1) = e \right\}.$$

Lemma 2 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $I \in C^1(X, \mathbb{R})$ satisfy $I(0) = 0$ and*

(i) *There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.*
(ii) *There is an $e \in X \setminus B_\rho$ such that $I(e) \leq 0$.*

Let C_M be characterized by

$$C_M = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I\left(\gamma(t)\right),$$

where

$$\Gamma = \left\{ \gamma \in C^0\left([0, 1], X\right), \gamma(0) = 0, \gamma(1) = e \right\}.$$

Then I possesses a $(\mathrm{C})_{C_M}$ sequence.

What motivates our work is the so-called Moser–Trudinger inequality which can be found in [33]. As we know, if $U \subset \mathbb{R}^N$ is a bounded domain, then the Sobolev embedding theorem states that $W_0^{1,2}(U) \subset L^p(U)$, for $1 \leq p \leq 2^* = \frac{2N}{N-2}$, or equivalently,

$$\sup_{u \in W_0^{1,2}(U), \|u\| \leq 1} \int_U |u|^p dx \leq C(U), \quad \text{for } 1 \leq p \leq 2^*,$$

while the supremum is infinite for $p > 2^*$. In the case $N = 2$, every polynomial growth is admitted, but one knows by easy examples that $W_0^{1,2}(U) \nsubseteq L^\infty(U)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to \mathbb{R}^+$ with maximal growth such that

$$\sup_{u \in W_0^{1,2}(U), \|u\| \le 1} \int_U g(u) dx < \infty.$$

It was shown by Trudinger [43] and Moser [33] that the maximal growth is of exponential type. More precisely, we have the following lemma.

Lemma 3 *Let* $u \in W_0^{1,2}(\Omega)$; *then* $\exp(|u|^2) \in L^p(\Omega)$ *for all* $1 \le p < \infty$. *Moreover,*

$$\sup_{u \in W_0^{1,2}(\Omega), \|u\| \le 1} \int_\Omega \exp\bigl(\alpha |u|^2\bigr) dx \le C(\Omega) \quad \textit{for } \alpha \le 4\pi.$$

The inequality is optimal: For any growth $\exp(\alpha |u|^2)$ *with* $\alpha > 4\pi$, *the corresponding supremum is* $+\infty$.

We also refer to the interested reader to the survey papers of A.S.Y. Chang and P. Yang [11] and the authors [24] for more applications of Moser–Trudinger inequalities in different directions.

3 Subcritical Exponential Growth—Proof of Theorem 3

In this section, we will study the problem (P) in the case $N = 2$ and f satisfies the (SCE). As far as we know, this is the first work without using the AR condition in subcritical exponential growth.

3.1 The Geometry of the Functional J

In this subsection, we will check the Mountain Pass properties of the functional J.

Lemma 4 *Let* f *satisfy* (L2). *Then* $J(tu) \to -\infty$ *as* $t \to \infty$ *for all nonnegative functions* $u \in W_0^{1,2}(\Omega) \setminus \{0\}$.

Proof Let $u \in W_0^{1,2}(\Omega) \setminus \{0\}$, $u \ge 0$. By (L2), there exist $M > \frac{\|u\|^2}{2\|u\|_2^2} > 0$ and A such that for all $(x, s) \in \Omega \times \mathbb{R}^+$

$$F(x, s) \ge Ms^2 - A. \tag{3.1}$$

Then

$$\begin{aligned} J(tu) &\le \frac{t^2}{2}\|u\|^2 - Mt^2 \int_\Omega |u|^2 dx + O(1) \\ &= t^2 \left(\frac{\|u\|^2}{2} - M \int_\Omega |u|^2 dx \right) + O(1). \end{aligned}$$

Since $M > \frac{\|u\|^2}{2\|u\|_2^2}$, we have $J(tu) \to -\infty$ as $t \to \infty$. $\square$

Lemma 5 *Let f satisfy* (L1), (L4), *and* (SCE). *Then there exist $\delta, \rho > 0$ such that*

$$J(u) \geq \delta \quad \text{if } \|u\| = \rho.$$

Proof By (L4) and (SCE), there exist $\kappa, \tau > 0$ and $q > 2$ such that

$$F(x,s) \leq \frac{1}{2}(\lambda_1 - \tau)|s|^2 + C\exp\big(\kappa|s|^2\big)|s|^q, \quad \forall (x,s) \in \Omega \times \mathbb{R}. \tag{3.2}$$

By Hölder's inequality and the Moser–Trudinger embedding, we have

$$\begin{aligned}\int_\Omega \exp\big(\kappa|u|^2\big)|u|^q dx &\leq \left(\int_\Omega \exp\left(\kappa r\|u\|^2\left(\frac{|u|}{\|u\|}\right)^2\right)dx\right)^{1/r}\left(\int_\Omega |u|^{r'q}dx\right)^{1/r'} \\ &\leq C\left(\int_\Omega |u|^{r'q}dx\right)^{1/r'},\end{aligned}$$

if $r > 1$ sufficiently close to 1 and $\|u\| \leq \sigma$, where $\kappa r\sigma^2 < 4\pi$. Thus by the definition of λ_1 and the Sobolev embedding,

$$J(u) \geq \frac{1}{2}\left(1 - \frac{(\lambda_1 - \tau)}{\lambda_1}\right)\|u\|^2 - C\|u\|^q.$$

Since $\tau > 0$ and $q > 2$, we may choose $\rho, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\| = \rho$. □

The following is the main lemma in this paper.

