EQUIVALENCE OF CRITICAL AND SUBCRITICAL SHARP TRUDINGER-MOSER-ADAMS INEQUALITIES

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Abstract. Sharp Trudinger-Moser inequalities on the first order Sobolev spaces and their analogous Adams inequalities on high order Sobolev spaces play an important role in geometric analysis, partial differential equations and other branches of modern mathematics. Such geometric inequalities have been studied extensively by many authors in recent years and there is a vast literature. There are two types of such optimal inequalities: critical and subcritical sharp inequalities, both are with best constants. Critical sharp inequalities are under the restriction of the full Sobolev norms for the functions under consideration, while the subcritical inequalities are under the restriction of the partial Sobolev norms for the functions under consideration. There are subtle differences between these two type of inequalities. Surprisingly, we prove in this paper that these critical and subcritical Trudinger-Moser and Adams inequalities are actually equivalent. Moreover, we also establish the asymptotic behavior of the supremum for the subcritical Trudinger-Moser and Adams inequalities on the entire Euclidean spaces (Theorem 1.1 and Theorem 1.3) and provide a precise relationship between the supremums for the critical and subcritical Trudinger-Moser and Adams inequalities (Theorem 1.2 and Theorem 1.4). Since the critical Trudinger-Moser and Adams inequalities can be easier to prove than subcritical ones in some occasions, and more difficult to establish in other occasions, our results and the method suggest a new approach to both the critical and subcritical Trudinger-Moser and Adams type inequalities.

1. Introduction

In this section, we will begin with giving an overview of the state of affairs of the best constants for sharp Trudinger and Adams inequalities. Subsection 1.1 concerns the sharp Trudinger-Moser inequalities and Subsection 1.2 discusses the sharp Adams inequalities involving high order derivatives. In Subection 1.3, we will state our main results on the equivalence between critical and subcritical Trudinger-Moser and Adams inequalities.

1.1. **Trudinger-Moser inequality.**

Motivated by the applications to the prescribed Gauss curvature problem on two dimensional sphere $\mathbb{S}^2$, J. Moser proved in [19] an exponential type inequality on $\mathbb{S}^2$ with an optimal constant. In the same paper, he sharpened an inequality on any bounded domain Ω in the Euclidean space $\mathbb{R}^N$ studied independently by Pohozaev [20], Trudinger [24] and Yudovich [25], namely the embedding $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$, where $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp\left(\alpha\, |t|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$. More precisely, using the Schwarz rearrangement, Moser proved the following inequality in [19]:

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Theorem A. *Let Ω be a domain with finite measure in Euclidean $N-$space $\mathbb{R}^N$, $n \geq 2$. Then there exists a constant $\alpha_N > 0$, such that*

$$\frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) dx \leq c_0 \tag{1.1}$$

for any $u \in W_0^{1,N}(\Omega)$ with $\int_\Omega |\nabla u|^N dx \leq 1$. The constant $\alpha_N = \omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the area of the surface of the unit $N-$ ball, is optimal in the sense that if we replace α_N by any number $\alpha > \alpha_N$, then the above inequality can no longer hold with some c_0 independent of u.

Moser used the following symmetrization argument: every function u is associated to a radially symmetric function u^* such that the sublevel-sets of u^* are balls with the same area as the corresponding sublevel-sets of u. Moreover, u is a positive and non-increasing function defined on $B_R(0)$ where $|B_R(0)| = |\Omega|$. Hence, by the layer cake representation, we can have that

$$\int_\Omega f(u)\, dx = \int_{B_R(0)} f(u^*)\, dx$$

for any function f that is the difference of two monotone functions. In particular, we obtain

$$\|u\|_p = \|u^*\|_p ;$$
$$\int_\Omega \exp\left(\alpha |u|^{\frac{n}{n-1}}\right) dx = \int_{B_R(0)} \exp\left(\alpha |u^*|^{\frac{n}{n-1}}\right) dx.$$

Moreover, the well-known Pólya-Szegö inequality

$$\int_{B_R(0)} |\nabla u^*|^p\, dx \leq \int_\Omega |\nabla u|^p\, dx \tag{1.2}$$

plays a crucial role in the approach of J. Moser.

As far as the existence of extremal functions of Moser's inequality, the first breakthrough was due to the celebrated work of Carleson and Chang [3] in which they proved that the supremum

$$\sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N dx \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) dx$$

can be achieved when Ω is an Euclidean ball. This result came as a surprise because it has been known that the Sobolev inequality does not have extremal functions supported on any finite ball. Subsequently, existence of extremal functions has been established on arbitrary domains in [6], [17], and on Riemannian manifolds in [15], etc.

We note when the volume of Ω is infinite, the Trudinger-Moser inequality (1.1) becomes meaningless. Thus, it becomes interesting and nontrivial to extend such inequalities to unbounded domains. Here we state the following two such results in the Euclidean spaces.

We first recall the subcritical Moser-Trudinger inequality in the Euclidean spaces established by Adachi and Tanaka [1].

Theorem B. *For any $\alpha < \alpha_N$, there exists a positive constant $C_{N,\alpha}$ such that $\forall u \in W^{1,N}\left(\mathbb{R}^N\right),\ \|\nabla u\|_N \leq 1:$*

$$\int_{\mathbb{R}^N} \phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq C_{N,\alpha} \|u\|_N^N, \tag{1.3}$$

where

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

The constant α_N is sharp in the sense that the supremum is infinity when $\alpha \geq \alpha_N$.

We note in the above theorem, we only impose the restriction on the norm $\int_{\mathbb{R}^N} |\nabla u|^N$ without restricting the full norm

$$\left[\int_{\mathbb{R}^N} |\nabla u|^N + \tau \int_{\mathbb{R}^N} |u|^N\right]^{1/N} \leq 1.$$

The method in [1] requires a symmetrization argument which is not available in many other non-Euclidean settings. The above inequality fails at the critical case $\alpha = \alpha_N$. So it is natural to ask when the above can be true when $\alpha = \alpha_N$. This is done by Ruf [21] and Li and Ruf [16] by using the restriction of the full norm of the Sobolev space $W^{1,N}\left(\mathbb{R}^N\right)$: $\left[\int_{\mathbb{R}^N} |\nabla u|^N + \tau \int_{\mathbb{R}^N} |u|^N\right]^{1/N}$.

Theorem C. *For all $0 \leq \alpha \leq \alpha_N$:*

$$\sup_{\|u\| \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) dx < \infty \tag{1.4}$$

where

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^N + |u|^N\right) dx\right)^{1/N}.$$

Moreover, this constant α_N is sharp in the sense that if $\alpha > \alpha_N$, then the supremum is infinity.

Sharp critical and subcritical Trudinger-Moser inequalities on infinite volume domains of the Heisenberg groups were also established in [11, 13] by using a symmetrization-free method.

The inequality (1.3) uses the seminorm $\|\nabla u\|_N$ and hence fails at the critical case $\alpha = \alpha_N$, the best constant. Thus, it can be considered as a sharp subcritical Trudinger-Moser inequality. In (1.4), when using the full norm of $W^{1,N}\left(\mathbb{R}^N\right)$, the best constant could be attained. Namely, the inequality holds at the critical case $\alpha = \alpha_N$. Hence, (1.4) is the sharp critical Trudinger-Moser inequality.

Nevertheless, the main purpose of this paper is to show that in fact, these two versions of critical and subcritical Trudinger-Moser type inequalities are indeed equivalent. Since the critical Trudinger-Moser type inequality is easier to study than the subcritical one in some occasions, and it is easier to investigate subcritical Trudinger-Moser type inequality than the critical one in other occasions, our paper suggests a new approach to both the critical and subcritical Trudinger-Moser type inequalities.

1.2. **Adams inequalities.** It is worthy noting that symmetrization has been a very useful and efficient (and almost inevitable) method when dealing with the sharp geometric inequalities. Thus, it is very fascinating to investigate such sharp geometric inequalities, in particular, the Trudinger-Moser type inequalities, in the settings where the symmetrization is not available such as on the higher order Sobolev spaces, the Heisenberg groups, Riemannian manifolds, sub-Riemannian manifolds, etc. Indeed, in these settings, an inequality like (1.2) is not available. In these situations, the first break-through came from the work of D. Adams [2] when he attempted to set up the Trudinger-Moser inequality in the higher order setting in Euclidean spaces. In fact, using a new idea that one can write a smooth function as a convolution of a (Riesz) potential with its derivatives, and then one can use the symmetrization for this convolution, instead of the symmetrization of the higher order derivatives, Adams proved the following inequality with boundary Dirichlet condition [2] which was extended to the Navier boundary condition in [23] when $\beta = 0$, and then the first two authors extended it to the case $0 \leq \beta < N$ [10]. The following is taken from [10].

