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Calculus of Variations

Factorizations and Hardy's type identities and inequalities on upper half spaces

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Abstract

Motivated and inspired by the improved Hardy inequalities studied in their well-known works by Brezis and Vázquez (Rev Mat Univ Complut Madrid 10:443–469, 1997) and Brezis and Marcus (Ann Scuola Norm Sup Pisa Cl Sci 25(1–2):217–237, 1997), we establish in this paper several identities that imply many sharpened forms of the Hardy type inequalities on upper half spaces $\{x_N > 0\}$. We set up these results for the distance to the origin, the distance to the boundary of any strip $\mathbb{R}^{N-1} \times (0, R)$ and the distance to the hyperplane $\{x_N = 0\}$, using both the usual full gradient and radial derivative (in the case of distance to the origin) or only the partial derivative $\frac{\partial u}{\partial x_N}$ (in the case of distance to the boundary of the strip or hyperplane). One of the applications of our main results is that when Ω is the strip $\mathbb{R}^{N-1} \times (0, 2R)$, the bound $\lambda(\Omega)$ given by Brezis and Marcus in Brezis and Marcus (1997) can be improved to $\frac{z_0^2}{R^2}$, where $z_0 = 2.4048\ldots$ is the first zero of the Bessel function $J_0(z)$. Our approach makes use of the notion of Bessel pairs introduced by Ghoussoub and Moradifam (Math Ann 349(1):1–57, 2011) and (Functional inequalities: new perspectives and new applications. Mathematical Surveys and Monographs, American Mathematical Society, Providence, 2013) and the method of factorizations of differential operators. In particular, our identities and inequalities offer sharpened and more precise estimates of the second remainder term in the existing Hardy type inequalities on upper half spaces in the literature, including the Hardy-Sobolev-Maz'ya type inequalities.

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1 Introduction

The classical Hardy inequalities

$$\int_{\Omega} |\nabla u|^2 \, dx \geq C_{opt} (\Omega) \int_{\Omega} \frac{|u|^2}{d (x)^2} dx, \, u \in C_0^{\infty} (\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, $d(x)$ is the distance from x to a certain surface, have attracted extensive attention because of their important roles in many branches of mathematics. There is a vast amount of literature on the Hardy type inequalities together with their variations and generalizations. It is impossible to give a complete list on the subject. Hence, we just refer the interested reader to [32,49] for historical backgrounds and to the monographs [3,33,34,55], that are standard references on the topic.

The principal aim of this note is to use the factorizations of differential operators to study the Hardy type inequalities on half-spaces. For a comprehensive review, the history and properties of the factorization method, the interested reader is referred to [24]. We note here that factorizing differential equations was used in [26] to prove the classical Hardy inequality and in [25] for the radial and logarithmic refinements of the Hardy inequality. In Gesztesy [24], the authors applied factorizations of singular, even-order partial differential operators to provide simple proofs for several Hardy-Rellich type inequalities. Recently in [38,39,42–45], the Hardy operator has been decomposed into or bounded by the product of differential operators and thus higher order Hardy-Sobolev-Maz'ya inequalities and their borderline case of Hardy-Adams inequalities have been established using the Fourier analysis techniques on hyperbolic spaces, among other techniques (see also the first order Hardy-Trudinger-Moser type inequalities in [41,61,63]). More recently, the factorization method was used to obtain Hardy, Hardy-Rellich and refined Hardy inequalities on general stratified groups and weighted Hardy inequalities on general homogeneous groups in [57].

1.1 Hardy inequalities with distance to the origin

When $0 \in \Omega$ and $d(x) = |x|$ is the distance to the origin, the Hardy inequalities (1.1) have been investigated extensively and intensively:

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx, \, u \in C_0^{\infty} (\Omega). \tag{1.2}$$

The fact that the optimal constant $\left(\frac{N-2}{2}\right)^2$, $N \geq 3$, is not achieved in $H_0^1(\Omega)$ has been well understood. On domains having 0 on their boundary, the situation is more complicated. Indeed, in this case, it was proved that the best constant $C_{opt}(\Omega)$ can be anywhere between $\left(\frac{N-2}{2}\right)^2$ and $\left(\frac{N}{2}\right)^2$. Moreover, it can be attained by nontrivial functions in $H_0^1(\Omega)$ as long as $C_{opt}(\Omega) < \left(\frac{N}{2}\right)^2$. In particular, on the half-spaces $\mathbb{R}_+^N = \left\{x \in \mathbb{R}^N : x_N > 0\right\}$, $C_{opt}\left(\mathbb{R}_+^N\right) = \left(\frac{N}{2}\right)^2$:

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 \, dx \geq \left(\frac{N}{2} \right)^2 \int_{\mathbb{R}_+^N} \frac{|u|^2}{|x|^2} dx, \, u \in C_0^{\infty} \left(\mathbb{R}_+^N\right). \tag{1.3}$$

See, for instance [14,17,19,53].

When $0 \in \Omega$, there has been considerable efforts to improve the Hardy inequalities. One possible way is to find extra terms to add to the potential $\left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}$. The first result in this direction appeared in [12] where in order to investigate the stability of certain singular solutions of nonlinear elliptic equations, Brezis and Vázquez proved on bounded domains that

Theorem A (Brezis-Vázquez [12]). *For any bounded domain $\Omega \subset R^N$, $N \geq 2$, and every $u \in H_0^1(\Omega)$,*

$$\int_{\Omega} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 \, dx \tag{1.4}$$

where $z_0 = 2.4048\ldots$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball but is not achieved in $H_0^1(\Omega)$.

It is also conjectured by Brezis and Vázquez that (see [12]) $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 \, dx$ is just the first term of an infinite series of extra terms that can be added to the RHS of (1.4). The results of Brezis and Vázquez in [12] have generated a noticeable amount of interest. We refer the reader to, for instance [1,6,7,23,29,52,62]. See also the book [28] for more discussions and developments in this aspect. We just state here a result by Ghoussoub and Moradifam in [27] that extends and unifies several results in this direction (see also their beautiful book [28]):

Theorem B (Ghoussoub-Moradifam [27]). *Assume $N \geq 1$. Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0, R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ and $\int_0^R r^{N-1} V(r) \, dr < \infty$. If $\left(r^{N-1}V, r^{N-1}W\right)$ is a Bessel pair on $(0, R)$, then for all $u \in C_0^\infty(B_R)$*

$$\int_{B_R} V(|x|) |\nabla u|^2 \, dx \geq \int_{B_R} W(|x|) |u|^2 \, dx. \tag{1.5}$$

Also, if (1.5) holds for all $u \in C_0^\infty(B_R)$, then $\left(r^{N-1}V, r^{N-1}cW\right)$ is a Bessel pair on $(0, R)$ for some $c > 0$.

Here a couple of C^1-functions (V, W) is a Bessel pair on $(0, R)$ if the ordinary differential equation

$$\left(V y'\right)' + W y = 0$$

has a positive solution on the interval $(0, R)$.

Another way to improve the Hardy type inequalities is to replace the usual ∇ by $\frac{x}{|x|} \cdot \nabla$. We note that $\frac{x}{|x|} \cdot \nabla$ is just the radial derivative. Indeed, in the polar coordinate $(r, \sigma) = \left(|x|, \frac{x}{|x|}\right)$, we have $\frac{x}{|x|} \cdot \nabla u(x) = \partial_r u(r\sigma) =: \mathcal{R}u$. We also mention here that the operator $\mathcal{R}$ has appeared naturally in the literature. Indeed, much research has been devoted to investigating the functional and geometric inequalities on general homogeneous groups. However, since these spaces do not have to be stratified or even graded, the concept of horizontal gradients does not make sense. Thus, one may want to work with the full gradient. On the other hand, unless the homogeneous groups are Abelian, the full gradient is not homogeneous. Nevertheless, on the homogeneous groups, the operator $\mathcal{R}$ is homogeneous of order -1 and thus, is reasonable to work on. Actually, the Hardy type inequalities with radial derivative have been studied intensively recently. See [31,35,36,58], to name just a few. For instance, in [9,47], the following equalities have been set up to provide a direct understanding to the validity as well as the nonexistence of optimizers for the Hardy inequalities:

$$\int_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}dx + \int_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot\nabla u + \frac{N-2}{2}\frac{u}{|x|}\right|^2 dx$$

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}dx + \int_{\mathbb{R}^N} \left|\nabla u + \frac{N-2}{2}\frac{u}{|x|}\frac{x}{|x|}\right|^2 dx.$$

A version of Theorem B with radial derivative was also set up recently in [16].

For the improvements of the Hardy type inequalities with boundary singularities, the interested reader is referred to [6–8,14,17,20,22,40,43,46,51,56,59,60], to name just a few. We note here that a sharp version of the Hardy inequality for fractional Sobolev spaces on half-spaces was investigated in [22]. In [6,7], Beckner established Hardy type inequalities on the entire spaces and half-spaces using Fourier analysis and a nonlinear generalization of the Stein-Weiss lemma for non-unimodular groups.

Our first primary goal of this article is to employ the method of factorization to investigate a version of Theorem B in the setting of half-spaces. More precisely, we will show that

Theorem 1.1 *Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0, R)$. If $\left(r^{N+1}V, r^{N+1}W\right)$ is a Bessel pair on $(0, R)$, then for $u \in C_0^\infty\left(B_R^{(N)} \cap \mathbb{R}_+^N\right)$:*

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\,|\nabla u|^2 dx - \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left[W(|x|) - \frac{V'(|x|)}{|x|}\right]|u|^2 dx$$
$$= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\nabla\left(\frac{u}{\varphi}\frac{1}{x_N}\right)\right|^2 \varphi^2 x_N^2 dx$$

and

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left[W(|x|) - \frac{V'(|x|)}{|x|} - \frac{(N-1)}{|x|^2}V(|x|)\right]|u|^2 dx$$
$$= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla\left(\frac{1}{\varphi}\frac{u}{x_N}\right)\right|^2 \varphi^2 x_N^2 dx.$$

Here $\varphi = \varphi_{r^{N+1}V,r^{N+1}W;R}$ is the positive solution of

$$\left(r^{N+1}V(r)\,y'(r)\right)' + r^{N+1}W(r)\,y(r) = 0$$

on the interval $(0, R)$.

As a consequence of Theorem 1.1, we obtain the following type inequalities on half-spaces.

