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EXISTENCE OF NONTRIVIAL SOLUTIONS TO POLYHARMONIC EQUATIONS WITH SUBCRITICAL AND CRITICAL EXPONENTIAL GROWTH

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Abstract. The main purpose of this paper is to establish the existence of nontrivial solutions to semilinear polyharmonic equations with exponential growth at the subcritical or critical level. This growth condition is motivated by the Adams inequality [1] of Moser-Trudinger type. More precisely, we consider the semilinear elliptic equation

$$(-\Delta)^m u = f(x,u),$$

subject to the Dirichlet boundary condition $u = \nabla u = ... = \nabla^{m-1} u = 0$, on the bounded domains $\Omega \subset \mathbb{R}^{2m}$ when the nonlinear term f satisfies exponential growth condition. We will study the above problem both in the case when f satisfies the well-known Ambrosetti-Rabinowitz condition and in the case without the Ambrosetti-Rabinowitz condition. This is one of a series of works by the authors on nonlinear equations of Laplacian in $\mathbb{R}^2$ and $N-$Laplacian in $\mathbb{R}^N$ when the nonlinear term has the exponential growth and with a possible lack of the Ambrosetti-Rabinowitz condition (see [23], [24]).

1. **Introduction.** Sharp Moser-Trudinger's inequality plays an important in geometric analysis and partial differential equations. In 1971, J. Moser [37] sharpened the result of Pohozaev [38] and Trudinger [42] and found the largest positive constant $\beta_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit $n-$ball, such that if Ω is an open subset of Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ with finite Lebesgue measure, then there is a constant C_0 depending only on n such that

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\beta |u(x)|^{\frac{n}{n-1}}\right) dx \leq C_0$$

for any $\beta \leq \beta_n$, any u in the Sobolev space $W_0^{1,n}(\Omega)$, provided $||\nabla u||_{L^n(\Omega)} \leq 1$. Moser also proved that if β exceeds β_n, then the above inequality can not hold with uniform C_0 independent of u.

In 1986, Carleson and Chang [9] proved that the following supremum

$$\sup_{u\in W_0^{1,n}(\Omega), ||\nabla u||_{L^n(\Omega)}\leq 1} \left\{\frac{1}{|\Omega|}\int_\Omega \exp\left(n\omega_{n-1}^{\frac{1}{n-1}}|u(x)|^{\frac{n}{n-1}}\right) dx\right\}$$

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has extremals for the case when Ω is a ball in $\mathbb{R}^n$ for $n \geq 2$. We refer the reader to the works of on existence of extremal functions by Flucher [15] on smooth domains in $\mathbb{R}^n$ when $n = 2$, by Lin [32] for the case $n > 2$, on existence of extremal functions with mean value zero by Leckband [27], Lu and Yang [33] and in L^p norms ([35]) and on existence of extremal functions on Riemannian manifolds by Y.X. Li [28, 29] and Yang [43], and on unbounded domains by Ruf in $\mathbb{R}^2$ [41] and by Y.X. Li and Ruf in $\mathbb{R}^n$ [31]. We also refer the reader to the survey article by S.Y. A. Chang and P. Yang on Moser-Trudinger inequalities and their applications in PDEs [10].

D. Adams was the first one who finds the sharp constants for higher order Moser's inequality [1]. To state Adams' result, we use the symbol $\nabla^m u$, m is a positive integer, to denote the $m-$th order gradient for $u \in C^m$, the class of $m-$th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{for } m \text{ even} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{for } m \text{ odd.} \end{cases}$$

where ∇ is the usual gradient operator and $\triangle$ is the Laplacian. We use $||\nabla^m u||_p$ to denote the L^p norm $(1 \leq p \leq \infty)$ of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $||u||_{L^p(\Omega)} + ||\nabla^k u||_{L^p(\Omega)}$. We will also use the notation $H_0^m(\Omega)$ to denote the space $W_0^{m,2}$ in the subsequent sections of this paper. Then Adams proved the following

Theorem A. *Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|} \int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \leq C_0$$

for all $\beta \leq \beta(n,m)$ where

$$\beta(n,m) = \begin{cases} \frac{n}{w_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\ \frac{n}{w_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is even.} \end{cases}$$

Furthermore, for any $\beta > \beta(n,m)$, the integral can be made arbitrarily large.

Note that $\beta(n,1)$ coincides with Moser's value of β_n and $\beta(2m,m) = 2^{2m}\pi^m\Gamma(m+1)$ for both odd and even m. In the particularly interesting case $n = 4$ and $m = 2$, $\beta(4,2) = 32\pi^2$ and the existence of the extremal function has been established in [30], [34]. The Adams inequality was extended to compact Riemannian manifolds without boundary by Fontana [16].

Motivated by the Adams inequality, we consider the following polyharmonic problem

$$(-\Delta)^m u = f(x,u) \text{ in } \Omega \subset \mathbb{R}^n \tag{1}$$

with Dirichlet boundary conditions

$$u = \nabla u = ... = \nabla^{m-1} u = 0 \text{ on } \partial\Omega$$

Here Ω is a sufficiently smooth (say C^{m-1}) bounded domain of $\mathbb{R}^n$, $n \geq 2$, $m \geq 1$, Δ stands for the Laplace operator and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies some regularity and growth conditions. More precisely, we are interested in existence of nontrivial solutions of (1) when the nonlinear term has the subcritical or critical exponential growth.

When $m = 1$, $n = 2$, problem (1) becomes the well-known Laplacian problem on bounded domains $\Omega \subset \mathbb{R}^2$ and has been investigated in [2, 13, 17, 23, 36]. In fact, it is well known that problems involving the Laplacian appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of propagation phenomena of solitary waves, Newtonian fluids and nonlinear elasticity problems. It also appears in the search for solitons of certain Lorentz-invariant nonlinear field equations. Those authors considered the maximal growth on the nonlinear term $f(x, u)$ which allowed them to treat the equation (1) (in the case $m = 1$) variationally. Here those maximal growths are given by Trudinger-Moser inequality [37, 42] which says that

$$\exp\left(\alpha u^2\right) \in L^1\left(\Omega\right),\ \forall u \in W_0^{1,2}\left(\Omega\right),\ \forall \alpha > 0$$

and

$$\sup_{u \in W_0^{1,2}(\Omega),\ \int_\Omega |\nabla u|^2 dx \leq 1} \int_\Omega \exp\left(\alpha u^2\right) \leq C\left(\Omega\right) < \infty, \text{ if } \alpha \leq 4\pi$$