Lemma 6 *Assume that* (L1), (L2), (L3), *and* (L4) *hold. If f has subcritical growth on Ω* (SCE) *then J satisfies* $(\mathrm{C})_c$ *for all $c \in \mathbb{R}$.*

Proof Let $\{u_n\}$ be a Cerami sequence in $W_0^{1,2}(\Omega)$ such that

$$\begin{aligned}\big(1 + \|u_n\|\big)\big\|DJ(u_n)\big\| &\to 0 \\ J(u_n) &\to c,\end{aligned}$$

i.e.,

$$\begin{aligned}\big(1 + \|u_n\|\big)\left|\int_\Omega \nabla u_n \nabla v dx - \int_\Omega f(x,u_n)v dx\right| &\leq \varepsilon_n\|v\| \\ \frac{1}{2}\|u_n\|^2 - \int_\Omega F(x,u_n)dx &\to c,\end{aligned} \tag{3.3}$$

where $\varepsilon_n \overset{n\to\infty}{\to} 0$. We first show that $\{u_n\}$ is bounded, which is our main purpose in this paper. Indeed, suppose that

$$\|u_n\| \to \infty. \tag{3.4}$$

Setting

$$v_n = \frac{u_n}{\|u_n\|},$$

then $\|v_n\| = 1$, so we can suppose that $v_n \rightharpoonup v$ in $W_0^{1,2}(\Omega)$. We may similarly show that $v_n^+ \rightharpoonup v^+$ in $W_0^{1,2}(\Omega)$, where $w^+ = \max\{w, 0\}$. Since Ω is bounded, Sobolev's embedding theorem implies that

$$\begin{cases} v_n^+(x) \to v^+(x) & \text{a.e. in } \Omega \\ v_n^+ \to v^+ & \text{in } L^p(\Omega),\ \forall p \geq 1. \end{cases}$$

We will prove that $v^+ = 0$ a.e. Ω. Indeed, suppose that $\mu(\Omega^+) = \mu\{x \in \Omega : v^+(x) > 0\} > 0$. Then in Ω^+, we have

$$\lim_{n\to\infty} u_n^+(x) = \lim_{n\to\infty} v_n^+(x)\|u_n\| = +\infty,$$

and thus by (L2),

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^2} = +\infty \quad \text{a.e. in } \Omega^+.$$

This means that

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^2} \left|v_n^+(x)\right|^2 = +\infty \quad \text{a.e. in } \Omega^+. \tag{3.5}$$

Also, by (3.3), we see that

$$\|u_n\|^2 = 2c + 2\int_\Omega F\big(x, u_n^+(x)\big)dx + o(1), \tag{3.6}$$

which implies that

$$\int_\Omega F\big(x, u_n^+(x)\big)dx \to +\infty. \tag{3.7}$$

Now, note that $F(x, s) \geq 0$, by Fatou's lemma and (3.5), (3.6), and (3.7):

$$\begin{aligned} +\infty &= \int_{\Omega^+} \liminf_{n\to\infty} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^2} \left|v_n^+(x)\right|^2 dx \\ &\leq \liminf_{n\to\infty} \int_{\Omega^+} \frac{F(x, u_n^+(x))}{|u_n^+(x)|^2} \left|v_n^+(x)\right|^2 dx \\ &\leq \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n^+(x))}{\|u_n\|^2} dx \\ &= \liminf_{n\to\infty} \frac{\int_{\Omega^+} F(x, u_n^+(x))dx}{2c + 2\int_\Omega F(x, u_n^+(x))dx + o(1)} \\ &= \frac{1}{2}. \end{aligned}$$

This is a contradiction. So we get $v \leq 0$ a.e.

In fact, we have $v = 0$ a.e. Indeed, since

$$\big(1 + \|u_n\|\big)\big|DJ(u_n)v\big| = \big(1 + \|u_n\|\big)\left|\int_\Omega \nabla u_n \nabla v dx - \int_\Omega f(x, u_n) v dx\right| \le \varepsilon_n \|v\|,$$

we get

$$\int_\Omega \nabla u_n \nabla v dx \le \int_\Omega \nabla u_n \nabla v dx - \int_\Omega f(x, u_n) v dx \le \frac{\varepsilon_n \|v\|}{(1 + \|u_n\|)} \to 0$$

by noticing that since $v \le 0$, $f(x, u_n)v \le 0$ a.e. Ω, thus $-\int_\Omega f(x, u_n)v \ge 0$. So we have

$$\int_\Omega \nabla v_n \nabla v dx = \frac{\int_\Omega \nabla u_n \nabla v dx}{\|u_n\|} \le \frac{\varepsilon_n \|v\|}{(1 + \|u_n\|)\|u_n\|} \to 0.$$

On the other hand, since $v_n \rightharpoonup v$ in $W_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla v_n \nabla v dx \to \int_\Omega |\nabla v|^2 dx,$$

which implies $v = 0$.

Next, let $t_n \in [0, 1]$ such that

$$J(t_n u_n) = \max_{t \in [0,1]} J(t u_n).$$

For all $R > 0$, by (SCE), there exists $C > 0$ such that

$$F(x, s) \le C|s| + \exp\left(\frac{4\pi}{R^2} s^2\right), \quad \forall (x, s) \in \Omega \times \mathbb{R}. \tag{3.8}$$

Also, since $\|u_n\| \to \infty$, we have

$$J(t_n u_n) \ge J\left(\frac{R}{\|u_n\|} u_n\right) = J(R v_n), \tag{3.9}$$

and by (3.8) and noting that $\|v_n\| = 1$,

$$2J(R v_n) \ge R^2 - 2CR \int_\Omega \big|v_n(x)\big| dx - 2\int_\Omega \exp\big(4\pi v_n^2(x)\big) dx. \tag{3.10}$$