Corollary 1.1 *If* $\left(r^{N+1}V, r^{N+1}W\right)$ *is a Bessel pair on* $(0,R)$*, then for* $u \in C_0^\infty\left(B_R^{(N)} \cap \mathbb{R}_+^N\right)$*:*

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\,|\nabla u|^2\,dx \geq \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left[W(|x|) - \frac{V'(|x|)}{|x|}\right]|u|^2\,dx$$

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \geq \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left[W(|x|) - \frac{V'(|x|)}{|x|} - \frac{(N-1)}{|x|^2}V(|x|)\right]|u|^2\,dx$$

with the ground state $x_N\varphi(x)$*.*

By applying Theorem 1.1 to the Bessel pair $\left(r^{N+1}, \left(\frac{N}{2}\right)^2 r^{N-1}\right)$ with $\varphi = r^{-\frac{N}{2}}$ on $(0,\infty)$, we deduce the following identities that provide a straightforward interpretation of (1.3):

Corollary 1.2 *For* $u \in C_0^\infty\left(\mathbb{R}_+^N\right)$*, there holds*

$$\int_{\mathbb{R}_+^N} |\nabla u|^2\,dx - \left(\frac{N}{2}\right)^2 \int_{\mathbb{R}_+^N} \frac{|u|^2}{|x|^2}dx = \int_{\mathbb{R}_+^N} \left|\nabla\left(|x|^{\frac{N}{2}}\frac{u}{x_N}\right)\right|^2 |x|^{-N}\,x_N^2 dx \tag{1.6}$$

$$\int_{\mathbb{R}_+^N} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}_+^N} \frac{|u|^2}{|x|^2}dx = \int_{\mathbb{R}_+^N} \left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{N}{2}}\frac{u}{x_N}\right)\right|^2 |x|^{-N}\,x_N^2 dx. \tag{1.7}$$

We note that $x_N\,|x|^{-\frac{N}{2}}$ is not in $L^2\left(\mathbb{R}_+^N\right)$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}_+^N} x_N^2\,|x|^{-N}\,dx &= \int_0^\infty \int_{\mathbb{R}^{N-1}} x_N^2\left|x_N^2+|y|^2\right|^{-\frac{N}{2}}\,dy dx_N \\ &= \left|\mathbb{S}^{N-2}\right| \int_0^\infty\int_0^\infty x_N^2\left|x_N^2+r^2\right|^{-\frac{N}{2}} r^{N-2} dr dx_N.\end{aligned}$$

By setting $r = x_N t$, we get

$$\begin{aligned}\int_0^\infty\int_0^\infty x_N^2\left|x_N^2+r^2\right|^{-\frac{N}{2}} r^{N-2}drdx_N &= \int_0^\infty\int_0^\infty \left|1+t^2\right|^{-\frac{N}{2}} t^{N-2}x_N dt dx_N \\ &= \int_0^\infty x_N dx_N \int_0^\infty \left|1+t^2\right|^{-\frac{N}{2}} t^{N-2}dt\end{aligned}$$

which is divergent. Hence, the best constants $\left(\frac{N}{2}\right)^2$ in (1.6) and $\left(\frac{N-2}{2}\right)^2$ in (1.7) cannot be attained by nontrivial functions in $H_0^1\left(\mathbb{R}_+^N\right)$. Nevertheless, from the equality (1.6), we deduce that $cx_N\,|x|^{-\frac{N}{2}}$, $c \in \mathbb{C}$, can play the role of the “virtual” ground state in the sense of Frank and Seiringer [21].

Remark 1.1 It is also worthy to note that the best constant for the Hardy inequality on half-spaces with radial derivative is exactly the same as the optimal constant of the one on the whole space. This is in contrast to the case of full gradient where the sharp constant of the Hardy inequality on half-spaces with usual gradient is bumped to $\left(\frac{N}{2}\right)^2$ on $\mathbb{R}^N_+$.

Using Theorem 1.1, we can derive many new and interesting Hardy type inequalities on half-spaces. Indeed, we can deduce as many the Hardy inequalities on $\mathbb{R}^N_+$ as we can form the Bessel pairs. The book [28] provides various examples and properties about Bessel pairs. In the next section, we will state a few typical versions of the Hardy type inequalities on half-spaces that seem new in the literature and have potential applications in the study of the stability of certain singular solutions of nonlinear elliptic equations on half-spaces.

We illustrate below some particular applications of our Theorem 1.1 which provide the following sharpened inequalities on half balls or half spaces in the spirit of Brezis and Vázquez [12].

Theorem 1.2 *For any* $0 < R \le \infty$*, we have for* $u \in C_0^\infty\left(B_R^{(N)} \cap \mathbb{R}^N_+\right)$:

$$
\begin{aligned}
&\int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} |\nabla u|^2\,dx - \left(\frac{N}{2}\right)^2 \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} \frac{|u|^2}{|x|^2}dx \\
&= \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} |u|^2\,dx + \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} \left|\nabla\left(\frac{|x|^{\frac{N}{2}}}{J_{0;R}(|x|)}\frac{u}{x_N}\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N}{2}}}x_N\right|^2 dx \\
&\ge \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} |u|^2\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{N-2}{2}\right)^2 \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} \frac{|u|^2}{|x|^2}dx \\
&= \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} |u|^2\,dx + \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} \left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N}{2}}}{J_{0;R}(|x|)}\frac{u}{x_N}\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N}{2}}}x_N\right|^2 dx \\
&\ge \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)}\cap\mathbb{R}^N_+} |u|^2\,dx.
\end{aligned}
$$

We can also obtain the following versions of the celebrated Heisenberg's uncertainty principle on half-spaces as applications of our results:

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N_+} |u|^2\,dx \\
&\qquad \le \left(\int\limits_{\mathbb{R}^N_+} \frac{|u|^2}{|x|^2}dx\right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^N_+} |x|^2\,|u|^2\,dx\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \frac{2}{N-2}\left(\int_{\mathbb{R}_+^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}_+^N}|x|^2|u|^2 dx\right)^{\frac{1}{2}}$$

and

$$\int_{\mathbb{R}_+^N}|u|^2 dx$$

$$\leq \left(\int_{\mathbb{R}_+^N}\frac{|u|^2}{|x|^2}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}_+^N}|x|^2|u|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \frac{2}{N}\left(\int_{\mathbb{R}_+^N}|\nabla u|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}_+^N}|x|^2|u|^2 dx\right)^{\frac{1}{2}}.$$

1.2 Hardy inequalities with distance to a surface of codimension 1

Another type of the Hardy inequalities that has also received much attention in the literature is the case when $d(x)$ is the distance to a surface K of codimention 1. The most notable example is $d(x) = d(x, \partial\Omega)$ is the distance to the boundary. Here, Ω is a domain in $\mathbb{R}^N$ with Lipschitz boundary. In this case, the optimal constant

$$C_{opt}(\Omega) := \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{|u|^2}{d(x,\partial\Omega)^2}dx}$$

depends on the domain. In general, $C_{opt}(\Omega) \leq \frac{1}{4}$ when $\partial\Omega$ has a tangent plane at least at one point. There exists smooth bounded domains such that $C_{opt}(\Omega) < \frac{1}{4}$, but for convex domains $C_{opt}(\Omega) = \frac{1}{4}$. It is interesting to mention that the optimal constant $C_{opt}(\Omega)$ can be attained if and only if $C_{opt}(\Omega) < \frac{1}{4}$. See, for instance [4,10,11,15,48]. Actually, Brezis and Marcus proved in [10] the following result to explain explicitly this situation on bounded convex domain:

Theorem C (Brezis-Marcus [10]). *For every smooth domain of class C^2, there exists $\lambda(\Omega) \in \mathbb{R}$ such that*

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega \frac{|u|^2}{d(x,\partial\Omega)^2}dx \geq \lambda(\Omega)\int_\Omega |u|^2 dx,\ u \in C_0^\infty(\Omega). \tag{1.8}$$

Moreover, for convex domains,

$$\lambda(\Omega) \geq \frac{1}{4diam^2(\Omega)}.$$

They also questioned in [10] whether the diameter could be replaced by the volume of Ω. An affirmative answer was posed in [30] where it was showed that $\lambda(\Omega) \geq \frac{N^{\frac{N-2}{N}}\omega_{N-1}^{\frac{2}{N}}}{4|\Omega|^{\frac{2}{N}}}$. In [18], the authors improved the bound in [10] to $\lambda(\Omega) \geq \frac{3}{D_{int}(\Omega)^2}$ where $D_{int}(\Omega) := 2\sup_{x\in\Omega} d(x, \partial\Omega)$ is the interior diameter. In [2], Avkhadiev and Wirths established a sharp bound $\lambda(\Omega) \geq \frac{\lambda_0^2}{(\sup_{x\in\Omega} d(x,\partial\Omega))^2}$, $\lambda_0 = 0.940\ldots$ is the first zero in $(0, \infty)$ of the function $J_0(x) + 2xJ_0'(x)$, using one-dimensional inequalities. We refer the interested reader to the monograph [3] for further developments in this direction. We brieftly note here that as a consequence of our main results, the constant $\lambda(\Omega)$ can be improved to $\frac{z_0^2}{R^2}$ when $\Omega = \mathbb{R}^{N-1} \times (0, 2R)$.