Therefore, from this result they have naturally associated notions of criticality and subcriticality, namely, they say that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has subcritical growth on $\Omega \subset \mathbb{R}^2$ if

$$\lim_{|u| \to \infty} \frac{|f(x, u)|}{\exp\left(\alpha u^2\right)} = 0, \text{ uniformly on } \Omega,\ \forall \alpha > 0$$

and f has critical growth on Ω if there exists $\alpha_0 > 0$ such that

$$\lim_{|u| \to \infty} \frac{|f(x, u)|}{\exp\left(\alpha u^2\right)} = 0, \text{ uniformly on } \Omega,\ \forall \alpha > \alpha_0$$

and

$$\lim_{|u| \to \infty} \frac{|f(x, u)|}{\exp\left(\alpha u^2\right)} = +\infty, \text{ uniformly on } \Omega,\ \forall \alpha < \alpha_0.$$

We should stress that in those works of Adimurthi, de Figueiredo-Miyagaki-Ruf, Miyagaki-Souto, J.M. do Ó, etc. [2, 13, 17, 23, 36], the Ambrosetti-Rabinowitz condition played an important role. Indeed, this well-known Ambrosetti-Rabinowitz condition ensures the boundedness of the Palais-Smale sequence which is very necessary for using the Mountain-Pass Theorem [3]. There have been some works trying to remove this condition, but only in the case of subcritical polynomial growth. The Ambrosetti-Rabinowtiz condition in the exponential growth (both subcritical and critical case) was first removed for the Laplacian operator in $\mathbb{R}^2$ in [23].

When $m \geq 2$, many authors have studied (1), specially in the case of the biharmonic $m = 2$. See for example, in [14, 6, 18, 20, 21, 26, 39, 40], where the authors treated problem in the case $2m < n$ and the nonlinearity has the subcritical and critical polynomial growth of power $\leq \frac{n+2m}{n-2m}$. It is worthwhile to notice that polyharmonic operators $(-\triangle)^m$ with Dirichlet boundary conditions in general do not satisfy the maximum principle if $m \geq 2$ (see [5] for existence of minimal solutions for biLaplacian).

We now consider in this paper the case $n = 2m$. So it's necessary to introduce the definition of the subcritical (exponential) and critical (exponential) growth in this case. By the Adams' inequalities (see [1]) for the high order derivatives (see Theorem A), namely,

$$\sup_{u\in H_0^m(\Omega),\ \int_\Omega |\nabla^m u|^p dx\le 1} \int_\Omega \exp\left(\alpha |u|^{p'}\right) dx \le C(m,n,\Omega) < \infty, \text{ for all } \alpha \le \beta(m,n)$$

where $p = n/m$ and $p' = p/(p-1)$. Therefore, it is natural to expect that the critical growth functions are, roughly speaking, the nonlinearities that behave like $\exp(|t|^{n/(n-m)})$ at infinity. Since we consider in this paper the case $n = 2m$, we have the same definition for the subcritical and critical (exponential) growth as in the case $m = 1$, $n = 2$. Note that $\beta(2m,m) = m!\,(4\pi)^m$. As we mentioned earlier, for the case of Laplace operator (that is $m = 1$ and $n = 2$), the answer of the existence of nontrivial solutions has been given in many works, such as in [2, 13, 17], etc. Nevertheless, in the case of polyharmonic operators, not much has been done about the existence of nontrivial solution when the nonlinear term $f(x,u)$ satisfies the subcritical and critical exponential growth in the sense of Adams' inequalities. In [22], O. Lakkis used the same method of using Nehari manifold as Adimurthi did when he treated the Laplace operator in [2]. As a result, the author in [22] must require much restrictive conditions on the nonlinear term f. For instance (among others), it was assumed that f is C^1 and satisfies

$$\frac{\partial f}{\partial u}(x,u) > \frac{f(x,u)}{u},\ \forall u \in \mathbb{R}\setminus\{0\},\ \forall x\in\Omega.$$

We also mention that in [4], the authors treated the case when $f(x,u) = e^u$, which is of exponential growth but not of the critical power implied by the Adams inequality.

In this paper, we will consider the problem in the case $n = 2m$ in bounded domains $\Omega \subset \mathbb{R}^n$. More precisely, we study the following problem in both the subcritical and critical cases of exponential growth:

$$\begin{cases} (-\Delta)^m u = f(x,u) \text{ in } \Omega\subset\mathbb{R}^{2m} \\ u = \nabla u = \ldots = \nabla^{m-1} u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{P}$$

We now recall the definition of the subcritical and critical growth for our case $n = 2m$. We say that f has subcritical growth if for all $\alpha > 0$

$$\lim_{t\to\infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = 0,\ \text{uniformly on } x\in\Omega\subset\mathbb{R}^n \tag{SCG}$$

and f has critical growth if there exists $\alpha_0 > 0$ such that

$$\lim_{t\to\infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = 0,\ \text{uniformly on } x\in\Omega,\ \forall\alpha>\alpha_0 \tag{CG}$$

$$\lim_{t\to\infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = \infty,\ \text{uniformly on } x\in\Omega,\ \forall\alpha<\alpha_0$$

As we pointed out earlier, this notion of criticality is motivated by the so-called Trudinger-Moser inequality and Adams' inequalities. We will prove results analogous to [22] under much less restrictive conditions than those in [22]. Our approach is similar to those approaches in [13, 17] for the Laplace operator in domains in $\mathbb{R}^2$ (i.e., $m = 1$ and $n = 2$). Indeed, again in the critical case, our Euler-Lagrange functional does not satisfy the Palais-Smale condition at all level. Similar to the idea used by Brezis and Nirenberg [8], we use the test functions related to the optimal Adams' inequalities to prove that our Euler-Lagrange functional satisfies the Palais-Smale condition in a certain level and that is sufficient to get the nontrivial solution thanks to the Ambrosetti-Rabinowitz (AR) condition that f satisfies.

However, results in [13, 17] do not include the case when f does not satisfies the Ambrosetti-Rabinowitz (AR) condition. In this paper, using the method similar to that in [23, 24], we also consider the existence of nontrivial solutions to the polyharmonic operators when f does not satisfy the well-known Ambrosetti-Rabinowitz condition.

We begin with two types of conditions on f that will be assumed in all theorems below. The (H)-type condition is when f satisfies the well-known Ambrosetti-Rabinowitz (AR) condition and the (L)-type condition is without the (AR) condition.

We first introduce the constant $\Lambda_{2m,m}$ which is defined by

$$\Lambda_{2m,m}(\Omega) = \inf_{H_0^m(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}$$

be the first eigenvalue of the poly-Laplacian operator Δ^m (see [19]). The constant $\mathcal{M}$ in the following conditions will be defined in Section 2.

1.1. **The (H)-type conditions.** This kind of condition includes the well-known Ambrosetti-Rabinowitz condition:

$(H1)$ $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,u) \geq 0$ on $\Omega \times [0,\infty)$ and $f(x,u) = 0$ on $\Omega \times (-\infty, 0]$.

$(H2)$ $\exists R > 0,\ \exists M > 0$ such that $\forall u \geq R,\ \forall x \in \Omega :$

$$0 < F(x,u) \leq M f(x,u)$$

$(H3)$ $\lim\limits_{u\to 0^+} \sup \frac{2F(x,u)}{|u|^2} < \Lambda_{2m,m}(\Omega)$, uniformly on $x \in \Omega$.