By the Moser–Trudinger inequality (Lemma 3), $\int_\Omega \exp(4\pi v_n^2(x))dx$ is bounded by a constant $C(\Omega) > 0$. Also, since $v_n \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$, $\int_\Omega |v_n(x)| dx \to 0$. Thus if we let $n \to \infty$ in (3.10), and then let $R \to \infty$ and using (3.9), we get

$$J(t_n u_n) \to \infty. \tag{3.11}$$

Note that $J(0) = 0$ and $J(u_n) \to c$; we can then suppose that $t_n \in (0, 1)$. Since $DJ(t_n u_n) t_n u_n = 0$, we have

$$t_n^2 \|u_n\|^2 = \int_\Omega f(x, t_n u_n) t_n u_n dx.$$

Also, by (3.3),

$$\int_\Omega \big[f(x,u_n)u_n - 2F(x,u_n)\big]dx = \|u_n\|^2 + 2c - \|u_n\|^2 + o(1)$$
$$= 2c + o(1).$$

So by (L3),

$$\begin{aligned} 2J(t_nu_n) &= t_n^2\|u_n\|^2 - 2\int_\Omega F(x,t_nu_n)dx \\ &= \int_\Omega \big[f(x,t_nu_n)t_nu_n - 2F(x,t_nu_n)\big]dx \\ &\le \theta\int_\Omega \big[f(x,u_n)u_n - 2F(x,u_n)\big]dx + O(1) \\ &\le O(1), \end{aligned}$$

which is a contradiction to (3.11). This proves that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Without loss of generality, we can suppose that

$$\begin{cases} \|u_n\| \le K & \\ u_n \rightharpoonup u & \text{in } W_0^{1,2}(\Omega) \\ u_n(x) \to u(x) & \text{a.e. } \Omega \\ u_n \longrightarrow u & \text{in } L^p(\Omega),\ \forall p \ge 1. \end{cases}$$

Now, since f has subcritical growth on Ω, we can find a constant $c_K > 0$ such that

$$f(x,s) \le c_K \exp\left(\frac{2\pi}{K^2}|s|^2\right), \quad \forall (x,s) \in \Omega \times \mathbb{R}.$$

Then by the Moser–Trudinger inequality,

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\le \int_\Omega \big|f(x,u_n)(u_n-u)\big|dx \\ &\le \left(\int_\Omega \big|f(x,u_n)\big|^2dx\right)^{1/2}\left(\int_\Omega |u_n-u|^2dx\right)^{1/2} \\ &\le C\left(\int_\Omega \exp\left(\frac{4\pi}{K^2}|u_n|^2\right)dx\right)^{1/2}\|u_n-u\|_2 \\ &\le C\left(\int_\Omega \exp\left(\frac{4\pi}{K^2}\|u_n\|^2\left|\frac{u_n}{\|u_n\|}\right|^2\right)dx\right)^{1/2}\|u_n-u\|_2 \\ &\le C\|u_n-u\|_2 \overset{n\to\infty}{\to} 0. \end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, $\int_\Omega f(x,u)(u_n-u)dx \to 0$. Thus we can conclude that

$$\int_\Omega \big(f(x,u_n)-f(x,u)\big)(u_n-u)dx \overset{n\to\infty}{\to} 0. \tag{3.12}$$

Also, by (3.3) we have

$$\big\langle DJ(u_n)-DJ(u),(u_n-u)\big\rangle \overset{n\to\infty}{\to} 0. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\|u_n-u\| \overset{n\to\infty}{\to} 0.$$

Thus $u_n \overset{n\to\infty}{\to} u$ strongly in $W_0^{1,2}(\Omega)$, which means that J satisfies $(\mathrm{C})_c$. □

3.2 Proof of Theorem 3

Using Lemmas 4, 5, 6, and the Mountain Pass Theorem (Lemma 1), we can easily deduce that the problem (P) has a nontrivial weak solution.

4 The Case of Critical Exponential Growth—Proof of Theorem 4

4.1 Proof of Theorem 4

In this subsection, we study the problem (P) where Ω is the bounded domain in $\mathbb{R}^2$ and f has critical growth (CG), say, at $\alpha_0>0$.

Note that the condition (L3) in this section is the condition (L3) in the previous two sections with $\theta=1$ and $C^*=0$.

Proof Similar to the previous two sections, by our conditions, we see that our Euler–Lagrange functional associated with the problem (P) has the Palais–Smale geometry properties (Lemmas 4 and 5). Now we consider the Moser functions

$$\widetilde{M}_n(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & 0\le |x|\le 1/n\\ \frac{\log(1/|x|)}{\sqrt{\log n}}, & 1/n\le |x|\le 1\\ 0, & 1\le |x|.\end{cases}$$

We see that $\widetilde{M}_n \in W_0^{1,2}(B_1(0))$ and $\|\widetilde{M}_n\|=1, \forall n\in\mathbb{N}$. Since d is the inner radius of Ω, we can find $x_0\in\Omega$ such that $B_d(x_0)\subset\Omega$. Letting $M_n(x)=\widetilde{M}_n(\frac{x-x_0}{d})$, which are in $W_0^{1,2}(\Omega)$, $\|M_n\|=1$, and $\operatorname{supp} M_n=B_d(x_0)$. As in the proof of Theorem 1.3 in [17], we can conclude that

$$\max\big\{J(tM_n): t\ge 0\big\} < \frac{2\pi}{\alpha_0}.$$

Now thanks to Lemmas 4, 5, and 2, there exists a Cerami sequence $\{u_n\}$ in $W_0^{1,2}(\Omega)$ such that

$$\begin{aligned}\big(1+\|u_n\|\big)\big\|DJ(u_n)\big\| &\to 0\\ J(u_n)\to C_M &< \frac{2\pi}{\alpha_0}.\end{aligned} \tag{4.1}$$

We again want to show that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Indeed, if we again suppose that $\{u_n\}$ is unbounded, then similarly to the previous two sections, we can get that

$$v_n \rightharpoonup 0 \quad \text{in } W_0^{1,2}(\Omega) \quad \text{where } v_n = \frac{u_n}{\|u_n\|}.$$

