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Existence and nonexistence of extremal functions for sharp Trudinger-Moser inequalities ☆

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ABSTRACT

Our main purpose in this paper is to establish the existence and nonexistence of extremal functions (also known as maximizers) and symmetry of extremals for several Trudinger-Moser type inequalities on the entire space $\mathbb{R}^n$, including both the critical and subcritical Trudinger-Moser inequalities (see Theorems 1.1, 1.2, 1.3, 1.4, 1.5). Most of earlier works on existence of maximizers in the literature rely on the complicated blow-up analysis of PDEs for the associated Euler-Lagrange equations of the corresponding Moser functionals. The new approaches developed in this paper are using the identities and relationship between the supremums of the subcritical Trudinger-Moser inequalities and the critical ones established by the same authors in [25], combining with the continuity of the supremum function that is observed for the first time in the literature. These allow us to establish the existence and nonexistence of the maximizers for the Trudinger-Moser inequalities in different ranges of the parameters (including those inequalities with the exact growth). This method is considerably simpler and also allows us to study the symmetry problem of the extremal functions and prove that the ex-

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tremal functions for the subcritical singular Truddinger-Moser inequalities are symmetric. Moreover, we will be able to calculate the exact values of the supremums of the Trudinger-Moser type in certain cases. These appear to be the first results in this direction.

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1. Introduction

It is well-known that the Trudinger-Moser type inequalities can be considered as the borderline case of the Sobolev embeddings. Indeed, since in this case $W^{1,N}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right),\ \forall q \in [N,\infty)$, but $W^{1,N}\left(\mathbb{R}^N\right) \not\subseteq L^\infty\left(\mathbb{R}^N\right)$, it was shown by Trudinger [45] (see also Yudovich [48] and Pohozaev [43]) that $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain with finite measure, $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp\left(\alpha |t|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$. More precisely, we have the following

Theorem A *(Trudinger). Let Ω be a domain with finite measure in Euclidean $N-$space $\mathbb{R}^N$, $N \geq 2$. Then there exist constants $\alpha > 0$ and $c_0 > 0$, such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq c_0$$

for any $u \in W_0^{1,N}(\Omega)$ with $\|\nabla u\|_N \leq 1$.

This result was sharpened in the 1971 paper [42] by J. Moser. In fact, Moser showed, using symmetrization method, that the best possible constant α in the above theorem is $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the area of the surface of the unit $N-$ball in the sense that if $\alpha > \alpha_N$, then the above inequality can no longer hold with some c_0 independent of u. It can be easily calculated that $\alpha_N = N\left(\frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\right)^{\frac{1}{N-1}}$. This inequality is by now known as the Trudinger-Moser inequality. The affine version of the Trudinger-Moser inequality on finite domains was established by Cianchi, Lutwak, Yang and Zhang using the affine norm of the gradient (see [8]). In another direction, we also note the Moser-Onofri inequality on the high dimensional sphere $\mathbb{S}^N$ established by Beckner [3] and Trudinger-Moser inequality on surfaces with conical singularity (see Chen [5]).

The Trudinger-Moser inequalities on domains with infinite measure are taken the following forms.

Theorem B. *Let $0 \leq \alpha < \alpha_N$. There hold*

$$\sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^N} \int_{\mathbb{R}^N} \phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) dx < \infty \tag{1.1}$$

$$\sup_{\|\nabla u\|_N^N + \|u\|_N^N \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha_N |u|^{\frac{N}{N-1}}\right) dx < \infty, \tag{1.2}$$

where

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}$$

The constant α_N is sharp in the sense that if $\alpha > \alpha_N$ then the supremum in (1.1) is infinite and if $\alpha > \alpha_N$ then the supremum in (1.2) is infinite.