$(H4)$ $\lim\limits_{u\to\infty} uf(x,u)\exp(-\alpha_0 |u|^2) \geq \beta_1 > \frac{32\pi^2}{\alpha_0 R^4 \mathcal{M}}$.

We should stress that as a consequence of (H)-type conditions, f automatically satisfies the following

$(H5)$ There is a positive constant C such that $\forall u \geq R_0,\ \forall x \in \Omega :$

$$F(x,u) \geq C \exp\left(\frac{1}{M} u\right)$$

and

$(H6)$ $\exists R_0 > 0,\ \exists \theta > 2$ such that $\forall |u| \geq R_0,\ \forall x \in \Omega :$

$$\theta F(x,u) \leq uf(x,u)$$

The above condition $(H6)$ is exactly the well-known Ambrosetti-Rabinowitz condition.

1.2. **The (L)-type conditions.** This kind of condition is without the well-known Ambrosetti-Rabinowitz condition:

$(L1)$ $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,u) \geq 0$ on $\Omega \times [0,\infty)$ and $f(x,u) = 0$ on $\Omega \times (-\infty, 0]$.

$(L2)$: $\lim\limits_{u\to+\infty} \frac{F(x,u)}{u^2} = +\infty$ uniformly on $x \in \Omega$.

$(L3)$: There is $C_* \geq 0,\ \theta \geq 1$ such that $H(x,t) \leq \theta H(x,s) + C_*$ for all $0 < t < s,\ \forall x \in \Omega$ where $H(x,u) = uf(x,u) - 2F(x,u)$.

$(L4)$ $\lim\limits_{u\to 0^+} \sup \frac{2F(x,u)}{|u|^2} < \Lambda_{2m,m}(\Omega)$, uniformly on $x \in \Omega$.

$(L5)$ $\lim\limits_{u\to\infty} uf(x,u)\exp(-\alpha_0 |u|^2) \geq \beta_1 > \frac{32\pi^2}{\alpha_0 R^4 \mathcal{M}}$.

$(L6)$ f is in the class (L_0), i.e., for any $\{u_k\}$ in $H_0^m(\Omega)$, if $\begin{cases} u_k \rightharpoonup 0 \text{ in } H_0^m(\Omega) \\ f(x,u_k) \to 0 \text{ in } L^1(\Omega) \end{cases}$ then $F(x,u_k) \to 0$ in $L^1(\Omega)$ (up to a subsequence).

We note that the $(L)-$ type condition does not imply the well-known Ambrosetti-Rabinowitz condition $(H6)$.

We end this introduction with the following remarks. This paper, along with a series of works by the authors [23], [24], is an attempt to study the existence of nontrivial nonnegative solutions to nonlinear equations when the nonlinear term satisfies the subcritical and critical exponential growth but does not satisfy the well-known Ambrosetti-Rabinowitz (AR) condition (see also [25] for $N-$Laplace equations with nonlinear terms without satisfying the (AR) condition on unbounded domains). The lack of this condition generates many more kinds of nonlinearity of exponential growth which was not studied in the literature. There are many interesting examples of nonlinear term f which does not satisfy the (AR) condition, but still allows us to have the existence of nontrivial and nonnegative solutions.

More precisely, in [23], we let Ω be a bounded smooth domain in $\mathbb{R}^2$ and consider the following class of semilinear elliptic problems

$$\begin{cases} -\Delta u = f(x,u) \\ u \in W_0^{1,2}(\Omega) \setminus \{0\} \end{cases} \tag{2}$$

Further in [24], we let Ω be a bounded domain in $\mathbb{R}^N$ and consider the following nonlinear elliptic equation of N-Laplacian type:

$$\begin{cases} -\Delta_N u = f(x,u) \\ u \in W_0^{1,2}(\Omega) \setminus \{0\} \end{cases} \tag{3}$$

Both in [23] and [24], the nonlinear term f is of subcritical or critical exponential growth without the Ambrosetti-Rabinowitz condition. Earlier works in the literature on the existence of nontrivial solutions to Laplacian in $\mathbb{R}^2$ and $N-$Laplacian in $\mathbb{R}^N$ when the nonlinear term f has the exponential growth only deal with the case when f satisfies the (AR) condition.

We mention in passing that in a recent paper of the authors [7], we consider the Bessel type polyharmonic equations in the whole space $\mathbb{R}^{2n}$ of the form

$$(I-\Delta)^n u = f(x,u) \text{ in } \mathbb{R}^{4n}.$$

We study the existence of the nontrivial solutions when the nonlinear terms have the critical exponential growth in the sense of Adams' inequalities Our approach is variational methods such as the Mountain Pass Theorem without Palais-Smale condition combining with a version of a result due to Lions for the critical growth case. Moreover, using the regularity lifting by contracting operators and regularity lifting by combinations of contracting and shrinking operators, we prove that our solutions are uniformly bounded and Lipschitz continuous. This appears to be the first work concerning existence and regularity of nontrivial nonnegative solutions of the Bessel type polyharmonic equation with exponential growth of the nonlinearity in the whole Euclidean space.

This paper is organized as follows:

In Section 2, we introduce some notations and state our main results (Theorems 2.1, 2.2, 2.3, 2.4). In Section 3, we establish the existence of nontrivial solutions to our problem under (H)-type conditions, namely the Ambrosetti-Rabinowitz condition holds. Section 4 deals with the existence of nontrivial solutions to our problem with the (L)-conditions, namely when the nonlinear term f does not satisfy the Ambrosetti-Rabinowitz condition.

2. **Notations and main results.** Let Ω be a bounded domain in $\mathbb{R}^{2m}$. Denote,

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla^m u|^2 \, dx - \int_{\Omega} F(x,u)dx$$

$$F(x,u) = \int_0^u f(x,s)ds$$

$$\|u\| = \left(\int_{\Omega} |\nabla^m u|^2 \, dx\right)^{1/2}$$

$$\beta_0 = \beta(2m,m) = m!\,(4\pi)^m$$

Let

$$\Lambda_{2m,m}(\Omega) = \inf_{H_0^m(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}$$

be the first eigenvalue of the poly-Laplacian operator Δ^m (see [19]). By a direct method of variation, one can show that $\Lambda_{2m,m}(\Omega) > 0$. We assume that there exist positive constants C and β such that

$$|f(x,t)| \leq C \exp\left(\beta t^2\right), \ \forall x \in \Omega, \ \forall t \in \mathbb{R}. \tag{4}$$

In particular, this is the case if f has subcritical or critical growth. Then J is a C^1 function: $H_0^m(\Omega) \to \mathbb{R}$ and

$$DJ(u)v = \int_{\Omega} \nabla^m u \nabla^m v dx - \int_{\Omega} f(x,u)v dx, \ \forall v \in H_0^m(\Omega)$$