The next goal of this paper is to investigate the general versions of the Hardy inequality with distance to the boundary on $\mathbb{R}_+^N$. We will use the method of factorizing differential operators and the notion of Bessel pairs to consolidate and improve many existing results in the literature. More precisely, let $0 < R \leq \infty$ and let $A_R^{(N)} = \mathbb{R}^{N-1} \times (0, R)$ be a strip in $\mathbb{R}^N$. Denote $d_R(x) = \min\{x_N, 2R - x_N\}$ (with the convention that $d_\infty(x) = x_N$). Note that $d_R(x)$ is the distance from x to the boundary of the strip $A_{2R}^{(N)}$. Then we will prove that

Theorem 1.3 *Let* $0 < R \leq \infty$. *If* (V, W) *is a Bessel pair on* $(0, R)$, *then for* $u \in C_0^\infty\left(A_{2R}^{(N)}\right)$:

$$\int_{A_{2R}^{(N)}} V(d_R(x))|\nabla u|^2 dx - \int_{A_{2R}^{(N)}} W(d_R(x))|u|^2 dx$$
$$= \int_{A_{2R}^{(N)}} V(d_R(x))\left|\nabla\left(\frac{u}{\varphi(d_R(x))}\right)\right|^2 \varphi^2(d_R(x))\,dx,$$

and

$$\int_{A_{2R}^{(N)}} V(d_R(x))\left|\frac{\partial u}{\partial x_N}\right|^2 dx - \int_{A_{2R}^{(N)}} W(d_R(x))|u|^2 dx$$
$$= \int_{A_{2R}^{(N)}} V(d_R(x))\left|\frac{\partial}{\partial x_N}\left(\frac{u}{\varphi(d_R(x))}\right)\right|^2 \varphi^2(d_R(x))\,dx.$$

We note that for any $0 < R < \infty$, $\left(1, \frac{1}{4}\frac{1}{r^2} + \frac{z_0^2}{R^2}\right)$ is a Bessel pair on $(0, R)$ with $\varphi = \sqrt{r}J_0\left(\frac{rz_0}{R}\right) = \sqrt{r}J_{0;R}(r)$. Hence we get the following Hardy inequalities in the spirit of Brezis-Marcus:

Theorem 1.4 *For any* $0 < R < \infty$, *we have for* $u \in C_0^\infty\left(A_{2R}^{(N)}\right)$:

$$\int_{A_{2R}^{(N)}} |\nabla u|^2 dx - \frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{|u|^2}{d_R(x)^2} dx$$
$$= \frac{z_0^2}{R^2}\int_{A_{2R}^{(N)}} |u|^2 dx + \int_{A_{2R}^{(N)}} \left|\nabla\left(\frac{u}{\sqrt{d_R(x)}J_{0;R}(d_R(x))}\right)\right|^2 \left|J_{0;R}(d_R(x))\right|^2 d_R(x)\,dx$$

and

$$\int_{A_{2R}^{(N)}} \left|\frac{\partial u}{\partial x_N}\right|^2 dx - \frac{1}{4} \int_{B_R^{(N)} \cap \mathbb{R}_+^N} \frac{|u|^2}{d_R(x)^2} dx$$
$$= \frac{z_0^2}{R^2} \int_{A_{2R}^{(N)}} |u|^2 dx + \int_{A_{2R}^{(N)}} \left|\frac{\partial}{\partial x_N}\left(\frac{u}{\sqrt{d_R(x)} J_{0;R}(d_R(x))}\right)\right|^2 \left|J_{0;R}(d_R(x))\right|^2 d_R(x)\, dx.$$

Hence, when the domain is the strip $A_{2R}^{(N)}$, we improve the constant $\lambda(\Omega)$ in Theorem C to $\frac{z_0^2}{R^2}$. This theorem can be considered as a special case of Corollary 3.1 in Sect. 3.

Another interesting application of our Theorem 1.3 is as follows: Since $\left(r^{\alpha+1}, \frac{\alpha^2}{4} r^{\alpha-1}\right)$ is a Bessel pair on $(0, \infty)$ with $\varphi = r^{-\frac{\alpha}{2}}$, we obtain the following identities:

Corollary 1.3 *For $u \in C_0^\infty\left(\mathbb{R}_+^N\right)$, there holds*

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 x_N^{\alpha+1} dx - \frac{\alpha^2}{4} \int_{\mathbb{R}_+^N} |u|^2 x_N^{\alpha-1} dx = \int_{\mathbb{R}_+^N} \left|\nabla\left(x_N^{\frac{\alpha}{2}} u\right)\right|^2 x_N dx$$

$$\int_{\mathbb{R}_+^N} \left|\frac{\partial u}{\partial x_N}\right|^2 x_N^{\alpha+1} dx - \frac{\alpha^2}{4} \int_{\mathbb{R}_+^N} |u|^2 x_N^{\alpha-1} dx = \int_{\mathbb{R}_+^N} \left|\frac{\partial\left(x_N^{\frac{\alpha}{2}} u\right)}{\partial x_N}\right|^2 x_N dx.$$

By choosing $\alpha = -1$, we get

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^N} \frac{|u|^2}{x_N^2} dx \geq 0.$$

Actually, we also have the following Hardy-Sobolev-Maz'ya inequality: (see, for instance [50]):

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^N} \frac{|u|^2}{x_N^2} dx \geq \frac{1}{C^2(N,1)} \left(\int_{\mathbb{R}_+^N} |u|^{\frac{2(N+1)}{N-1}} x_N^{-\frac{2}{N-1}} dx\right)^{\frac{N-1}{N+1}}, u \in C_0^\infty\left(\mathbb{R}_+^N\right). \tag{1.9}$$

Here $C(N,1)$ is sharp and is given in (1.10). We will show that by combining the identities in Corollary 1.3 and the optimal Sobolev inequality with monomial weights, we can derive a sharp version of the Hardy-Sobolev-Maz'ya type inequalities.

Theorem 1.5 *Let $\alpha \in \mathbb{R}$. Then we have*

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 x_N^{\alpha+1} dx - \frac{\alpha^2}{4} \int_{\mathbb{R}_+^N} |u|^2 x_N^{\alpha-1} dx$$
$$\geq \frac{1}{C^2(N,1)} \left(\int_{\mathbb{R}_+^N} |u|^{\frac{2(N+1)}{N-1}} x_N^{\frac{N-1+\alpha(N+1)}{N-1}} dx\right)^{\frac{N-1}{N+1}}.$$

The optimal constant $C(N,1)$ given in (1.10) is optimal and can be achieved when $u(x) = x_N^{-\alpha/2}\left(a+b|x|^2\right)^{(1-N)/2}$, a and b are arbitrary positive constants.

Indeed, the optimal Sobolev inequality with monomial weights in [13,54] asserts that

$$\left(\int_{\mathbb{R}_+^N} |\nabla v|^2 x_N^A dx\right)^{\frac{1}{2}} \geq \frac{1}{C(N,A)}\left(\int_{\mathbb{R}_+^N} |v|^{2_A^*} x_N^A dx\right)^{\frac{1}{2_A^*}}$$

where $2_A^* = \frac{2(N+A)}{N+A-2}$ and the best constant $C(N,A)$ is achieved by $U_{a,b}(x) = \left(a+b|x|^2\right)^{1-\frac{N+A}{2}}$, a and b are arbitrary positive constants:

$$\begin{aligned} C(N,A) &= (N+A)\left(\frac{\Gamma\left(\frac{A+1}{2}\right)}{2\Gamma\left(1+\frac{N+A}{2}\right)}\right)^{\frac{1}{N+A}} \\ &\quad \times (N+A)^{\frac{1}{N+A}-\frac{3}{2}}\left(\frac{1}{N+A-2}\right)^{\frac{1}{2}}\left(\frac{2\Gamma(N+A)}{\Gamma\left(\frac{N+A}{2}\right)\Gamma\left(\frac{N+A}{2}\right)}\right)^{\frac{1}{N+A}}. \end{aligned} \tag{1.10}$$

Using this result, we get for any $\alpha \in \mathbb{R}$ that

$$\begin{aligned} &\int_{\mathbb{R}_+^N} |\nabla u|^2 x_N^{\alpha+1} dx - \frac{\alpha^2}{4}\int_{\mathbb{R}_+^N} |u|^2 x_N^{\alpha-1} dx \\ &= \int_{\mathbb{R}_+^N} \left|\nabla\left(x_N^{\frac{\alpha}{2}} u\right)\right|^2 x_N dx \\ &\geq \frac{1}{C^2(N,1)}\left(\int_{\mathbb{R}_+^N} \left|x_N^{\frac{\alpha}{2}} u\right|^{\frac{2(N+1)}{N-1}} x_N dx\right)^{\frac{N-1}{N+1}} \\ &= \frac{1}{C^2(N,1)}\left(\int_{\mathbb{R}_+^N} |u|^{\frac{2(N+1)}{N-1}} x_N^{1+\frac{\alpha(N+1)}{N-1}} dx\right)^{\frac{N-1}{N+1}}. \end{aligned}$$

Remark 1.2 We remark here that the method in [50, Section 2.7] is dimension reduction. Hence the argument there doesn't work for arbitrary real number α.

Our final goal of this paper is to set up Hardy type inequalities with distance to the hyperplane $\{x_N = 0\}$ on $\mathbb{R}_+^N$.

Theorem 1.6 *If (V, W) is a Bessel pair on $(0, R)$, then for $u \in C_0^\infty\left(A_R^{(N)}\right)$:*

$$\int_{A_R^{(N)}} V(x_N)|\nabla u|^2\,dx - \int_{A_R^{(N)}} W(x_N)|u|^2\,dx$$
$$= \int_{A_R^{(N)}} V(x_N)\left|\nabla\left(\frac{u}{\varphi(x_N)}\right)\right|^2 \varphi^2(x_N)\,dx,$$

and

$$\int_{A_R^{(N)}} V(x_N)\left|\frac{\partial u}{\partial x_N}\right|^2 - \int_{A_R^{(N)}} W(x_N)|u|^2\,dx$$
$$= \int_{A_R^{(N)}} V(x_N)\left|\frac{\partial}{\partial x_N}\left(\frac{u}{\varphi(x_N)}\right)\right|^2 \varphi^2(x_N)\,dx.$$

For instance, we can get the following result in the spirit of Brezis-Vázquez and Brezis-Marcus:

Corollary 1.4 *For any $0 < R < \infty$, we have for $u \in C_0^\infty\left(A_R^{(N)}\right)$:*

$$\int_{A_R^{(N)}} |\nabla u|^2\,dx - \frac{1}{4}\int_{A_R^{(N)}} \frac{|u|^2}{x_N^2}dx$$
$$= \frac{z_0^2}{R^2}\int_{A_R^{(N)}} |u|^2\,dx + \int_{A_R^{(N)}} \left|\nabla\left(\frac{u}{\sqrt{x_N}J_{0;R}(x_N)}\right)\right|^2 \left|J_{0;R}(x_N)\right|^2 x_N dx$$

and

$$\int_{A_R^{(N)}} \left|\frac{\partial u}{\partial x_N}\right|^2 dx - \frac{1}{4}\int_{A_R^{(N)}} \frac{|u|^2}{x_N^2}dx$$
$$= \frac{z_0^2}{R^2}\int_{A_R^{(N)}} |u|^2\,dx + \int_{A_R^{(N)}} \left|\frac{\partial\left(\frac{u}{\sqrt{x_N}J_{0;R}(x_N)}\right)}{\partial x_N}\right|^2 \left|J_{0;R}(x_N)\right|^2 x_N dx.$$

Basically, the above results assert that on $A_R^{(N)}$, if we replace the distance to the boundary by the distance to the hyperplane $\{x_N = 0\}$, then the constant $\lambda(\Omega)$ in Theorem C can be improved to $\frac{z_0^2}{R^2}$.

Furthermore, as another application of Theorem 1.6, we will be able to establish Hardy type identities and inequalities with as many remainder terms as we wish (See Corollary 4.2) in Sect. 4.