The first inequality (1.1) above is called the subcritical Trudinger-Moser inequality in $\mathbb{R}^N$ and was established by Adachi-Tanaka [1] and do O [10], and the second one above is called the critical Trudinger-Moser inequality in $\mathbb{R}^N$ and was proved by Y. X. Li and Ruf [31]. We note that the Trudinger-Moser type inequality can only be established for the subcritical case when only the seminorm $\|\nabla u\|_N$ is used in the restriction of function class. Thus the inequality holds only for $\alpha < \alpha_N$ and it is sharp. To achieve the critical case of the inequality (1.2) for $\alpha = \alpha_N$, one needs to use the full norm in $W^{1,N}$, namely, $\left(\|u\|_N^N + \|\nabla u\|_N^N\right)^{\frac{1}{N}}$. Again α_N is sharp. In all the proofs of the above inequalities (1.1) and (1.2) in the literature, the classic Pólya-Szegö inequality and the Schwartz symmetrization argument is very crucial. An alternative proof of (1.2) without using symmetrization has been given by the first two authors in [20] (see Section 6 there). Different proofs of (1.1) and (1.2) have also been given without using symmetrization in settings such as on the Heisenberg group where symmetrization is not available (see the work by the first two authors [19], and by Tang and the first two authors [22] and by Li, Lu and Zhu [27] which extend the earlier work by Cohn and Lu [9] on finite domains.) The arguments used in these works [19,20,22] are from local inequalities to global ones using the level sets of functions under consideration. Such an argument has also been used in [6,23,49].

In the recent paper [24], it was proved that these two versions of the above sharp subcritical and critical Trudinger-Moser type inequalities are indeed equivalent. Indeed, an identity of these supremums was also given. Moreover, it was also showed in [24] that for $a, b > 0$:

$$\sup_{\|\nabla u\|_N^a + \|u\|_N^b \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha_N |u|^{\frac{N}{N-1}}\right) dx < \infty \text{ if and only if } b \leq N.$$

Concerning the existence of extremal functions for Trudinger-Moser inequalities on bounded domains, the first breakthrough was due to Carleson and Chang [4] using sym-

metrization method. Namely, they proved that there exists a function $u \in W_0^{1,N}(\Omega)$ with $\|\nabla u\|_N \leq 1$ such that the following supremum is achieved:

$$\sup_{u \in W_0^{1,N}(\Omega), \|\nabla u\|_N \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) dx.$$

These are surprising results since it is well-known that in the case of the Sobolev embeeding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}$, for $1 \leq p < N$ and $\Omega \subset \mathbb{R}^n$, the best constant

$$S_N^{p^*} = \sup_{\|\nabla u\|_p = 1} \int_\Omega |u|^{p^*} dx$$

is not attained unless $\Omega = \mathbb{R}^N$. The result by Carleson and Chang [4] was subsequently extended to other cases, see Flucher [13] and Lin [33] on domains on finite measure, and on Riemannian manifolds by Y. X. Li [28,29]. Malchiodi and Martinazzi [40] further investigated the blow-up of a sequence of critical points of the Trudinger-Moser functionals on the planar disk. Hardy-Trudinger-Moser inequalities and existence of extremals on planar domains have been considered in Wang and Ye [47], Lu and Q. Yang [35] and Wang [46]. On domains of infinite volume, existence of extremal functions for Trudinger-Moser inequalities were studied by Ruf [44], Li-Ruf [31], Ishiwata [15], Dong, Lam and Lu [12], Dong and Lu [11], Lam [18,16,17], Lu and Zhu [39], etc. As far as the existence of extremals for Adams inequalities established by Adams [2] (see also recent improvement to Hardy-Adams inequalities in [26,36] which are the borderline cases of the higher order Hardy-Sobolev-Maz'ya inequalities on upper half spaces [37]), for the second order derivatives on compact Riemannian manifolds and bounded domains in dimension four, existence of extremal functions have been established by Y. X. Li and Ndiaye [30] and Lu and Y. Yang [38] respectively. In particular, the methods used in the above works [28,29,31,39,30,38] rely on the complicated blow-up analysis of the PDEs for the Euler-Lagrange equations associated with the Moser functionals. Existence of extremal functions for weighted Adams inequalities on the entire space have also been recently established by Chen, Lu and Zhang [7].