Now, we will construct particular functions, namely the Adams functions. Denote by $\mathbb{B}$ the unit ball $\mathbb{B}(0,1)$ in $\mathbb{R}^{2m}$ and by $\mathbb{B}_l := \mathbb{B}(0,l)$ whenever $l \in (0,1)$. We have the following result (see [1, 22]):

Claim 1. *For all $l \in (0,1)$ there exists*

$$U_l \in \omega := \{u \in H_0^m(\mathbb{B}) : \ u|_{\mathbb{B}_l} = 1\}$$

such that

$$\|U_l\|^2 = C_{m,2}(\mathbb{B}_l;\mathbb{B}) \leq \frac{\beta_0}{2m\log\left(\frac{1}{l}\right)}$$

$C_{m,2}(K;E)$ denotes the $(m,2)$-conductor capacity of K in E, whenever E is an open set and K a relatively compact subset of E; it is defined as

$$C_{m,2}(K;E) := \inf\left\{\|\nabla^m u\|_2^2 : \ u \in C_0^\infty(E), \ u|_K = 1\right\}.$$

Now, let $x_0 \in \Omega$, $R \leq R_0 = dist\,(x_0, \ \partial\Omega)$, the Adams function is the function

$$\widetilde{A}_r(x) = \begin{cases} \sqrt{\frac{2m\log\left(\frac{R}{r}\right)}{\beta_0}}\, U_{r/R}\left(\frac{x-x_0}{R}\right), \text{ if } |x-x_0| < R \\ 0, \text{ if } |x-x_0| \geq R \end{cases}$$

where $0 < r < R$ and U_l is as in Claim 1.

It's easy to check that the Adams functions satisfy $\left\|\widetilde{A}_r\right\| \leq 1$. Put

$$\mathcal{M} = \lim_{k\to\infty} \int_{\frac{1}{k} \leq |x-x_0| \leq 1} \exp\left(2m\log k \left|U_{R/k}(x)\right|^2\right) dx$$

Note that $\mathcal{M} > 0$.

Then we have the following results:

Theorem 2.1. *Assume that f has subcritical growth and satisfies the (H)-type conditions: $(H1)$, $(H2)$, $(H3)$. Then, problem (2) has a nontrivial solution.*

Theorem 2.2. *Assume that f has critical growth and satisfies the (H)-type conditions: $(H1)$, $(H2)$, $(H3)$, $(H4)$. Then, problem (2) has a nontrivial solution.*

Theorem 2.3. *Assume that f has subcritical growth and satisfies the (L)-type conditions: $(L1)$, $(L2)$, $(L3)$, $(L4)$. Then, problem (2) has a nontrivial solution.*

Theorem 2.4. *Assume that f has critical growth and satisfies the (L)-type conditions: $(L1)$, $(L2)$, $(L3)$ with $C_* = 0$, $\theta = 1$, $(L4)$, $(L5)$, $(L6)$. Then, problem (2) has a nontrivial solution.*

Note that the superlinear condition $(L2)$ is just a consequence of the critical growth condition of the nonlinear term f.

3. Proof of Theorems 2.1 and 2.2: Existence under the (H)-type condition - with the Ambrosetti-Rabinowitz condition.

3.1. Mountain pass geometry.

Lemma 3.1. *$J(tu) \to -\infty$ as $t \to \infty$ for all $u \in H_0^m(\Omega) \setminus \{0\}$ with $u \geq 0$.*

Proof. Let $u \in H_0^m(\Omega) \setminus \{0\}$, $u \geq 0$. By $(H5)$, for $p > 2$, there exist $M > 0$ and A such that for all $(x, s) \in \Omega \times \mathbb{R}^+$

$$F(x, s) \geq M s^p - A.$$

Then

$$J(tu) \leq \frac{t^2}{2} \|u\|^2 - M t^p \int_\Omega |u|^p \, dx + O(1)$$

Since $p > 2$, we have $J(tu) \to -\infty$ as $t \to \infty$. □

Lemma 3.2. *There exist $\delta, \rho > 0$ such that*

$$J(u) \geq \delta \text{ if } \|u\| = \rho$$

Proof. By $(H1)$, $(H3)$ and (4), there exist $\kappa, \tau > 0$ and $q > 2$ such that

$$F(x, s) \leq \frac{1}{2} (\Lambda_{2m,m}(\Omega) - \tau) |s|^2 + C \exp\left(\kappa |s|^2\right) |s|^q, \ \forall (x, s) \in \Omega \times \mathbb{R}$$

By Holder's inequality and the Adams' inequalities, we have:

$$\begin{aligned} \int_\Omega \exp\left(\kappa |u|^2\right) |u|^q \, dx &\leq \left(\int_\Omega \exp\left(\kappa r \|u\|^2 \left(\frac{|u|}{\|u\|}\right)^2\right) dx\right)^{1/r} \left(\int_\Omega |u|^{r'q} \, dx\right)^{1/r'} \\ &\leq C \left(\int_\Omega |u|^{r'q} \, dx\right)^{1/r'} \end{aligned}$$

if $r > 1$ sufficiently close to 1 and $\|u\| \leq \sigma$, where $\kappa r \sigma^2 < \beta_0$. Thus by the definition of $\Lambda_{2m,m}(\Omega)$ and the Sobolev embedding:

$$J(u) \geq \frac{1}{2}\left(1 - \frac{(\Lambda_{2m,m}(\Omega) - \tau)}{\Lambda_{2m,m}(\Omega)}\right) \|u\|^2 - C \|u\|^q$$

Since $\tau > 0$ and $q > 2$, we may choose $\rho, \delta > 0$ such that $J(u) \geq \delta$ if $\|u\| = \rho$. □

3.2. **The subcritical case-Proof of Theorem 2.1.**

Lemma 3.3. *The functional J satisfies $(PS)_c$ for all $c \in \mathbb{R}$.*

Proof. Let $\{u_k\} \subset H_0^m(\Omega)$ be a Palais-Smale sequence, i.e.

$$J(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega F(x,u_k)dx \to c \tag{5}$$

$$|DJ(u_k)v| = \left|\int_\Omega \nabla^m u_k \nabla^m v dx - \int_\Omega f(x,u_k)v dx\right| \le \varepsilon_n \|v\| \tag{6}$$

where $\varepsilon_n \to 0$. Choose $v = u_k$ in (6), we get

$$\frac{1}{2}\|u_k\|^2 - \int_\Omega F(x,u_k)dx \to c$$
$$\left|\|u_k\|^2 - \int_\Omega f(x,u_k)u_k dx\right| \le \varepsilon_n \|u_k\|$$

which thus yields

$$\left(\frac{\theta}{2}-1\right)\|u_k\|^2 + \int_\Omega [f(x,u_k)u_k - \theta F(x,u_k)]\,dx \le O(1) + \varepsilon_n\|u_k\|$$

