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Sharp Moser–Trudinger inequality on the Heisenberg group at the critical case and applications [☆]

Nguyen Lam [a], Guozhen Lu [b,a,∗]

[a] *Department of Mathematics, Wayne State University, Detroit, MI 48202, USA*
[b] *School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China*

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Abstract

Let $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ be the n-dimensional Heisenberg group, $Q = 2n + 2$ be the homogeneous dimension of $\mathbb{H}$, $Q' = \frac{Q}{Q-1}$, and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ be the homogeneous norm of $\xi = (z, t) \in \mathbb{H}$. Then we prove the following sharp Moser–Trudinger inequality on $\mathbb{H}$ (Theorem 1.6): there exists a positive constant $\alpha_Q = Q\left(2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1}\right)^{Q'-1}$ such that for any pair β, α satisfying $0 \leq \beta < Q$, $0 < \alpha \leq \alpha_Q(1 - \frac{\beta}{Q})$ there holds

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)}\right) - \sum_{k=0}^{Q-2} \frac{\alpha^k}{k!} |u|^{kQ/(Q-1)} \right\}$$
$$\leq C(Q, \beta, \tau) < \infty.$$

The constant $\alpha_Q(1 - \frac{\beta}{Q})$ is best possible in the sense that the supremum is infinite if $\alpha > \alpha_Q(1 - \frac{\beta}{Q})$. Here τ is any positive number, and $\|u\|_{1,\tau} = \left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q\right]^{1/Q}$.

Our result extends the sharp Moser–Trudinger inequality by Cohn and Lu (2001) [19] on domains of finite measure on $\mathbb{H}$ and sharpens the recent result of Cohn et al. (2012) [18] where such an inequality was studied for the subcritical case $\alpha < \alpha_Q(1 - \frac{\beta}{Q})$. We carry out a completely different and much simpler

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∗ Corresponding author.
E-mail addresses: nguyenlam@wayne.edu (N. Lam), gzlu@math.wayne.edu (G. Lu).

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argument than that in Cohn et al. (2012) [18] to conclude the critical case. Our method avoids using the rearrangement argument which is not available in an optimal way on the Heisenberg group and can be used in more general settings such as Riemanian manifolds, appropriate metric spaces, etc. As applications, we establish the existence and multiplicity of nontrivial nonnegative solutions to certain nonuniformly subelliptic equations of Q-Laplacian type on the Heisenberg group (Theorems 1.8, 1.9, 1.10 and 1.11):

$$-\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}}u|^{Q-2}\nabla_{\mathbb{H}}u\right)+V(\xi)\,|u|^{Q-2}\,u=\frac{f(\xi,u)}{\rho(\xi)^{\beta}}+\varepsilon h(\xi)$$

with nonlinear terms f of maximal exponential growth $\exp(\alpha|u|^{\frac{Q}{Q-1}})$ as $|u|\to\infty$. In particular, when $\varepsilon=0$, the existence of a nontrivial solution is also given.

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1. Introduction

Analysis and study of partial differential equations on the Heisenberg group has received great attention in the past decades. Heisenberg group is the simplest example of noncommutative nilpotent Lie groups which has a close connection with several complex variables and CR geometry. Sharp geometric inequalities on the Heisenberg group have particularly played an important role in harmonic analysis, partial differential equations and differential geometry. A good example of this role is the identification of the sharp constant and extremal functions for the L^2 Sobolev inequality on the Heisenberg group. This was achieved in a series of celebrated works of Jerison and Lee in conjunction with the solution of the CR Yamabe problem [36–38]. To state our main theorems, we first introduce some preliminaries on the Heisenberg group.

Let $\mathbb{H}$ be the n-dimensional Heisenberg group

$$\mathbb{H}=\mathbb{C}^n\times\mathbb{R}$$

whose group structure is given by

$$(z,t)\cdot(z',t')=(z+z',t+t'+2\operatorname{Im}(z\cdot\overline{z'})),$$

for any two points (z,t) and (z',t') in $\mathbb{H}$.

The Lie algebra of $\mathbb{H}$ is generated by the left invariant vector fields

$$T=\frac{\partial}{\partial t},\qquad X_i=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t},\qquad Y_i=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t}$$

for $i=1,\dots,n$. These generators satisfy the non-commutative relationship

$$[X_i,Y_j]=-4\delta_{ij}T.$$

Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r\in\mathbb{R}$, there is a dilation naturally associated with the Heisenberg group structure which is usually denoted as

$$\delta_r u=\delta_r(z,t)=(rz,r^2t).$$

However, for simplicity we will write ru to denote $\delta_r u$. The Jacobian determinant of δ_r is r^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}$.

The anisotropic dilation structure on $\mathbb{H}$ introduces a homogeneous norm

$$|u| = |(z,t)| = (|z|^4 + t^2)^{\frac{1}{4}}.$$

With this norm, we can define the Heisenberg ball centered at $u = (z,t)$ with radius R

$$B(u,R) = \{v \in \mathbb{H} : |u^{-1} \cdot v| < R\}.$$

The volume of such a ball is $C_Q R^Q$ for some constant depending on Q.

The subelliptic gradient on the Heisenberg group is denoted by

$$\nabla_{\mathbb{H}} f(z,t) = \sum_{j=1}^{n} \left((X_j f(z,t)) X_j + (Y_j f(z,t)) Y_j \right).$$

The following Sobolev inequality on the Heisenberg group is well known: for $f \in C_0^\infty(\mathbb{H})$

$$\left(\int_{\mathbb{H}} |f(z,t)|^q dz dt \right)^{\frac{1}{q}} \le C_{p,q} \left(\int_{\mathbb{H}} | \nabla_{\mathbb{H}} f(z,t)|^p dz dt \right)^{\frac{1}{p}} \tag{1.1}$$

provided that $1 \le p < Q = 2n + 2$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$. This inequality was first proved by Folland–Stein [27,28]; see also [31,58]. In the above inequality, we have used $| \nabla_{\mathbb{H}} f|$ to express the (Euclidean) norm of the subelliptic gradient of f:

$$| \nabla_{\mathbb{H}} f| = \left(\sum_{i=1}^{n} \left((X_i f)^2 + (Y_i f)^2 \right) \right)^{\frac{1}{2}}.$$

It is then clear that the above inequality is also true for functions in the anisotropic Sobolev space $W_0^{1,p}(\mathbb{H})$ ($p \ge 1$), where $W_0^{1,p}(\Omega)$ for open set $\Omega \subset \mathbb{H}$ is the completion of $C_0^\infty(\Omega)$ under the norm

$$\|f\|_{L^p(\Omega)} + \| \nabla_{\mathbb{H}} f\|_{L^p(\Omega)}.$$

Nevertheless, much less is known about sharp constants for Sobolev inequality (1.1) for the Heisenberg group than for Euclidean space. In fact, the first major breakthrough came after the works by Jerison and Lee [37] on the sharp constants for the Sobolev inequality and extremal functions on the Heisenberg group in conjunction with the solution to the CR Yamabe problem (we should note the well-known results of Talenti [68] and Aubin [7] for sharp constants and extremal functions in the isotropic case). More precisely, in a series of papers [36,38,37], the Yamabe problem on CR manifolds was first studied. In particular, Jerison and Lee study the problem of conformally changing the contact form to one with constant Webster curvature in the compact setting.

In particular, the best constant $C_{p,q}$ for the Sobolev inequality (1.1) on $\mathbb{H}$ for $p = 2$ was found and the extremal functions were identified in [37].

Theorem 1.1 (*Jerison and Lee [37]*). *The best constant for the inequality* (1.1) *on* $\mathbb{H}$ *is*

$$C_{2,\frac{2n+2}{n}} = (4\pi)^{-1} n^{-2} \left[\Gamma(n+1) \right]^{\frac{1}{n+1}}$$

and all the extremals of (1.1) are obtained by dilations and left translations of the function

$$K\left|\left(t+i(|z|^2+1)\right)\right|^{-n}.$$

Furthermore, the extremals in (1.1) are constant multiples of images under the Cayley transform of extremals for the Yamabe functional on the sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$.

The sharp Sobolev inequality on the Heisenberg group for $p = 2$ is closely related to the sharp Hardy–Littlewood–Sobolev inequality, also known as the HLS inequality (see [35]):

$$\left|\iint_{\mathbb{H}\times\mathbb{H}} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^{\lambda}}dudv\right| \le C_{r,\lambda,n}\|f\|_r\|g\|_s. \tag{1.2}$$

In fact, the result of Jerison and Lee is equivalent to the sharp version of HLS inequality (1.2) when $\lambda = Q-2$ and $r = s = 2Q/(2Q-\lambda) = 2Q/(Q+2)$.

Very recently, in a remarkable paper of Frank and Lieb [32], they have succeeded to establish the sharp constants and extremal functions of the HLS inequality on the Heisenberg group for all $0<\lambda<Q$ and $r = s = \frac{2Q}{2Q-\lambda}$, an analogue to Lieb's celebrated result in Euclidean spaces [54]. We can state the result in [32] as the following theorem.

Theorem 1.2 (*Frank and Lieb, Theorem 2.1 in [32]*)**.** *Let $0<\lambda<Q$ and $r = 2Q/(2Q-\lambda)$. Then for any $f, g \in L^r(\mathbb{H})$,*

$$\left|\iint_{\mathbb{H}\times\mathbb{H}} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^{\lambda}}dudv\right| \le \left(\frac{\pi^{n+1}}{2^{n-1}n!}\right)^{\frac{\lambda}{Q}} \frac{n!\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)}\|f\|_r\|g\|_r, \tag{1.3}$$

with equality if and only if

$$f(u) = cH(\delta(a^{-1}u)), \qquad g(v) = c'H(\delta(a^{-1}v))$$

for some $c, c' \in \mathbb{C}$, $\delta > 0$, $a \in \mathbb{H}$ (unless $f \equiv 0$ or $g \equiv 0$), and

$$H = \left[(1+|z|^2)^2+t^2\right]^{-\frac{2Q-\lambda}{4}}.$$

The explicit formula in Theorem 1.2 about the extremal functions for the sharp Hardy–Littlewood–Sobolev inequality (1.3) is consistent with Branson, Fontana and Morpurgo's natural guess initially made in their work [14] where among other things, important progress has been made on sharp Onfri–Beckner inequalities on CR spheres (see also earlier work of Cohn and the second author in this direction on sharp Moser–Trudinger inequalities on CR sphere [21]). We also mention the recent work of Han, Zhu and the second author [34] in which weighted Hardy–Littlewood–Sobolev inequalities (namely, Stein–Weiss inequalities) on the Heisenberg group have been established. Existence of extremal functions of Hardy–Littlewood–Sobolev inequalities on the Heisenberg group has been recently proved by Han [33] using Lions' concentration compactness argument [56,57]. We refer the reader to [11] for some nonexistence result of extremal functions for Stein–Weiss inequalities on the Heisenberg group by Beckner (see also [10,12] for Stein–Weiss inequalities in the Euclidean spaces).[1]

[1] Added in Proof: Sharp constants for Stein–Weiss inequalities on the Heisenberg group in some special cases were identified very recently by Beckner [9].