We end up this introduction with the following remarks. In our recent work [37], we establish general geometric Hardy's inequalities on domains in $\mathbb{R}^N$ in the spirit of their works by Brezis-Vázquez [12] and Brezis-Marcus [10]. More precisely, we use the notion of Bessel pairs introduced by Ghoussoub and Moradifam [27,28] to establish several Hardy identities and inequalities with the general distance functions $d(x)$ from x to a surface of any codimension.

Theorem D *Let* $0 < R \leq \infty$*,* V *and* W *be positive* C^1-*functions on* $(0, R)$*. Assume that* $|\nabla d(x)| = 1$ *and for some* $\alpha \in \mathbb{R}$*,* $\Delta d(x) - \frac{\alpha-1}{d(x)}$ *exists on* $\{0 < d(x) < R\}$ *in the sense of distributions and* $\left(r^{\alpha-1}V, r^{\alpha-1}W\right)$ *is a Bessel pair on* $(0, R)$*. Then for* $u \in C_0^\infty(\{0 < d(x) < R\})$:

$$\begin{aligned}&\int\limits_{0<d(x)<R} V(d(x))|\nabla u(x)|^2 dx - \int\limits_{0<d(x)<R} W(d(x))|u(x)|^2 dx \\ &= \int\limits_{0<d(x)<R} V(d(x))\varphi^2(d(x))\left|\nabla\left(\frac{u(x)}{\varphi(d(x))}\right)\right|^2 dx \\ &\quad - \int\limits_{0<d(x)<R} V(d(x))|u(x)|^2\left[\Delta d(x) - \frac{\alpha-1}{d(x)}\right]\frac{\varphi'(d(x))}{\varphi(d(x))}dx\end{aligned}$$

and

$$\begin{aligned}&\int\limits_{0<d(x)<R} V(d(x))|\nabla d(x)\cdot\nabla u(x)|^2 dx - \int\limits_{0<d(x)<R} W(d(x))|u(x)|^2 dx \\ &= \int\limits_{0<d(x)<R} V(d(x))\varphi^2(d(x))\left|\nabla d(x)\cdot\nabla\left(\frac{u(x)}{\varphi(d(x))}\right)\right|^2 dx \\ &\quad - \int\limits_{0<d(x)<R} V(d(x))|u(x)|^2\left[\Delta d(x) - \frac{\alpha-1}{d(x)}\right]\frac{\varphi'(d(x))}{\varphi(d(x))}dx.\end{aligned}$$

Here φ *is the positive solution of*

$$\left(r^{\alpha-1}V(r)y'(r)\right)' + r^{\alpha-1}W(r)y(r) = 0$$

on the interval $(0, R)$.

It is worth noting that our distance functions can be understood as the distance to the surfaces of codimention $\alpha \in \mathbb{R}$, and include the distance to the origin ($\alpha = N$), the distance to the boundary ($\alpha = 1$), and even the distance to surfaces of codimension $k \in \mathbb{N}$ with $1 \leq k \leq N$, as special cases. As applications of our main results, we are able to obtain the improved Hardy inequalities in the sense of Brezis-Vázquez [12] and Brezis-Marcus [10] for general distance functions. We also prove a version of the Hardy-Sobolev-Maz'ya inequality on $(\mathbb{R}_+)^N$ for the distance function $\min\{x_1, \ldots, x_N\}$. Moreover, the critical Hardy inequalities with general distance functions are also considered. Our results also extend and sharpen earlier results on Hardy's inequalities with distance functions to surfaces of codimension $1 \leq k \leq N$ in the literature by adding nonnegative remainder terms.

The organization of the paper is as follows. In Sect. 2, we will establish the Hardy identities and inequalities with distance to the origin on half-spaces and prove Theorem 1.1. We will also apply this theorem to derive several Corollaries of Hardy type identities and inequalities on half spaces in the spirit of Brezis and Vázquez. In Sect. 3, we will prove the Hardy type identities and inequalities with distance to the boundary of the strips $A_{2R}^{(N)}$ on half-spaces, namely Theorem 1.3. We will also give some applications of Theorem 1.3. As a consequence, we will improve the bounds $\lambda(\Omega)$ in Theorem C of Brezis and Marcus when Ω is a strip $A_{2R}^{(N)}$. We will also get an extension and improvement of the Hardy-Sobolev-Maz'ya inequality to arbitrary $\alpha \leq 0$. Finally, in Sect. 4, we will prove the Hardy type identities and inequalities with distance to the hyperplane $\{x_N = 0\}$, namely Theorem 1.6. We will also give some applications of Theorem 1.6.

2 Hardy identities and inequalities with distance to the origin on half-spaces

In this section, we will establish the Hardy identities and inequalities with distance to the origin on half-spaces and prove Theorem 1.1.

2.1 Proof of Theorem 1.1

Proof of Theorem 1.1 Let $T = \sqrt{V}\nabla - \sqrt{V}\frac{\nabla\varphi}{\varphi} - \sqrt{V}\frac{e_N}{x_N}$. Then its formal adjoint is $T^+ = -\operatorname{div}(\sqrt{V}\cdot) - \sqrt{V}\frac{\nabla\varphi}{\varphi}\cdot -\sqrt{V}\frac{e_N}{x_N}\cdot$.

Also,

$$\begin{aligned}
T^+T &= -\operatorname{div}(V\nabla) + \operatorname{div}\left(V\frac{\nabla\varphi}{\varphi}\right) + V\frac{\nabla\varphi}{\varphi}\cdot\nabla + \operatorname{div}\left(V\frac{e_N}{x_N}\right) + V\frac{e_N}{x_N}\cdot\nabla \\
&\quad - V\frac{\nabla\varphi}{\varphi}\cdot\nabla + V\frac{\nabla\varphi}{\varphi}\cdot\frac{\nabla\varphi}{\varphi} + V\frac{\nabla\varphi}{\varphi}\cdot\frac{e_N}{x_N} \\
&\quad - V\frac{e_N}{x_N}\cdot\nabla + V\frac{\nabla\varphi}{\varphi}\cdot\frac{e_N}{x_N} + V\frac{1}{x_N^2} \\
&= -\operatorname{div}(V\nabla) + \operatorname{div}\left(V\frac{\nabla\varphi}{\varphi}\right) + \frac{V'}{|x|} \\
&\quad + V\left(\frac{\varphi'}{\varphi}\right)^2 + V\frac{\varphi'}{\varphi}\frac{1}{|x|} \\
&\quad + V\frac{\varphi'}{\varphi}\frac{1}{|x|}
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \overline{u(x)}\left(T^+Tu\right)(x)\,dx \\
&= -\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \overline{u(x)}\operatorname{div}(V\nabla u(x))\,dx + \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \operatorname{div}\left(V\frac{\nabla\varphi}{\varphi}\right)|u(x)|^2\,dx \\
&\quad + \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{V'}{|x|}|u(x)|^2\,dx \\
&\quad + \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V\left(\frac{\varphi'}{\varphi}\right)^2|u(x)|^2\,dx + 2\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V\frac{\varphi'}{\varphi}\frac{1}{|x|}|u(x)|^2\,dx \\
&= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\,|\nabla u|^2\,dx + \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{V\varphi'' + V\frac{N+1}{|x|}\varphi' + V'\varphi'}{\varphi}|u(x)|^2\,dx
\end{aligned}$$

$$+\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{V'}{|x|}|u(x)|^2\,dx$$
$$=\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)|\nabla u|^2\,dx-\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\left[W(|x|)-\frac{V'(|x|)}{|x|}\right]|u|^2\,dx.$$

On the other hand,

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \overline{u(x)}\left(T^+Tu\right)(x)\,dx$$
$$=\int_{B_R^{(N)}\cap\mathbb{R}_+^N}|Tu|^2\,dx$$
$$=\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\nabla\left(\frac{u}{\varphi}\frac{1}{x_N}\right)\right|^2\varphi^2x_N^2\,dx.$$

Hence

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)|\nabla u|^2\,dx-\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\left[W(|x|)-\frac{V'(|x|)}{|x|}\right]|u|^2\,dx$$
$$=\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\nabla\left(\frac{u}{\varphi}\frac{1}{x_N}\right)\right|^2\varphi^2x_N^2\,dx.$$

Next, we set $S=\sqrt{V}\frac{x}{|x|}\cdot\nabla-\sqrt{V}\frac{\varphi'}{\varphi}-\sqrt{V}\frac{1}{|x|}$. Then its formal adjoint is

$$S^+=-\sqrt{V}\frac{x}{|x|}\cdot\nabla-\sqrt{V}\frac{N-1}{|x|}-\frac{V'}{2\sqrt{V}}-\sqrt{V}\frac{\varphi'}{\varphi}-\sqrt{V}\frac{1}{|x|}$$
$$=-\sqrt{V}\frac{x}{|x|}\cdot\nabla-\sqrt{V}\frac{N}{|x|}-\frac{V'}{2\sqrt{V}}-\sqrt{V}\frac{\varphi'}{\varphi}$$

Then we have

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}(x)\left(S^+Su\right)(x)\,dx$$
$$=-\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\sqrt{V}\frac{x}{|x|}\cdot\nabla\left(\sqrt{V}\frac{x}{|x|}\cdot\nabla u\right)dx+\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\sqrt{V}\frac{x}{|x|}\cdot\nabla\left(\sqrt{V}\frac{\varphi'}{\varphi}u\right)dx$$
$$+\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\sqrt{V}\frac{x}{|x|}\cdot\nabla\left(\sqrt{V}\frac{1}{|x|}u\right)dx$$
$$-\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\left(\sqrt{V}\frac{N}{|x|}+\frac{V'}{2\sqrt{V}}+\sqrt{V}\frac{\varphi'}{\varphi}\right)\sqrt{V}\frac{x}{|x|}\cdot\nabla u\,dx$$

$$+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\left(\sqrt{V}\frac{N}{|x|}+\frac{V'}{2\sqrt{V}}+\sqrt{V}\frac{\varphi'}{\varphi}\right)\sqrt{V}\frac{\varphi'}{\varphi}\,|u|^2\,dx$$
$$+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\left(\sqrt{V}\frac{N}{|x|}+\frac{V'}{2\sqrt{V}}+\sqrt{V}\frac{\varphi'}{\varphi}\right)\sqrt{V}\frac{1}{|x|}\,|u|^2\,dx.$$