Let $t_n \in [0,1]$ such that

$$J(t_n u_n) = \max_{t\in[0,1]} J(tu_n).$$

Let $R \in (0, \sqrt{\frac{4\pi}{\alpha_0}})$ and choose $\varepsilon = \frac{4\pi}{R^2} - \alpha_0 > 0$; by (CG), there exists $C>0$ such that

$$F(x,s) \le C|s| + \left|\frac{4\pi}{R^2} - \alpha_0\right| \exp\big((\alpha_0+\varepsilon)s^2\big), \quad \forall (x,s)\in \Omega\times\mathbb{R}. \tag{4.2}$$

Also, since $\|u_n\|\to\infty$, we have

$$J(t_n u_n) \ge J\left(\frac{R}{\|u_n\|}u_n\right) = J(Rv_n), \tag{4.3}$$

and by (4.2) and note $\|v_n\|=1$,

$$2J(Rv_n) \ge R^2 - 2CR\int_\Omega \big|v_n(x)\big|dx - 2\left|\frac{4\pi}{R^2}-\alpha_0\right| \int_\Omega \exp\big((\alpha_0+\varepsilon)R^2 v_n^2(x)\big)dx. \tag{4.4}$$

By the Moser–Trudinger inequality (Lemma 3),

$$\int_\Omega \exp\big((\alpha_0+\varepsilon)R^2v_n^2(x)\big)dx = \int_\Omega \exp\left(\frac{4\pi}{R^2}R^2 v_n^2(x)\right)dx$$

is bounded by a universal constant $C(\Omega)>0$ thanks to the choice of ε and Lemma 3. Also, since $v_n \rightharpoonup 0$ in $W_0^{1,2}(\Omega)$, $\int_\Omega |v_n(x)|dx \to 0$. Thus if we let $n\to\infty$ in (4.4), and then let $R \to \sqrt{\frac{4\pi}{\alpha_0}}^{-}$ and using (4.3), we get

$$\liminf_{n\to\infty} J(t_n u_n) \ge \frac{2\pi}{\alpha_0} > C_M. \tag{4.5}$$

Note that $J(0)=0$ and $J(u_n)\to C_M$; we can suppose that $t_n\in(0,1)$. Thus since $DJ(t_nu_n)t_nu_n = 0$,

$$t_n^2\|u_n\|^2 = \int_\Omega f(x,t_nu_n)t_nu_n dx.$$

Also, by (4.1)

$$\int_{\Omega}\big[f(x,u_n)u_n - 2F(x,u_n)\big]dx = \|u_n\|^2 + 2C_M - \|u_n\|^2 + o(1)$$
$$= 2C_M + o(1).$$

So by (L3),

$$\begin{aligned} 2J(t_n u_n) &= t_n^2\|u_n\|^2 - 2\int_{\Omega} F(x,t_n u_n)dx \\ &= \int_{\Omega}\big[f(x,t_n u_n)t_n u_n - 2F(x,t_n u_n)\big]dx \\ &\le \int_{\Omega}\big[f(x,u_n)u_n - 2F(x,u_n)\big]dx \\ &= 2C_M + o(1), \end{aligned}$$

which is a contradiction to (4.5). This proves that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Without loss of generality, we can suppose that

$$\begin{cases} \|u_n\| \le K & \\ u_n \rightharpoonup u & \text{in } W_0^{1,2}(\Omega) \\ u_n(x) \to u(x) & \text{a.e. } \Omega \\ u_n \longrightarrow u & \text{in } L^p(\Omega),\ \forall p \ge 1. \end{cases}$$

Now, following the proof of Lemma 4 in [34] for the case $N = 2$, we can prove that u is a weak solution of (P). So the last remaining thing that we need to show is the nontriviality of u. However, we can get this thanks to our technical assumption (L6). Indeed, suppose $u = 0$. Similarly as in [34] for the case $N = 2$, we get $f(x,u_n) \to 0$ in $L^1(\Omega)$. Thanks to (L6), $F(x,u_n) \to 0$ in $L^1(\Omega)$ and we can get

$$\lim_{n\to\infty} \|u_n\|^2 = 2C_M < \frac{4\pi}{\alpha_0}$$

and again, following the proof in [34], we have a contradiction. The proof is now completed. □

5 System of Equations—Proof of Theorem 6

5.1 Abstract Framework

The functional associated with system (S) is

$$I : E := W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$$

$$I(u,v)=\int_\Omega \nabla u\nabla v-\int_\Omega F(u)-\int_\Omega G(v).$$

The norm of an element $z=(u,v)$ in E is defined by $\|z\|=(\|u\|^2+\|v\|^2)^{1/2}$. Again, since we are interested in positive solutions, we define f and g to be zero on $(-\infty,0]$. Moreover, it is easy to check that I is well defined and C^1 with

$$\begin{aligned}DI(u,v)(\varphi,\psi)=&\int_\Omega [\nabla u\nabla\psi+\nabla v\nabla\varphi]\\ &-\int_\Omega \big[f(u)\varphi+g(v)\psi\big]dx,\quad \text{for all } (\varphi,\psi)\in E.\end{aligned}$$

As a consequence, the weak solutions of (S) are the critical points of I. We will find these critical points using the Linking Theorem.

Lemma 7 *The following inequality holds.*

$$st\le\begin{cases}(e^{t^2}-1)+s(\log^+ s)^{1/2}, & \textit{for all } t\ge 0 \textit{ and } s\ge e^{1/4}\\ (e^{t^2}-1)+\frac{1}{2}s^2, & \textit{for all } t\ge 0 \textit{ and } 0\le s\le e^{1/4}\end{cases}$$

For the proof of this lemma, see [18].

5.2 Proof of Theorem 6

By results in [18], we know that I satisfies the geometry of the Linking Theorem. So now we will prove that under our new condition (K3$'$) instead of (K3), the Cerami sequence is bounded. For the similar result with the Palais–Smale sequence, see Proposition 2.3 in [18].