The work of Jerison and Lee [37] raised two natural questions. What is the best constant $C_{p,q}$ for the L^p to L^q Sobolev inequality (1.1) for all $1 \le p < Q$ and $q = \frac{Qp}{Q-p}$ when $p \neq 2$? What is the sharp constant for the borderline case $p = Q$? While the first question still seems to be open, the second question was answered in the work of the second author with Cohn in [19] on domains of finite measure in the Heisenberg group. Namely, we proved in [19] the sharp Moser–Trudinger inequality on any domain Ω with $|\Omega| < \infty$ on the Heisenberg group.

To familiarize the reader with the borderline case inequality, we first recall the well known Moser–Trudinger inequality in Euclidean space. When Ω is an open set in $\mathbb{R}^n$ with $|\Omega| < \infty$, it is showed by Judovich [39], Pohozaev [64] and Trudinger [70] independently that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(|t|^{n/(n-1)}\right) - 1$. Using the method of symmetrization, Moser [63] finds the largest positive real number $\beta_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit n-ball, such that if Ω is a domain with finite n-measure in Euclidean n-space $\mathbb{R}^n$, $n \ge 2$, then there is a constant c_0 depending only on n such that

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\beta\,|u|^{\frac{n}{n-1}}\right) dx \le c_0$$

for any $\beta \le \beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n\,dx \le 1$. Moreover, this constant β_n is sharp in the sense that if $\beta > \beta_n$, then the above inequality can no longer hold with some c_0 independent of u. However, when Ω has infinite volume, the result of Moser is meaningless. In this case, the sharp Moser–Trudinger type inequality was obtained in [65] in dimension two and in [53] in general dimension.

As has been the case in most proofs of sharp constants in Euclidean spaces, one often attempts to use the radial non-increasing rearrangement u^* of functions u (in terms of a certain norm) on the Heisenberg group. However, it is not known whether or not the L^p norm of the subelliptic gradient of the rearrangement of a function is dominated by the L^p norm of the subelliptic gradient of the function. In other words, an inequality like

$$\| \nabla_{\mathbb{H}} u^*\|_{L^p} \le \| \nabla_{\mathbb{H}} u\|_{L^p} \tag{1.4}$$

is not available on the Heisenberg group. In fact, the work of Jerison–Lee on the best constant and extremals [37] indicates that this inequality fails to hold for the case $p = 2$. Thus, in the works of Jerison and Lee [37] and Frank and Lieb [32], substantially new ideas are needed in deriving sharp Sobolev and Hardy–Littlewood–Sobolev inequalities on the Heisenberg group.

As for the Moser–Trudinger inequality on bounded domains on the Heisenberg group, the borderline case of the Sobolev inequality when $p = Q$, we also have to avoid the rearrangement argument due to the unavailability of the symmetrization inequality (1.4) when $p = Q$. This was carried out in the work of Cohn and the second author [19]. In fact, we can adapt Adams' idea [2] in deriving the Moser–Trudinger inequality for higher order derivatives in Euclidean space [2], which requires, roughly speaking, an optimal bound on the size of a function in terms of the potential of its gradient, namely a sharp representation formula. By using this one parameter representation formula on the Heisenberg group, we are able to avoid considering the subelliptic gradient of the rearrangement function. Instead, we will consider the rearrangement of the convolution of the subelliptic gradient with an optimal kernel (see [19] for more details). Adams inequality on bounded domains [2] was extended to compact Riemannian manifolds in [29], to bounded domains with Navier boundary value conditions by Tarsi [69], and Fontana and Morpurgo [30] on metric measure spaces. Adams inequality has been recently generalized to unbounded domains on high order Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for any even order by Ruf–Sani [66] and to any odd order by the authors [42,47]. In particular, we developed in [43] a rearrangement-free argument to show all integral orders m and also on fractional Sobolev spaces $W^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$ of arbitrary fractional order $0 < \alpha < n$. The method used in [43] avoids the symmetrization which is not available in general settings such as Riemannian or sub-Riemannian manifolds or deep comparison principle of solutions to polyharmonic operators as used in [66,42,47]. We also refer the reader to [46] for detailed descriptions of results proved in [43]. We also mention that existence of extremal functions for Adams inequalities has only been proved in dimension four (see [52,60]).

The sharp constant for the Moser–Trudinger inequality on domains of finite measure in the Heisenberg group established in [19] is stated as follows. Throughout the remainder of this paper, we use $\xi = (z,t)$ to denote any point $(z,t) \in \mathbb{H}$ and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ to denote the homogeneous norm of $\xi \in \mathbb{H}$.

Theorem 1.3. *Let* $\alpha_Q = Q\left(2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1}\right)^{Q'-1}$. *Then there exists a uniform constant C_0 depending only on Q such that for all* $\Omega \subset \mathbb{H}$, $|\Omega| < \infty$ *and* $\alpha \le \alpha_Q$

$$\sup_{u\in W_0^{1,Q}(\Omega),\|\nabla_{\mathbb{H}}u\|_{L^Q}\le 1} \frac{1}{|\Omega|}\int_\Omega \exp(\alpha|u(\xi)|^{Q'})d\xi \le C_0 < \infty. \tag{1.5}$$

The constant α_Q is the best possible in the sense that if $\alpha > \alpha_Q$, then the supremum in the inequality (1.5) is infinite.

It is clear that when $|\Omega| = \infty$, the above inequality (1.5) in Theorem 1.3 is not meaningful. We also remark that using the similar ideas of representation formulas and rearrangement of convolutions as done on the Heisenberg group in [19], Theorem 1.3 was extended to the groups of Heisenberg type in [20] and to general stratified groups in [8]. We refer to [13] for more introduction of stratified groups. The best constant for the sharp Moser–Trudinger inequality on CR sphere was also identified in [21].

Using the sharp representation formula in [19], the authors established the following version of sharp singular Moser–Trudinger inequality on domains of finite measure on the Heisenberg group in [48].

Theorem 1.4. *Let* $\Omega \subset \mathbb{H}$, $|\Omega| < \infty$ *and* $0 \le \beta < Q$. *Then there exists a uniform constant* $C_0 < \infty$ *depending only on* Q, β *such that*

$$\sup_{u\in W_0^{1,Q}(\Omega),\|\nabla_{\mathbb{H}}u\|_{L^Q}\le 1} \frac{1}{|\Omega|^{1-\frac{\beta}{Q}}}\int_\Omega \frac{\exp\left(\alpha_Q\left(1-\frac{\beta}{Q}\right)|u(\xi)|^{Q'}\right)d\xi}{\rho(\xi)^\beta} \le C_0.$$

The constant $\alpha_Q\left(1-\frac{\beta}{Q}\right)$ *is sharp in the sense that if* $\alpha_Q\left(1-\frac{\beta}{Q}\right)$ *is replaced by any larger number, then the supremum is infinite.*

The situation is more complicated when dealing with unbounded domains on the Heisenberg group. Before we state the Moser–Trudinger inequality on the entire Heisenberg group, we need to recall some preliminaries.

Let $u : \mathbb{H} \to \mathbb{R}$ be a nonnegative function in $W^{1,Q}(\mathbb{H})$, and u^* be the decreasing rearrangement of u, namely

$$u^*(\xi) := \sup\left\{s \ge 0 : \xi \in \{u > s\}^*\right\}$$

where

$$\{u > s\}^* = B_r = \{\xi : \rho(\xi) \le r\}$$

such that $|\{u > s\}| = |B_r|$. It is known from a result of Manfredi and Vera De Serio [62] that there exists a constant $c \ge 1$ depending only on Q such that

$$\int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} u^*\right|^Q d\xi \le c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q d\xi \tag{1.6}$$

for all $u \in W^{1,Q}(\mathbb{H})$. Thus we can define

$$c^* = \inf\left\{c^{1/(Q-1)} : \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} u^*\right|^Q d\xi \le c \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q d\xi, u \in W^{1,Q}(\mathbb{H})\right\} \ge 1.$$

We can now state the following version of the Moser–Trudinger type inequality (see [18]).

Theorem 1.5. *Let $\alpha^* = \alpha_Q/c^*$. Then for any pair β, α satisfying $0 \le \beta < Q$ and $\alpha \le \alpha^*(1 - \frac{\beta}{Q})$, there holds*

$$\sup_{\|u\|_{W^{1,Q}(\mathbb{H})} \le 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{\exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u)\right\} < \infty \tag{1.7}$$

where

$$S_{Q-2}(\alpha, u) = \sum_{k=0}^{Q-2} \frac{\alpha^k}{k!} |u|^{kQ/(Q-1)}.$$

Moreover, the supremum is infinite if $\alpha > \alpha_Q(1 - \frac{\beta}{Q})$.

We mention in passing that inequality (1.7) in Euclidean spaces when $\beta = 0$ was established in two dimensional case $\mathbb{R}^2$ in [65] and high dimensional case $\mathbb{R}^N$ in [53], while the singular case $0 \le \beta < N$ was treated in [6]. A subcritical case was studied first in two dimension $\mathbb{R}^2$ in [16] and in high dimension in [1].

We briefly outline the proof of Theorem 1.5 given in [18]. By using the rearrangement inequality (1.6), we can reduce the inequality to the case where the functions are radial in terms of the homogeneous norm on the Heisenberg group. Then we break the integral over the space $\mathbb{H}$ into two parts, the interior of a large ball and the exterior of the ball. Over the finite ball, we can use the sharp Moser–Trudinger inequality on finite domains proved in [19]. On the exterior of the ball, we will then use the radial lemma for radial functions on the Heisenberg group. However, we should note that, in the above Theorem 1.5, we cannot exhibit the best constant $\alpha^*(1 - \frac{\beta}{Q})$ due to the loss of the non-optimal rearrangement inequality in the Heisenberg group. In fact, in the inequality controlling the norm of the subelliptic gradient of the rearranged function u^*, the constant c^* is not known to be 1. Therefore, the constant $\frac{\alpha_Q}{c^*}(1 - \frac{\beta}{Q})$ is not known to be equal to $\alpha_Q(1 - \frac{\beta}{Q})$. Thus, the considerably more difficult critical case $\alpha = \alpha_Q(1 - \frac{\beta}{Q})$ is still left open from [18].