Using the polar coordinate $(r,\sigma)=\left(|x|\,,\,\frac{x}{|x|}\right)$ and the fact that $\frac{x}{|x|}\cdot\nabla=\partial_r$, we get by a direct computation that

$$\begin{aligned}
&-\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\sqrt{V}\frac{x}{|x|}\cdot\nabla\left(\sqrt{V}\frac{x}{|x|}\cdot\nabla u\right)dx\\
&\quad=\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}V\left(|x|\right)\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\left(\frac{x}{|x|}\cdot\nabla u\right)\frac{V'}{2}dx\\
&\qquad+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\left(\frac{x}{|x|}\cdot\nabla u\right)V\frac{N-1}{|x|}dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\sqrt{V}\frac{x}{|x|}\cdot\nabla\left(\sqrt{V}\frac{\varphi'}{\varphi}u\right)dx\\
&\quad=\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\left(\frac{x}{|x|}\cdot\nabla u\right)V\frac{\varphi'}{\varphi}dx+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}V\left(\frac{\varphi''\varphi-\left(\varphi'\right)^2}{\varphi^2}\right)|u|^2\,dx\\
&\qquad+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{\varphi'}{\varphi}\frac{V'}{2}\,|u|^2\,dx.
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\sqrt{V}\frac{x}{|x|}\cdot\nabla\left(\sqrt{V}\frac{1}{|x|}u\right)dx\\
&\quad=\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\left(\frac{V'}{2\,|x|}-\frac{V}{|x|^2}\right)|u|^2\,dx+\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\left(\frac{x}{|x|}\cdot\nabla u\right)\frac{V}{|x|}dx
\end{aligned}$$

Hence

$$\begin{aligned}
&\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\overline{u}\left(x\right)\left(S^+Su\right)(x)\,dx\\
&\quad=\int\limits_{B_R^{(N)}\cap\mathbb{R}_+^N}\left[\frac{V\varphi''+V\frac{N+1}{|x|}\varphi'+V'\varphi'}{\varphi}+\frac{V'}{|x|}+V\frac{N-1}{|x|^2}\right]|u|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \\
&= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left[-W(|x|)+\frac{V'(|x|)}{|x|}+\frac{(N-1)}{|x|^2}V(|x|)\right]|u|^2\,dx \\
&+ \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \overline{u}(x)\left(S^+Su\right)(x)\,dx \\
&\quad= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} |Su|^2\,dx \\
&\quad= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla\left(\frac{1}{\varphi}\frac{u}{x_N}\right)\right|^2\varphi^2x_N^2dx.
\end{aligned}
$$

Hence, we deduce

$$
\begin{aligned}
&\int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \int_{B_R^{(N)}\cap\mathbb{R}_+^N}\left[W(|x|)-\frac{V'(|x|)}{|x|}-\frac{(N-1)}{|x|^2}V(|x|)\right]|u|^2\,dx \\
&= \int_{B_R^{(N)}\cap\mathbb{R}_+^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla\left(\frac{1}{\varphi}\frac{u}{x_N}\right)\right|^2\varphi^2x_N^2dx.
\end{aligned}
$$

□

2.2 Applications of Theorem 1.1

The pair $\left(r^{N+1}, r^{N+1}\left(\left(\frac{N}{2}\right)^2 r^{-2}+\frac{z_0^2}{R^2}\right)\right)$, z_0 is the first zero of the Bessel function J_0, is a Bessel pair on $(0, R)$ with $\varphi = r^{-\frac{N}{2}}J_0\left(\frac{rz_0}{R}\right) = r^{-\frac{N}{2}}J_{0;R}(r)$. Hence, as a consequence of our Theorem 1.1, we get the following version of the Hardy inequality on $\mathbb{R}_+^N$ in the spirit of Brezis and Vázquez [12]:

Corollary 2.1 *For any $R>0$, we have for $u\in C_0^\infty\left(B_R^{(N)}\cap\mathbb{R}_+^N\right)$:*

$$
\int_{B_R^{(N)}\cap\mathbb{R}_+^N} |\nabla u|^2\,dx - \left(\frac{N}{2}\right)^2\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^2}dx
$$

$$= \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} |u|^2 \, dx + \int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} \left| \nabla \left(\frac{|x|^{\frac{N}{2}}}{J_{0;R}(|x|)} \frac{u}{x_N} \right) \right|^2 \left| \frac{J_{0;R}\left(|x|\right)}{|x|^{\frac{N}{2}}} x_N \right|^2 dx$$

$$\geq \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} |u|^2 \, dx, \tag{2.1}$$

and

$$\int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \left(\frac{N-2}{2} \right)^2 \int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} \frac{|u|^2}{|x|^2} dx$$

$$= \frac{z_0^2}{R^2} \int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} |u|^2 \, dx + \int\limits_{B_R^{(N)} \cap \mathbb{R}_+^N} \left| \frac{x}{|x|} \cdot \nabla \left(\frac{|x|^{\frac{N}{2}}}{J_{0;R}(|x|)} \frac{u}{x_N} \right) \right|^2 \left| \frac{J_{0;R}\left(|x|\right)}{|x|^{\frac{N}{2}}} x_N \right|^2 dx.$$

A version of (2.1) on the bounded domain containing 0 was proved in [12] and was used to study the stability of certain singular solutions of nonlinear elliptic equations. Hence, our results in the above corollary can be used to investigate the stability of singular solutions of certain nonlinear elliptic equations on half-spaces.

Assume $0 \leq \lambda \leq N$. Then $\left(r^{N+1-\lambda}, \left(\frac{N-\lambda}{2}\right)^2 r^{N-1-\lambda} \right)$ is a Bessel pair on $(0, \infty)$ with $\varphi = r^{-\frac{N-\lambda}{2}}$. Hence, we get from Theorem 1.1 that

Corollary 2.2 *Assume* $0 \leq \lambda \leq N$. *We have for* $u \in C_0^\infty\left(\mathbb{R}_+^N\right)$:

$$\int\limits_{\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |\nabla u|^2 \, dx - \left[\left(\frac{N-\lambda}{2} \right)^2 + \lambda \right] \int\limits_{\mathbb{R}_+^N} \frac{1}{|x|^{\lambda+2}} |u|^2 \, dx$$

$$= \int\limits_{\mathbb{R}_+^N} \left| \nabla \left(\frac{|x|^{\frac{N-\lambda}{2}} u}{x_N} \right) \right|^2 \frac{x_N^2}{|x|^N} dx$$

and

$$\int\limits_{\mathbb{R}_+^N} \frac{1}{|x|^\lambda} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \left(\frac{N-\lambda-2}{2} \right)^2 \int\limits_{\mathbb{R}_+^N} \frac{1}{|x|^{\lambda+2}} |u|^2 \, dx$$

$$= \int\limits_{\mathbb{R}_+^N} \left| \frac{x}{|x|} \cdot \nabla \left(\frac{|x|^{\frac{N-\lambda}{2}} u}{x_N} \right) \right|^2 \frac{x_N^2}{|x|^N} dx.$$

Assume $0 \leq \lambda \leq N$. Then $\left(r^{N+1-\lambda}, \left(\frac{N-\lambda}{2}\right)^2 r^{N-1-\lambda} + \frac{z_0^2}{R^2} r^{N+1-\lambda} \right)$ is a Bessel pair on $(0, R)$ with $\varphi = r^{-\frac{N-\lambda}{2}} J_0\left(\frac{r z_0}{R}\right) = r^{-\frac{N-\lambda}{2}} J_{0;R}\left(r\right)$. Hence by Theorem 1.1, we obtain

Corollary 2.3 *Assume* $0 \le \lambda \le N$. *We have for* $u \in C_0^\infty\left(B_R^{(N)} \cap \mathbb{R}_+^N\right)$:

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |\nabla u|^2 \, dx - \left[\left(\frac{N-\lambda}{2}\right)^2 + \lambda\right] \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^{\lambda+2}} |u|^2 \, dx$$
$$= \frac{z_0^2}{R^2} \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |u|^2 \, dx + \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left|\nabla\left(\frac{|x|^{\frac{N-\lambda}{2}}}{J_{0;R}(|x|)} \frac{u}{x_N}\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda}{2}}} x_N\right|^2 dx$$
$$\ge \frac{z_0^2}{R^2} \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |u|^2 \, dx$$

and

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx - \left(\frac{N-2-\lambda}{2}\right)^2 \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^{\lambda+2}} |u|^2 \, dx$$
$$= \frac{z_0^2}{R^2} \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |u|^2 \, dx + \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left|\frac{x}{|x|} \cdot \nabla\left(\frac{|x|^{\frac{N-\lambda}{2}}}{J_{0;R}(|x|)} \frac{u}{x_N}\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda}{2}}} x_N\right|^2 dx$$
$$\ge \frac{z_0^2}{R^2} \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |u|^2 \, dx.$$

For any $R > 0$, $0 \le \lambda \le N$, and $k \in \mathbb{N}$, the pair $\left(r^{N+1-\lambda}, \left(\frac{N-\lambda}{2}\right)^2 r^{N-1-\lambda} + r^{N+1-\lambda} P_{k,R}(r)\right)$ is a Bessel pair on $(0, R)$ with $\varphi = r^{-\frac{N-\lambda}{2}} \left[X_1\left(\frac{r}{R}\right) \cdots X_k\left(\frac{r}{R}\right)\right]^{-\frac{1}{2}}$ where

$$P_{k,R}(r) = \frac{1}{4}\frac{1}{r^2}\sum_{j=1}^k X_1^2\left(\frac{r}{R}\right) \cdots X_j^2\left(\frac{r}{R}\right)$$
$$X_1(r) = \frac{1}{1-\ln r}, \; X_i(r) = X_1(X_{i-1}(r)).$$

See [28, Proposition 1.1.1 and Proposition 1.3.1]. Hence, we deduce

Corollary 2.4 *For any* $R > 0$, $0 \le \lambda \le N$, *and* $k \in \mathbb{N}$, *we have*