Theorem D. *Let Ω be an open and bounded set in $\mathbb{R}^N$. If m is a positive integer less than N, $0 \leq \beta < N$, then there exists a constant $C_0 = C(N, m, \beta) > 0$ such that for any $u \in W_N^{m,\frac{N}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{N}{m}}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|^{1-\frac{\beta}{N}}} \int_\Omega \exp(\alpha \left(1 - \frac{\beta}{N}\right) |u(x)|^{\frac{N}{N-m}}) \frac{dx}{|x|^\beta} \leq C_0$$

for all $\beta \leq \beta(N, m)$ where

$$\beta(N,\ m) \ = \begin{cases} \frac{N}{w_{N-1}} \left[\frac{\pi^{N/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{N-m+1}{2})}\right]^{\frac{N}{N-m}} & \text{when } m \text{ is odd} \\ \frac{N}{w_{N-1}} \left[\frac{\pi^{N/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})}\right]^{\frac{N}{N-m}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, the constant $\beta(N, m)$ is optimal in the sense that for any $\alpha > \beta(N, m)$, the integral can be made as large as possible.

Adams inequalities have been extended to compact Riemannian manifolds in [7]. The Adams inequalities with optimal constants for high order derivatives on domains of infinite volume were recently established by Ruf and Sani in [22] in the case of even order derivatives and by Lam and Lu for all order of derivatives including fractional orders [9, 12]. The idea of [22] is to use the comparison principle for polyharmonic equations (thus could deal with the case of even order of derivatives) and thus involves some difficult construction of auxiliary functions. The argument in [9, 12] uses the representation of the Bessel potentials and thus avoids dealing with such a comparison principle. In particular, the method developed in [12] adapts the idea of deriving the sharp Moser-Trudinger-Adams inequalities on domains of finite measure to the entire spaces using the level sets of the functions under consideration. Thus, the argument in [12] does not use the symmetrization method and thus also works for the sub-Riemannian setting such as the Heisenberg groups [11, 13]. The following general version is taken from [12].

Theorem (Lam-Lu, 2013) Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$ and $\tau > 0$. There holds

$$\sup_{u \in W^{\gamma,p}(\mathbb{R}^n), \left\| (\tau I - \Delta)^{\frac{\gamma}{2}} u \right\|_p \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta_0(n,\gamma)\,|u|^{p'}\right) dx < \infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{j_p - 2} \frac{t^j}{j!},$$
$$j_p = \min\{j \in \mathbb{N} : j \geq p\} \geq p.$$

Furthermore this inequality is sharp in the sense that if $\beta_0(n,\gamma)$ is replaced by any $\beta > \beta_0(n,\gamma)$, then the supremum is infinite.

Very little is known for existence of extremals for Adams inequalities. Existence of extremal functions for the Adams inequality on bounded domains in Euclidean spaces has been established in [18] and compact Riemannian manifolds by [14] only when $N = 4$ and $m = 2$ and is still widely open in other cases.

1.3. **Our Main Results.** Though the Adachi-Tanaka type inequality in unbounded domains has been known for quite some time, it is still not known what the following supremum is:

$$\sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta}.$$

In particular, we do not even know how the supremum behaves asymptotically when α goes to α_N.

The following theorem answers this question and provides the lower and upper bounds asymptotically for the supremum.

Theorem 1.1. *Let $N \geq 2$, $\alpha_N = N\left(\frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\right)^{\frac{1}{N-1}}$, $0 \leq \beta < N$ and $0 \leq \alpha < \alpha_N$. Denote*

$$AT(\alpha,\beta) = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^\beta}.$$

Then there exist positive constants $c = c(N,\beta)$ and $C = C(N,\beta)$ such that when α is close enough to α_N :

$$\frac{c(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}} \leq AT(\alpha,\beta) \leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}. \tag{1.5}$$

Moreover, the constant α_N is sharp in the sence that $AT(\alpha_N,\beta) = \infty$.

We note that we do not assume a priori the validity of the critical Trudinger-Moser inequality with the restriction on the full norm (i.e., the inequality (1.4)) in order to derive the above asymptotic behavior of the supremum $AT(\alpha,\beta)$. We also mention that the upper bound in (1.5) in dimension two in the nonsingular case $\beta = 0$ has also been given in [4] using the sharp critical Trudinger-Moser inequality in $\mathbb{R}^2$.

Next, we like to know how the supremum $AT(\alpha,\beta)$ we established in Theorem 1.1 will provide a proof to the sharp critical Trudinger-Moser inequality. Thus, this gives a new proof of the sharp critical Trudinger-Moser inequality in all dimension N. We also answer the question under for which a and b the critical Trudinger-Moser inequality holds under the restriction of the full norm $\|\nabla u\|_N^a + \|u\|_N^b \leq 1$. Moreover, we establish the precise relationship between the supremums for the critical and subcritical Trudinger-Moser inequalities.

Theorem 1.2. *Let $N \geq 2$, $0 \leq \beta < N$, $0 < a,\ b$. Denote*

$$\begin{aligned} MT_{a,b}(\beta) &= \sup_{\|\nabla u\|_N^a + \|u\|_N^b \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}; \\ MT(\beta) &= MT_{N,N}(\beta). \end{aligned}$$

Then $MT_{a,b}(\beta) < \infty$ if and only if $b \leq N$. The constant α_N is sharp. Moreover, we have the following identity:

$$MT_{a,b}(\beta) = \sup_{\alpha \in (0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}\right)^{\frac{N-\beta}{b}} AT(\alpha,\beta). \tag{1.6}$$

In particular, $MT(\beta) < \infty$ and

$$MT(\beta) = \sup_{\alpha \in (0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}{\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}\right)^{\frac{N-\beta}{N}} AT(\alpha,\beta).$$

We now consider the sharp subcritical and critical Adams inequalities on $W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)$, $N \geq 3$. Our first result is the following sharp subcritical Adams inequality which provides the asymptotic behavior of the supremum (lower and upper bounds) in this case.

Theorem 1.3. *Let $N \geq 3$, $0 \leq \beta < N$ and $0 \leq \alpha < \beta(N,2)$. Denote*

$$ATA(\alpha,\beta) = \sup_{\|\Delta u\|_{\frac{N}{2}} \leq 1} \frac{1}{\|u\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}} dx;$$

$$\phi_{N,2}(t) = \sum_{j\in\mathbb{N}: j \geq \frac{N-2}{2}} \frac{t^j}{j!}.$$

Then there exist positive constants $c = c(N,\beta)$ and $C = C(N,\beta)$ such that when α is close enough to $\beta(N,2)$:

$$\frac{c(N,\beta)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}} \leq ATA(\alpha,\beta) \leq \frac{C(N,\beta)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}}. \tag{1.7}$$

Moreover, the constant $\beta(N,2)$ is sharp in the sence that $AT(\alpha_N,\beta) = \infty$.

The next theorem offers a precise relationship between the supremums of critical and subcritical Adams inequalities. Thus, it also provides a new approach of proving one of the critical and subcritical Adams inequalities from the other.

Theorem 1.4. *Let $N \geq 3$, $0 \leq \beta < N$, $0 < a$, b. We denote:*

$$A_{a,b}(\beta) = \sup_{\|\Delta u\|_{\frac{N}{2}}^{a} + \|u\|_{\frac{N}{2}}^{b} \leq 1} \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^{\beta}} dx;$$

$$A_{\frac{N}{2},\frac{N}{2}}(\beta) = A(\beta);$$

Then $A_{a,b}(\beta) < \infty$ if and only if $b \leq \frac{N}{2}$. The constant $\beta(N,2)$ is sharp. Moreover, we have the following identity:

$$A_{a,b}(\beta) = \sup_{\alpha\in(0,\beta(N,2))} \left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}} ATA(\alpha,\beta). \tag{1.8}$$

In particular, $A(\beta) < \infty$ and

$$A(\beta) = \sup_{\alpha\in(0,\beta(N,2))} \left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}\right)^{\frac{N-\beta}{N}} ATA(\alpha,\beta).$$

Finally, we will study the following improved sharp critical Adams inequality under the assumption that a version of the sharp subcritical Adams inequality holds for the factional order derivatives:

Theorem 1.5. *Let $0 < \gamma < N$ be an arbitrary real positive number, $p = \frac{N}{\gamma}$, $0 \leq \alpha < \beta_0(N,\gamma) = \frac{N}{\omega_{N-1}} \left[\frac{\pi^{\frac{N}{2}} 2^{\gamma} \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{N-\gamma}{2}\right)} \right]^{\frac{p}{p-1}}$, $0 \leq \beta < N$, $0 < a$, b. We note*

$$GATA(\alpha,\beta) = \sup_{u \in W^{\gamma,p}(\mathbb{R}^N) : \left\| (-\Delta)^{\frac{\gamma}{2}} u \right\|_p \leq 1} \frac{1}{\|u\|_p^{p\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^{\beta}} dx;$$

$$GA_{a,b}(\beta) = \sup_{u \in W^{\gamma,p}(\mathbb{R}^N) : \left\| (-\Delta)^{\frac{\gamma}{2}} u \right\|_p^a + \|u\|_p^b \leq 1} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^{\beta}} dx$$

where

$$\phi_{N,\gamma}(t) = \sum_{j \in \mathbb{N} : j \geq p-1} \frac{t^j}{j!}.$$

Assume that $GATA(\alpha,\beta) < \infty$ and there exists a constant $C(N,\gamma,\beta) > 0$ such that

$$GATA(\alpha,\beta) \leq \frac{C(N,\gamma,\beta)}{\left(1 - \left(\frac{\alpha}{\beta_0(N,\gamma)}\right)^{\frac{p-1}{p}}\right)} \tag{1.9}$$

Then when $b \leq p$, we have $GA_{a,b}(\beta) < \infty$. In particular $GA_{p,p}(\beta) < \infty$.