Thus, our main purpose in this paper is to establish the Moser–Trudinger type inequalities in the critical case $\alpha = \alpha_Q(1 - \frac{\beta}{Q})$. Our new argument is completely different from and much simpler than those used in [18]. In addition, our method avoids using the symmetrization (i.e., the non-optimal rearrangement inequality (1.6)) on the Heisenberg group. Most importantly, our method allows us to derive the best constant $\alpha = \alpha_Q(1 - \frac{\beta}{Q})$. We also mention that sharp Adams inequalities on high order Sobolev spaces $W^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$ for any arbitrary fractional order $0 < \alpha < n$ have been established in [43] on unbounded domains without using any symmetrization or comparison principle for solutions to polyharmonic operators, significantly simpler than the method used in [66,42,47].

Indeed, our first main result concerning the best constant for the Moser–Trudinger inequality on the entire Heisenberg group $\mathbb{H}$ can be read as follows:

Theorem 1.6. *Let τ be any positive real number. Then for any pair β, α satisfying $0 \le \beta < Q$ and $0 < \alpha \le \alpha_Q(1 - \frac{\beta}{Q})$, there holds*

$$\sup_{\|u\|_{1,\tau} \le 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u) \right\} < \infty. \tag{1.8}$$

When $\alpha > \alpha_Q(1 - \frac{\beta}{Q})$, the integral in (1.8) is still finite for any $u \in W^{1,Q}(\mathbb{H})$, but the supremum is infinite. Here

$$\|u\|_{1,\tau} = \left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q \right]^{1/Q}.$$

We remark in passing that we have proved in [49] the following different version of Moser–Trudinger inequality in the spirit of Adachi–Tanaka [1] with less restriction on the norm by only assuming $\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \le 1$ instead of $\|u\|_{1,\tau} \le 1$.

Theorem 1.7. *For any pair β, α satisfying $0 \le \beta < Q$ and $0 < \alpha < \alpha_Q(1 - \frac{\beta}{Q})$ there exists a constant $0 < C_{\alpha,\beta} < \infty$ such that the following inequality holds*

$$\sup_{\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \le 1} \frac{1}{\|u\|_{L^Q(\mathbb{H})}^{Q-\beta}} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u) \right\} \le C_{\alpha,\beta}. \tag{1.9}$$

The above result is sharp in the sense when $\alpha \ge \alpha_Q(1 - \frac{\beta}{Q})$, the integral in (1.9) is still finite for any $u \in W^{1,Q}(\mathbb{H})$, but the supremum is infinite.

We note here that our proof of Theorem 1.7 does not rely on the method of symmetrization which was used in [1] the Euclidean space. As a matter of fact, such a symmetrization is not available on the Heisenberg group $\mathbb{H}$. Therefore, the argument in [1] does not work on $\mathbb{H}$.

It is interesting to note that there is a sharp difference between Theorems 1.6 and 1.7. The inequality (1.8) in Theorem 1.6 holds for all $\alpha \le (1 - \frac{\beta}{Q})\alpha_Q$, while the inequality (1.9) in Theorem 1.7 only holds for $\alpha < (1 - \frac{\beta}{Q})\alpha_Q$. This indicates the restriction of Sobolev norms on the functions under consideration has a substantial impact on the sharp constants for the geometric inequalities.

As applications of our Theorem 1.6, we study a class of partial differential equations of exponential growth on the Heisenberg group. More precisely, we consider the existence of nontrivial weak solutions for the non-uniformly subelliptic equations of Q-sub-Laplacian type of the form (see [25] for the study of such type of non-uniformly equations in Euclidean spaces):

$$-\operatorname{div}_{\mathbb{H}}(a(\xi, \nabla_{\mathbb{H}} u)) + V(\xi)|u|^{Q-2} u = \frac{f(\xi, u)}{\rho(\xi)^\beta} + \varepsilon h(\xi) \tag{NU}$$

where

$$|a(\xi,\tau)| \le c_0\left(h_0(\xi) + h_1(\xi)|\tau|^{Q-1}\right)$$

for any τ in $\mathbb{R}^{Q-2}$ and a.e. ξ in $\mathbb{H}$, $h_0 \in L^{Q'}(\mathbb{H})$ and $h_0 \in L^{\infty}_{\mathrm{loc}}(\mathbb{H})$, $0 \le \beta < Q$, $V : \mathbb{H} \to \mathbb{R}$ is a continuous potential, $f : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp\left(\alpha|u|^{Q'}\right)$ when $|u| \to \infty$ and satisfy those assumptions (V1),(V2), (V3) and (f1), (f2), (f3) in Section 3, and $h \in \left(W^{1,Q}(\mathbb{H})\right)^*$, $h \neq 0$ and ε is a positive parameter. The main features of this class of problems are that it is defined in the whole space $\mathbb{H}$ and involves critical growth and the nonlinear operator Q-sub-Laplacian type. In spite of a possible failure of the Palais–Smale (PS) compactness condition, in this article we apply the mountain-pass theorem to obtain the weak solution of (NU) in the suitable subspace E of $W^{1,Q}(\mathbb{H})$. Moreover, in the case of Q-sub-Laplacian, i.e.,

$$a(\xi,\nabla_{\mathbb{H}}u) = |\nabla_{\mathbb{H}}u|^{Q-2}\nabla_{\mathbb{H}}u,$$

we will apply the mountain-pass theorem combined with minimization and Ekelands variational principle to obtain multiplicity of weak solutions to the nonhomogeneous problem

$$-\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}}u|^{Q-2}\nabla_{\mathbb{H}}u\right) + V(\xi)|u|^{Q-2}u = \frac{f(\xi,u)}{\rho(\xi)^{\beta}} + \varepsilon h(\xi). \tag{NH}$$

We mention that the existence of nontrivial nonnegative solutions to the equation (NH) was established in [18]. However, the multiplicity of solutions was not treated in [18]. In fact, in the case with no perturbation term, i,e., $\epsilon = 0$, the existence of a nontrivial solution for the following nonlinear equation is established in this paper (see Theorem 1.11):

$$-\operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}}u|^{Q-2}\nabla_{\mathbb{H}}u\right) + V(\xi)|u|^{Q-2}u = \frac{f(\xi,u)}{\rho(\xi)^{\beta}}. \tag{1.10}$$

To establish the existence of a nontrivial solution for this Eq. (1.10), the sharp constant $\alpha = \alpha_Q(1-\frac{\beta}{Q})$ for the Moser–Trudinger inequality on the entire Heisenberg group at the critical case (Inequality (1.8) in Theorem 1.6) plays an important role. It is also worth noticing that the Moser–Trudinger type inequalities in Euclidean spaces are crucial in the study of elliptic partial differential equations of the exponential growth. Here we just mention some of them, we refer the reader to [17,26,55,3–5,22–24,51,50,40,61,67,71] and the references therein. Existence of solutions to polyharmonic operators and N-Laplacian equations of exponential growth without satisfying the standard Ambrosetti–Rabinowitz condition have been studied by the authors in [41,44,45].

We next state our main results concerning the existence and multiplicity of nontrivial nonnegative solutions to the Q-sub-Laplacian (NH) on the Heisenberg group.

Theorem 1.8. *Suppose that* (V1) *and* V(2) *(or* (V3)*) and* (f1)–(f3) *are as stated in Section* 3 *and* $\lambda_1(Q)$ *is as defined in* (3.2) *in Section* 3. *Furthermore, assume that*

(f4) $\limsup_{s\to 0+} \dfrac{F(\xi,s)}{k_0|s|^Q} < \lambda_1(Q)$ *uniformly in* $\xi \in \mathbb{H}$.

Then there exists $\varepsilon_1 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_1$, *(NU) has a weak solution of mountain-pass type.*

Theorem 1.9. *In addition to the hypotheses in Theorem 1.8, assume that*

$$\text{(f5)}\ \lim_{s\to\infty} sf(\xi,s)\exp\left(-\alpha_0 |s|^{Q/(Q-1)}\right) = +\infty$$

uniformly on compact subsets of $\mathbb{H}$*. Then, there exists* $\varepsilon_2 > 0$*, such that for each* $0 < \varepsilon < \varepsilon_2$*, problem (NH) has at least two weak solutions and one of them has a negative energy.*

In the case where the function h does not change sign, we have the following.

Theorem 1.10. *Under the assumptions in Theorems 1.8 and 1.9, if* $h(\xi) \geq 0$ $(h(\xi) \leq 0)$ *a.e., then the solutions of problem (NH) are nonnegative (nonpositive).*

The perturbation term $\varepsilon h(\xi)$ in Eq. (NH) helps us to establish the existence result. However, when the perturbation term disappears, it is harder to do. Nevertheless, using the best constant $\alpha = \alpha_Q(1-\frac{\beta}{Q})$ we can still succeed to conclude the existence result.

We state this existence of a nontrivial solution result as follows.

Theorem 1.11. *Under the same hypotheses in Theorems 1.8 and 1.9, problem (NH) with* $\varepsilon = 0$ *has a nontrivial weak solution.*

Our paper is organized as follows. In Section 2, we will prove one of our main theorems, the sharp Moser–Trudinger inequality (Theorem 1.6). In Section 3, we will briefly discuss about nonlinear equations of exponential growth. We also provide the assumptions on the nonlinearity f and potential V. We will also discuss the variational framework concerning Eqs. (NU) and (NH). In Section 4, we prove some basic lemmas that are useful in proving Theorems 1.8 and 1.9. We will investigate in Section 5 the existence of nontrivial solution to Eq. (NU) (Theorem 1.8). Section 6 is devoted to the study of multiplicity of solutions to Eq. (NH) (Theorem 1.9). We also establish Theorems 1.10 and 1.11 in this section.

2. Proof of Theorem 1.6: the sharp Moser–Trudinger inequality

The primary purpose of this section is to offer a completely different and much simpler proof of the best constant $\alpha_Q(1-\frac{\beta}{Q})$ for the Moser–Trudinger inequality on unbounded domains in the Heisenberg group $\mathbb{H}$. All existing proofs on the Heisenberg group only give the subcritical case for $\alpha < \alpha_Q(1-\frac{\beta}{Q})$. Our proof does not rely on the special structure of the Heisenberg group and applies to much more general cases including the stratified groups [13], Euclidean spaces, complete Riemannian manifolds, even appropriate metric measure spaces, etc. However, for its simplicity and clarity, we only present it on the Heisenberg group.