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^\lambda} |\nabla u|^2 \, dx - \left[\left(\frac{N-\lambda}{2}\right)^2 + \lambda\right] \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^{\lambda+2}} |u|^2 \, dx$$
$$= \frac{1}{4} \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{|u|^2}{|x|^{\lambda+2}} \sum_{j=1}^k X_1^2\left(\frac{|x|}{R}\right) \cdots X_j^2\left(\frac{|x|}{R}\right) dx$$
$$+ \int_{B_R^{(N)}\cap\mathbb{R}_+^N} \left|\nabla\left(|x|^{\frac{N-\lambda}{2}} \left[X_1\left(\frac{|x|}{R}\right) \cdots X_k\left(\frac{|x|}{R}\right)\right]^{\frac{1}{2}} \frac{u}{x_N}\right)\right|^2$$

$$\left|\frac{x_N}{|x|^{\frac{N-\lambda}{2}}\left[X_1\left(\frac{|x|}{R}\right)\cdots X_k\left(\frac{|x|}{R}\right)\right]^{\frac{1}{2}}}\right|^2 dx$$

$$\geq \frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}}\sum_{j=1}^{k}X_1^2\left(\frac{|x|}{R}\right)\cdots X_j^2\left(\frac{|x|}{R}\right)dx \tag{2.2}$$

and

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx-\left(\frac{N-\lambda-2}{2}\right)^2\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{1}{|x|^{\lambda+2}}|u|^2 dx$$

$$=\frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}}\sum_{j=1}^{k}X_1^2\left(\frac{|x|}{R}\right)\cdots X_j^2\left(\frac{|x|}{R}\right)dx$$

$$+\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{N-\lambda}{2}}\left[X_1\left(\frac{|x|}{R}\right)\cdots X_k\left(\frac{|x|}{R}\right)\right]^{\frac{1}{2}}\frac{u}{x_N}\right)\right|^2$$

$$\left|\frac{x_N}{|x|^{\frac{N-\lambda}{2}}\left[X_1\left(\frac{|x|}{R}\right)\cdots X_k\left(\frac{|x|}{R}\right)\right]^{\frac{1}{2}}}\right|^2 dx$$

$$\geq \frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}}\sum_{j=1}^{k}X_1^2\left(\frac{|x|}{R}\right)\cdots X_j^2\left(\frac{|x|}{R}\right)dx. \tag{2.3}$$

The inequality (2.2) was derived in ([59]) in the case $\lambda = 0$ using the spherical harmonic decomposition. Our results here provide the exact remainders and the ground states. The case $0 < \lambda \leq N$ is new. (2.3) is completely new in the literature.

Assume $0 \leq \lambda \leq N$, $R > 0$ and $\rho > Re$. $\left(r^{N+1-\lambda}, \left(\frac{N-\lambda}{2}\right)^2 r^{N-1-\lambda} + \frac{1}{4}r^{N-1-\lambda}\left(\log\frac{\rho}{r}\right)^{-2}\right)$ is a Bessel pair on $(0, R)$ with $\varphi = r^{-\frac{N-\lambda}{2}}\left(\log\frac{\rho}{r}\right)^{\frac{1}{2}}$. Hence

Corollary 2.5 *Assume* $0 \leq \lambda \leq N$, $R > 0$ *and* $\rho > Re$. *We have*

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{1}{|x|^{\lambda}}|\nabla u|^2 dx-\left[\left(\frac{N-\lambda}{2}\right)^2+\lambda\right]\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{1}{|x|^{\lambda+2}}|u|^2 dx$$

$$=\frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}\left(\log\frac{\rho}{|x|}\right)^2}dx+\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\left|\nabla\left(\frac{|x|^{\frac{N-\lambda}{2}}}{\left(\log\frac{\rho}{|x|}\right)^{\frac{1}{2}}}\frac{u}{x_N}\right)\right|^2\left|\frac{\left(\log\frac{\rho}{|x|}\right)^{\frac{1}{2}}}{|x|^{\frac{N-\lambda}{2}}}x_N\right|^2 dx$$

$$\geq \frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}\left(\log\frac{\rho}{|x|}\right)^2}dx$$

and

$$\int_{B_R^{(N)}\cap\mathbb{R}_+^N} \frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx-\left(\frac{N-2-\lambda}{2}\right)^2\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{1}{|x|^{\lambda+2}}|u|^2\,dx$$

$$=\frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}\left(\log\frac{\rho}{|x|}\right)^2}dx$$

$$+\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-\lambda}{2}}}{\left(\log\frac{\rho}{|x|}\right)^{\frac{1}{2}}}\frac{u}{x_N}\right)\right|^2\left|\frac{\left(\log\frac{\rho}{|x|}\right)^{\frac{1}{2}}}{|x|^{\frac{N-\lambda}{2}}}x_N\right|^2 dx$$

$$\geq\frac{1}{4}\int_{B_R^{(N)}\cap\mathbb{R}_+^N}\frac{|u|^2}{|x|^{\lambda+2}\left(\log\frac{\rho}{|x|}\right)^2}dx.$$

3 Hardy identities and inequalities with distance to the boundary of domains on half-spaces

In this section, we will provide the proof for the general Hardy inequality with distance to the boundary of the strip $A_{2R}^{(N)}$. We also derive a general result that implies that the constant $\lambda(\Omega)$ in Theorem C can be improved to $\frac{z_0^2}{R^2}$ when Ω is the strip $A_{2R}^{(N)}$.

3.1 Proof of Theorem 1.3

Proof of Theorem 1.3 Let $T=\sqrt{V(d_R(x))}\nabla-\sqrt{V(d_R(x))}\frac{\varphi'(d_R(x))}{\varphi(d_R(x))}\nabla d_R(x)$. Then its formal adjoint is $T^+=-\operatorname{div}(\sqrt{V(d_R(x))}\cdot)-\sqrt{V(d_R(x))}\frac{\varphi'(d_R(x))}{\varphi(d_R(x))}\nabla d_R(x)\cdot$.

Hence,

$$T^+Tu=-\operatorname{div}\left(V(d_R(x))\left[\nabla u-\frac{\varphi'(d_R(x))}{\varphi(d_R(x))}u\nabla d_R(x)\right]\right)$$

$$-V(d_R(x))\frac{\varphi'(d_R(x))}{\varphi(d_R(x))}\nabla d_R(x)\cdot\nabla u+V(d_R(x))\left(\frac{\varphi'(d_R(x))}{\varphi(d_R(x))}\right)^2u$$

$$=-V(d_R(x))\Delta u-V'(d_R(x))\nabla d_R(x)\cdot\nabla u$$

$$+V'(d_R(x))\frac{\varphi'(d_R(x))}{\varphi(d_R(x))}u+V(d_R(x))\frac{\varphi''(d_R(x))}{\varphi(d_R(x))}u.$$

Thus

$$\int_{A_{2R}^{(N)}}\overline{u(x)}\left(T^+Tu\right)(x)\,dx$$

$$=\int_{A_{2R}^{(N)}}V(d_R(x))|\nabla u|^2dx+\int_{A_{2R}^{(N)}}\frac{V'(d_R(x))\varphi'(d_R(x))+V(d_R(x))\varphi''(d_R(x))}{\varphi(d_R(x))}|u|^2\,dx$$

$$= \int_{A_{2R}^{(N)}} V(d_R(x)) |\nabla u|^2 \, dx - \int_{A_{2R}^{(N)}} W(d_R(x)) |u|^2 \, dx.$$

On the other hand,

$$\begin{aligned}
&\int_{A_{2R}^{(N)}} \overline{u(x)} \left(T^+ T u\right)(x) \, dx \\
&\quad = \int_{A_{2R}^{(N)}} |Tu|^2 \, dx \\
&\quad = \int_{A_{2R}^{(N)}} V(d_R(x)) \left| \nabla \left(\frac{u}{\varphi(d_R(x))} \right) \right|^2 \varphi^2(d_R(x)) \, dx.
\end{aligned}$$

Hence, we deduce

$$\begin{aligned}
&\int_{A_{2R}^{(N)}} V(d_R(x)) |\nabla u|^2 \, dx - \int_{A_{2R}^{(N)}} W(d_R(x)) |u|^2 \, dx \\
&\quad = \int_{A_{2R}^{(N)}} V(d_R(x)) \left| \nabla \left(\frac{u}{\varphi(d_R(x))} \right) \right|^2 \varphi^2(d_R(x)) \, dx.
\end{aligned}$$

Now, let $Su = \sqrt{V(d_R(x))} \frac{\partial u}{\partial x_N} - \sqrt{V(d_R(x))} \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} u$. Then its formal adjoint is $S^+ v = -\frac{\partial}{\partial_{x_N}} \left(\sqrt{V(d_R(x))} v\right) - \sqrt{V(d_R(x))} \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} v$.

Therefore,

$$\begin{aligned}
&S^+ S u \\
&= -\frac{\partial}{\partial x_N} \left(V(d_R(x)) \frac{\partial u}{\partial x_N} - V(d_R(x)) \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} u \right) \\
&\quad - V(d_R(x)) \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} \frac{\partial u}{\partial x_N} + V(d_R(x)) \left(\frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \right)^2 u \\
&= -V(d_R(x)) \left(\frac{\partial^2 u}{\partial x_N^2} - \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} \frac{\partial u}{\partial x_N} \right. \\
&\quad \left. - \frac{\varphi''(d_R(x)) \varphi(d_R(x)) - \left(\varphi'(d_R(x))\right)^2}{\varphi^2(d_R(x))} u \right) \\
&\quad - V'(d_R(x)) \frac{\partial d_R(x)}{\partial x_N} \left(\frac{\partial u}{\partial x_N} - \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} u \right) \\
&\quad - V(d_R(x)) \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \frac{\partial d_R(x)}{\partial x_N} \frac{\partial u}{\partial x_N} + V(d_R(x)) \left(\frac{\varphi'(d_R(x))}{\varphi(d_R(x))} \right)^2 u \\
&= -V(d_R(x)) \left(\frac{\partial^2 u}{\partial x_N^2} - \frac{\varphi''(d_R(x))}{\varphi(d_R(x))} u \right) - V'(d_R(x)) \left(\frac{\partial d_R(x)}{\partial x_N} \frac{\partial u}{\partial x_N} - \frac{\varphi'(d_R(x))}{\varphi(d_R(x))} u \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{A_{2R}^{(N)}} \overline{u(x)}\left(S^{+}Su\right)(x)\,dx\\
&\quad=\int_{A_{2R}^{(N)}} V\left(d_R(x)\right)\left|\frac{\partial u}{\partial x_N}\right|^2 dx\\
&\qquad+\int_{A_{2R}^{(N)}} \frac{V'\left(d_R(x)\right)\varphi'\left(d_R(x)\right)+V\left(d_R(x)\right)\varphi''\left(d_R(x)\right)}{\varphi\left(d_R(x)\right)}|u|^2\,dx.\\
&\quad=\int_{A_{2R}^{(N)}} V\left(d_R(x)\right)|\nabla u|^2\,dx-\int_{A_{2R}^{(N)}} W\left(d_R(x)\right)|u|^2\,dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{A_{2R}^{(N)}} \overline{u(x)}\left(S^{+}Su\right)(x)\,dx\\
&\quad=\int_{A_{2R}^{(N)}} |Su|^2\,dx\\
&\quad=\int_{A_{2R}^{(N)}} V\left(d_R(x)\right)\left|\frac{\partial u}{\partial x_N}-\frac{\varphi'\left(d_R(x)\right)}{\varphi\left(d_R(x)\right)}\frac{\partial d_R(x)}{\partial x_N}u\right|^2 dx\\
&\quad=\int_{A_{2R}^{(N)}} V\left(d_R(x)\right)\left|\frac{\partial}{\partial x_N}\left(\frac{u}{\varphi\left(d_R(x)\right)}\right)\right|^2\varphi^2\left(d_R(x)\right)dx.
\end{aligned}$$