Lemma 8 *Let $(u_n,v_n)\in E$ such that*

(I1) $I(u_n,v_n)=c+\delta_n$, *where* $\delta_n\to 0$ *as* $n\to\infty$.
(I2) $(1+\|(u_n,v_n)\|)|DI(u_n,v_n)(\varphi,\psi)|\le\varepsilon_n\|(\varphi,\psi)\|$ *for* $(\varphi,\psi)\in E$, *where* $\varepsilon_n\to 0$ *as* $n\to\infty$.

Then

$$\|u_n\|\le C,\qquad \|v_n\|\le C$$

$$\int_\Omega f(u_n)u_n\le C,\qquad \int_\Omega g(v_n)v_n\le C$$

$$\int_\Omega F(u_n)\le C,\qquad \int_\Omega G(v_n)\le C.$$

Proof From (I1), we have

$$\int_\Omega \nabla u_n\nabla v_n-\int_\Omega F(u_n)-\int_\Omega G(v_n)=c+\delta_n. \tag{5.1}$$

Choosing $(\varphi, \psi) = (u_n, 0)$ and $(\varphi, \psi) = (0, v_n)$, we get from (I2),

$$\left| \int_\Omega \nabla u_n \nabla v_n - \int_\Omega f(u_n) u_n \right| \leq \varepsilon_n$$
$$\left| \int_\Omega \nabla u_n \nabla v_n - \int_\Omega g(v_n) v_n \right| \leq \varepsilon_n. \tag{5.2}$$

Finally, choosing $(\varphi, \psi) = (v_n, 0)$ and $(\varphi, \psi) = (0, u_n)$, we receive

$$\left| \|v_n\|^2 - \int_\Omega f(u_n) v_n \right| \leq \varepsilon_n$$
$$\left| \|u_n\|^2 - \int_\Omega g(v_n) u_n \right| \leq \varepsilon_n. \tag{5.3}$$

From (5.1) and (5.2), we have

$$\begin{aligned} 2c + 2\delta_n &= \int_\Omega \nabla u_n \nabla v_n - 2\int_\Omega F(u_n) + \int_\Omega \nabla u_n \nabla v_n - 2\int_\Omega G(v_n) \\ &\geq \int_\Omega f(u_n) u_n - 2\int_\Omega F(u_n) + \int_\Omega g(v_n) v_n - 2\int_\Omega G(v_n) - 2\varepsilon_n. \end{aligned}$$

Using (K3$'$), we get

$$\begin{aligned} O(1) &\geq \int_\Omega g(v_n) v_n - 2\int_\Omega G(v_n) \\ &\geq O(1) + \left(1 - \frac{2}{\theta}\right) \int_\Omega g(v_n) v_n, \end{aligned}$$

which means that $\int_\Omega g(v_n) v_n$ is bounded.

Now, since f, g have subcritical or critical growth, we can find constants $D, \alpha > 0$ such that $f(t), g(t) \leq De^{\alpha u^2}, \forall u \geq 0$.

Using Lemma 7 for $t = \frac{u_n}{\|u_n\|}$ and $s = \frac{g(v_n)}{D}$ and the Moser–Trudinger inequality, we have

$$\begin{aligned} \int_\Omega g(v_n) \frac{u_n}{\|u_n\|} &= D \int_\Omega \frac{g(v_n)}{D} \frac{u_n}{\|u_n\|} \\ &\leq D \int_\Omega \exp\left[\left(\frac{u_n}{\|u_n\|}\right)^2\right] \\ &\quad + D \int_{\Omega \cap \{\frac{g(v_n)}{D} \geq e^{1/4}\}} \frac{g(v_n)}{D} \left[\log^+ \frac{g(v_n)}{D}\right]^{1/2} dx \\ &\quad + \frac{1}{2} \int_{\Omega \cap \{\frac{g(v_n)}{D} \leq e^{1/4}\}} \left(\frac{g(v_n)}{D}\right)^2 \\ &\leq O(1) + \sqrt{\alpha} \int_\Omega g(v_n) v_n. \end{aligned}$$

From this estimate and (5.3), we can conclude that $\|u_n\|$ is bounded.

So far, we have proved that $\int_\Omega g(v_n)v_n, \int_\Omega f(u_n)u_n, \int_\Omega F(u_n), \int_\Omega G(v_n)$, and $\|u_n\|$ are bounded. It remains to prove that $\|v_n\|$ are bounded. Indeed, using Lemma 7 again with $t = \frac{v_n}{\|v_n\|}$ and $s = \frac{f(v_n)}{D}$, we have by using the same argument as above that

$$\int_\Omega f(u_n)\frac{v_n}{\|v_n\|} \le O(1) + \sqrt{\alpha}\int_\Omega f(u_n)u_n.$$

Again, using (5.3) and the fact that $\int_\Omega f(u_n)u_n$ is bounded, we get $\|v_n\|$ is bounded.
Thus, the proof is completed. □

Now, following the proof in [18], we can prove the existence of nontrivial solutions to (S).

Appendix A: Examples of Weaker Nonlinearity

In this section, we will discuss and compare the conditions (H2) in [17] and (L3) in our work and also give some examples to illustrate that our condition (L3) is weaker than (H2) in [17] and the strict inequality of (A.2) of [1]: $f'(x,u) > \frac{f(x,u)}{u}$. Therefore, it is worthwhile to study the existence of nontrivial solutions to problem (P) under our condition (L3).