In the recent paper [25], the authors studied several sharp versions of the Trudinger-Moser type inequalities under different norms for functions in $D^{N,q}(\mathbb{R}^N)$, $q \geq 1$, the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\nabla u\|_N + \|u\|_q$. Throughout the paper, we will use the following notation: $TMSC$ represents the supremum for a type of subcritical Trudinger-Moser inequality on $\mathbb{R}^N$. The subscripts are used to distinguish the different supremums. For instance, $TMSC_{p,q,N,\alpha,\beta}$, $TMSC_{q,N,\alpha,\beta}$, $TMSC_{N,\alpha,\beta}$ denote the corresponding supremums for the subcritical Trudinger-Moser inequalities under different norms. We will also use TMC to denote the supremum of the critical Trudinger-Moser inequalities and the subscripts to distinguish them under different norms. Similarly, we will use TME to express the supremum of the Trudinger-Moser inequality with the exact growth and use the subscripts to reflect the corresponding norms.

From now on, we assume that $N \geq 2$, $\alpha_N = N\left(\frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\right)^{\frac{1}{N-1}}$, $0 \leq \beta < N$, $0 \leq \alpha < \alpha_N$ and $q \geq 1$.

Then we have established the following results in [25].

Theorem C *(Sharp subcritical Trudinger-Moser inequality). Let $p > q\left(1-\frac{\beta}{N}\right)$ ($p \geq q$ if $\beta = 0$). The following supremums are finite:*

$$TMSC_{p,q,N,\alpha,\beta} = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta};$$

$$TMSC_{q,N,\alpha,\beta} = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta};$$

$$TMSC_{N,\alpha,\beta} = \sup_{\|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^{N-\beta}} \int\limits_{\mathbb{R}^N} \Phi_{N,N,0}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta}.$$

Moreover, this constant α_N is sharp. Here

$$\Phi_{N,q,\beta}(t) = \begin{cases} \sum\limits_{j\in\mathbb{N},\ j>\frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)} \frac{t^j}{j!} & \text{if } \beta > 0. \\ \sum\limits_{j\in\mathbb{N},\ j\geq\frac{q(N-1)}{N}} \frac{t^j}{j!} & \text{if } \beta = 0. \end{cases}.$$

Theorem D *(Sharp critical Trudinger-Moser inequality). Let $a, b > 0$. Then*

$$TMC_{a,b}(q,N,\beta) = \sup_{\|\nabla u\|_N^a + \|u\|_q^b \leq 1} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx < \infty$$

if and only if $b \leq N$. The constant α_N is sharp.

Next, we extend the results of [14] (see also [21,25,34,41]) to the singular cases.

Theorem E *(Sharp critical Trudinger-Moser inequality with exact growth). Let $p \geq q \geq 1$. Then there exists a constant $C = C(N,p,q,\beta) > 0$ such that for all $u \in D^{N,q}(\mathbb{R}^N)$: $\|\nabla u\|_N \leq 1$, there holds*

$$\int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha_N\left(1-\frac{\beta}{N}\right)u^{\frac{N}{N-1}}\right)}{\left(1+|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx \leq TME_{p,q,N,\beta}\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}. \tag{1.3}$$

Moreover, the inequality does not hold when $p < q$. We have used on the right hand side the constant $TME_{p,q,N,\beta}$ to express the best constant.

Remark 1. When $\beta = 0$, we note that the inequality (1.3) still holds when we replace $\Phi_{N,q,0}$ by a function Φ such that there exists $C_{N,q} > 0$:

$$\Phi\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \leq C_{N,q} \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \ \forall u;$$
$$\Phi\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \leq C_{N,q} |u|^q \text{ for every } |u| \leq 1.$$

Our main purpose in this paper is to study the existence and nonexistence of extremal functions for several Trudinger-Moser type inequalities. Our first aim is to prove the existence of maximizers of the above subcritical Trudinger-Moser inequalities. More precisely, we will prove

Theorem 1.1. *Let $0 \leq \beta < N$, $0 \leq \alpha < \alpha_N$, $q > 1$ and $p > q\left(1 - \frac{\beta}{N}\right)$ ($p \geq q$ if $\beta = 0$). Then the supremums of the subcritical Trudinger-Moser inequalities $TMSC_{p,q,N,\alpha,\beta}$, $TMSC_{N,\alpha,\beta}$, $TMSC_{q,N,\alpha,\beta}$ are attained. Moreover, all the maximizers u satisfy $\|\nabla u\|_N = \|\nabla u^{\#}\|_N = 1$, and can be rescaled such that $\|\nabla u\|_N = \|\nabla u^{\#}\|_N = \|u\|_q = 1$.*

Moreover, in the singular case, we will provide more information on the maximizers. Indeed, we also show that the maximizers are radially symmetric.