Then we can get

$$\begin{aligned}
&\int_{A_{2R}^{(N)}} V\left(d_R(x)\right)\left|\frac{\partial u}{\partial x_N}\right|^2 dx-\int_{A_{R}^{(N)}} W\left(d_R(x)\right)|u|^2\,dx\\
&\quad=\int_{A_{2R}^{(N)}} V\left(d_R(x)\right)\left|\frac{\partial}{\partial x_N}\left(\frac{u}{\varphi\left(d_R(x)\right)}\right)\right|^2\varphi^2\left(d_R(x)\right)dx.
\end{aligned}$$

□

3.2 Applications of Theorem 1.3

For $R>0$, we note that $\left(r^{\alpha+1},\left(\frac{\alpha}{2}\right)^2 r^{\alpha-1}+\frac{z_0^2}{R^2}r^{\alpha+1}\right)$ is a Bessel pair on $(0,R)$ with $\varphi=r^{-\frac{\alpha}{2}}J_0\left(\frac{rz_0}{R}\right)=r^{-\frac{\alpha}{2}}J_{0;R}(r)$. Hence we get the following Hardy type inequality

Corollary 3.1 *For any $R>0$, we have for $u\in C_0^\infty\left(A_{2R}^{(N)}\right)$:*

$$\int\limits_{A_{2R}^{(N)}}|\nabla u|^2 d_R(x)^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int\limits_{A_{2R}^{(N)}}|u|^2 d_R(x)^{\alpha-1}dx$$
$$=\frac{z_0^2}{R^2}\int\limits_{A_{2R}^{(N)}}|u|^2dx+\int\limits_{A_{2R}^{(N)}}\left|\nabla\left(\frac{d_R(x)^{\frac{\alpha}{2}}u}{J_{0;R}(d_R(x))}\right)\right|^2\left|\frac{J_{0;R}(d_R(x))}{d_R(x)^{\frac{\alpha}{2}}}\right|^2dx$$

and

$$\int\limits_{A_{2R}^{(N)}}\left|\frac{\partial u}{\partial x_N}\right|^2 d_R(x)^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int\limits_{A_{2R}^{(N)}}|u|^2 d_R(x)^{\alpha-1}dx$$
$$=\frac{z_0^2}{R^2}\int\limits_{A_{2R}^{(N)}}|u|^2dx+\int\limits_{A_{2R}^{(N)}}\left|\frac{\partial}{\partial x_N}\left(\frac{d_R(x)^{\frac{\alpha}{2}}u}{J_{0;R}(d_R(x))}\right)\right|^2\left|\frac{J_{0;R}(d_R(x))}{d_R(x)^{\frac{\alpha}{2}}}\right|^2dx.$$

For any $R>0$ and $k\in\mathbb{N}$, the pair $\left(r^{\alpha+1},\ \left(\frac{\alpha}{2}\right)^2r^{\alpha-1}+r^{\alpha+1}P_{k,R}(r)\right)$ is a Bessel pair on $(0,R)$ with $\varphi=r^{-\frac{\alpha}{2}}\left[X_1\left(\frac{r}{R}\right)\cdots X_k\left(\frac{r}{R}\right)\right]^{-\frac{1}{2}}$ where

$$P_{k,R}(r)=\frac{1}{4}\frac{1}{r^2}\sum_{j=1}^{k}X_1^2\left(\frac{r}{R}\right)\cdots X_j^2\left(\frac{r}{R}\right)$$
$$X_1(r)=\frac{1}{1-\ln r},\ X_i(r)=X_1(X_{i-1}(r)).$$

Hence, we deduce

Corollary 3.2 *For any $R>0$ and $k\in\mathbb{N}$, we have*

$$\int\limits_{A_{2R}^{(N)}}\left|\frac{\partial u}{\partial x_N}\right|^2 d_R(x)^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int\limits_{A_{2R}^{(N)}}|u|^2 d_R(x)^{\alpha-1}dx$$
$$=\frac{1}{4}\int\limits_{A_{2R}^{(N)}}|u|^2 d_R(x)^{\alpha-1}\sum_{j=1}^{k}X_1^2\left(\frac{d_R(x)}{R}\right)\cdots X_j^2\left(\frac{d_R(x)}{R}\right)dx$$
$$+\int\limits_{A_{2R}^{(N)}}\left|\frac{\partial}{\partial x_N}\left(d_R(x)^{\frac{\alpha}{2}}\left[X_1\left(\frac{d_R(x)}{R}\right)\cdots X_k\left(\frac{d_R(x)}{R}\right)\right]^{\frac{1}{2}}u\right)\right|^2$$
$$\times\left|\frac{1}{d_R(x)^{\frac{\alpha}{2}}\left[X_1\left(\frac{d_R(x)}{R}\right)\cdots X_k\left(\frac{d_R(x)}{R}\right)\right]^{\frac{1}{2}}}\right|^2dx$$
$$\geq\frac{1}{4}\int\limits_{A_{2R}^{(N)}}|u|^2 d_R(x)^{\alpha-1}\sum_{j=1}^{k}X_1^2\left(\frac{d_R(x)}{R}\right)\cdots X_j^2\left(\frac{d_R(x)}{R}\right)dx$$

and

$$\begin{aligned}
&\int_{A_{2R}^{(N)}} |\nabla u|^2 d_R(x)^{\alpha+1} dx - \left(\frac{\alpha}{2}\right)^2 \int_{A_{2R}^{(N)}} |u|^2 d_R(x)^{\alpha-1} dx \\
&= \frac{1}{4} \int_{A_{2R}^{(N)}} |u|^2 d_R(x)^{\alpha-1} \sum_{j=1}^{k} X_1^2\left(\frac{d_R(x)}{R}\right) \cdots X_j^2\left(\frac{d_R(x)}{R}\right) dx \\
&\quad + \int_{A_{2R}^{(N)}} \left|\nabla\left(d_R(x)^{\frac{\alpha}{2}}\left[X_1\left(\frac{d_R(x)}{R}\right)\cdots X_k\left(\frac{d_R(x)}{R}\right)\right]^{\frac{1}{2}} u\right)\right|^2 \\
&\quad \times \left|\frac{1}{d_R(x)^{\frac{\alpha}{2}}\left[X_1\left(\frac{d_R(x)}{R}\right)\cdots X_k\left(\frac{d_R(x)}{R}\right)\right]^{\frac{1}{2}}}\right|^2 dx \\
&\geq \frac{1}{4} \int_{A_{2R}^{(N)}} |u|^2 d_R(x)^{\alpha-1} \sum_{j=1}^{k} X_1^2\left(\frac{d_R(x)}{R}\right) \cdots X_j^2\left(\frac{d_R(x)}{R}\right) dx
\end{aligned}$$

Hence, the Hardy inequality with distance to the boundary of the strip $A_{2R}^{(N)}$ can be improved by adding an infinite number of terms.

4 Hardy identities and inequalities with distance to the hyperplane $\{x_N = 0\}$ on half-spaces

In this section, we will prove the Hardy type identities and inequalities with distance to the hyperplane $\{x_N = 0\}$, namely Theorem 1.6. We mention that Hardy-Sobolev-Maz'ya inequalities in half spaces have been studied, for example, in [5,8,43,50]. We will also give some applications of Theorem 1.6.

4.1 Proof of Theorem 1.6

Proof Let $T = \sqrt{V(x_N)}\nabla - \sqrt{V(x_N)}\frac{\varphi'(x_N)}{\varphi(x_N)}e_N$. Then its formal adjoint is $T^+ = -\operatorname{div}(\sqrt{V(x_N)}\cdot) - \sqrt{V(x_N)}\frac{\varphi'(x_N)}{\varphi(x_N)}e_N\cdot$.

Hence,

$$\begin{aligned}
T^+Tu &= -\operatorname{div}\left(\sqrt{V(x_N)}\left[\sqrt{V(x_N)}\nabla u - \sqrt{V(x_N)}\frac{\varphi'(x_N)}{\varphi(x_N)}ue_N\right]\right) \\
&\quad - \sqrt{V(x_N)}\frac{\varphi'(x_N)}{\varphi(x_N)}e_N \cdot \left[\sqrt{V(x_N)}\nabla u - \sqrt{V(x_N)}\frac{\varphi'(x_N)}{\varphi(x_N)}ue_N\right] \\
&= -\operatorname{div}\left(V(x_N)\left[\nabla u - \frac{\varphi'(x_N)}{\varphi(x_N)}ue_N\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&- V\left(x_N\right)\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\frac{\partial u}{\partial x_N} + V\left(x_N\right)\left(\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\right)^2 u \\
&= -V\left(x_N\right)\Delta u - V'\left(x_N\right)\frac{\partial u}{\partial x_N} + V'\left(x_N\right)\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u + V\left(x_N\right)\frac{\varphi''\left(x_N\right)}{\varphi\left(x_N\right)}u.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{A_R^{(N)}} \overline{u\left(x\right)}\left(T^+Tu\right)\left(x\right)dx \\
&= \int_{A_R^{(N)}} V\left(x_N\right)|\nabla u|^2\,dx + \int_{A_R^{(N)}} \frac{V'\left(x_N\right)\varphi'\left(x_N\right) + V\left(x_N\right)\varphi''\left(x_N\right)}{\varphi\left(x_N\right)}|u|^2\,dx \\
&= \int_{A_R^{(N)}} V\left(x_N\right)|\nabla u|^2\,dx - \int_{A_R^{(N)}} W\left(x_N\right)|u|^2\,dx.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{A_R^{(N)}} \overline{u\left(x\right)}\left(T^+Tu\right)\left(x\right)dx \\
&= \int_{A_R^{(N)}} |Tu|^2\,dx \\
&= \int_{A_R^{(N)}} V\left(x_N\right)\left|\nabla\left(\frac{u}{\varphi\left(x_N\right)}\right)\right|^2\varphi^2\left(x_N\right)dx.
\end{aligned}
$$

Hence, we deduce

$$
\begin{aligned}
&\int_{A_R^{(N)}} V\left(x_N\right)|\nabla u|^2 - \int_{A_R^{(N)}} W\left(x_N\right)|u|^2\,dx \\
&= \int_{A_R^{(N)}} V\left(x_N\right)\left|\nabla\left(\frac{u}{\varphi\left(x_N\right)}\right)\right|^2\varphi^2\left(x_N\right)dx.
\end{aligned}
$$

Now, let $Su = \sqrt{V\left(x_N\right)}\frac{\partial u}{\partial x_N} - \sqrt{V\left(x_N\right)}\frac{\varphi'(x_N)}{\varphi(x_N)}u$. Then its formal adjoint is $S^+v = -\frac{\partial}{\partial x_N}\left(\sqrt{V\left(x_N\right)}v\right) - \sqrt{V\left(x_N\right)}\frac{\varphi'(x_N)}{\varphi(x_N)}v$.