By the Ambrosetti-Rabinowitz condition $(H6)$, we have

$$\left(\frac{\theta}{2}-1\right)\|u_k\|^2 \le O(1) + \varepsilon_n\|u_k\|$$

and thus $\{u_k\}$ is bounded. WLOG, we suppose that

$$\begin{aligned} \|u_k\| &\le K \\ u_k &\rightharpoonup u_0 \text{ weakly in } H_0^m(\Omega) \\ u_k &\to u_0 \text{ strongly in } L^p(\Omega),\ \forall p \ge 1. \\ u_k(x) &\to u_0(x) \text{ a.e. } \Omega \end{aligned}$$

Now, since f has the subcritical growth on Ω, we can find a constant $c_K > 0$ such that

$$f(x,s) \le c_K \exp\left(\frac{\beta_0}{2K^2}|s|^2\right),\ \forall (x,s) \in \Omega \times \mathbb{R}$$

then by the Holder's inequality and Adams' inequalities,

$$\begin{aligned} \left|\int_\Omega f(x,u_k)(u_k-u)\,dx\right| &\le \int_\Omega |f(x,u_k)(u_k-u)|\,dx \\ &\le \left(\int_\Omega |f(x,u_k)|^2\,dx\right)^{1/2}\left(\int_\Omega |u_k-u|^2\,dx\right)^{1/2} \\ &\le C\left(\int_\Omega \exp\left(\frac{\beta_0}{K^2}|u_k|^2\right)dx\right)^{1/2}\|u_k-u\|_2 \\ &\le C\left(\int_\Omega \exp\left(\frac{\beta_0}{K^2}\|u_k\|^2\left|\frac{u_k}{\|u_k\|}\right|^2\right)dx\right)^{1/2}\|u_k-u\|_2 \\ &\le C\|u_k-u\|_2 \overset{n\to\infty}{\to} 0. \end{aligned}$$

Similarly, since $u_k \rightharpoonup u$ weakly in $H_0^m(\Omega)$, $\int_\Omega f(x,u)(u_k-u)\,dx \to 0$. Thus we can conclude that

$$\int_\Omega (f(x,u_k)-f(x,u))(u_k-u)\,dx \overset{n\to\infty}{\to} 0 \tag{7}$$

Also, by (6) we have

$$\langle DJ(u_k)-DJ(u),(u_k-u)\rangle \overset{n\to\infty}{\to} 0 \tag{8}$$

From (7) and (8), we get

$$\|u_k-u\| \overset{n\to\infty}{\to} 0$$

thus $u_k \overset{n\to\infty}{\to} u$ strongly in $H_0^m(\Omega)$ which means that J satisfies $(PS)_c$. $\square$

By the above lemma, it's easy to deduce Theorem 2.1 by the well-known Mountain Pass Theorem.

3.3. **The critical case-Proof of Theorem 2.2.**

Lemma 3.4. *There exists k such that*

$$\max\{J(tA_k): t\geq 0\} < \frac{\beta_0}{2\alpha_0}$$

where $A_k = \widetilde{A}_{R/k}$.

Proof. Suppose for the sake of contradiction that for all k we have

$$\max\{J(tA_k): t\geq 0\} \geq \frac{\beta_0}{2\alpha_0}$$

So for all k, we can choose $t_k>0$ such that

$$J(t_kA_k) = \max\{J(tA_k): t\geq 0\}$$

which thus

$$J(t_kA_k) = \frac{t_k^2\|A_k\|^2}{2} - \int_\Omega F(x,t_kA_k)dx \geq \frac{\beta_0}{2\alpha_0}.$$

and

$$t_k^2\|A_k\|^2 = \int_\Omega t_kA_kf(x,t_kA_k)dx \tag{9}$$

Since $F(x,s)\geq 0$ and $\|A_k\|^2\leq 1$, we get

$$t_k^2 \geq \frac{\beta_0}{\alpha_0} \tag{10}$$

On the other hand, given $\varepsilon>0$, there exists $R_\varepsilon>0$ such that

$$uf(x,u)\geq(\beta_1-\varepsilon)\exp\left(\alpha_0|u|^2\right),\ \forall u\geq R_\varepsilon.$$

Thus, we have

$$\begin{aligned}
t_k^2 &\geq (\beta_1-\varepsilon)\int_{\mathbb{B}(x_0,R/k)}\exp\left(\alpha_0|t_kA_k|^2\right)dx\\
&\geq (\beta_1-\varepsilon)\frac{\omega_{2m-1}}{2m}\left(\frac{R}{k}\right)^{2m}\exp\left(\alpha_0|t_k|^2\frac{2m\log k}{\beta_0}\right)\\
&\geq (\beta_1-\varepsilon)\frac{\omega_{2m-1}}{2m}R^{2m}\exp\left(\left[\frac{\alpha_0|t_k|^2}{\beta_0}-1\right]2m\log k\right)
\end{aligned}$$

for k large, which implies that (t_k) is bounded and moreover, by (10),

$$t_k^2 \to \frac{\beta_0}{\alpha_0} \tag{11}$$
$$\|A_k\| \to 1$$

It's also easy to see that

$$A_k(x) \to 0 \text{ a.e. } x \in \Omega.$$

Now, let

$$X_k = \{x \in \Omega : t_k A_k \geq R_\varepsilon\} \text{ and } Y_k = \Omega \setminus X_k$$

then the characteristic functions $\chi_{Y_k} \to 1$ a.e. $x \in \Omega$. Therefore, in view of the Lebesgue Dominated Convergence Theorem, we have

$$\int_{Y_n} t_k A_k f(x, t_k A_k) dx \to 0$$
$$\int_{Y_k} \exp\left(\alpha_0 |t_k A_k|^2\right) dx \to \frac{\omega_{2m-1}}{2m} R^{2m}$$

Also,

$$\begin{aligned}
\int_{\mathbb{B}(x_0,R)} \exp\left(\alpha_0 |t_k A_k|^2\right) dx \geq& \int_{\mathbb{B}(x_0,R)} \exp\left(\beta_0 |A_n|^2\right) dx \\
&= R^{2m} \int_{|x-x_0| \leq 1/k} \exp\left(\beta_0 |A_k|^2\right) dx \\
&+ R^{2m} \int_{\frac{1}{k} \leq |x-x_0| \leq 1} \exp\left(\beta_0 |A_k|^2\right) dx \\
&= \frac{\omega_{2m-1}}{2m} R^{2m} \\
&+ R^{2m} \int_{\frac{1}{k} \leq |x-x_0| \leq 1} \exp\left(2m \log k \left|U_{R/k}(x)\right|^2\right) dx
\end{aligned}$$