Though we have to assume a sharp subcritical Adams inequality (1.9), the main idea of Theorem 1.5 is that since $GATA(\alpha,\beta)$ is actually subcritical, i.e. α is strictly less than the critical level $\beta_0(N,\gamma)$, it is easier to study than $GA_{a,b}(\beta)$. Hence, it suggests a new approach in the study of $GA_{a,b}(\beta)$.

2. Some lemmata

Lemma 2.1.

$$AT(\alpha,\beta) = \sup_{\|\nabla u\|_N \leq 1; \|u\|_N = 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right) \frac{dx}{|x|^{\beta}}.$$

Proof. For any $u \in W^{1,N}\left(\mathbb{R}^N\right) : \|\nabla u\|_N \leq 1$, we define

$$v(x) = u(\lambda x)$$
$$\lambda = \|u\|_N .$$

Then,

$$\nabla v(x) = \lambda \nabla u(\lambda x).$$

Hence

$$\|\nabla v\|_N = \|\nabla u\|_N \leq 1; \|v\|_N = 1,$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|v(x)|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta} \\
&= \int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta} \\
&= \frac{1}{\lambda^{N-\beta}}\int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-1}}\right)\frac{d(\lambda x)}{|\lambda x|^\beta} \\
&= \frac{1}{\|u\|_N^{N-\beta}}\int_{\mathbb{R}^N} \phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.
\end{aligned}
$$

□

By Lemma 2.1, we can always assume $\|u\|_N = 1$ in the sharp subcritical Trudinger-Moser inequality.

Lemma 2.2. *The sharp subcritical Moser-Trudinger inequality is a consequence of the sharp critical Moser-Trudinger inequality. More precisely, if $MT_{a,b}(\beta)$ is finite, then $AT(\alpha,\beta)$ is finite. Moreover,*

$$
AT(\alpha,\beta) \le \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}\right)^{\frac{N-\beta}{b}} MT_{a,b}(\beta). \tag{2.1}
$$

In particular,

$$
AT(\alpha,\beta) \le \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}\right)^{1-\frac{\beta}{N}} MT(\beta).
$$

Proof. Let $u \in W^{1,N}\left(\mathbb{R}^N\right):\ \|\nabla u\|_N \le 1;\ \|u\|_N = 1$. Set

$$
v(x) = \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}} u(\lambda x)
$$

$$
\lambda = \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}\right)^{1/b}.
$$

then

$$
\|\nabla v\|_N^a = \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}\|\nabla u\|_N^a \le \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}
$$

$$
\|v\|_N^b = \left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}\frac{1}{\lambda^b}\|u\|_N^b = 1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}.
$$

Hence $\|\nabla v\|_N^a + \|v\|_N^b \leq 1$. By the definition of $MT_{a,b}(\beta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_N\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-1)}\right)\frac{dx}{|x|^\beta} \\
&= \int_{\mathbb{R}^N} \phi_N\left(\alpha(1-\frac{\beta}{N})\,|u(\lambda x)|^{N/(N-1)}\right)\frac{d(\lambda x)}{|\lambda x|^\beta} \\
&= \lambda^{N-\beta}\int_{\mathbb{R}^N} \phi_N\left(\alpha_N(1-\frac{\beta}{N})\,|v|^{N/(N-1)}\right)\frac{dx}{|x|^\beta} \\
&\leq \left(\frac{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}\right)^{\frac{N-\beta}{b}} MT_{a,b}(\beta).
\end{aligned}$$

□

Lemma 2.3.

$$ATA(\alpha,\beta) = \sup_{\|\Delta u\|_{\frac{N}{2}}\leq 1;\|u\|_{\frac{N}{2}}=1}\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx.$$

Proof. Let $u \in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right) : \|\Delta u\|_{\frac{N}{2}} \leq 1$ and set

$$v(x) = u(\lambda x);$$
$$\lambda = \|u\|_{\frac{N}{2}}^{\frac{1}{2}}$$

Then it is easy to check that

$$\Delta v(x) = \lambda^2 \Delta u(\lambda x)$$

and

$$\begin{aligned}
&\|\Delta v\|_{\frac{N}{2}} = \|\Delta u\|_{\frac{N}{2}};\\
&\|v\|_{\frac{N}{2}}^{\frac{N}{2}} = \int_{\mathbb{R}^N}|v(x)|^{\frac{N}{2}}dx = \int_{\mathbb{R}^N}|u(\lambda x)|^{\frac{N}{2}}dx = \frac{1}{\lambda^N}\int_{\mathbb{R}^N}|u(x)|^{\frac{N}{2}}dx = 1.
\end{aligned}$$

Moreover

$$\begin{aligned}
\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx &= \int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx \\
&= \frac{1}{\lambda^{N-\beta}}\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u(x)|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx \\
&= \frac{1}{\|u\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx.
\end{aligned}$$

□

Lemma 2.4. *Assume $A_{a,b}(\beta) < \infty$, then $ATA(\alpha,\beta) < \infty$. Moreover,*

$$ATA(\alpha,\beta) \le \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}} \right)^{\frac{N-\beta}{2b}} A_{a,b}(\beta). \tag{2.2}$$

In particular, if $A(\beta) < \infty$, then

$$ATA(\alpha,\beta) \le \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}} \right)^{\frac{N-\beta}{N}} A(\beta).$$

Proof. Let $u \in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right) : \|\Delta u\|_{\frac{N}{2}} \le 1$ and $\|u\|_{\frac{N}{2}} = 1$. We define

$$v(x) = \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}} u(\lambda x)$$

$$\lambda = \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}} \right)^{\frac{1}{2b}}.$$

then

$$\|\Delta v\|_{\frac{N}{2}} = \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}} \|\Delta u\|_{\frac{N}{2}} \le \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}}$$

$$\|v\|_{\frac{N}{2}}^{b} = \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b} \frac{1}{\lambda^{2b}} \|u\|_{\frac{N}{2}}^{b} = 1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}.$$

Hence $\|\Delta v\|_{\frac{N}{2}}^{a} + \|v\|_{\frac{N}{2}}^{b} \le 1$. By the definition of $A_{a,b}(\beta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-2)}\right)\frac{dx}{|x|^{\beta}} \\
&= \int_{\mathbb{R}^N} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\,|u(\lambda x)|^{N/(N-2)}\right)\frac{d(\lambda x)}{|\lambda x|^{\beta}} \\
&= \lambda^{N-\beta}\int_{\mathbb{R}^N} \phi_{N}\left(\alpha_N(1-\frac{\beta}{N})\,|v|^{N/(N-2)}\right)\frac{dx}{|x|^{\beta}} \\
&\le \left(\frac{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}} \right)^{\frac{N-\beta}{2b}} A_{a,b}(\beta).
\end{aligned}$$

□

3. Asymptotic behavior of the supremums in subcritical Trudinger-Moser inequalities and relationship with the critical supremums

In this section, we will prove the improved sharp subcritical Trudinger-Moser inequality. In particular, we will establish the asymptotic behavior for the supremum $AT(\alpha,\beta)$ for the subcritical Trudinger-Moser inequality (Theorem 1.1). We would like to note here that we don't assume the critical $MT(\beta)<\infty$ in the proof of Theorem 1.1. Moreover, we also establish the relationship between the supremums $AT(\alpha,\beta)$ and $MT(\beta)$ of the critical and subcritical Trudinger-Moser inequalities (Theorem 1.2).