Proof. It suffices to prove that for any β,τ satisfying $0 \leq \beta < Q$ and $\tau > 0$, there exists a constant $C = C(\beta,\tau,Q)$ such that for all $u \in C_0^\infty(\mathbb{H})\setminus\{0\}$, $u \geq 0$ and $\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q \leq 1$, there holds

$$\int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta}\left\{\exp\left(\alpha_Q\left(1-\frac{\beta}{Q}\right)|u|^{Q/(Q-1)}\right) - S_{Q-2}\left(\alpha_Q\left(1-\frac{\beta}{Q}\right),u\right)\right\} \leq C(\beta,\tau,Q). \tag{2.1}$$

We fix some notations here:

$$A(u) = 2^{-\frac{1}{Q(Q-1)}}\tau^{\frac{1}{Q}}\|u\|_Q$$

$$\Omega(u) = \{\xi \in \mathbb{H} : u(\xi) > A(u)\}.$$

Then, it is clear that

$$A(u) < 1. \tag{2.2}$$

Moreover, since

$$\begin{aligned}\int_{\mathbb{H}} |u|^Q &\geq \int_{\Omega(u)} |u|^Q \\ &\geq \int_{\Omega(u)} |A(u)|^Q \\ &= 2^{-\frac{1}{(Q-1)}} \tau \, \|u\|_Q^Q \, |\Omega(u)|\end{aligned}$$

we have

$$|\Omega(u)| \leq 2^{\frac{1}{(Q-1)}} \frac{1}{\tau}. \tag{2.3}$$

Now, we write

$$\int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)}\right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u\right) \right\} = I_1 + I_2$$

where

$$I_1 = \int_{\Omega(u)} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)}\right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u\right) \right\}$$

and

$$I_2 = \int_{\mathbb{H}\setminus\Omega(u)} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)}\right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u\right) \right\}.$$

We will prove that both I_1 and I_2 are bounded by a constant $C = C(\beta, \tau, Q)$.

Indeed, from (2.2), we see

$$\begin{aligned} I_2 &\leq \int_{\{u(\xi)<1\}} \frac{1}{\rho(\xi)^\beta} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} |u|^{kQ/(Q-1)} \\ &\leq \int_{\{u(\xi)<1\}} \frac{1}{\rho(\xi)^\beta} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} |u|^{Q} \\ &\leq \int_{\{\rho(\xi)\geq 1\}} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} |u|^{Q} \\ &\quad + \int_{\{\rho(\xi)<1\}} \frac{1}{\rho(\xi)^\beta} \sum_{k=Q-1}^{\infty} \frac{\left[\alpha_Q\left(1-\frac{\beta}{Q}\right)\right]^k}{k!} \\ &\leq C(\beta, \tau, Q). \end{aligned}$$

Now, to estimate I_1, we first notice that if we set

$$v(\xi) = u(\xi) - A(u) \quad \text{in } \Omega(u),$$

then $v \in W_0^{1,Q}(\Omega(u))$. Moreover, in $\Omega(u)$:

$$\begin{aligned}
|u|^{Q'} &= (|v| + A(u))^{Q'} \\
&\le |v|^{Q'} + Q'2^{Q'-1}\left(|v|^{Q'-1} A(u) + |A(u)|^{Q'}\right) \\
&\le |v|^{Q'} + Q'2^{Q'-1}\frac{|v|^{Q'}\,|A(u)|^{Q}}{Q} + Q'2^{Q'-1}\left(\frac{1}{Q'} + |A(u)|^{Q'}\right) \\
&\le |v|^{Q'}\left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right) + C(Q)
\end{aligned}$$

where we did use Young's inequality and the following elementary inequality:

$$(a+b)^q \le a^q + q2^{q-1}\left(a^{q-1}b + b^q\right) \quad \text{for all } q \ge 1 \text{ and } a,\ b \ge 0.$$

Let

$$w(\xi) = \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)^{\frac{Q-1}{Q}} v(\xi) \quad \text{in } \Omega(u),$$

then it is clear that

$$w \in W_0^{1,Q}(\Omega) \quad \text{and} \quad |u|^{Q'} \le |w|^{Q'} + C(Q). \tag{2.4}$$

Moreover, we have

$$\nabla_{\mathbb{H}} w = \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)^{\frac{Q-1}{Q}} \nabla_{\mathbb{H}} v.$$

Thus

$$\begin{aligned}
\int_{\Omega(u)} |\nabla_{\mathbb{H}} w|^{Q} &= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)^{Q-1} \int_{\Omega(u)} |\nabla_{\mathbb{H}} v|^{Q} \\
&= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)^{Q-1} \int_{\Omega(u)} |\nabla_{\mathbb{H}} u|^{Q} \\
&\le \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)^{Q-1} \left[1 - \tau \int_{\mathbb{H}} |u|^{Q}\right].
\end{aligned}$$

Then

$$\begin{aligned}
\left(\int_{\Omega(u)} |\nabla_{\mathbb{H}} w|^{Q}\right)^{\frac{1}{Q-1}} &= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)\left[1 - \tau \int_{\mathbb{H}} |u|^{Q}\right]^{\frac{1}{Q-1}} \\
&\le \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}|A(u)|^{Q}\right)\left(1 - \frac{\tau}{Q-1}\int_{\mathbb{H}} |u|^{Q}\right) \\
&= \left(1 + \frac{2^{\frac{1}{Q-1}}}{Q-1}2^{-\frac{1}{(Q-1)}}\tau \|u\|_Q^Q\right)\left(1 - \frac{\tau}{Q-1}\int_{\mathbb{H}} |u|^{Q}\right)
\end{aligned}$$

$$= \left(1 + \frac{\tau}{Q-1} \int_{\mathbb{H}} |u|^Q\right) \left(1 - \frac{\tau}{Q-1} \int_{\mathbb{H}} |u|^Q\right)$$
$$\leq 1. \tag{2.5}$$

Here, we used the inequality

$$(1-x)^q \leq 1 - qx \quad \text{for all } 0 \leq x \leq 1,\ 0 < q \leq 1.$$

From (2.4) and (2.5), using Theorem 1.4 and (2.3), we get

$$\begin{aligned} I_1 &\leq \int_{\Omega(u)} \frac{\exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} \\ &\leq e^{\alpha_Q \left(1-\frac{\beta}{Q}\right) C(Q)} \int_{\Omega(u)} \frac{\exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |w|^{Q/(Q-1)}\right)}{\rho(\xi)^\beta} \\ &\leq e^{\alpha_Q \left(1-\frac{\beta}{Q}\right) C(Q)} C_0 \left|\Omega(u)\right|^{1-\frac{\beta}{Q}} \\ &\leq C(\beta, \tau, Q). \end{aligned}$$

The proof is then completed. □

3. Assumptions on the nonlinearity and the potential and variational framework

In this section, we will provide conditions on the nonlinearity and potential of Eqs. (NU) and (NH). Motivated by the Moser–Trudinger inequality (Theorem 1.6), we consider here the maximal growth on the nonlinear term $f(\xi, u)$ which allows us to treat Eqs. (NU) and (NH) variationally in a subspace of $W^{1,Q}(\mathbb{H})$. We assume that $f : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(\xi, 0) = 0$ and f behaves like $\exp\left(\alpha |u|^{Q/(Q-1)}\right)$ as $|u| \to \infty$. More precisely, we assume the following growth conditions on the nonlinearity $f(\xi, u)$:

(f1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(\xi, u) \in \mathbb{H} \times \mathbb{R}$,

$$|f(\xi, u)| \leq b_1 |u|^{Q-1} + b_2 \left[\exp\left(\alpha_0 |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0, u)\right],$$

(f2) There exists $p > Q$ such that for all $\xi \in \mathbb{H}$ and $s > 0$,

$$0 < pF(\xi, s) = p \int_0^s f(\xi, \tau) d\tau \leq sf(\xi, s)$$

(f3) There exist constant $R_0, M_0 > 0$ such that for all $\xi \in \mathbb{H}$ and $s \geq R_0$,

$$F(\xi, s) \leq M_0 f(\xi, s).$$

Since we are interested in nonnegative weak solutions, we will assume

$$f(\xi, u) = 0 \quad \text{for all } (\xi, u) \in \mathbb{H} \times (-\infty, 0].$$

Let A be a measurable function on $\mathbb{H} \times \mathbb{R}^{Q-2}$ such that $A(\xi, 0) = 0$ and $a(\xi, \tau) = \nabla_\tau A(\xi, \tau)$ is a Caratheodory function on $\mathbb{H} \times \mathbb{R}^{Q-2}$. Assume that there are positive real numbers c_0, c_1, k_0, k_1 and two nonnegative measurable functions h_0, h_1 on $\mathbb{H}$ such that $h_1 \in L^\infty_{\text{loc}}(\mathbb{H})$, $h_0 \in L^{Q/(Q-1)}(\mathbb{H})$, $h_1(\xi) \geq 1$ for a.e. ξ in $\mathbb{H}$ and the following conditions hold:

(A1) $|a(\xi,\tau)| \le c_0\left(h_0(\xi) + h_1(\xi)|\tau|^{Q-1}\right) \quad \forall \tau \in \mathbb{R}^{Q-2},\ \text{a.e.}\ \xi \in \mathbb{H}$

(A2) $c_1|\tau - \tau_1|^Q \le \langle a(\xi,\tau) - a(\xi,\tau_1), \tau - \tau_1\rangle \quad \forall \tau, \tau_1 \in \mathbb{R}^{Q-2},\ \text{a.e.}\ \xi \in \mathbb{H}$

(A3) $0 \le a(\xi,\tau).\tau \le QA(\xi,\tau) \quad \forall \tau \in \mathbb{R}^{Q-2},\ \text{a.e.}\ \xi \in \mathbb{H}$

(A4) $A(\xi,\tau) \ge k_0 h_1(\xi)|\tau|^Q \quad \forall \tau \in \mathbb{R}^{Q-2},\ \text{a.e.}\ \xi \in \mathbb{H}.$

Then A verifies the growth condition:

$$|A(\xi,\tau)| \le c_0\left(h_0(\xi)|\tau| + h_1(\xi)|\tau|^Q\right) \quad \forall \tau \in \mathbb{R}^{Q-2},\ \text{a.e.}\ \xi \in \mathbb{H}. \tag{3.1}$$

For examples of A, we can consider $A(\xi,\tau) = h(\xi)\frac{|\tau|^Q}{Q}$ where $h \in L^\infty_{\text{loc}}(\mathbb{H})$.

We also propose the following conditions on the potential:

(V1) V is a continuous function such that $V(\xi) \ge V_0 > 0$ for all $\xi \in \mathbb{H}$, and one of the following two conditions:

(V2) $V(\xi) \to \infty$ as $\rho(\xi) \to \infty$; or more generally, for every $M > 0$,

$$\mu(\{\xi \in \mathbb{H} : V(\xi) \le M\}) < \infty$$

or

(V3) The function $[V(\xi)]^{-1}$ belongs to $L^1(\mathbb{H})$.