So, since

$$\begin{aligned}
t_k^2 \geq& (\beta_1 - \varepsilon) \int_{\mathbb{B}(x_0,R)} \exp\left(\alpha_0 |t_k A_k|^2\right) dx + \int_{Y_k} t_k A_k f(x, t_k A_k) dx \\
&- (\beta_1 - \varepsilon) \int_{Y_k} \exp\left(\alpha_0 |t_k A_k|^2\right) dx
\end{aligned}$$

we will get

$$\frac{\beta_0}{\alpha_0} \geq (\beta_1 - \varepsilon) R^{2m} \mathcal{M}$$

and

$$\frac{\beta_0}{\alpha_0 R^{2m} \mathcal{M}} \geq \beta_1$$

which is a contradiction. □

Lemma 3.5. *Let $\{u_k\} \subset H_0^m(\Omega)$ be a Palais-Smale sequence. Then $\{u_k\}$ has a subsequence, still denoted by $\{u_k\}$, and $u \in H_0^m(\Omega)$ such that*

$$\begin{cases} u_k \rightharpoonup u \text{ weakly in } H_0^m(\Omega) \\ f(x, u_k) \longrightarrow f(x, u) \text{ strongly in } L^1(\Omega) \end{cases}$$

Proof. Let $\{u_k\} \subset H_0^m(\Omega)$ be a Palais-Smale sequence, i.e.

$$J(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega F(x,u_k)dx \to c$$
$$|DJ(u_k)v| = \left|\int_\Omega \nabla^m u_k \nabla^m v dx - \int_\Omega f(x,u_k)v dx\right| \le \varepsilon_n \|v\|$$

where $\varepsilon_n \to 0$. Similarly as in the Lemma 3.3, we can prove that $\{u_k\}$ is bounded thanks to the (AR) condition. Moreover, $\int_\Omega F(x,u_k)dx$ and $\int_\Omega f(x,u_k)u_k dx$ are also bounded. So WLOG, we suppose that

$$\begin{aligned}
\|u_k\| \le K;\ \int_\Omega F(x,u_k)dx \le K;\ &\int_\Omega f(x,u_k)u_k dx \le K\\
u_k \rightharpoonup u &\text{ weakly in } H_0^m(\Omega)\\
u_k \to u &\text{ strongly in } L^p(\Omega),\ \forall p \ge 1.\\
u_k(x) \to u(x) &\text{ a.e. } \Omega
\end{aligned}$$

Use the argument as in [13], Lemma 4, we get the lemma. □

Now, thanks to the Mountain Pass Geometry of the functional J and Lemma 3.4, we can find a Palais-Smale sequence $\{u_k\}$ at the level $0 < C_M < \frac{\beta_0}{2\alpha_0}$. More precisely, we have

$$J(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega F(x,u_k)dx \to c \tag{12}$$
$$|DJ(u_k)v| = \left|\int_\Omega \nabla^m u_k \nabla^m v dx - \int_\Omega f(x,u_k)v dx\right| \le \varepsilon_n \|v\| \tag{13}$$

By Lemma 3.5, there exists u in $H_0^m(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly in } H_0^m(\Omega) \\ f(x,u_k) \longrightarrow f(x,u) \text{ strongly in } L^1(\Omega) \end{cases}$$

Moreover, it's easy to check that

$$DJ(u)v = 0,\ \forall v \in C_0^\infty(\Omega)$$

which means that u is a weak solution of (2). So it is remainder to prove that u is not trivial. Suppose for the sake of contradiction that $u = 0$, then by $(H2)$ and the generalized Lebesgue Dominated Convergence Theorem, we have thanks to $f(x,u_k) \longrightarrow 0$ strongly in $L^1(\Omega)$:

$$\int_\Omega F(x,u_k)dx \longrightarrow 0 \text{ strongly in } L^1(\Omega)$$

So from (12), we have

$$\|u_k\|^2 \to 2C_M < \frac{\beta_0}{\alpha_0}$$

It means that we can choose $q > 1$ sufficiently close to 1 such that for n sufficiently large

$$q\alpha_0 \|u_k\|^2 < \beta_0$$

Notice that f has critical growth (at α_0) and the Adams' inequalities, we can conclude that

$$\int_\Omega |f(x,u_k(x))|^q\,dx \le C\int_\Omega \exp\left[q\alpha_0 |u_k|^2\right]dx$$
$$\le C\int_\Omega \exp\left[q\alpha_0 \|u_k\|^2 \left|\frac{u_k}{\|u_k\|}\right|^2\right]dx \le O(1)$$

By (13) with $v=u_k$, we have $\|u_k\|^2 \to 0$ and it's a contradiction.

4. **Proof of Theorems 2.3 and 2.4: Existence under the (L)-type condition - without the Ambrosetti-Rabinowitz condition.** In this case, we still use the variational method to find the solution for (2). However, since we don't have the (AR) condition, we need to use a modified version of Mountain Pass Theorem which was introduced in [11, 12].

Definition 4.1. Let $(X,\|\cdot\|_X)$ be a real Banach space with its dual space $(X^*,\|\cdot\|_{X^*})$ and $I\in C^1(X,\mathbb{R})$. For $c\in\mathbb{R}$, we say that I satisfies the $(C)_c$ condition if for any sequence $\{x_k\}\subset X$ with

$$I(x_k)\to c,\ \|DI(x_k)\|_{X^*}(1+\|x_k\|_X)\to 0$$

there is a subsequence $\{x_{k_l}\}$ such that $\{x_{k_l}\}$ converges strongly in X.

We have the following versions of the Mountain Pass Theorem (see [3, 11, 12]):

Lemma 4.2. *Let $(X,\|\cdot\|_X)$ be a real Banach space and $I\in C^1(X,\mathbb{R})$ satisfies the $(C)_c$ condition for any $c\in\mathbb{R}$, $I(0)=0$ and*

(i) There are constants $\rho,\ \alpha>0$ such that $I|_{\partial B_\rho}\ge\alpha$.
(ii) There is an $e\in X\setminus B_\rho$ such that $I(e)\le 0$.
Then $c=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t))\ge\alpha$ is a critical value of I where

$$\Gamma=\left\{\gamma\in C^0([0,1],X),\ \gamma(0)=0,\ \gamma(1)=e\right\}.$$

Lemma 4.3. *Let $(X,\|\cdot\|_X)$ be a real Banach space and $I\in C^1(X,\mathbb{R})$ satisfies $I(0)=0$ and*

(i) There are constants $\rho,\ \alpha>0$ such that $I|_{\partial B_\rho}\ge\alpha$.
(ii) There is an $e\in X\setminus B_\rho$ such that $I(e)\le 0$.
Let C_M be characterized by

$$C_M=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t))$$

where

$$\Gamma=\left\{\gamma\in C^0([0,1],X),\ \gamma(0)=0,\ \gamma(1)=e\right\}.$$

Then I possesses a $(C)_{C_M}$ sequence.

4.1. **The geometry of the functional J.** In this subsection, again we will check the Mountain Pass properties of the functional J.