Proof of Theorem 1.1. Suppose that $u\in C_0^{\infty}\left(\mathbb{R}^N\right)\setminus\{0\}$, $u\geq 0$, $\|\nabla u\|_N\leq 1$ and $\|u\|_N=1$. Let

$$\Omega=\left\{x:u\left(x\right)>\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}}\right\}.$$

Then the volume of Ω can be estimated as follows:

$$|\Omega|=\int\limits_{\Omega}1dx\leq\int\limits_{\Omega}\frac{u\left(x\right)^N}{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}dx\leq\frac{1}{1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}}.$$

We have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N\setminus\Omega}\frac{\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^{\beta}}dx\\
&\leq\int\limits_{\{u\leq 1\}}\frac{\phi_N\left(\alpha\,|u|^{N/(N-1)}\right)}{|x|^{\beta}}dx\\
&\leq e^{\alpha}\int\limits_{\{u\leq 1\}}\frac{u^N}{|x|^{\beta}}dx\\
&\leq e^{\alpha}\int\limits_{\{u\leq 1;|x|\geq 1\}}\frac{u^N}{|x|^{\beta}}dx+e^{\alpha}\int\limits_{\{u\leq 1;|x|<1\}}\frac{u^N}{|x|^{\beta}}dx\\
&\leq\frac{C\left(N,\beta\right)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}.
\end{aligned}$$

Now, consider

$$
\begin{aligned}
I &= \int_{\Omega} \frac{\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^{\beta}} dx \\
&\leq \int_{\Omega} \frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^{\beta}} dx.
\end{aligned}
$$

On Ω, we set

$$
v(x) = u(x) - \left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}}.
$$

Then it is clear that $v \in W_0^{1,N}(\Omega)$ and $\|\nabla v\|_N \leq 1$. Also, on Ω, with $\varepsilon = \frac{\alpha_N}{\alpha} - 1$:

$$
\begin{aligned}
|u|^{N/(N-1)} &\leq \left(|v| + \left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}}\right)^{N/(N-1)} \\
&\leq (1+\varepsilon)|v|^{N/(N-1)} + (1-\frac{1}{(1+\varepsilon)^{N-1}})^{\frac{1}{1-N}} | \left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{\frac{1}{N}} |^{N/(N-1)} \\
&= \frac{\alpha_N}{\alpha}|v|^{N/(N-1)} + 1.
\end{aligned}
$$

Hence, by Moser-Trudinger inequality on bounded domains:

$$
\begin{aligned}
I &\leq \int_{\Omega} \frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)}{|x|^{\beta}} dx \\
&\leq \int_{\Omega} \frac{\exp\left(\alpha_N\left(1-\frac{\beta}{N}\right)|v|^{N/(N-1)} + \alpha\right)}{|x|^{\beta}} dx \\
&\leq C(N,\beta)|\Omega|^{1-\frac{\beta}{N}} \\
&\leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}.
\end{aligned}
$$

In conclusion, we have

$$
AT(\alpha,\beta) \leq \frac{C(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}.
$$

Next, we will show that $AT\left(\alpha_N,\beta\right)=\infty$. Indeed, consider the following sequence:

$$u_n(x)=\left\{\begin{array}{c} 0 \text{ if } |x|\geq 1, \\ \left(\frac{N-\beta}{\omega_{N-1}n}\right)^{1/N}\log\left(\frac{1}{|x|}\right) \text{ if } e^{-\frac{n}{N-\beta}}<|x|<1 \\ \left(\frac{1}{\omega_{N-1}}\right)^{\frac{1}{N}}\left(\frac{n}{N-\beta}\right)^{\frac{N-1}{N}} \text{ if } 0\leq|x|\leq e^{-\frac{n}{N-\beta}} \end{array}\right. .$$

Then we can see easily that

$$\left\|\nabla u_n\right\|_N=1;\ \left\|u_n\right\|_N=o_n(1).$$

However

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N}\frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)\left|u_n\right|^{N/(N-1)}\right)}{|x|^{\beta}}dx \\
&\geq \int\limits_{\left\{0\leq|x|\leq e^{-\frac{n}{N-\beta}}\right\}}\frac{\phi_N\left(n\right)}{|x|^{\beta}}dx \\
&=\omega_{N-1}\phi_N\left(n\right)\int\limits_0^{e^{-\frac{n}{N-\beta}}}r^{N-1-\beta}dr \\
&=\frac{\omega_{N-1}\phi_N\left(n\right)}{e^n\left(N-\beta\right)}\rightarrow\frac{\omega_{N-1}}{N-\beta}\text{ as }n\rightarrow\infty.
\end{aligned}$$

Now, it is clear that there exists a large constant M_1, such that when $n\geq M_1$,

$$\begin{aligned}
\|u_n\|_N^N&=\int_0^{e^{-\frac{n}{N-\beta}}}(\frac{1}{\omega_{N-1}})^{N/N}(\frac{n}{N-\beta})^{\frac{N(N-1)}{N}}r^{N-1}dr+\int_{e^{-\frac{n}{N-\beta}}}^1(\frac{N-\beta}{\omega_{N-1}n})^{N/N}(\log(\frac{1}{r}))^Nr^{N-1}dr \\
&\approx n^{N-1}\int_0^{e^{-\frac{n}{N-\beta}}}r^{N-1}dr+\frac{1}{n}\int_0^{\frac{n}{N-\beta}}y^Ne^{-Ny}dy \\
&\approx n^Ne^{-\frac{nN}{N-\beta}}+\frac{1}{n}\approx\frac{1}{n}
\end{aligned}$$

So

$$\|u_n\|_N^{N-\beta}\approx\frac{1}{n^{N-\beta}}\quad\text{when}\quad n\geq M_1.$$

Now we consider the following integral

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{\phi_N(\alpha(1-\beta/N)|u_n|^{\frac{N}{N-1}})}{|x|^{\beta}}dx \\
&\gtrsim\int_0^{e^{-\frac{n}{N-\beta}}}\phi_N\left(\alpha(1-\beta/N)(\frac{1}{\omega_{N-1}})^{\frac{1}{N-1}}(\frac{n}{N-\beta})\right)r^{N-1-\beta}dr \\
&\gtrsim\int_0^{e^{-\frac{n}{N-\beta}}}\phi_N\left(\frac{\alpha}{\alpha_N}n\right)r^{N-1-\beta}dr\gtrsim\phi_N\left(\frac{\alpha}{\alpha_N}n\right)e^{-n}
\end{aligned}$$

We note that there exists a large constant M_2 independent of α such that for $n \geq M_2$

$$\phi_N\left(\frac{\alpha}{\alpha_N}n\right) \approx e^{(\frac{\alpha}{\alpha_N})n}$$

as long as $\frac{\alpha}{\alpha_N} \geq \frac{1}{2}$.
Now we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{\phi_N(\alpha(1-\beta/N)|u|^{\frac{N}{N-1}})}{|x|^\beta}dx \\ &\gtrsim e^{\left(\frac{\alpha}{\alpha_N}n\right)}e^{-n} = e^{-(1-\frac{\alpha}{\alpha_N})n}\end{aligned}$$

Now for α that is close enough to α_N we can pick n such that $1 \leq (1-\frac{\alpha}{\alpha_N})n \leq 2$, i.e

$$\alpha \approx (1-\frac{1}{n})\alpha_N \geq \left(1-\frac{1}{\max(M_1, M_2)}\right)\alpha_N$$

or

$$\max(M_1, M_2) \leq n \approx \frac{1}{1-\frac{\alpha}{\alpha_N}},$$

Then

$$\begin{aligned}&\frac{1}{\|u_n\|_N^{N-\beta}}\int_{\mathbb{R}^N} \frac{\Phi_N(\alpha(1-\beta/N)|u_n|^{\frac{N}{N-1}})}{|x|^\beta}dx \\ &\gtrsim n^{N-\beta}e^{-2} \\ &\approx \left(\frac{1}{1-\frac{\alpha}{\alpha_N}}\right)^{N-\beta} \end{aligned} \tag{3.1}$$

And note that when α is close enough to α_N, we have

$$\frac{1-(\frac{\alpha}{\alpha_N})^{N-1}}{1-\frac{\alpha}{\alpha_N}} \approx 1,$$

which implies

$$AT(\alpha,\beta) \geq \frac{c(N,\beta)}{\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{N-1}\right)^{(N-\beta)/N}}$$

when α is close enough to α_N. □

Now, we will provide a proof of the sharp critical Trudinger-Moser inequality, namely Theorem 1.2, using the above improved sharp subcritical Trudinger-Moser inequality (1.5). This suggests a new approach to and another look at the study of the sharp Trudinger-Moser inequality:

Proof of Theorem 1.2. First assume that $b \leq N$. Let $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : \|\nabla u\|_N^a + \|u\|_N^b \leq 1$. Assume that

$$\|\nabla u\|_N = \theta \in (0,1); \ \|u\|_N^b \leq 1-\theta^a.$$

If $\frac{1}{2} < \theta < 1$, then we set

$$v(x) = \frac{u(\lambda x)}{\theta}$$
$$\lambda = \frac{(1-\theta^a)^{\frac{1}{b}}}{\theta} > 0.$$

Hence

$$\|\nabla v\|_N = \frac{\|\nabla u\|_N}{\theta} = 1;$$
$$\|v\|_N^N = \int_{\mathbb{R}^N} |v|^N dx = \frac{1}{\theta^N} \int_{\mathbb{R}^N} |u(\lambda x)|^N dx = \frac{1}{\theta^N \lambda^N} \|u\|_N^N \le \frac{(1-\theta^a)^{\frac{N}{b}}}{\theta^N \lambda^N} = 1.$$

By Theorem 1.1, we get

$$\int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-1}}\right)}{|\lambda x|^\beta} d(\lambda x)$$
$$\le \lambda^{N-\beta} \int_{\mathbb{R}^N} \frac{\phi_N\left(\theta^{\frac{N}{N-1}}\alpha_N(1-\frac{\beta}{N})|v|^{N/(N-1)}\right)}{|x|^\beta} dx$$
$$\le \lambda^{N-\beta} AT\left(\theta^{\frac{N}{N-1}}\alpha_N, \beta\right) \le \left(\frac{(1-\theta^a)^{\frac{N}{b}}}{\theta^N}\right)^{1-\frac{\beta}{N}} \frac{C(N,\beta)}{\left(1-\left(\frac{\theta^{\frac{N}{N-1}}\alpha_N}{\alpha_N}\right)^{N-1}\right)^{1-\frac{\beta}{N}}}$$
$$\le \frac{\left((1-\theta^a)^{\frac{N}{b}}\right)^{1-\frac{\beta}{N}}}{(1-\theta^N)^{1-\frac{\beta}{N}}} C(N,\beta) \le C(N,\beta,a,b) \text{ since } b \le N.$$

If $0 < \theta \le \frac{1}{2}$, then with

$$v(x) = 2u(2x),$$

we have

$$\|\nabla v\|_N = 2\|\nabla u\|_N \le 1$$
$$\|v\|_N \le 1.$$

By Theorem 1.1:

$$\int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx \le 2^N \int_{\mathbb{R}^N} \frac{\phi_N\left(\frac{\alpha_N(1-\frac{\beta}{N})}{2^{\frac{N}{N-1}}}|v|^{N/(N-1)}\right)}{|x|^\beta} dx$$
$$\le C(N,\beta).$$

Next, we will verify that the constant $\alpha_N(1-\frac{\beta}{N})$ is our best possible. Indeed, we choose the sequence $\{u_k\}$ as follows

$$u_n(x)=\begin{cases} 0 \text{ if } |x|\geq 1, \\ \left(\frac{N-\beta}{\omega_{N-1}n}\right)^{1/N}\log\left(\frac{1}{|x|}\right) \text{ if } e^{-\frac{n}{N-\beta}}<|x|<1 \\ \left(\frac{1}{\omega_{N-1}}\right)^{\frac{1}{N}}\left(\frac{n}{N-\beta}\right)^{\frac{N-1}{N}} \text{ if } 0\leq |x|\leq e^{-\frac{n}{N-\beta}} \end{cases}. \tag{3.2}$$

Then,

$$\|\nabla u_n\|_N=1;\ \|u_n\|_N=O(\frac{1}{n^{\frac{1}{N}}}).$$

Set

$$\begin{aligned} &w_n(x)=\lambda_n u_n(x) \text{ where } \lambda_n\in(0,1) \text{ is a solution of } \lambda_n^a+\lambda_n^b\|u_n\|_N^b=1. \\ &\lambda_n=1-O\left(\frac{1}{n^{\frac{b}{aN}}}\right)\rightarrow_{k\rightarrow\infty}1. \end{aligned}$$

Then

$$\|\nabla w_n\|_N^a+\|w_n\|_N^b=1.$$

Also, for $\alpha>\alpha_N$:

$$\begin{aligned} &\int_{\mathbb{R}^N}\frac{\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)}{|x|^\beta}dx \\ &\geq \int\limits_{\left\{0\leq|x|\leq e^{-\frac{n}{N-\beta}}\right\}}\frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)-\sum_{j=0}^{N-2}\frac{\left[\alpha\left(1-\frac{\beta}{N}\right)\right]^j}{j!}|w_n|^{\frac{N}{N-1}j}}{|x|^\beta}dx \\ &\geq\left[\exp\left(\frac{\alpha n\left(1-O\left(\frac{1}{n^{\frac{b}{a(N-1)}}}\right)\right)}{\alpha_N}\right)-O\left(k^{N-1}\right)\right]\frac{\omega_{N-1}\exp\left(-n\right)}{N-\beta} \\ &\rightarrow\infty \text{ as } n\rightarrow\infty. \end{aligned}$$

Now, we will show that

$$MT_{a,b}(\beta)=\sup_{\alpha\in(0,\alpha_N)}\left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}}\right)^{\frac{N-\beta}{b}}AT(\alpha,\beta)$$

when $MT_{a,b}(\beta) < \infty$. Indeed, by (2.1), we have

$$\sup_{\alpha\in(0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}} \right)^{\frac{N-\beta}{b}} AT(\alpha,\beta) \le MT_{a,b}(\beta).$$

Now, let (u_n) be the maximizing sequence of $MT_{a,b}(\beta)$, i.e., $u_n \in W^{1,N}\left(\mathbb{R}^N\right) \setminus \{0\}$: $\|\nabla u_n\|_N^a + \|u_n\|_N^b \le 1$ and

$$\int_{\mathbb{R}^N} \phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u_n|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta} \to_{n\to\infty} MT_{a,b}(\beta).$$

We define

$$v_n(x) = \frac{u(\lambda_n x)}{\|\nabla u_n\|_N}$$
$$\lambda_n = \left(\frac{1-\|\nabla u_n\|_N^a}{\|\nabla u_n\|_N^b}\right)^{1/b} > 0.$$

Hence

$$\|\nabla v_n\|_N = 1 \text{ and } \|v_n\|_N \le 1.$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u_n|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}\\
&= \lambda_n^{N-\beta}\int_{\mathbb{R}^N} \frac{\phi_N\left(\|\nabla u_n\|_N^{\frac{N}{N-1}}\alpha_N(1-\frac{\beta}{N})\,|v_n|^{N/(N-1)}\right)}{|x|^\beta}dx\\
&\le \lambda_n^{N-\beta} AT\left(\|\nabla u_n\|_N^{\frac{N}{N-1}}\alpha_N,\beta\right) \le \sup_{\alpha\in(0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}} \right)^{\frac{N-\beta}{b}} AT(\alpha,\beta).
\end{aligned}$$

Hence, we receive

$$MT_{a,b}(\beta) = \sup_{\alpha\in(0,\alpha_N)} \left(\frac{1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}}{\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}b}} \right)^{\frac{N-\beta}{b}} AT(\alpha,\beta)$$

when $MT_{a,b}(\beta) < \infty$.

Now, if there exists some $b > N$ such that $MT_{a,b}(\beta) < \infty$. Then we have

$$\overline{\lim}_{\alpha\to\alpha_N^-}\left(1-\left(\frac{\alpha}{\alpha_N}\right)^{\frac{N-1}{N}a}\right)^{\frac{N-\beta}{b}} AT(\alpha,\beta) < \infty.$$

Also, since $MT(\beta) < \infty$:

$$\overline{\lim}_{\alpha \to \alpha_N^-} \left(1 - \left(\frac{\alpha}{\alpha_N} \right)^{N-1} \right)^{\frac{N-\beta}{N}} AT(\alpha, \beta) < \infty.$$

By Theorem 1.1, we can show that

$$\overline{\lim}_{\alpha \to \alpha_N^-} \left(1 - \left(\frac{\alpha}{\alpha_N} \right)^{N-1} \right)^{\frac{N-\beta}{N}} AT(\alpha, \beta) > 0. \tag{3.3}$$

Hence

$$\underline{\lim}_{\alpha \to \alpha_N^-} \frac{\left(1 - \left(\frac{\alpha}{\alpha_N} \right)^{\frac{N-1}{N} a} \right)^{\frac{N-\beta}{b}}}{\left(1 - \left(\frac{\alpha}{\alpha_N} \right)^{N-1} \right)^{\frac{N-\beta}{N}}} < \infty$$

which is impossible since $b > N$. The proof is now completed. $\square$

4. Asymptotic behavior of subcritical Adams inequalities and relationship with the critical ones

4.1. **Sharp Adams inequalities on** $W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)$**.** In this subsection, we establish the asymptotic behavior of the supremums in the subcritical Adams inequalities, namely Theorem 1.3. Again, it is worthy noticing that no version of Theorem 1.4 is assumed in order to prove Theorem 1.3. Moreover, we also establish the relationship between the supremums for the critical and subcritical Adams inequalities (Theorem 1.4).