We introduce some notations:

$$E = \left\{u \in W_0^{1,Q}(\mathbb{H}) : \int_{\mathbb{H}} h_1(\xi)|\nabla_{\mathbb{H}} u|^Q d\xi + \int_{\mathbb{H}} V(\xi)|u|^Q < \infty\right\}$$

$$\|u\| = \left(\int_{\mathbb{H}}\left(h_1(\xi)|\nabla_{\mathbb{H}} u|^Q + \frac{1}{k_0 Q}V(\xi)|u|^Q\right)d\xi\right)^{1/Q}, \quad u \in E$$

$$\lambda_1(Q) = \inf\left\{\frac{\|u\|^Q}{\int_{\mathbb{H}} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi} : u \in E \setminus \{0\}\right\}.$$

Under the condition on the potential (V1), we can see that E is a reflexive Banach space when endowed with the norm

$$\|u\| = \left(\int_{\mathbb{H}}\left(h_1(\xi)|\nabla_{\mathbb{H}} u|^Q + \frac{1}{k_0 Q}V(\xi)|u|^Q\right)d\xi\right)^{1/Q}$$

and for all $Q \le q < \infty$,

$$E \hookrightarrow W^{1,Q}(\mathbb{H}) \hookrightarrow L^q(\mathbb{H})$$

with continuous embedding. Furthermore,

$$\lambda_1(Q) = \inf\left\{\frac{\|u\|^Q}{\int_{\mathbb{H}} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi} : u \in E \setminus \{0\}\right\} > 0 \quad \text{for any } 0 \le \beta < Q. \tag{3.2}$$

By the assumptions (V2) or (V3), we can get the compactness of the embedding

$$E \hookrightarrow L^p(\mathbb{H}) \quad \text{for all } p \ge Q.$$

Following from $(f1)$, we can conclude for all $(\xi, u) \in \mathbb{H} \times \mathbb{R}$,

$$|F(\xi, u)| \leq b_3 \left[\exp\left(\alpha_1 |u|^{Q/(Q-1)} \right) - S_{Q-2}(\alpha_1, u) \right]$$

for some constants $\alpha_1, b_3 > 0$. Thus, by the Moser–Trudinger type inequalities, we have $F(\xi, u) \in L^1(\mathbb{H})$ for all $u \in W^{1,Q}(\mathbb{H})$. Define the functional $E,\ T,\ J_\varepsilon : E \to \mathbb{R}$ by

$$\begin{aligned}
E(u) &= \int_{\mathbb{H}} A(\xi, \nabla_H u) d\xi + \frac{1}{Q} \int_{\mathbb{H}} V(\xi) |u|^Q d\xi \\
T(u) &= \int_{\mathbb{H}} \frac{F(\xi, u)}{\rho(\xi)^\beta} d\xi \\
J_\varepsilon(u) &= E(u) - T(u) - \varepsilon \int_{\mathbb{H}} hu d\xi
\end{aligned}$$

then the functional J_ε is well-defined. Moreover, J_ε is a C^1 functional on E with

$$\begin{aligned}
DJ_\varepsilon(u)v = &\int_{\mathbb{H}} a(\xi, \nabla_H u) \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi) |u|^{Q-2} v d\xi \\
&- \int_{\mathbb{H}} \frac{f(\xi, u) v}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} hv d\xi, \quad \forall u, v \in E.
\end{aligned}$$

Therefore, $DJ_\varepsilon(u) = 0$ if and only if $u \in E$ is a weak solution to Eq. (NU).

4. Some basic lemmas

First, we recall what we call the Radial Lemma (see [18]) which asserts:

$$\left| u^*(\xi) \right|^Q \leq \frac{Q}{\omega_{Q-1}} \frac{\|u^*\|_Q^Q}{\rho(\xi)^Q}, \quad \forall \xi \in \mathbb{H} \setminus \{0\}$$

where u^* is the decreasing rearrangement of $|u|$ and $\omega_{Q-1} = \int_{\rho(\xi)=1} d\xi$. Using this Radial Lemma, we can prove the following two lemmas (see [24,18]).

Lemma 4.1. *For $\kappa > 0$ and $\|u\|_E \leq M$ with M sufficiently small and $q > Q$, we have*

$$\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa |u|^{Q/(Q-1)} \right) - S_{Q-2}(\kappa, u) \right] |u|^q}{\rho(\xi)^\beta} d\xi \leq C(Q, \kappa) \|u\|^q .$$

Proof. The proof is analogous to the proof of Theorem 1.1 in [18]. For the completeness, we give the details here.

Set

$$R(\alpha, u) = \exp\left(\alpha |u|^{Q/(Q-1)} \right) - S_{Q-2}(\alpha, u) .$$

Assume that u^* is the decreasing rearrangement of $|u|$. We have by the Hardy–Littlewood inequality that

$$\int_{\mathbb{H}} \frac{R(\kappa, u) |u|^q}{\rho(\xi)^\beta} d\xi \leq \int_{\mathbb{H}} \frac{R(\kappa, u^*) |u^*|^q}{\rho(\xi)^\beta} d\xi. \tag{4.1}$$

Let γ be a positive number to be chosen later, we estimate

$$\begin{aligned}
&\int_{\rho(\xi)\le\gamma} \frac{R(\kappa,u^*)\,|u^*|^q}{\rho(\xi)^\beta}d\xi \\
&\le \left(\int_{\rho(\xi)\le\gamma} \frac{(R(\kappa,u^*))^p}{\rho(\xi)^\beta}d\xi\right)^{1/p}\left(\int_{\rho(\xi)\le\gamma}\frac{1}{\rho(\xi)^{\beta s}}d\xi\right)^{1/p's} \\
&\quad\times\left(\int_{\rho(\xi)\le\gamma}\left|u^*\right|^{qp's'}d\xi\right)^{1/p's'} \\
&\le C\left(\int_{\rho(\xi)\le\gamma}\frac{R(p\kappa,u^*)}{\rho(\xi)^\beta}d\xi\right)^{1/p}\left(\int_{\rho(\xi)\le\gamma}\left|u^*\right|^{qp's'}d\xi\right)^{1/p's'}
\end{aligned}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{p'}=1,\ 1<s<\frac{Q}{\beta}$, and $\frac{1}{s}+\frac{1}{s'}=1$. This together with Moser–Trudinger type inequalities and the continuous embedding of $E\hookrightarrow L^t(\mathbb{H})\,,\ t\ge Q$ implies

$$\int_{\rho(\xi)\le\gamma}\frac{R(\kappa,u^*)\,|u^*|^q}{\rho(\xi)^\beta}d\xi\le C\,\|u\|^q \tag{4.2}$$

for some constant $C=C(Q,\kappa,\gamma)$, provided that $\|u\|_E$ is sufficiently small such that $p\kappa\,\|u\|_E^{Q/(Q-1)}\le\alpha^*$.

On the other hand, choosing γ sufficiently large such that $\left(Q/\omega_{Q-1}\right)^{1/Q}\gamma^{-1}\|u\|_E<1/2$, we obtain by the Radial lemma and the continuous embedding of $E\hookrightarrow L^q(\mathbb{H})$,

$$\begin{aligned}
\int_{\rho(\xi)\ge\gamma}\frac{R(\kappa,u^*)\,|u^*|^q}{\rho(\xi)^\beta}d\xi &\le \frac{R(\kappa,u^*(\gamma))}{\gamma^\beta}\int_{\rho(\xi)\ge\gamma}\left|u^*\right|^q d\xi \\
&\le \frac{R(\kappa,1/2)}{\gamma^\beta}\left\|u^*\right\|_q^q\le C\,\|u\|_E^q
\end{aligned}\tag{4.3}$$

for some constant C. By (4.1)–(4.3), we then complete the proof of the lemma. □

Lemma 4.2. *If $\kappa>0,0\le\beta<Q,u\in E$ and $\|u\|_E\le M$ with $\kappa M^{Q/(Q-1)}<\left(1-\frac{\beta}{Q}\right)\alpha_Q$, then*

$$\int_{\mathbb{H}}\frac{\left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right)-S_{Q-2}(\kappa,u)\right]|u|}{\rho(\xi)^\beta}d\xi\le C(Q,M,\kappa)\,\|u\|_s$$

for some $s>Q$.

Proof. First, recall the following inequality: For $\alpha\ge0,r\ge1$, we have

$$\left(e^\alpha-\sum_{k=0}^{Q-2}\frac{\alpha^k}{k!}\right)^r\le e^{r\alpha}-\sum_{k=0}^{Q-2}\frac{(r\alpha)^k}{k!}. \tag{4.4}$$

Now, using the Holder inequality, (4.4) and Theorem 1.6, we have

$$\begin{aligned}
&\int_{\mathbb{H}}\frac{\left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right)-S_{Q-2}(\kappa,u)\right]|u|}{\rho(\xi)^\beta}d\xi \\
&\le\left[\int_{\mathbb{H}}\frac{\left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right)-S_{Q-2}(\kappa,u)\right]^r}{\rho(\xi)^{r\beta}}d\xi\right]^{1/r}\left[\int_{\mathbb{H}}|u|^s\right]^{1/s}
\end{aligned}$$

$$\leq \left[\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa r \left|u\right|^{Q/(Q-1)}\right) - S_{Q-2}\left(\kappa r, u\right)\right]}{\rho(\xi)^{r\beta}} d\xi \right]^{1/r} \|u\|_s$$
$$\leq C\left(Q, M, \kappa\right) \|u\|_s$$

where $r, s \geq 1$, $\frac{1}{r} + \frac{1}{s} = 1$ and r is sufficiently close to 1. □

We also have the following lemma (for Euclidean case, see [56]).

Lemma 4.3. *Let $\{w_k\} \subset E$, $\|w_k\|_E = 1$. If $w_k \to w \neq 0$ weakly and almost everywhere, $\nabla_{\mathbb{H}} w_k \to \nabla_{\mathbb{H}} w$ almost everywhere, then $\frac{R(\alpha, w_k)}{\rho(\xi)^\beta}$ is bounded in $L^1(\mathbb{H})$ for $0 < \alpha < \alpha_Q \left(1 - \frac{\beta}{Q}\right)\left(1 - \|w\|_E^Q\right)^{-1/(Q-1)}$.*

Proof. Using the Brezis–Lieb lemma in [15], we deduce that

$$\|w_k\|_E^Q - \|w_k - w\|_E^Q \to \|w\|_E^Q.$$

Thus for k large enough and $\delta > 0$ small enough:

$$0 < \alpha\left(1 + \delta\right) \|w_k - w\|_E^{Q/(Q-1)} < \alpha_Q \left(1 - \frac{\beta}{Q}\right).$$

Now, by noticing that the function $e^x - \sum_{k=0}^{Q-2} \frac{x^k}{k!}$ is increasing and convex in $x \geq 0$ and the fact that for all $\varepsilon > 0$ sufficiently small, there exists $C(\varepsilon) > 0$ such that for all real numbers a, b:

$$|a + b|^{Q'} \leq (1 + \varepsilon) |a|^{Q'} + C(\varepsilon) |b|^{Q'},$$

we have

$$\int_{\mathbb{H}} \frac{R(\alpha, w_k)}{\rho(\xi)^\beta} d\xi \leq \frac{1}{p} \int_{\mathbb{H}} \frac{R((1 + \varepsilon) p\alpha, w_k - w)}{\rho(\xi)^\beta} d\xi + \frac{1}{q} \int_{\mathbb{H}} \frac{R(qC(\varepsilon)\alpha, w)}{\rho(\xi)^\beta} d\xi$$

where $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now, by choosing p sufficiently close to 1 and ε small enough such that $(1 + \varepsilon) p < (1 + \delta)$ and using Theorem 1.6, we get the conclusion. □

5. The existence of the solution to the problem (NU)

The existence of the nontrivial solution to Eq. (NU) will be proved by a mountain-pass theorem without a compactness condition such as the one of the Palais–Smale (PS) type. This version of the mountain-pass theorem is a consequence of Ekeland's variational principle. First of all, we will check that the functional J_ε satisfies the geometric conditions of the mountain-pass theorem.