Theorem 1.2. *Let $0 < \beta < N$, $0 < \alpha < \alpha_N$, $q > 1$ and $p > q\left(1 - \frac{\beta}{N}\right)$. Assume that u is a maximizer of the supremum $TMSC_{p,q,N,\alpha,\beta}$ (or the supremum $TMSC_{N,\alpha,\beta}$ or $TMSC_{q,N,\alpha,\beta}$). Then $|u| = (|u|)^{\#}$. Namely, the absolute values of the maximizers are radially symmetric.*

Remark 2. This appears to be the first result about the radial symmetry of the maximizers for the Trudinger-Moser type inequalities.

Our next main aim is to study the existence of extremals for $TMC_{a,b}(q, N, \beta)$ in the subcritical cases. We will propose here a new approach to study this problem. Our method makes use of the relation between the supremums $TMSC_{q,N,\alpha,\beta}$ and $TMC_{a,b}(q, N, \beta)$, combining with the continuity of the supremum function $TMSC_{q,N,\alpha,\beta}$ that is observed for the first time in the literature. More precisely, the following identity has been established in [25, Theorem 1.2]:

$$TMC_{a,b}(N, N, \beta) = \sup_{s \in (0, \alpha_N)} \left(\frac{1 - \left(\frac{s}{\alpha_N}\right)^{\frac{N-1}{N} a}}{\left(\frac{s}{\alpha_N}\right)^{\frac{N-1}{N} b}} \right)^{\frac{N-\beta}{b}} TMSC_{N,N,s,\beta}.$$

Using the same approach, we could prove that for $0 < \alpha \leq \alpha_N$:

$$TMC_{a,b,q,N,\beta}(\alpha) = \sup_{s\in(0,\alpha)} \left[\frac{1 - \left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}} \right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} TMSC_{q,N,\alpha,\beta}.$$

Hence, if the vanishing phenomenon $\lim_{s\downarrow 0}$ and the concentration level $\lim_{s\uparrow\alpha}$ cannot reach $TMC_{a,b,q,N,\beta}(\alpha)$, then the supremum becomes the maximum and is achieved at some $s \in (0,\alpha)$. Then, by using the existence of maximizers for $TMSC_{q,N,\alpha,\beta}$, we will show the existence of extremal functions for the supremum $TMC_{a,b,q,N,\beta}(\alpha)$. One of the advantages of our new method is that we can give the characterizations of the attainability and inattainability for the supremum $TMC_{a,b}(q,N,\beta)$ in the subcritical cases.

Theorem 1.3. *Let $0 \le \beta < N$, $a, b > 0$ and $q > 1$. Define the supremum function*

$$\begin{aligned} TMC_{a,b,q,N,\beta}(\alpha) &= \sup_{\|\nabla u\|_N^a + \|u\|_q^b \le 1} \int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx \\ &= \sup_{\|\nabla u\|_N^a + \|u\|_q^b = 1} \int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx. \end{aligned}$$

Then we conclude that

1) If $\beta > 0$, then the supremum $TMC_{a,b,q,N,\beta}(\alpha)$ can be attained for all $(a,b,\alpha) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,\alpha_N)$. Moreover, the absolute value of these maximizers are symmetric.

2) If $\beta = 0$ and $\frac{q(N-1)}{N} \notin \mathbb{N}$, then the supremum $TMC_{a,b,q,N,0}(\alpha)$ can be attained for all $(a,b,\alpha) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,\alpha_N)$.

3) If $\beta = 0$ and $\frac{q(N-1)}{N} \in \mathbb{N}$, then the supremum $TMC_{a,b,q,N,0}(\alpha)$ is NOT attained only if

$$TMC_{a,b,q,N,0}(\alpha) = \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

We also give the following characterizations for the attainability and inattainability of $TMC_{a,b,q,N,0}(\alpha)$ when $\frac{q(N-1)}{N} \in \mathbb{N}$.