Proof of Theorem 1.3. Let $u \in C_0^\infty\left(\mathbb{R}^N\right) \setminus \{0\}$, $u \geq 0$, $\|\Delta u\|_{\frac{N}{2}} \leq 1$ and $\|u\|_{\frac{N}{2}} = 1$. Set

$$\Omega(u) = \left\{ x \in \mathbb{R}^n : u(x) > \left[1 - \left(\frac{\alpha}{\beta(N,2)} \right)^{\frac{N-2}{2}} \right]^{\frac{2}{N}} \right\}.$$

Since $u \in C_0^\infty(\mathbb{R}^n)$, we have that $\Omega(u)$ is a bounded set. Moreover, we have

$$|\Omega(u)| \leq \int_{\Omega(u)} \frac{|u|^{\frac{N}{2}}}{1 - \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}} dx \leq \frac{1}{1 - \left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}.$$

Now, consider

$$\begin{aligned} I &= \int\limits_{\Omega(u)} \frac{\phi_{N,2}\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{N/(N-2)}\right)}{|x|^\beta} dx \\ &\leq \int\limits_{\Omega(u)} \frac{\exp\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{N/(N-2)}\right)}{|x|^\beta} dx. \end{aligned}$$

On $\Omega(u)$, we set

$$v\left(x\right)=u\left(x\right)-\left[1-\left(\frac{\alpha}{\beta\left(N,2\right)}\right)^{\frac{N-2}{2}}\right]^{\frac{2}{N}}.$$

Then it is clear that $v\in W_N^{2,\frac{N}{2}}\left(\Omega(u)\right)$ and $\|\Delta v\|_{\frac{N}{2}}\leq 1$. Also, on $\Omega(u)$, with $\varepsilon=\frac{\beta(N,2)}{\alpha}-1$:

$$\begin{aligned}
|u|^{N/(N-2)} &\leq \left(|v|+\left(1-\left(\frac{\alpha}{\beta\left(N,2\right)}\right)^{\frac{N-2}{2}}\right)^{\frac{2}{N}}\right)^{N/(N-2)} \\
&\leq (1+\varepsilon)|v|^{N/(N-2)}+(1-\frac{1}{(1+\varepsilon)^{\frac{N-2}{2}}})^{\frac{2}{2-N}}|\left(1-\left(\frac{\alpha}{\beta\left(N,2\right)}\right)^{\frac{N-2}{2}}\right)^{\frac{2}{N}}|^{N/(N-2)} \\
&=\frac{\beta\left(N,2\right)}{\alpha}|v|^{N/(N-2)}+1.
\end{aligned}$$

Hence, by Adams inequality on bounded domains (Theorem D):

$$\begin{aligned}
I &\leq \int\limits_{\Omega(u)}\frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-2)}\right)}{|x|^{\beta}}dx \\
&\leq \int\limits_{\Omega(u)}\frac{\exp\left(\beta\left(N,2\right)\left(1-\frac{\beta}{N}\right)|v|^{N/(N-2)}+\alpha\right)}{|x|^{\beta}}dx \\
&\leq C\left(N,\beta\right)|\Omega(u)|^{1-\frac{\beta}{N}} \\
&\leq C\left(N,\beta\right)\left(\frac{1}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}\right)^{1-\frac{\beta}{N}}.
\end{aligned}$$

We also have the following estimate:

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega(u)} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-2)}\right)\frac{dx}{|x|^{\beta}} \\
&\leq \int_{\{u\leq 1\}} \phi_{N,2}\left(\alpha(1-\frac{\beta}{N})\,|u|^{N/(N-2)}\right)\frac{dx}{|x|^{\beta}} \\
&\leq C\left(N\right)\int_{\{u\leq 1\}} \frac{|u|^{\frac{N}{2}}}{|x|^{\beta}}dx \\
&\leq C\left(N\right)\left(\int_{\{u\leq 1;|x|\geq 1\}} \frac{|u|^{\frac{N}{2}}}{|x|^{\beta}}dx+\int_{\{u\leq 1;|x|<1\}} \frac{|u|^{\frac{N}{2}}}{|x|^{\beta}}dx\right) \\
&\leq C\left(N,\beta\right).
\end{aligned}$$

In conclusion, we have

$$ATA\left(\alpha,\beta\right)\leq \frac{C\left(N,\beta\right)}{\left[1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}}.$$

We now show that $ATA\left(\beta\left(N,2\right),\beta\right)=\infty$. Indeed, let $\psi\in C^{\infty}\left([0,1]\right)$ be such that

$$\psi\left(0\right)=\psi'\left(0\right)=0;\ \psi\left(1\right)=\psi'\left(1\right)=1.$$

For $0<\varepsilon<\frac{1}{2}$ we set

$$H\left(t\right)=\begin{cases}\varepsilon\psi\left(\frac{t}{\varepsilon}\right) & 0<t\leq\varepsilon\\ t & \varepsilon<t\leq 1-\varepsilon\\ 1-\varepsilon\psi\left(\frac{1-t}{\varepsilon}\right) & 1-\varepsilon<t\leq 1\\ 0 & 1<t\end{cases}$$

and consider Adams' test functions

$$\psi_r\left(|x|\right)=H\left(\frac{\log\frac{1}{|x|}}{\log\frac{1}{r}}\right).$$

By construction, $\psi_r \in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)$ and $\psi_r(|x|) = 1$ for $x \in B_r$. Moreover, by [2]:

$$\begin{aligned}
\|\Delta\psi_r\|_{\frac{N}{2}}^{\frac{N}{2}} &\leq \omega_{N-1} a(N,2)^{\frac{N}{2}} \log\left(\frac{1}{r}\right)^{1-\frac{N}{2}} A_r; \\
\|\psi_r\|_{\frac{N}{2}}^{\frac{N}{2}} &= o\left(\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)^{\frac{N-2}{2}}\right) \\
a(N,2) &= \frac{\beta(N,2)^{\frac{N-2}{N}}}{N\sigma_N^{\frac{2}{N}}}; \\
A_r = A_r(N,2) &= \left[1 + 2\varepsilon\left(\|\psi'\|_\infty + O\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right)^{\frac{N}{2}}\right];
\end{aligned}$$

Now, we set

$$u_r(|x|) = \left(\log\left(\frac{1}{r}\right)\right)^{\frac{N-2}{N}} \psi_r(|x|).$$

Then

$$\begin{aligned}
u_r(|x|) &= \left(\log\left(\frac{1}{r}\right)\right)^{\frac{N-2}{N}} \text{ for } x \in B_r \\
\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{2}} &\leq \omega_{N-1} a(N,2)^{\frac{N}{2}} A_r \text{ and} \\
\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{N-2}} &\leq \frac{\beta(N,2)}{N} A_r^{\frac{2}{N-2}}.
\end{aligned}$$

Now,

$$\begin{aligned}
ATA(\beta(N,2),\beta) &\geq \lim_{r\to 0^+} \frac{1}{\left\|\frac{u_r}{\|\Delta u_r\|_{\frac{N}{2}}}\right\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{B_r} \phi_{N,2}\left(\beta(N,2)(1-\frac{\beta}{N})\left|\frac{u_r}{\|\Delta u_r\|_{\frac{N}{2}}}\right|^{N/(N-2)}\right)\frac{dx}{|x|^\beta} \\
&\geq \lim_{r\to 0^+} \frac{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}{\|u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{B_r} \phi_{N,2}\left(\frac{\beta(N,2)(1-\frac{\beta}{N})\log\left(\frac{1}{r}\right)}{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{N-2}}}\right)\frac{dx}{|x|^\beta} \\
&\geq \lim_{r\to 0^+} \frac{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}}{\|u_r\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \omega_{N-1}\frac{r^{N-\beta}}{N-\beta}\phi_{N,2}\left(\frac{(N-\beta)\log\left(\frac{1}{r}\right)}{\left[1+2\varepsilon\left(\|\psi'\|_\infty + O\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right)^{\frac{N}{2}}\right]^{\frac{2}{N-2}}}\right) \\
&\to \infty \text{ as } r \to 0^+.
\end{aligned}$$