Lemma 5.1. *Suppose that* (V1), (f1) *and* (f4) *hold. Then there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$, there exists $\rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $\|u\|_E = \rho_\varepsilon$. Furthermore, ρ_ε can be chosen such that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. From (f4), there exist $\tau, \delta > 0$ such that $|u| \leq \delta$ implies

$$F(\xi, u) \leq k_0 \left(\lambda_1(Q) - \tau\right) |u|^Q \tag{5.1}$$

for all $\xi \in \mathbb{H}$. Moreover, using (f1) for each $q > Q$, we can find a constant $C = C(q, \delta)$ such that

$$F(\xi, u) \leq C\, |u|^q \left[\exp\left(\kappa\, |u|^{Q/(Q-1)}\right) - S_{Q-2}\,(\kappa, u)\right] \tag{5.2}$$

for $|u| \geq \delta$ and $\xi \in \mathbb{H}$. From (5.1) and (5.2) we have

$$F(\xi, u) \leq k_0\,(\lambda_1(Q) - \tau)\, |u|^Q + C\, |u|^q \left[\exp\left(\kappa\, |u|^{Q/(Q-1)}\right) - S_{Q-2}\,(\kappa, u)\right]$$

for all $(\xi, u) \in \mathbb{H} \times \mathbb{R}$. Now, by $(A4)$, Lemma 4.1, (3.2) and the continuous embedding $E \hookrightarrow L^Q(\mathbb{H})$, we obtain

$$\begin{aligned} J_\varepsilon(u) &\geq k_0\, \|u\|_E^Q - k_0\,(\lambda_1(Q) - \tau) \int_{\mathbb{H}} \frac{|u|^Q}{\rho\,(\xi)^\beta} d\xi - C\, \|u\|_E^q - \varepsilon\, \|h\|_*\, \|u\|_E \\ &\geq k_0 \left(1 - \frac{(\lambda_1(Q) - \tau)}{\lambda_1(Q)}\right) \|u\|_E^Q - C\, \|u\|_E^q - \varepsilon\, \|h\|_*\, \|u\|_E\,. \end{aligned}$$

Thus

$$J_\varepsilon(u) \geq \|u\|_E \left[k_0 \left(1 - \frac{(\lambda_1(Q) - \tau)}{\lambda_1(Q)}\right) \|u\|_E^{Q-1} - C\, \|u\|_E^{q-1} - \varepsilon\, \|h\|_*\right]. \tag{5.3}$$

Since $\tau > 0$ and $q > Q$, we may choose $\rho > 0$ such that $k_0 \left(1 - \frac{(\lambda_1(Q)-\tau)}{\lambda_1(Q)}\right) \rho^{Q-1} - C\rho^{q-1} > 0$. Thus, if ε is sufficiently small then we can find some $\rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $\|u\| = \rho_\varepsilon$ and even $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. □

Lemma 5.2. *There exists $e \in E$ with $\|e\|_E > \rho_\varepsilon$ such that $J_\varepsilon(e) < \inf_{\|u\|=\rho_\varepsilon} J_\varepsilon(u)$.*

Proof. Let $u \in E \setminus \{0\}$, $u \geq 0$ with compact support $\Omega = \operatorname{supp}(u)$. By (f2) and (f3), we have that for $p > Q$, there exists a positive constant $C > 0$ such that

$$\forall s \geq 0, \quad \forall \xi \in \Omega : F\,(\xi, s) \geq cs^p - d. \tag{5.4}$$

Then by (3.1), we get

$$\begin{aligned} J_\varepsilon(tu) \leq\ & Ct \int_\Omega h_0\,(\xi)\, |\nabla_H u|\, d\xi + Ct^Q\, \|u\|_E^Q \\ & - Ct^p \int_\Omega \frac{|u|^p}{\rho\,(\xi)^\beta} d\xi + C + \varepsilon t \left| \int_\Omega hu d\xi \right|. \end{aligned}$$

Since $p > Q$, we have $J_\varepsilon(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t sufficiently large, we get the conclusion. □

In studying this class of sub-elliptic problems involving critical growth and unbounded domains, the loss of the (PS) compactness condition raises many difficulties. In the following lemmas, we will analyze the compactness of (PS) sequences of J_ε.

Lemma 5.3. *Let $(u_k) \subset E$ be an arbitrary (PS) sequence of J_ε, i.e.,*

$$J_\varepsilon\,(u_k) \to c, \qquad DJ_\varepsilon\,(u_k) \to 0 \quad \text{in } E' \text{ as } k \to \infty.$$

Then there exists a subsequence of (u_k) (still denoted by (u_k)) and $u \in E$ such that

$$\begin{cases} \dfrac{f(\xi, u_k)}{\rho(\xi)^\beta} \to \dfrac{f(\xi, u)}{\rho(\xi)^\beta} & \text{strongly in } L^1_{\mathrm{loc}}(\mathbb{H}) \\ \nabla_{\mathbb{H}} u_k(\xi) \to \nabla_{\mathbb{H}} u(\xi) & \text{almost everywhere in } \mathbb{H} \\ a(\xi, \nabla_{\mathbb{H}} u_k) \rightharpoonup a(\xi, \nabla_{\mathbb{H}} u) & \text{weakly in } \left(L^{Q/(Q-1)}_{\mathrm{loc}}(\mathbb{H})\right)^{Q-2} \\ u_k \rightharpoonup u & \text{weakly in } E. \end{cases}$$

Furthermore u is a weak solution of (NU).

In order to prove this lemma, we need the following two lemmas.

Lemma 5.4. *Let $B_r(\xi^*)$ be a Heisenberg ball centered at $(\xi^*) \in \mathbb{H}$ with radius r. Then there exists a positive ε_0 depending only on Q such that*

$$\sup_{\int_{B_r(\xi^*)} |\nabla_H u|^Q d\xi \le 1, \int_{B_r(\xi^*)} u d\xi = 0} \frac{1}{|B_r(\xi^*)|} \int_{B_r(\xi^*)} \exp\left(\varepsilon_0 |u|^{Q/(Q-1)}\right) d\xi \le C_0$$

for some constant C_0 depending only on Q.

The proof of this lemma follows from the representation formula on stratified groups derived in [58] together with a general theorem of exponential integrability for fractional integrals in [59].

Lemma 5.5. *Let (u_n) be in $L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and let f be a continuous function. Then $\frac{f(\xi, u_n)}{\rho(\xi)^\beta} \to \frac{f(\xi, u)}{\rho(\xi)^\beta}$ in $L^1(\Omega)$, provided that $\frac{f(\xi, u_n(\xi))}{\rho(\xi)^\beta} \in L^1(\Omega)$ $\forall n$ and $\int_\Omega \frac{|f(\xi, u_n(\xi)) u_n(\xi)|}{\rho(\xi)^\beta} d\xi \le C_1$.*

We refer the reader to [18] for a proof.

Now we are ready to prove Lemma 5.3.

Proof. By the assumption, we have

$$\int_{\mathbb{H}} A(\xi, \nabla_H u_k) d\xi + \frac{1}{Q} \int_{\mathbb{H}} V(\xi) |u_k|^Q d\xi - \int_{\mathbb{H}} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} h u_k d\xi \overset{k\to\infty}{\to} c \quad (5.5)$$

and

$$\left| \int_{\mathbb{H}} a(\xi, \nabla_H u_k) \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi) |u_k|^{Q-2} u_k v d\xi - \int_{\mathbb{H}} \frac{f(\xi, u_k) v}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}} h v d\xi \right| \le \tau_k \|v\|_E \quad (5.6)$$

for all $v \in E$, where $\tau_k \to 0$ as $k \to \infty$.

Choosing $v = u_k$ in (5.6) and by (A3), we get

$$\int_{\mathbb{H}} \frac{f(\xi, u_k) u_k}{\rho(\xi)^\beta} d\xi + \varepsilon \int_{\mathbb{H}} h u_k d\xi - Q \int_{\mathbb{H}} A(\xi, \nabla_H u_k) d\xi - \int_{\mathbb{H}} V(\xi) |u_k|^{Q-2} u_k d\xi \le \tau_k \|u_k\|_E .$$

This together with (5.5), (f2) and (A4) leads to

$$\left(\frac{p}{Q} - 1\right) \|u_k\|_E^Q \le C(1 + \|u_k\|_E)$$

and hence $\|u_k\|_E$ is bounded and thus

$$\int_{\mathbb{H}} \frac{f(\xi, u_k)u_k}{\rho(\xi)^\beta} d\xi \le C, \qquad \int_{\mathbb{H}} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi \le C. \tag{5.7}$$

Note that the embedding $E \hookrightarrow L^q(\mathbb{H})$ is compact for all $q \ge Q$, by extracting a subsequence, we can assume that

$$u_k \to u \quad \text{weakly in } E \text{ and for almost all } \xi \in \mathbb{H}.$$

Thanks to Lemma 5.5, we have

$$\frac{f(\xi, u_n)}{\rho(\xi)^\beta} \to \frac{f(\xi, u)}{\rho(\xi)^\beta} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{H}). \tag{5.8}$$

Now, arguing as done in [18], up to a subsequence, we define an energy concentration set for any fixed $\delta > 0$,

$$\Sigma_\delta = \left\{\xi \in \mathbb{H} : \lim_{r \to 0} \lim_{k \to \infty} \int_{B_r(\xi)} \left(|u_k|^Q + |\nabla_{\mathbb{H}} u_k|^Q\right) \ge \delta\right\}$$

Since (u_k) is bounded, Σ_δ must be a finite set. For any $\xi^* \in \mathbb{H} \setminus \Sigma_\delta$, there exist $r : 0 < r <$ dist (ξ^*, Σ_δ) such that

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} \left(|u_k|^Q + |\nabla_{\mathbb{H}} u_k|^Q\right) d\xi < \delta$$

so for large k:

$$\int_{B_r(\xi^*)} \left(|u_k|^Q + |\nabla_{\mathbb{H}} u_k|^Q\right) d\xi < \delta. \tag{5.9}$$