Lemma 4.4. *Then $J(tu)\to-\infty$ as $t\to\infty$ for all nonnegative function $u\in H_0^m(\Omega)\setminus\{0\}$*

Proof. Let $u\in H_0^m(\Omega)\setminus\{0\},\ u\ge 0$. By $(L2)$, there exist $M>\frac{\|u\|^2}{2\|u\|_2^2}>0$ and A such that for all $(x,s)\in\Omega\times\mathbb{R}^+$

$$F(x,s)\ge Ms^2-A.$$

Then

$$J(tu) \leq \frac{t^2}{2}\|u\|^2 - Mt^2\int_\Omega |u|^2\,dx + O(1)$$
$$= t^2\left(\frac{\|u\|^2}{2} - M\int_\Omega |u|^2\,dx\right) + O(1)$$

Since $M > \frac{\|u\|^2}{2\|u\|_2^2}$, we have $J(tu) \to -\infty$ as $t \to \infty$. □

Lemma 4.5. *There exist $\delta, \rho > 0$ such that*

$$J(u) \geq \delta \text{ if } \|u\| = \rho$$

Proof. It's similar to Lemma 3.2. □

4.2. **The subcritical case-Proof of Theorem 2.3.**

Lemma 4.6. *The functional J satisfies $(C)_c$ for all $c \in \mathbb{R}$.*

Proof. Let $\{u_k\} \subset H_0^m(\Omega)$ be a Palais-Smale sequence, i.e.

$$J(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega F(x,u_k)dx \to c \tag{14}$$

$$(1+\|u_k\|)\left|DJ(u_k)v\right| = (1+\|u_k\|)\left|\int_\Omega \nabla^m u_k \nabla^m v dx - \int_\Omega f(x,u_k)v dx\right| \leq \varepsilon_k \|v\| \tag{15}$$

where $\varepsilon_n \to 0$. Choose $v = u_k$ in (15), we get

$$\frac{1}{2}\|u_k\|^2 - \int_\Omega F(x,u_k)dx \to c$$
$$\left|\|u_k\|^2 - \int_\Omega f(x,u_k)u_k dx\right| \leq \varepsilon_n \frac{\|u_k\|}{1+\|u_k\|}$$

We first show that $\{u_k\}$ is bounded which is our main purpose in this section. Indeed, suppose that

$$\|u_k\| \to \infty \tag{16}$$

Setting

$$v_k = \frac{u_k}{\|u_k\|}$$

then $\|v_k\| = 1$ so we can suppose that $v_k \rightharpoonup v$ in $H_0^m(\Omega)$. We may similarly show that $v_k^+ \rightharpoonup v^+$ in $H_0^m(\Omega)$, where $w^+ = \max\{w,0\}$. Since Ω is bounded, Sobolev's imbedding theorem implies that $\begin{cases} v_k^+(x) \to v^+(x) \text{ a.e. in } \Omega \\ v_k^+ \to v^+ \text{ in } L^p(\Omega),\ \forall p \geq 1 \end{cases}$. We will prove that $v^+ = 0$ a.e. Ω. Indeed, suppose $\mu(\Omega^+) = \mu\{x \in \Omega : v^+(x) > 0\} > 0$. Then in Ω^+, we have

$$\lim_{k\to\infty} u_k(x) = \lim_{k\to\infty} v_k^+(x)\|u_k\| = +\infty$$

and thus by $(L2)$:

$$\lim_{k\to\infty} \frac{F(x,u_k(x))}{|u_k(x)|^2} = +\infty \text{ a.e. in } \Omega^+$$

This means that

$$\lim_{k\to\infty} \frac{F(x,u_k(x))}{|u_k(x)|^2}\left|v_k^+(x)\right|^2 = +\infty \text{ a.e. in } \Omega^+ \tag{17}$$

Also, by (14), we see that

$$\|u_k\|^2 = 2c + 2\int_\Omega F(x, u_k(x))dx + o(1) \tag{18}$$

which implies that

$$\int_\Omega F(x, u_k(x))dx \to +\infty \tag{19}$$

Now, note that $F(x,s) \geq 0$, by Fatou's lemma and (17), (18) and (19):

$$\begin{aligned}
+\infty &= \int_{\Omega+} \liminf_{k\to\infty} \frac{F(x,u_k(x))}{|u_k(x)|^2} \left|v_k^+(x)\right|^2 dx \\
&\leq \liminf_{k\to\infty} \int_{\Omega+} \frac{F(x,u_k(x))}{|u_k(x)|^2} \left|v_k^+(x)\right|^2 dx \\
&\leq \liminf_{k\to\infty} \int_{\Omega} \frac{F(x,u_k(x))}{\|u_k\|^2} dx \\
&= \liminf_{k\to\infty} \frac{\int_{\Omega+} F(x,u_k(x))\, dx}{2c + 2\int_\Omega F(x,u_k(x))dx + o(1)} \\
&= \frac{1}{2}
\end{aligned}$$

This is a contradiction. So we get $v \leq 0$ a.e.

In fact, we have $v = 0$ a.e. Indeed, since

$$(1 + \|u_k\|)\, |DJ(u_k)v| = (1 + \|u_k\|) \left| \int_\Omega \nabla^m u_k \nabla^m v dx - \int_\Omega f(x,u_k)\, v dx \right| \leq \varepsilon_k \|v\|$$

we get

$$\int_\Omega \nabla^m u_k \nabla^m v dx \leq \int_\Omega \nabla^m u_k \nabla^m v dx - \int_\Omega f(x,u_k)\, v dx \leq \frac{\varepsilon_k \|v\|}{(1 + \|u_k\|)} \to 0$$

by noticing that since $v \leq 0$, $f(x,u_k)\, v \leq 0$ a.e. Ω, thus $-\int_\Omega f(x,u_k)\, v \geq 0$. So we have

$$\int_\Omega \nabla^m v_k \nabla^m v dx = \frac{\int_\Omega \nabla^m u_k \nabla^m v dx}{\|u_k\|} \leq \frac{\varepsilon_k \|v\|}{(1 + \|u_k\|)\, \|u_k\|} \to 0$$

On the other hand, since $v_k \rightharpoonup v$ in $H_0^m(\Omega)$,

$$\int_\Omega \nabla^m v_k \nabla^m v dx \to \int_\Omega |\nabla^m v|^2\, dx$$

which implies $v = 0$.