Let us see what the condition (H2) in [17] really means. First, we recall the (H2) condition:

(H2) $\exists t_0 > 0, \exists M > 0$ such that $\forall |u| \ge t_0, \forall x \in \Omega$,

$$0 < F(x,u) = \int_0^u f(x,t)dt \le M\big|f(x,u)\big|.$$

From

$$0 < F(x,u) \le M\big|f(x,u)\big|,$$

we have

$$0 < \frac{1}{M} \le \frac{|f(x,u)|}{F(x,u)},$$

which is

$$\left[\ln\left(\frac{F(x,u)}{e^{u/M}}\right)\right]' \ge 0 \tag{A.1}$$

when $f(x,u)$ is nonnegative. Thus the function $\frac{F(x,u)}{e^{Cu}}$ is nondecreasing for some small positive constant C when u is big enough. So, if $F(x,u) = P(u)e^{\alpha u^2}$ where P is a polynomial; the condition (H2) is satisfied since both terms $P(u)$ and $e^{\alpha u^2 - cu}$ are increasing when u is big enough. However, if we have periodic terms or decreasing terms in the nonlinear terms as the following example shows, (A.1) may not be satisfied and thus the condition (H2) may not hold anymore.

Example 1 The nonlinearity $f(x,u) = e^u \cos u + (\sqrt{2} + \sin u)e^u$ doesn't satisfy the condition (H2) in [17]. Indeed, it's easy to see that in this case, $F(u) = (\sqrt{2} + \sin u)e^u$. So if there exists a constant $M > 0$ such that

$$(\sqrt{2} + \sin u)e^u \le M(\sqrt{2} + \sin u + \cos u)e^u,$$

then

$$\sqrt{2} + \sin u \le M(\sqrt{2} + \sin u + \cos u),$$

i.e.,

$$(1 - M)\sin u - M\cos u \le \sqrt{2}(M - 1).$$

However, we can choose u such that

$$\begin{aligned} \big[(1-M)\sin u - M\cos u\big]^2 &\approx \big[(1-M)^2 + M^2\big]\big[\sin^2 u + \cos^2 u\big] \\ &= 2M^2 - 2M + 1 \\ &> 2(M-1)^2, \end{aligned}$$

which is a contradiction. This shows that $f(x,u)$ does not satisfy condition (H2) in [17].

Next, let us discuss what our condition (L3) means. We recall that

(L3) There are $C_* \ge 0, \theta \ge 1$ such that $H(x,t) \le \theta H(x,s) + C_*$ for all $0 < t < s$, $\forall x \in \Omega$, where $H(x,u) = uf(x,u) - 2F(x,u)$.

The condition (L3) suggests a sort of "weak" nondecreasing property of the function $H(x,t)$. In particular, a nondecreasing function $H(x,t)$ in t variable satisfies our condition (L3) (with $\theta = 1$ and $C_* = 0$). Now suppose that f' (in terms of u) exists, then $H(x,t)$ being nondecreasing is equivalent to $(uf(x,u) - 2F(x,u))' \ge 0$, which is in turn equivalent to

$$f^{'}(x,u) \ge \frac{f(x,u)}{u} \quad \text{for all } 0 < u, \quad \forall x \in \Omega. \tag{A.2}$$

This kind of condition was assumed in the work of Adimurthi [1] with strict inequality in (A.2) in order to get the existence of positive solutions of the semilinear Dirichlet problem with critical exponential growth. Indeed, as mentioned in [34], Adimurthi assumed that f is C^1 and satisfies $f^{'}(x,u) > \frac{f(x,u)}{u}$ for all $u \ne 0, \forall x \in \Omega$ in his paper [1]. In other words, our condition (L3) (even with $\theta = 1$ and $C_* = 0$) is weaker than the condition of Adimurthi. In the following example, we will give an example of a nonlinearity which satisfies our condition (L3) but does not satisfy Adimurthi's condition.

Example 2 Consider the function $f(x,u) = u(u-1)^3 e^u$, which implies

$$\big(f(x,u)\big)' = \big[u(u-1)^3 + (u-1)^3 + 3u(u-1)^2\big]e^u.$$

So we can see that $f'(x,u) \geq \frac{f(x,u)}{u}$ for all $0 < u, \forall x \in \Omega$. Therefore, f satisfies (A.2) and thus our condition (L3). However, when $u = 1$, the equality holds, which means that f does not satisfy Adimurthi's condition of strict inequality [1].

If we further assume that $f(x,u)$ is positive, then (A.2) gives

$$\frac{f'(x,u)}{f(x,u)} \geq \frac{1}{u} \quad \text{for all } 0 < u, \quad \forall x \in \Omega,$$

which thus implies that the function $\frac{f(x,u)}{u}$ is nondecreasing for all $0 < u, \forall x \in \Omega$. The assumption that the function $\frac{f(x,u)}{u}$ is nondecreasing for all $0 < u, \forall x \in \Omega$ is also a standard condition and is assumed in many works. In fact, our condition (L3) (even with $\theta = 1$ and $C_* = 0$) is weaker than this standard condition. Indeed, let $g(x,u) = \frac{f(x,u)}{u}$, which is nondecreasing for all $0 < u, \forall x \in \Omega$. We get with $0 < u$, $x \in \Omega$:

$$F(x,u) = \int_0^u sg(x,s)ds \leq g(x,u) \int_0^u sds = \frac{u^2 g(x,u)}{2} = \frac{uf(x,u)}{2},$$

which thus means that $H(x,u) \geq 0$. Moreover, with $0 < u < v, x \in \Omega$, we have

$$\begin{aligned} 2\int_0^v sg(x,s)ds - 2\int_0^u sg(x,s)ds = 2\int_u^v sg(x,s)ds \\ &\leq 2g(x,v)\int_u^v sdx \\ &= v^2 g(x,v) - u^2 g(x,v) \\ &\leq v^2 g(x,v) - u^2 g(x,u), \end{aligned}$$

from which we can conclude that

$$H(x,u) \leq H(x,v).$$

Example 3 Consider the function $F(x,u) = u^2 e^{\sqrt{u}}$ and then $f(x,u) = (2u + \frac{u\sqrt{u}}{2})e^{\sqrt{u}}$. We have $\frac{f(x,u)}{u} = (2 + \frac{\sqrt{u}}{2})e^{\sqrt{u}}$, which is a nondecreasing function. This shows that $f(x,u)$ satisfies our condition (L3). Moreover, for every small positive constant C, then $\frac{F(x,u)}{e^{Cu}} = u^2 e^{\sqrt{u}(1-C\sqrt{u})}$ is not always increasing when u is big enough. This means that $f(x,u)$ does not satisfy the condition (H2).