Now, consider the following sequence

$$u_k(x) = \begin{cases} \left[\frac{1}{\beta(N,2)}\ln k\right]^{1-\frac{2}{N}} - \frac{|x|^2}{\left(\frac{\ln k}{k}\right)^{\frac{2}{N}}} + \frac{1}{(\ln k)^{\frac{2}{N}}} & \text{if } 0 \le |x| \le \left(\frac{1}{k}\right)^{\frac{1}{N}} \\ N\beta(N,2)^{\frac{2}{N}-1}(\ln k)^{-\frac{2}{N}}\ln\frac{1}{|x|} & \text{if } \left(\frac{1}{k}\right)^{\frac{1}{N}} \le |x| \le 1. \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then, we can check that

$$1 \le \|\Delta u_k\|_{\frac{N}{2}}^{\frac{N}{2}} \le 1 + O\left(\frac{1}{\ln k}\right).$$

Also,

$$\begin{aligned} \|u_k\|_{\frac{N}{2}}^{\frac{N}{2}} &\le \omega_{N-1}\left(N\beta(N,2)^{\frac{2}{N}-1}(\ln k)^{-\frac{2}{N}}\right)^{\frac{N}{2}} \int_0^1 r^{N-1}\ln\frac{1}{r}dr \\ &\quad + \frac{\omega_{N-1}}{N}\left(\left[\frac{1}{\beta(N,2)}\ln k\right]^{1-\frac{2}{N}} + \frac{1}{\left(\frac{\ln k}{k}\right)^{\frac{2}{N}}}\right)^{\frac{N}{2}} \frac{1}{k} \\ &\le A(\ln k)^{-1} + B(\ln k)^{\frac{N-2}{2}}\frac{1}{k} \end{aligned}$$

for some constants $A, B > 0$.

Let

$$v_k = \frac{u_k}{\|\Delta u_k\|_{\frac{N}{2}}}$$

then

$$\|\Delta v_k\|_{\frac{N}{2}} = 1$$

and

$$\|v_k\|_{\frac{N}{2}}^{\frac{N}{2}} \le \|u_k\|_{\frac{N}{2}}^{\frac{N}{2}} \le A(\ln k)^{-1} + B(\ln k)^{\frac{N-2}{2}}\frac{1}{k}.$$

By the definition of $ATA(\alpha,\beta)$, we get

$$\begin{aligned} ATA(\alpha,\beta) &\ge \frac{1}{\|v_k\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|v_k|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^\beta} \\ &\ge \frac{1}{\|v_k\|_{\frac{N}{2}}^{\frac{N}{2}\left(1-\frac{\beta}{N}\right)}} \int_{|x|\le\left(\frac{1}{k}\right)^{\frac{1}{N}}} \phi_{N,2}\left(\alpha\left(1-\frac{\beta}{N}\right)|v_k|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^\beta} \\ &\ge C\frac{\exp\left(\frac{\alpha}{\beta(N,2)}\left(1-\frac{\beta}{N}\right)\left(\frac{1}{\|\Delta u_k\|_{\frac{N}{2}}^{\frac{2}{N-2}}} - \frac{\beta(N,2)}{\alpha}\right)\ln k\right)}{\left(A(\ln k)^{-1} + B(\ln k)^{\frac{N-2}{2}}\frac{1}{k}\right)^{1-\frac{\beta}{N}}} \end{aligned}$$

Note that when k (independent of α) is large

$$\frac{1}{\|\Delta u_k\|_{\frac{N}{2}}^{\frac{2}{N-2}}} - \frac{\beta(N,2)}{\alpha} \approx 1 - \frac{\beta(N,2)}{\alpha}.$$

So we have

$$ATA(\alpha,\beta) \gtrsim \exp\left\{\left(1-\frac{\beta}{N}\right)\left(\frac{\alpha}{\beta(N,2)}-1\right)\ln k\right\} \cdot (\ln k)^{1-\frac{\beta}{N}}$$

When α is close enough to $\beta(N,2)$, we are able to choose k large enough as required before such that

$$\ln k \approx \frac{1}{1-\frac{\alpha}{\beta(N,2)}}$$

or

$$\left(1-\frac{\beta}{N}\right)\left(\frac{\alpha}{\beta(N,2)}-1\right)\ln k \approx 1.$$

Then

$$ATA(\alpha,\beta) \gtrsim C \cdot \left(\frac{1}{1-\frac{\alpha}{\beta(N,2)}}\right)^{1-\frac{\beta}{N}} \approx \left(\frac{1}{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{2}}}\right)^{1-\frac{\beta}{N}}$$

when α is close enough to $\beta(N,2)$. □

We now offer another proof to Theorem 1.4 using the improved sharp subcritical Adams inequality (1.7).

Proof of Theorem 1.4. Assume $0 < b \leq \frac{N}{2}$. Let $u \in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)\backslash\{0\} : \|\Delta u\|_{\frac{N}{2}}^{a} + \|u\|_{\frac{N}{2}}^{b} \leq 1$. Assume that

$$\|\Delta u\|_{\frac{N}{2}} = \theta \in (0,1)\,;\ \|u\|_{\frac{N}{2}}^{b} \leq 1-\theta^{a}.$$

If $\frac{1}{4} < \theta < 1$, then we set

$$v(x) = \frac{u(\lambda x)}{\theta}$$
$$\lambda = \frac{(1-\theta^{a})^{\frac{1}{2b}}}{\theta^{\frac{1}{2}}} > 0.$$

Hence

$$\|\Delta v\|_{\frac{N}{2}} = \frac{\|\Delta u\|_{\frac{N}{2}}}{\theta} = 1;$$
$$\|v\|_{\frac{N}{2}}^{\frac{N}{2}} = \int_{\mathbb{R}^N} |v|^{\frac{N}{2}}\, dx = \frac{1}{\theta^{\frac{N}{2}}}\int_{\mathbb{R}^N} |u(\lambda x)|^{\frac{N}{2}}\, dx = \frac{1}{\theta^{\frac{N}{2}}\lambda^N}\|u\|_{\frac{N}{2}}^{\frac{N}{2}} \leq \frac{(1-\theta^{a})^{\frac{N}{2b}}}{\theta^{\frac{N}{2}}\lambda^N} = 1.$$

By Theorem 1.3, we get

$$\int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx = \int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{N}{N-2}}\right)}{|\lambda x|^\beta}d(\lambda x)$$
$$\leq \lambda^{N-\beta}\int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\theta^{\frac{N}{N-2}}\beta(N,2)\left(1-\frac{\beta}{N}\right)|v|^{N/(N-2)}\right)}{|x|^\beta}dx$$
$$\leq \lambda^{N-\beta} ATA\left(\theta^{\frac{N}{N-2}}\beta(N,2),\beta\right) \leq \left(\frac{(1-\theta^a)^{\frac{1}{2b}}}{\theta^{\frac{1}{2}}}\right)^{N-\beta} \frac{C(N,\beta)}{\left[1-\left(\frac{\theta^{\frac{N}{N-2}}\beta(N,2)}{\beta(N,2)}\right)^{\frac{N-2}{2}}\right]^{1-\frac{\beta}{N}}}$$
$$\leq \frac{\left((1-\theta^a)^{\frac{N}{2b}}\right)^{1-\frac{\beta}{N}}}{\left(1-\theta^{\frac{N}{2}}\right)^{1-\frac{\beta}{N}}} C(N,\beta) \leq C(N,\beta,a,b) \text{ since } b \leq \frac{N}{2}.$$

If $0<\theta\leq\frac{1}{4}$, then with

$$v(x) = 2^2 u(2x),$$

we have

$$\|\Delta v\|_{\frac{N}{2}} = 4\|\Delta u\|_{\frac{N}{2}} \leq 1$$
$$\|v\|_{\frac{N}{2}} \leq 1.$$

By Theorem 1.3:

$$\int_{\mathbb{R}^N} \frac{\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx \leq 4^N \int_{\mathbb{R}^N} \frac{\phi_N\left(\frac{\beta(N,2)(1-\frac{\beta}{N})}{4^{\frac{N}{N-2}}}|v|^{N/(N-2)}\right)}{|x|^\beta}dx$$
$$\leq C(N,\beta).$$

We now also consider the Adams' test functions as in the proof of Theorem 1.3. Let $\beta > \beta(N,2)$. Set

$$w_r(|x|) = \lambda_r \frac{u_r(|x|)}{\|\Delta u_r\|_{\frac{N}{2}}} \text{ where } \lambda_r \in (0,1) \text{ is a solution of } \lambda_r^a + \frac{\lambda_r^b\|u_r\|_{\frac{N}{2}}^b}{\|\Delta u_r\|_{\frac{N}{2}}^b} = 1.$$
$$\lambda_r \to_{r\to 0^+} 1.$$