By results in [18], we have:

$$\begin{aligned} &\int_{B_r(\xi^*)} \frac{|f(\xi, u_k)|\,|u_k - u|}{\rho(\xi)^\beta} d\xi \\ &\quad \le \left\| \frac{f(\xi, u_k)}{\rho(\xi)^{\beta/q}} \right\|_{L^q} \left\| \frac{1}{\rho(\xi)^\beta} \right\|_{L^s}^{1/q'} \|u_k - u\|_{L^{q's'}} \le C \|u_k - u\|_{L^{q's'}} \to 0 \end{aligned} \tag{5.10}$$

and for any compact set $K \subset\subset \mathbb{H} \setminus \Sigma_\delta$,

$$\lim_{k \to \infty} \int_K \frac{|f(\xi, u_k)\, u_k - f(\xi, u)\, u|}{\rho(\xi)^\beta} d\xi = 0. \tag{5.11}$$

So now, we will prove that for any compact set $K \subset\subset \mathbb{H} \setminus \Sigma_\delta$,

$$\lim_{k \to \infty} \int_K |\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u|^Q \, d\xi = 0. \tag{5.12}$$

It is enough to prove for any $\xi^* \in \mathbb{H} \setminus \Sigma_\delta$, and r given by (5.9), there holds

$$\lim_{k \to \infty} \int_{B_{r/2}(\xi^*)} |\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u|^Q \, d\xi = 0. \tag{5.13}$$

For this purpose, we take $\phi \in C_0^\infty(B_r(\xi^*))$ with $0 \le \phi \le 1$ and $\phi = 1$ on $B_{r/2}(\xi^*)$. Obviously ϕu_k is a bounded sequence. Choose $v = \phi u_k$ and $v = \phi u$ in (5.6), we have:

$$\begin{aligned}
&\int_{B_r(\xi^*)} \phi\left(a\left(\xi, \nabla_{\mathbb{H}} u_k\right) - a\left(\xi, \nabla_{\mathbb{H}} u\right)\right)\left(\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u\right) d\xi \\
&\quad \le \int_{B_r(\xi^*)} a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \nabla_{\mathbb{H}} \phi\left(u - u_k\right) d\xi \\
&\qquad + \int_{B_r(\xi^*)} \phi a\left(\xi, \nabla_{\mathbb{H}} u\right)\left(\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} u_k\right) d\xi + \int_{B_r(\xi^*)} \phi\left(u_k - u\right) \frac{f\left(\xi, u_k\right)}{\rho(\xi)^\beta} d\xi \\
&\qquad + \tau_k \left\|\phi u_k\right\|_E + \tau_k \|\phi u\|_E - \varepsilon \int_{B_r(\xi^*)} \phi h\left(u_k - u\right) d\xi .
\end{aligned}$$

Note that by the Holder inequality and the compact embedding of $E \hookrightarrow L^Q(\Omega)$, we get

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \nabla_{\mathbb{H}} \phi\left(u - u_k\right) d\xi = 0. \tag{5.14}$$

Since $\nabla_{\mathbb{H}} u_k \rightharpoonup \nabla_{\mathbb{H}} u$ and $u_k \rightharpoonup u$, there holds

$$\begin{aligned}
&\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi a\left(\xi, \nabla_{\mathbb{H}} u\right)\left(\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} u_k\right) d\xi = 0 \quad \text{and} \\
&\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi h\left(u_k - u\right) d\xi = 0.
\end{aligned} \tag{5.15}$$

The Holder inequality and (5.10) imply that

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi\left(u_k - u\right) f\left(\xi, u_k\right) d\xi = 0.$$

So we can conclude that

$$\lim_{k \to \infty} \int_{B_r(\xi^*)} \phi\left(a\left(\xi, \nabla_{\mathbb{H}} u_k\right) - a\left(\xi, \nabla_{\mathbb{H}} u\right)\right)\left(\nabla_{\mathbb{H}} u_k - \nabla_{\mathbb{H}} u\right) d\xi = 0$$

and hence we get (5.13) by (A2). So we have (5.12) by a covering argument. Since Σ_δ is finite, it follows that $\nabla_{\mathbb{H}} u_k$ converges to $\nabla_{\mathbb{H}} u$ almost everywhere. This immediately implies, up to a subsequence, $a\left(\xi, \nabla_{\mathbb{H}} u_k\right) \rightharpoonup a\left(\xi, \nabla_{\mathbb{H}} u\right)$ weakly in $\left(L_{\mathrm{loc}}^{Q/(Q-1)}(\Omega)\right)^{Q-2}$. Let k tend to infinity in (5.6) and combine with (5.8), we obtain

$$\left\langle D J_\varepsilon(u), h\right\rangle = 0 \quad \forall h \in C_0^\infty(\Omega).$$

This completes the proof of the lemma. □

5.1. The proof of Theorem 1.8

Proposition 5.1. *Under the assumptions* (V1) *and* (V2) *(or* V(3)*), and* (f1)–(f4)*, there exists* $\varepsilon_1 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_1$*, the problem (NU) has a solution* u_M *via mountain-pass theorem.*

Proof. For ε sufficiently small, by Lemmas 4.1 and 4.2, J_ε satisfies the hypotheses of the mountain-pass theorem except possibly for the (PS) condition. Thus, using the mountain-pass theorem without the (PS) condition, we can find a sequence (u_k) in E such that

$$J_\varepsilon(u_k) \to c_M > 0 \quad \text{and} \quad \|DJ_\varepsilon(u_k)\| \to 0$$

where c_M is the mountain-pass level of J_ε. Now, by Lemma 5.3, the sequence (u_k) converges weakly to a weak solution u_M of (NU) in E. Moreover, $u_M \neq 0$ since $h \neq 0$. □

6. The multiplicity results to the problem (NH): Theorem 1.9

In this section, we study the problem (NH). Note that (NH) is a special case of the problem (NU) where $A(\xi,\tau) = \frac{|\tau|^Q}{Q}$. As a consequence, there exists a nontrivial solution of standard "mountain-pass" type as in Theorem 1.8. Now, we will prove the existence of the second solution.

Lemma 6.1. *There exists $\eta > 0$ and $v \in E$ with $\|v\|_E = 1$ such that $J_\varepsilon(tv) < 0$ for all $0 < t < \eta$. In particular, $\inf_{\|u\|_E \le \eta} J_\varepsilon(u) < 0$.*

Proof. Let $v \in E$ be a solution of the problem

$$-\mathrm{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} v|^{Q-2}\nabla_{\mathbb{H}} v\right) + V(\xi)|v|^{Q-2}v = h \quad \text{in } \mathbb{H}.$$

Then, for $h \neq 0$, we have $\int_{\mathbb{H}} hv = \|v\|_E^Q > 0$. Moreover,

$$\frac{d}{dt}J_\varepsilon(tv) = t^{Q-1}\|v\|_E^Q - \int_{\mathbb{H}} \frac{f(\xi,tv)\,v}{\rho(\xi)^\beta}d\xi - \varepsilon\int_{\mathbb{H}} hv\,d\xi$$

for $t > 0$. Since $f(\xi,0) = 0$, by continuity, it follows that there exists $\eta > 0$ such that $\frac{d}{dt}J_\varepsilon(tv) < 0$ for all $0 < t < \eta$ and thus $J_\varepsilon(tv) < 0$ for all $0 < t < \eta$ since $J_\varepsilon(0) = 0$. □

Next, we define the Moser Functions (see [18,40]):

$$\widetilde{m}_l(\xi,r) = \frac{1}{\sigma_Q^{1/Q}}\begin{cases} (\log l)^{(Q-1)/Q} & \text{if } \rho(\xi) \le \frac{r}{l} \\ \dfrac{\log\frac{r}{\rho(\xi)}}{(\log l)^{1/Q}} & \text{if } \frac{r}{l} \le \rho(\xi) \le r \\ 0 & \text{if } \rho(\xi) \ge r. \end{cases}$$

Using the fact that $|\nabla_{\mathbb{H}}\rho(\xi)| = \frac{|z|}{\rho(\xi)}$ where $\xi = (z,t) \in \mathbb{H}$, we can conclude that $\widetilde{m}_l(.,r) \in W^{1,Q}(\mathbb{H})$, the support of $\widetilde{m}_l(\xi,r)$ is the ball B_r,

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}}\widetilde{m}_l(\xi,r)|^Q\,d\xi = 1, \quad \text{and} \quad \|\widetilde{m}_l\|_{W^{1,Q}(\mathbb{H})} = 1 + O(1/\log l). \tag{6.1}$$

Let $m_l(\xi,r) = \widetilde{m}_l(\xi,r)/\|\widetilde{m}_l\|_E$. Then by straightforward calculation, we have

$$m_l^{Q/(Q-1)}(\xi,r) = \sigma_Q^{-1/(Q-1)}\log l + d_l \quad \text{for } \rho(\xi) \le r/l, \tag{6.2}$$

where $d_l = \sigma_Q^{-1/(Q-1)}\log l\left(\|\widetilde{m}_l\|^{-1/(Q-1)} - 1\right)$. Moreover, we have

$$\|\widetilde{m}_l\| \to 1 \quad \text{as } l \to \infty$$
$$\frac{d_l}{\log l} \to 0 \quad \text{as } l \to \infty.$$

It is now standard to check the following lemma (for the Euclidean case, see [24,40]):

Lemma 6.2. *Suppose that* (V1) *and* (f1)–(f5) *hold. Then there exists* $k \in \mathbb{N}$ *such that*

$$\max_{t\geq 0}\left\{\frac{t^Q}{Q}-\int_{\mathbb{H}}\frac{F(\xi,tm_k)}{\rho(\xi)^\beta}d\xi\right\}<\frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Corollary 6.1. *Under the hypotheses* (V1) *and* (f1)–(f5)*, if* ε *is sufficiently small then*

$$\begin{aligned}\max_{t\geq 0} J_\varepsilon(tm_k) &= \max_{t\geq 0}\left\{\frac{t^Q}{Q}-\int_{\mathbb{H}}\frac{F(\xi,tm_k)}{\rho(\xi)^\beta}d\xi - t\int_{\mathbb{H}}\varepsilon h m_k d\xi\right\}\\ &< \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.\end{aligned}$$

Proof. Since $\left|\int_H \varepsilon h m_k d\xi\right| \leq \varepsilon \|h\|_*$, taking ε sufficiently small and using Moser–Trudinger type inequalities, the result follows. □

Note that we can conclude by inequality (5.3) and Lemma 6.1 that

$$-\infty < c_0 = \inf_{\|u\|_E\leq\rho_\varepsilon} J_\varepsilon(u) < 0. \tag{6.3}$$

Next, we will prove that this infimum is achieved and generate a solution. In order to obtain convergence results, we need to improve the estimate of Lemma 6.2.