In other words, from Example 3, we can see that there exist nonlinearities that satisfy our condition (L3) but do not satisfy the condition (H2).

A.1 About the Critical Growth

We will finish this paper by analyzing the critical growth of the nonlinearity term $f(x,u)$. We will see that in some cases, we don't need to assume the condition (H2)-type or (H5)-type as in [17]. More precisely, we consider the following three cases: $\lim_{u\to+\infty}\frac{|f(x,u)|}{\exp(\alpha_0|u|^2)}=0$; $\lim_{u\to+\infty}\frac{|f(x,u)|}{\exp(\alpha_0|u|^2)}=c\in(0,\infty)$, and $\lim_{u\to+\infty}\frac{|f(x,u)|}{\exp(\alpha_0|u|^2)}=\infty$.

A.1.1 Case 1

In this subsection, we will discuss the first case,

$$\lim_{u\to+\infty}\frac{f(x,u)}{\exp(\alpha_0|u|^2)}=0,\quad \text{uniformly on } x\in\Omega.$$

This case is easy to study. Indeed, by l'Hôpital's rule, we also get

$$\lim_{u\to+\infty}\frac{F(x,u)}{\exp(\alpha_0|u|^2)}=\lim_{u\to+\infty}\frac{f(x,u)}{2\alpha_0 u\exp(\alpha_0|u|^2)}=0,\quad \text{uniformly on } x\in\Omega.$$

Using l'Hôpital's rule again, we get

$$\lim_{u\to+\infty}\frac{uF(x,u)}{\exp(\alpha_0|u|^2)}=\lim_{u\to+\infty}\frac{uf(x,u)+F(x,u)}{2\alpha_0 u\exp(\alpha_0|u|^2)}=0,\quad \text{uniformly on } x\in\Omega.$$

So if we have the condition of (H5) type, i.e.,

$$\lim_{u\to+\infty}\frac{uf(x,u)}{\exp(\alpha_0|u|^2)}\ge\beta>0,\quad \text{uniformly on } x\in\Omega,$$

we can easily deduce the condition of (H2) type (so we have the AR condition) by noticing that

$$\lim_{u\to+\infty}\frac{uF(x,u)}{\exp(\alpha_0|u|^2)}=0<\beta\le\lim_{u\to+\infty}\frac{uf(x,u)}{\exp(\alpha_0|u|^2)},\quad \text{uniformly on } x\in\Omega.$$

A.1.2 Case 2

Now we will consider the case

$$\lim_{u\to+\infty}\frac{f(x,u)}{\exp(\alpha_0|u|^2)}=c\in(0,\infty),\quad \text{uniformly on } x\in\Omega.$$

In this case, it's clear that

$$\lim_{u\to+\infty}\frac{uf(x,u)}{\exp(\alpha_0|u|^2)}=\infty,\quad \text{uniformly on } x\in\Omega,$$

which means that the condition of (H5) type is satisfied automatically. Also, by l'Hôpital's rule again, we get

$$\lim_{u\to+\infty}\frac{F(x,u)}{\exp(\alpha_0|u|^2)}=\lim_{u\to+\infty}\frac{f(x,u)}{2\alpha_0 u\exp(\alpha_0|u|^2)}=0,\quad \text{uniformly on } x\in\Omega,$$

so the condition of (H2) type is also satisfied automatically by again noticing that

$$\lim_{u\to+\infty}\frac{F(x,u)}{\exp(\alpha_0|u|^2)}=0<c=\lim_{u\to+\infty}\frac{f(x,u)}{\exp(\alpha_0|u|^2)},\quad \text{uniformly on } x\in\Omega.$$

A.1.3 Case 3

We consider the last case,

$$\lim_{u\to+\infty}\frac{f(x,u)}{\exp(\alpha_0|u|^2)}=\infty,\quad \text{uniformly on } x\in\Omega.$$

In this case, the condition of (H5) type is satisfied automatically.

References

1. Adimurthi: Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **17**(3), 393–413 (1990)
2. Adimurthi, Druet, O.: Blow-up analysis in dimension 2 and a sharp form of Trudinger–Moser inequality. Commun. Partial Differ. Equ. **29**(1), 295–322 (2004)
3. Adimurthi, Prashanth, S.: Failure of Palais-Smale condition and blow-up analysis for the critical exponent problem in $\mathbb{R}^2$. Proc. Indian Acad. Sci. Math. Sci. **107**(3), 283–317 (1997)
4. Adimurthi, Yadava, S.L.: Multiplicity results for semilinear elliptic equations in a bounded domain of R^2 involving critical exponent. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **XVII**, 481–504 (1990)
5. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
6. Atkinson, F.V., Peletier, L.A.: Elliptic equations with critical growth. Math. Inst. Univ. Leiden, Rep. 21 (1986)
7. Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. Pure Appl. Math. **36**, 437–477 (1983)
8. Cao, D.: Nontrivial solution of semilinear elliptic equation with critical exponent in R^2. Commun. Partial Differ. Equ. **17**(3–4), 407–435 (1992)
9. Carleson, L., Chang, S.Y.A.: On the existence of an extremal function for an inequality of J. Moser. Bull. Sci. Math. **110**, 113–127 (1986)
10. Chang, K.C.: Critical Point Theory and Its Applications. Modern Mathematics Series. Shanghai Kexue Jishu Chubanshe, Shanghai (1986). 316 pp.
11. Chang, S.Y.A., Yang, P.: The inequality of Moser and Trudinger and applications to conformal geometry. Commun. Pure Appl. Math. **56**(8), 1135–1150 (2003)
12. Cerami, G.: An existence criterion for the critical points on unbounded manifolds. Istit. Lombardo Accad. Sci. Lett. Rend. A **112**(2), 332–336 (1978) (Italian)
13. Cerami, G.: On the existence of eigenvalues for a nonlinear boundary value problem. Ann. Mat. Pura Appl. **124**, 161–179 (1980) (Italian)
14. Costa, D.G., Magalhães, C.A.: Variational elliptic problems which are nonquadratic at infinity. Nonlinear Anal. **23**(11), 1401–1412 (1994)
15. de Figueiredo, D.G.: Semilinear elliptic systems: existence, multiplicity, symmetry of solutions. In: Handbook of Differential Equations: Stationary Partial Differential Equations, vol. V, pp. 1–48
16. de Figueiredo, D.G., do Ó, J.M., Ruf, B.: On an inequality by N. Trudinger and J. Moser and related elliptic equations. Commun. Pure Appl. Math. **55**(2), 135–152 (2002)
17. de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. Calc. Var. Partial Differ. Equ. **3**(2), 139–153 (1995)
18. de Figueiredo, D.G., do Ó, J.M., Ruf, B.: Critical and subcritical elliptic systems in dimension two. Indiana Univ. Math. J. **53**(4), 1037–1054 (2004)