Next, let $t_k \in [0,1]$ such that

$$J(t_k u_k) = \max_{t\in[0,1]} J(tu_k)$$

For all $R > 0$, by (*SCG*), there exists $C > 0$ such that

$$F(x,s) \leq C|s| + \exp\left(\frac{\beta_0}{R^2} s^2\right),\ \forall (x,s) \in \Omega \times \mathbb{R}. \tag{20}$$

Also since $\|u_k\| \to \infty$, we have

$$J(t_k u_k) \geq J\left(\frac{R}{\|u_k\|} u_k\right) = J(Rv_k) \tag{21}$$

and by (20) and noting that $\|v_k\| = 1$:

$$2J\left(Rv_k\right) \geq R^2 - 2CR\int_\Omega |v_k(x)|\, dx - 2\int_\Omega \exp\left(\beta_0 v_k^2(x)\right) dx \tag{22}$$

By Adams' inequalities, $\int_\Omega \exp\left(\beta_0 v_k^2(x)\right) dx$ is bounded by a constant $C\left(\Omega\right) > 0$. Also, since $v_k \rightharpoonup 0$ in $H_0^m\left(\Omega\right)$, $\int_\Omega |v_k(x)|\, dx \to 0$. Thus if we let $k \to \infty$ in (22), and then let $R \to \infty$ and using (21), we get

$$J\left(t_k u_k\right) \to \infty \tag{23}$$

Note that $J(0) = 0$ and $J(u_k) \to c$, we can then suppose that $t_k \in (0,1)$. Since $DJ(t_k u_k)t_k u_k = 0$, we have

$$t_k^2 \left\|u_k\right\|^2 = \int_\Omega f\left(x, t_k u_k\right) t_k u_k dx$$

Also, by (14) and (15):

$$\begin{aligned}\int_\Omega \left[f\left(x,u_k\right)u_k - 2F\left(x,u_k\right)\right] dx &= \left\|u_k\right\|^2 + 2c - \left\|u_k\right\|^2 + o(1)\\ &= 2c + o(1)\end{aligned}$$

So by $(L3)$:

$$\begin{aligned}2J\left(t_k u_k\right) &= t_k^2 \left\|u_k\right\|^2 - 2\int_\Omega F\left(x, t_k u_k\right) dx\\ &= \int_\Omega \left[f\left(x, t_k u_k\right) t_k u_k - 2F\left(x, t_k u_k\right)\right] dx\\ &\leq \theta \int_\Omega \left[f\left(x,u_k\right)u_k - 2F\left(x,u_k\right)\right] dx + O(1)\\ &\leq O(1)\end{aligned}$$

which is a contraction to (23). This proves that $\{u_k\}$ is bounded in $H_0^m\left(\Omega\right)$. Now, similarly as in Lemma 3.3, we can conclude that J satisfies $(C)_c$. □

By the above lemma, it's easy to deduce Theorem 2.3 by Lemma 4.1.

4.3. **The critical case-Proof of Theorem 2.4.** Thanks to the Geometry of the functional J, the Lemma 3.4 and Lemma 4.2, we can find a Cerami sequence $\{u_k\}$ in $H_0^m\left(\Omega\right)$ such that

$$\begin{aligned}\left(1 + \left\|u_k\right\|\right)\left\|DJ(u_k)\right\| \to 0 \qquad (24)\\ J(u_k) \to C_M < \frac{\beta_0}{2\alpha_0}\end{aligned}$$

We again want to show that $\{u_k\}$ is bounded in $H_0^m\left(\Omega\right)$. Indeed, again if we suppose that $\{u_k\}$ is unbounded, then similarly to the Lemma 4.5, we can get that

$$v_k \rightharpoonup 0 \text{ in } H_0^m\left(\Omega\right) \text{ where } v_k = \frac{u_k}{\left\|u_k\right\|}.$$

Let $t_k \in [0,1]$ such that

$$J\left(t_k u_k\right) = \max_{t\in[0,1]} J\left(tu_k\right)$$

Let $R \in \left(0, \sqrt{\frac{\beta_0}{\alpha_0}}\right)$ and choose $\varepsilon = \frac{\beta_0}{R^2} - \alpha_0 > 0$, by (CG), there exists $C > 0$ such that

$$F(x,s) \leq C\,|s| + \left|\frac{\beta_0}{R^2} - \alpha_0\right| \exp\left(\left(\alpha_0 + \varepsilon\right) s^2\right),\ \forall\,(x,s) \in \Omega \times \mathbb{R}. \tag{25}$$

Also since $\|u_k\| \to \infty$, we have

$$J\left(t_k u_k\right) \geq J\left(\frac{R}{\|u_k\|} u_k\right) = J\left(R v_k\right) \tag{26}$$

and by (25) and note $\|v_k\| = 1$:

$$2J\left(R v_k\right) \geq R^2 - 2CR \int_\Omega \left|v_k(x)\right| dx - 2\left|\frac{\beta_0}{R^2} - \alpha_0\right| \int_\Omega \exp\left(\left(\alpha_0 + \varepsilon\right) R^2 v_k^2(x)\right) dx \tag{27}$$

By the Adams' inequalities,

$$\int_\Omega \exp\left(\left(\alpha_0 + \varepsilon\right) R^2 v_k^2(x)\right) dx = \int_\Omega \exp\left(\frac{\beta_0}{R^2} R^2 v_k^2(x)\right) dx$$

is bounded by an universal constant $C\left(\Omega\right) > 0$ thanks to the choice of ε. Also, since $v_k \rightharpoonup 0$ in $H_0^m\left(\Omega\right)$, $\int_\Omega \left|v_k(x)\right| dx \to 0$. Thus if we let $k \to \infty$ in (27), and then let $R \to \sqrt{\frac{\beta_0}{\alpha_0}}^{-}$ and using (26), we get

$$\liminf_{k \to \infty} J\left(t_k u_k\right) \geq \frac{\beta_0}{2\alpha_0} > C_M. \tag{28}$$

Note that $J(0) = 0$ and $J(u_k) \to C_M$, we can suppose that $t_k \in (0,1)$. Thus since $DJ(t_k u_k) t_k u_k = 0$,

$$t_k^2 \left\|u_k\right\|^2 = \int_\Omega f\left(x, t_k u_k\right) t_k u_k dx$$

Also, by (24)

$$\begin{aligned} \int_\Omega \left[f\left(x, u_k\right) u_k - 2F\left(x, u_k\right)\right] dx &= \left\|u_k\right\|^2 + 2C_M - \left\|u_k\right\|^2 + o(1) \\ &= 2C_M + o(1) \end{aligned}$$

So from $(L3)$ with $C_* = 0,\ \theta = 1$:

$$\begin{aligned} 2J\left(t_k u_k\right) &= t_k^2 \left\|u_k\right\|^2 - 2\int_\Omega F\left(x, t_k u_k\right) dx \\ &= \int_\Omega \left[f\left(x, t_k u_k\right) t_k u_k - 2F\left(x, t_k u_k\right)\right] dx \\ &\leq \int_\Omega \left[f\left(x, u_k\right) u_k - 2F\left(x, u_k\right)\right] dx \\ &= 2C_M + o(1) \end{aligned}$$

which is a contradiction to (28). This proves that $\{u_k\}$ is bounded in $H_0^m\left(\Omega\right)$. Now as in the proof of Theorem 2.2, we get Theorem 2.4.

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