Corollary 6.2. *Under the hypotheses* (V1) *and* (f1)–(f5)*, there exist* $\varepsilon_2 \in (0,\varepsilon_1]$ *and* $u \in W^{1,Q}(\mathbb{H})$ *with compact support such that for all* $0<\varepsilon<\varepsilon_2$*,*

$$J_\varepsilon(tu) < c_0 + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1} \quad \textit{for all } t\geq 0.$$

Proof. It is possible to raise the infimum c_0 by reducing ε. By Lemma 5.1, $\rho_\varepsilon \overset{\varepsilon\to 0}{\to} 0$. Consequently, $c_0 \overset{\varepsilon\to 0}{\to} 0$. Thus there exists $\varepsilon_2>0$ such that if $0<\varepsilon<\varepsilon_2$ then, by Corollary 6.1, we have

$$\max_{t\geq 0} J_\varepsilon(tm_k) < c_0 + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Taking $u = m_k \in W^{1,Q}(\mathbb{H})$, the result follows. □

Now, similarly as in the Euclidean case (see [6,24,40]), we have the following lemma.

Lemma 6.3. *If* (u_k) *is a (PS) sequence for* J_ε *at any level with*

$$\liminf_{k\to\infty}\|u_k\|_E < \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{(Q-1)/Q} \tag{6.4}$$

then (u_k) *possesses a subsequence which converges strongly to a solution* u_0 *of (NH).*

6.1. Proof of Theorem 1.9

The proof of the existence of the second solution of (NH) follows by a minimization argument and Ekeland's variational principle. To this end, we first prove the following.

Proposition 6.1. *There exists $\varepsilon_2 > 0$ such that for each ε with $0 < \varepsilon < \varepsilon_2$, Eq. (NH) has a minimum type solution u_0 with $J_\varepsilon(u_0) = c_0 < 0$, where c_0 is defined in (6.3).*

Proof. Let ρ_ε be as in Lemma 5.1. We can choose $\varepsilon_2 > 0$ sufficiently small such that

$$\rho_\varepsilon < \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{(Q-1)/Q}.$$

Since $\overline{B}_{\rho_\varepsilon}$ is a complete metric space with the metric given by the norm of E, convex and the functional J_ε is of class C^1 and bounded below on $\overline{B}_{\rho_\varepsilon}$, by Ekeland's variational principle there exists a sequence (u_k) in $\overline{B}_{\rho_\varepsilon}$ such that

$$J_\varepsilon(u_k) \to c_0 = \inf_{\|u\|_E \le \rho_\varepsilon} J_\varepsilon(u) \quad \text{and} \quad \|DJ_\varepsilon(u_k)\| \to 0.$$

Observing that

$$\|u_k\|_E \le \rho_\varepsilon < \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{(Q-1)/Q}$$

by Lemma 6.3, it follows that there exists a subsequence of (u_k) which converges to a solution u_0 of (NH). Therefore, $J_\varepsilon(u_0) = c_0 < 0$. □

Remark 6.1. By Corollary 6.2, we can conclude that

$$0 < c_M < c_0 + \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}.$$

Proposition 6.2. *If $\varepsilon_2 > 0$ is small enough, then the solutions of (NH) obtained in Propositions 5.1 and 6.1 are distinct.*

Proof. By Propositions 5.1 and 6.1, there exist sequences (u_k), (v_k) in E such that

$$u_k \to u_0, \qquad J_\varepsilon(u_k) \to c_0 < 0, \qquad DJ_\varepsilon(u_k)\,u_k \to 0$$

and

$$v_k \rightharpoonup u_M, \qquad J_\varepsilon(v_k) \to c_M > 0, \qquad DJ_\varepsilon(v_k)\,v_k \to 0,$$
$$\nabla_{\mathbb{H}} v_k(\xi) \to \nabla_{\mathbb{H}} u_M(\xi) \quad \text{almost everywhere in } \mathbb{H}.$$

Now, suppose by contradiction that $u_0 = u_M$. As in the proof of Lemma 5.3 we obtain

$$\frac{f(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{f(\xi, u_0)}{\rho(\xi)^\beta} \quad \text{in } L^1(B_R) \text{ for all } R > 0. \tag{6.5}$$

From this, we have by (f2), (f3) and the generalized Lebesgue dominated convergence theorem:

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{F(\xi, u_0)}{\rho(\xi)^\beta} \quad \text{in } L^1(B_R) \text{ for all } R > 0.$$

Now, recall the following inequalities: there exists $c > 0$ such that for all $(\xi, s) \in \mathbb{H} \times \mathbb{R}^+$:

$$F(\xi, s) \le c\,|s|^Q + cf(\xi, s) \tag{6.6}$$
$$F(\xi, s) \le c\,|s|^Q + cR(\alpha_0, s)\,s$$
$$\int_{\mathbb{H}} \frac{f(\xi, v_k)v_k}{\rho(\xi)^\beta} d\xi \le C, \qquad \int_{\mathbb{H}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \le C.$$

We will prove that for arbitrary $\delta > 0$, we can find $R > 0$ such that

$$\int_{\rho(\xi)>R} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \leq 3\delta \quad \text{and} \quad \int_{\rho(\xi)>R} \frac{F(\xi, u_0)}{\rho(\xi)^\beta} d\xi \leq \delta.$$

As a consequence, we get

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to \frac{F(\xi, u_0)}{\rho(\xi)^\beta} \quad \text{in } L^1(\mathbb{H}). \tag{6.7}$$

First, we have

$$\begin{aligned}
\int_{\substack{\rho(\xi)>R \\ |v_k|>A}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi &\leq c \int_{\substack{\rho(\xi)>R \\ |v_k|>A}} \frac{|v_k|^Q}{\rho(\xi)^\beta} d\xi + c \int_{\substack{\rho(\xi)>R \\ |v_k|>A}} \frac{f(\xi, v_k)}{\rho(\xi)^\beta} d\xi \\
&\leq \frac{c}{R^\beta A} \int_{\rho(\xi)>R} |v_k|^{Q+1} d\xi + c\frac{1}{A} \int_{\mathbb{H}} \frac{f(\xi, v_k)v_k}{\rho(\xi)^\beta} d\xi \\
&\leq \frac{c}{R^\beta A} \|v_k\|_E^{Q+1} + c\frac{1}{A} \int_{\mathbb{H}} \frac{f(\xi, v_k)v_k}{\rho(\xi)^\beta} d\xi.
\end{aligned}$$

Hence, since $\|v_k\|_E$ is bounded and using (6.6), we can choose A and R such that

$$\int_{\substack{\rho(\xi)>R \\ |v_k|>A}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \leq 2\delta.$$

Next, we have

$$\begin{aligned}
\int_{\substack{\rho(\xi)>R \\ |v_k|\leq A}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi &\leq \frac{C(\alpha_0, A)}{R^\beta} \int_{\substack{\rho(\xi)>R \\ |v_k|\leq A}} |v_k|^Q d\xi \\
&\leq \frac{2^{Q-1}C(\alpha_0, A)}{R^\beta} \left(\int_{\substack{\rho(\xi)>R \\ |v_k|\leq A}} |v_k - u_0|^Q d\xi + \int_{\substack{\rho(\xi)>R \\ |v_k|\leq A}} |u_0|^Q d\xi \right).
\end{aligned}$$

Now, using the compactness of embedding $E \hookrightarrow L^q(\mathbb{H})$, $q \geq Q$ and noticing that $v_k \rightharpoonup u_0$, again we can choose R such that

$$\int_{\substack{\rho(\xi)>R \\ |v_k|\leq A}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \leq \delta.$$

Thus we have

$$\int_{\rho(\xi)>R} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi \leq 3\delta.$$

Similarly, we also have

$$\int_{\rho(\xi)>R} \frac{F(\xi, u_0)}{\rho(\xi)^\beta} d\xi \leq 3\delta.$$

Thus, we can get (6.7).

Now, by standard arguments (see [24,40]), we can deduce a contradiction. □

6.2. Proof of Theorem 1.10

Corollary 6.3. *There exists $\varepsilon_3 > 0$ such that if $0 < \varepsilon < \varepsilon_3$ and $h(\xi) \geq 0$ for all $\xi \in \mathbb{H}$, then the weak solutions of (NH) are nonnegative.*

Proof. Let u be a weak solution of (NH), that is,

$$\int_H \left(|\nabla_H u|^{Q-2} \nabla_H u \nabla_H v + V(\xi) |u|^{Q-2} uv \right) d\xi - \int_H \frac{f(\xi, u) v}{\rho(\xi)^\beta} d\xi - \int_H \varepsilon h v d\xi = 0$$

for all $v \in E$. Taking $v = u^- \in E$ and observing that $f(\xi, u(\xi)) u^-(\xi) = 0$ a.e., we have

$$\|u^-\|_E^Q = -\int_H \varepsilon h u^- d\xi \leq 0.$$

Consequently, $u = u^+ \geq 0$. □

6.3. Proof of Theorem 1.11

This is similar to the proof of Theorems 1.8 and 1.9. First, we can find a sequence (v_k) in E such that

$$J_0(v_k) \to c_M > 0 \quad \text{and} \quad DJ_0(v_k) \to 0$$

where c_M is the Mountain-pass level of J_0. Moreover, we have that the sequence (v_k) converges weakly to a weak solution v of (NH) with $\varepsilon = 0$. It is now enough to show that $v \neq 0$. Indeed, suppose that $v = 0$. Similarly as in the previous part, we get

$$\frac{F(\xi, v_k)}{\rho(\xi)^\beta} \to 0 \quad \text{in } L^1(\mathbb{H}). \tag{6.8}$$

Thus

$$\|v_k\|_E^Q \to Qc_M > 0. \tag{6.9}$$

Also, we have from the previous sections that $c_M \in \left(0, \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}\right)$. Hence, we can find $\delta > 0$ and $K \in \mathbb{N}$ such that

$$\|v_k\|_E^Q \leq \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0} - \delta\right)^{Q-1} \quad \text{for all } k \geq K. \tag{6.10}$$

Now, if we choose $\tau > 1$ sufficiently close to 1, then by $(f1)$ we have

$$|f(\xi, v_k)v_k| \leq b_1 |v_k|^Q + b_2 \left[\exp\left(\alpha_0 |v_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0, v_k)\right] |v_k|.$$

Hence

$$\int_H \frac{|f(\xi, v_k)v_k|}{\rho(\xi)^\beta} \leq b_1 \int_H \frac{|v_k|^Q}{\rho(\xi)^\beta} + b_2 \int_H \frac{\left[\exp\left(\alpha_0 |v_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0, v_k)\right] |v_k|}{\rho(\xi)^\beta}.$$

Using the Holder inequality, Theorem 1.6, Lemma 4.2 and (6.10), we can conclude that

$$\int_H \frac{|f(\xi, v_k)v_k|}{\rho(\xi)^\beta} \to 0.$$

Since $DJ_0(v_k) \to 0$, we get $\|v_k\|_E \to 0$ and it is a contradiction.

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