19. Iturriaga, L., Lorca, S., Ubilla, P.: A quasilinear problem without the Ambrosetti-Rabinowitz-type condition. Proc. R. Soc. Edinb. A **140**(2), 391–398 (2010)
20. Jeanjean, L.: On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb. A **129**(4), 787–809 (1999)
21. Lam, N., Lu, G.: Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in R^N. J. Funct. Anal. **262**, 1132–1165 (2012)
22. Lam, N., Lu, G.: N-Laplacian equations in R^N with subcritical and critical growth without the Ambrosetti-Rabinowitz condition (2010). arXiv:1012.5489v1. To appear in Adv. Nonlinear Stud.
23. Lam, N., Lu, G.: Existence of nontrivial solutions to polyharmonic equations with subcritical and critical exponential growth. Discrete Contin. Dyn. Syst. **32**(6), 2187–2205 (2012)
24. Lam, N., Lu, G.: The Moser-Trudinger and Adams inequalities and elliptic and subelliptic equations with nonlinearity of exponential growth. In: Recent Developments in Geometry and Analysis. Advanced Lectures in Mathematics, vol. 23, pp. 179–251 (2012)
25. Li, Y.X.: Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. Partial Differ. Equ. **14**(2), 163–192 (2001)
26. Li, Y.X., Liu, P.: A Moser-Trudinger inequality on the boundary of a compact Riemann surface. Math. Z. **250**(2), 363–386 (2005)
27. Li, Y.X., Liu, P., Yang, Y.: Moser-Trudinger inequalities of vector bundle over a compact Riemannian manifold of dimension 2. Calc. Var. Partial Differ. Equ. **28**(1), 59–83 (2007)
28. Liu, Z., Wang, Z.: On the Ambrosetti-Rabinowitz superlinear condition. Adv. Nonlinear Stud. **4**(4), 563–574 (2004)
29. Li, Y.Q., Wang, Z., Zeng, J.: Ground states of nonlinear Schrödinger equations with potentials. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **23**(6), 829–837 (2006)
30. Lu, G., Yang, Y.: Sharp constant and extremal function for the improved Moser-Trudinger inequality involving L^p norm in two dimension. Discrete Contin. Dyn. Syst. **25**(3), 963–979 (2009)
31. Lu, G., Yang, Y.: A sharpened Moser-Pohozaev-Trudinger inequality with mean value zero in R^2. Nonlinear Anal. **70**(8), 2992–3001 (2009)
32. Miyagaki, O.H., Souto, M.A.S.: Superlinear problems without Ambrosetti and Rabinowitz growth condition. J. Differ. Equ. **245**(12), 3628–3638 (2008)
33. Moser, J.: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. **20**, 1077–1092 (1970/71)
34. do Ó, J.M.: Semilinear Dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range. Differ. Integral Equ. **9**(5), 967–979 (1996)
35. do Ó, J.M., Medeiros, E., Severo, U.: On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$. J. Differ. Equ. **246**(4), 1363–1386 (2009)
36. Papageorgiou, N.S., Rocha, E.M.: Pairs of positive solutions for p-Laplacian equations with sublinear and superlinear nonlinearities which do not satisfy the AR-condition. Nonlinear Anal. **70**(11), 3854–3863 (2009)
37. Rabinowitz, P.: Minimax methods in critical point theory with applications to differential equations. In: CBMS, vol. 65. AMS, Providence (1986)
38. Ruf, B.: A sharp Trudinger-Moser type inequality for unbounded domains in R^2. J. Funct. Anal. **219**(2), 340–367 (2005)
39. Schechter, M., Zou, W.: Superlinear problems. Pac. J. Math. **214**(1), 145–160 (2004)
40. Shaw, M.C.: Eigenfunctions of the nonlinear equation $\Delta u + \nu f(x, u) = 0$ in R^2. Pac. J. Math. **129**(2), 349–356 (1987)
41. Struwe, M., Tarantello, G.: On multivortex solutions in Chern-Simons gauge theory. Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. **1**(1), 109–121 (1998)
42. Szulkin, A., Zou, W.: Homoclinic orbits for asymptotically linear Hamiltonian systems. J. Funct. Anal. **187**(1), 25–41 (2001)
43. Trudinger, N.S.: On imbeddings into Orlicz spaces and some applications. J. Math. Mech. **17**, 473–483 (1967)
44. Willem, M., Zou, W.: On a Schrödinger equation with periodic potential and spectrum point zero. Indiana Univ. Math. J. **52**(1), 109–132 (2003)
45. Zhou, H.: Positive solution for a semilinear elliptic equation which is almost linear at infinity. Z. Angew. Math. Phys. **49**(6), 896–906 (1998)