Then

$$\|\Delta w_r\|_{\frac{N}{2}}^a + \|w_r\|_{\frac{N}{2}}^b = 1$$

and

$$\begin{aligned}
&\lim_{r\to 0^+}\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\beta\left(1-\frac{\beta}{N}\right)|w_r|^{\frac{N}{N-2}}\right)}{|x|^\beta}dx\\
&\geq \lim_{r\to 0^+}\int_{B_r}\phi_{N,2}\left(\frac{\beta\left(1-\frac{\beta}{N}\right)\lambda_r^{\frac{N}{N-2}}|u_r|^{\frac{N}{N-2}}}{\|\Delta u_r\|_{\frac{N}{2}}^{\frac{N}{N-2}}}\right)\frac{dx}{|x|^\beta}\\
&\geq \lim_{r\to 0^+}\omega_{N-1}\frac{r^{N-\beta}}{N-\beta}\phi_{N,2}\left(\frac{\beta}{\beta(N,2)}\frac{(N-\beta)\log\left(\frac{1}{r}\right)}{\left[1+2\varepsilon\left(\|\psi'\|_\infty+O\left(\frac{1}{\log\left(\frac{1}{r}\right)}\right)\right)^{\frac{N}{2}}\right]^{\frac{2}{N-2}}}\right)\\
&\quad\to\infty \text{ as } r\to 0^+ \text{ if we choose } \varepsilon \text{ small enough.}
\end{aligned}$$

It now remains to show that

$$A_{a,b}(\beta)=\sup_{\alpha\in(0,\beta(N,2))}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}}ATA(\alpha,\beta).$$

By (2.2):

$$\sup_{\alpha\in(0,\beta(N,2))}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}}ATA(\alpha,\beta)\leq A_{a,b}(\beta).$$

Now, let (u_n) be the maximizing sequence of $A_{a,b}(\beta)$, i.e., $u_n\in W^{2,\frac{N}{2}}\left(\mathbb{R}^N\right)\setminus\{0\}$: $\|\Delta u_n\|_{\frac{N}{2}}^a+\|u_n\|_{\frac{N}{2}}^b\leq 1$ and

$$\int_{\mathbb{R}^N}\phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u_n|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^\beta}\to_{n\to\infty}A_{a,b}(\beta).$$

We define a new sequence:

$$v_n(x)=\frac{u(\lambda_n x)}{\|\Delta u_n\|_{\frac{N}{2}}}$$

$$\lambda_n=\left(\frac{1-\|\Delta u_n\|_{\frac{N}{2}}^a}{\|\Delta u_n\|_{\frac{N}{2}}^b}\right)^{\frac{1}{2b}}>0.$$

Hence

$$\|\Delta v_n\|_{\frac{N}{2}}=1 \text{ and } \|v_n\|_{\frac{N}{2}}\leq 1.$$

Also,

$$\int_{\mathbb{R}^N} \phi_{N,2}\left(\beta(N,2)\left(1-\frac{\beta}{N}\right)|u_n|^{\frac{N}{N-2}}\right)\frac{dx}{|x|^{\beta}}$$
$$=\lambda_n^{N-\beta}\int_{\mathbb{R}^N}\frac{\phi_{N,2}\left(\|\Delta u_n\|_{\frac{N}{2}}^{N/(N-2)}\beta(N,2)\left(1-\frac{\beta}{N}\right)|v_n|^{N/(N-2)}\right)}{|x|^{\beta}}dx$$
$$\leq \lambda_n^{N-\beta}ATA\left(\|\Delta u_n\|_{\frac{N}{2}}^{N/(N-2)}\beta(N,2),\beta\right)\leq \sup_{\alpha\in(0,\beta(N,2))}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}}ATA(\alpha,\beta).$$

Now, we assume that there is some $b>\frac{N}{2}$ such that $A_{a,b}(\beta)<\infty$. Then

$$A_{a,b}(\beta)=\sup_{\alpha\in(0,\beta(N,2))}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}}ATA(\alpha,\beta)$$

and so

$$\overline{\lim}_{\alpha\uparrow\beta(N,2)}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{2b}}ATA(\alpha,\beta)<\infty.$$

Also, by Theorem 1.3:

$$\overline{\lim}_{\alpha\uparrow\beta(N,2)}\left(\frac{1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}}{\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}b}}\right)^{\frac{N-\beta}{N}}ATA(\alpha,\beta)>0, \tag{4.1}$$

Hence:

$$\underline{\lim}_{\alpha\uparrow\beta(N,2)}\frac{\left(1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}\right)^{\frac{N-\beta}{2b}}}{\left(1-\left(\frac{\alpha}{\beta(N,2)}\right)^{\frac{N-2}{N}a}\right)^{\frac{N-\beta}{N}}}>0$$

which is impossible since $b>\frac{N}{2}$. The proof is now completed. □

4.2. **Adams inequalities on $W^{\gamma,\frac{N}{\gamma}}\left(\mathbb{R}^N\right)$-Proof of Theorem 1.5.** Let $u\in W^{\gamma,p}\left(\mathbb{R}^N\right)\setminus\{0\}:\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p^a+\|u\|_p^b\leq 1$. We set

$$\left\|(-\Delta)^{\frac{\gamma}{2}}u\right\|_p=\theta\in(0,1);\ \|u\|_p^b\leq 1-\theta^a.$$

If $\frac{1}{2^\gamma} < \theta < 1$, then by define a new function

$$v(x) = \frac{u(\lambda x)}{\theta}$$
$$\lambda = \frac{(1-\theta^a)^{\frac{1}{\gamma b}}}{\theta^{\frac{1}{\gamma}}} > 0.$$

we get

$$(-\Delta)^{\frac{\gamma}{2}} v(x) = \frac{\lambda^\gamma}{\theta} \left((-\Delta)^{\frac{\gamma}{2}} u \right) (\lambda x).$$

Hence

$$\left\| (-\Delta)^{\frac{\gamma}{2}} v \right\|_p = \frac{\left\| (-\Delta)^{\frac{\gamma}{2}} u \right\|_p}{\theta} = 1;$$
$$\|v\|_p^p = \int_{\mathbb{R}^N} |v|^p \, dx = \frac{1}{\theta^p} \int_{\mathbb{R}^N} |u(\lambda x)|^p \, dx = \frac{1}{\theta^p \lambda^N} \|u\|_p^p \le \frac{(1-\theta^a)^{\frac{p}{b}}}{\theta^p \lambda^N} = 1.$$

By the definition of $GATA(\alpha, \beta)$, we get

$$\int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|u(\lambda x)|^{\frac{p}{p-1}}\right)}{|\lambda x|^\beta} d(\lambda x)$$
$$\le \lambda^{N-\beta} \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\theta^{\frac{p}{p-1}}\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|v|^{\frac{p}{p-1}}\right)}{|x|^\beta} dx$$
$$\le \lambda^{N-\beta} GATA\left(\theta^{\frac{p}{p-1}}\beta_0(N,\gamma), \beta\right) \le \left(\frac{(1-\theta^a)^{\frac{1}{\gamma b}}}{\theta^{\frac{1}{\gamma}}} \right)^{N-\beta} \frac{C(N,\beta)}{\left[1 - \left(\frac{\theta^{\frac{p}{p-1}}\beta_0(N,\gamma)}{\beta_0(N,\gamma)}\right)^{\frac{p-1}{p}}\right]^{1-\frac{\beta}{N}}}$$
$$\le \frac{\left((1-\theta^a)^{\frac{N}{\gamma b}}\right)^{1-\frac{\beta}{N}}}{(1-\theta)^{1-\frac{\beta}{N}}} C(N,\beta) \le C(N,\beta,a,b) \text{ since } b \le p.$$

If $0 < \theta \le \frac{1}{2^\gamma}$, then with

$$v(x) = 2^\gamma u(2x),$$

we have

$$\left\| (-\Delta)^{\frac{\gamma}{2}} v \right\|_p = 2^\gamma \left\| (-\Delta)^{\frac{\gamma}{2}} u \right\|_p \le 1$$
$$\|v\|_p \le 1.$$

By the definition of $GATA(\alpha, \beta)$:

$$\int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\beta_0(N,\gamma)\left(1-\frac{\beta}{N}\right)|u|^{\frac{p}{p-1}}\right)}{|x|^\beta} dx \le 2^N \int_{\mathbb{R}^N} \frac{\phi_{N,\gamma}\left(\frac{\beta_0(N,\gamma)}{2^{\gamma\frac{p}{p-1}}}\left(1-\frac{\beta}{N}\right)|v|^{\frac{p}{p-1}}\right)}{|x|^\beta} dx$$
$$\le C(N,\beta).$$

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