Therefore,

$$
\begin{aligned}
S^+Su &= -\frac{\partial}{\partial x_N}\left[\sqrt{V\left(x_N\right)}\left(\sqrt{V\left(x_N\right)}\frac{\partial u}{\partial x_N} - \sqrt{V\left(x_N\right)}\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u\right)\right] \\
&\quad - \sqrt{V\left(x_N\right)}\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\left(\sqrt{V\left(x_N\right)}\frac{\partial u}{\partial x_N} - \sqrt{V\left(x_N\right)}\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u\right) \\
&= -\frac{\partial}{\partial x_N}\left(V\left(x_N\right)\frac{\partial u}{\partial x_N} - V\left(x_N\right)\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u\right)
\end{aligned}
$$

$$
\begin{aligned}
&-V\left(x_N\right)\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\frac{\partial u}{\partial x_N}+V\left(x_N\right)\left(\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\right)^2 u\\
&=-V\left(x_N\right)\left(\frac{\partial^2 u}{\partial x_N^2}-\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\frac{\partial u}{\partial x_N}-\frac{\varphi''\left(x_N\right)\varphi\left(x_N\right)-\left(\varphi'\left(x_N\right)\right)^2}{\varphi^2\left(x_N\right)}u\right)\\
&-V'\left(x_N\right)\left(\frac{\partial u}{\partial x_N}-\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u\right)-V\left(x_N\right)\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\frac{\partial u}{\partial x_N}+V\left(x_N\right)\left(\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}\right)^2 u\\
&=-V\left(x_N\right)\left(\frac{\partial^2 u}{\partial x_N^2}-\frac{\varphi''\left(x_N\right)}{\varphi\left(x_N\right)}u\right)-V'\left(x_N\right)\left(\frac{\partial u}{\partial x_N}-\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{A_R^{(N)}}\overline{u\left(x\right)}\left(S^+Su\right)\left(x\right)dx\\
&=\int_{A_R^{(N)}}V\left(x_N\right)\left|\frac{\partial u}{\partial x_N}\right|^2 dx+\int_{A_R^{(N)}}\frac{V'\left(x_N\right)\varphi'\left(x_N\right)+V\left(x_N\right)\varphi''\left(x_N\right)}{\varphi\left(x_N\right)}|u|^2 dx.\\
&=\int_{A_R^{(N)}}V\left(x_N\right)|\nabla u|^2 dx-\int_{A_R^{(N)}}W\left(x_N\right)|u|^2 dx.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{A_R^{(N)}}\overline{u\left(x\right)}\left(S^+Su\right)\left(x\right)dx\\
&=\int_{A_R^{(N)}}|Su|^2 dx\\
&=\int_{A_R^{(N)}}V\left(x_N\right)\left|\frac{\partial u}{\partial x_N}-\frac{\varphi'\left(x_N\right)}{\varphi\left(x_N\right)}u\right|^2 dx\\
&=\int_{A_R^{(N)}}V\left(x_N\right)\left|\frac{\partial}{\partial x_N}\left(\frac{u}{\varphi\left(x_N\right)}\right)\right|^2\varphi^2\left(x_N\right)dx.
\end{aligned}
$$

Then we can get

$$
\begin{aligned}
&\int_{A_R^{(N)}}V\left(x_N\right)\left|\frac{\partial u}{\partial x_N}\right|^2 dx-\int_{A_R^{(N)}}W\left(x_N\right)|u|^2 dx\\
&=\int_{A_R^{(N)}}V\left(x_N\right)\left|\frac{\partial}{\partial x_N}\left(\frac{u}{\varphi\left(x_N\right)}\right)\right|^2\varphi^2\left(x_N\right)dx.
\end{aligned}
$$

□

4.2 Applications of Theorem 1.6

For $R>0$, we note that $\left(r^{\alpha+1},\left(\frac{\alpha}{2}\right)^2 r^{\alpha-1}+\frac{z_0^2}{R^2}r^{\alpha+1}\right)$ is a Bessel pair on $(0,R)$ with $\varphi=r^{-\frac{\alpha}{2}}J_0\left(\frac{rz_0}{R}\right)=r^{-\frac{\alpha}{2}}J_{0;R}(r)$. Hence we get the following Hardy type inequality in the spirit of Brezis-Vázquez and Brezis-Marcus:

Corollary 4.1 *For any $R>0$, we have for $u\in C_0^\infty\left(A_R^{(N)}\right)$:*

$$\int_{A_R^{(N)}}|\nabla u|^2 x_N^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int_{A_R^{(N)}}|u|^2 x_N^{\alpha-1}dx$$

$$=\frac{z_0^2}{R^2}\int_{A_R^{(N)}}|u|^2 x_N^{\alpha+1}dx+\int_{A_R^{(N)}}\left|\nabla\left(\frac{x_N^{\frac{\alpha}{2}}u}{J_{0;R}(x_N)}\right)\right|^2\left|\frac{J_{0;R}(x_N)}{x_N^{\frac{\alpha}{2}}}\right|^2dx$$

and

$$\int_{A_R^{(N)}}\left|\frac{\partial u}{\partial x_N}\right|^2 x_N^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int_{A_R^{(N)}}|u|^2 x_N^{\alpha-1}dx$$

$$=\frac{z_0^2}{R^2}\int_{A_R^{(N)}}|u|^2 x_N^{\alpha+1}dx+\int_{A_R^{(N)}}\left|\frac{\partial}{\partial x_N}\left(\frac{x_N^{\frac{\alpha}{2}}u}{J_{0;R}(x_N)}\right)\right|\left|\frac{J_{0;R}(x_N)}{x_N^{\frac{\alpha}{2}}}\right|^2dx.$$

For any $R>0$ and $k\in\mathbb{N}$, the pair $\left(r^{\alpha+1},\left(\frac{\alpha}{2}\right)^2 r^{\alpha-1}+r^{\alpha+1}P_{k,R}(r)\right)$ is a Bessel pair on $(0,R)$ with $\varphi=r^{-\frac{\alpha}{2}}\left[X_1\left(\frac{r}{R}\right)\cdots X_k\left(\frac{r}{R}\right)\right]^{-\frac{1}{2}}$ where

$$P_{k,R}(r)=\frac{1}{4}\frac{1}{r^2}\sum_{j=1}^{k}X_1^2\left(\frac{r}{R}\right)\cdots X_j^2\left(\frac{r}{R}\right)$$

$$X_1(r)=\frac{1}{1-\ln r},\ X_i(r)=X_1\left(X_{i-1}(r)\right).$$

Hence, we deduce

Corollary 4.2 *For any $R>0$ and $k\in\mathbb{N}$, we have*

$$\int_{A_R^{(N)}}|\nabla u|^2 x_N^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int_{A_R^{(N)}}|u|^2 x_N^{\alpha-1}dx$$

$$=\frac{1}{4}\int_{A_R^{(N)}}|u|^2 x_N^{\alpha-1}\sum_{j=1}^{k}X_1^2\left(\frac{x_N}{R}\right)\cdots X_j^2\left(\frac{x_N}{R}\right)dx$$

$$+\int_{A_R^{(N)}} \left|\nabla\left(x_N^{\frac{\alpha}{2}}\left[X_1\left(\frac{x_N}{R}\right)\cdots X_k\left(\frac{x_N}{R}\right)\right]^{\frac{1}{2}}u\right)\right|^2 \left|\frac{1}{x_N^{\frac{\alpha}{2}}\left[X_1\left(\frac{x_N}{R}\right)\cdots X_k\left(\frac{x_N}{R}\right)\right]^{\frac{1}{2}}}\right|^2 dx$$

$$\geq \frac{1}{4}\int_{A_R^{(N)}} |u|^2 x_N^{\alpha-1}\sum_{j=1}^{k} X_1^2\left(\frac{x_N}{R}\right)\cdots X_j^2\left(\frac{x_N}{R}\right)dx$$

and

$$\int_{A_R^{(N)}} \left|\frac{\partial u}{\partial x_N}\right|^2 x_N^{\alpha+1}dx-\left(\frac{\alpha}{2}\right)^2\int_{A_R^{(N)}} |u|^2 x_N^{\alpha-1}dx$$

$$=\frac{1}{4}\int_{A_R^{(N)}} |u|^2 x_N^{\alpha-1}\sum_{j=1}^{k} X_1^2\left(\frac{x_N}{R}\right)\cdots X_j^2\left(\frac{x_N}{R}\right)dx$$

$$+\int_{A_R^{(N)}} \left|\frac{\partial}{\partial x_N}\left(x_N^{\frac{\alpha}{2}}\left[X_1\left(\frac{x_N}{R}\right)\cdots X_k\left(\frac{x_N}{R}\right)\right]^{\frac{1}{2}}u\right)\right|^2 \left|\frac{1}{x_N^{\frac{\alpha}{2}}\left[X_1\left(\frac{x_N}{R}\right)\cdots X_k\left(\frac{x_N}{R}\right)\right]^{\frac{1}{2}}}\right|^2 dx$$

$$\geq \frac{1}{4}\int_{A_R^{(N)}} |u|^2 x_N^{\alpha-1}\sum_{j=1}^{k} X_1^2\left(\frac{x_N}{R}\right)\cdots X_j^2\left(\frac{x_N}{R}\right)dx.$$

Hence, the Hardy inequality with distance to the hyperplane $\{x_N = 0\}$ can be improved by adding an infinite number of terms.

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