Remark 3. Set

$$\begin{aligned} A_{a,b} &= \{\alpha \in (0,\alpha_N) : TMC_{a,b,q,N,0}(\alpha) \text{ is attained}\} \\ B_{a,b} &= \{\alpha \in (0,\alpha_N) : TMC_{a,b,q,N,0}(\alpha) \text{ is NOT attained}\} \\ X_{a,b} &= \left\{\alpha \in (0,\alpha_N) : TMC_{a,b,q,N,0}(\alpha) > \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\right\} \end{aligned}$$

$$Y_{a,b} = \left\{ \alpha \in (0, \alpha_N) : TMC_{a,b,q,N,0}(\alpha) = \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!} \right\}$$

Then

$$X_{a,b} \subset A_{a,b};\ B_{a,b} \subset Y_{a,b}.$$
$$X_{a,b} \cup Y_{a,b} = A_{a,b} \cup B_{a,b} = (0, \alpha_N).$$

Moreover,

$$X_{a,b} = A_{a,b} = (0, \alpha_N);\ B_{a,b} = Y_{a,b} = \emptyset$$

or

$$X_{a,b} = A_{a,b} = \emptyset;\ B_{a,b} = Y_{a,b} = (0, \alpha_N)$$

or there exists $\alpha_{a,b} \in (0, \alpha_N)$ such that

$$X_{a,b} = (\alpha_{a,b}, \alpha_N) \text{ and } Y_{a,b} = (0, \alpha_{a,b}].$$

In this case

$$A_{a,b} = (\alpha_{a,b}, \alpha_N) \text{ and } B_{a,b} = (0, \alpha_{a,b}]$$

or

$$A_{a,b} = [\alpha_{a,b}, \alpha_N) \text{ and } B_{a,b} = (0, \alpha_{a,b}).$$

Remark 4. It is very interesting to note that, by a result of Ishiwata [15], $TMC_{2,2,2,2,0}(\alpha)$ is not attained when α is small. Hence, by our Theorem 1.3,

$$TMC_{2,2,2,2,0}(\alpha) = \sup_{\|\nabla u\|_2^2 + \|u\|_2^2 \le 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 \right) dx = \alpha \text{ when } \alpha \text{ is small enough.}$$

This is the first result that can exhibit the supremum of the Trudinger-Moser type inequality.

Our next theorem will give some conditions for a triple $(a, b, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0, \alpha_N)$ such that $\alpha \in X_{a,b}$

Theorem 1.4. *1) If* $a > \frac{N}{N-1}$, *then* $X_{a,b} = A_{a,b} = (0, \alpha_N)$.

2) If $a = \frac{N}{N-1}$ *and if there exists a* $v \in D^{N,q}\left(\mathbb{R}^N\right) : \|v\|_q = 1$ *and*

$$\alpha_N > \alpha > q\left(\frac{q\,(N-1)}{N} + 1\right)\frac{a}{b}\frac{(N-1)}{N}\frac{\|\nabla v\|_N^a}{\|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}},$$

then $\alpha \in X_{a,b}$ *(hence* $TMC_{a,b,q,N,0}\,(\alpha)$ *can be achieved).*

3) When $a \leq \frac{N}{N-1}$*, then* $TMC_{a,b,q,N,0}\,(\alpha)$ *cannot be achieved when* α *is small enough. In this case*

$$TMC_{a,b,q,N,0}\,(\alpha) = \sup_{\|\nabla u\|_N^a + \|u\|_q^b \leq 1} \int_{\mathbb{R}^N} \Phi_{N,q}\left(\alpha\,|u|^{\frac{N}{N-1}}\right) dx = \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

Our last goal in this paper is to study the existence of maximizers of the Trudinger-Moser inequality with exact growth in the subcritical case $p > q$. We will prove

Theorem 1.5. *Let* $\lambda > 0,\ 0 \leq \beta < N,\ q > 1,\ 0 < \alpha \leq \alpha_N$ *and* $p > 0$ *(*$p > q$ *when* $\alpha = \alpha_N$*). Denote*

$$TME_{p,q,N,\lambda,\alpha,\beta} = \sup_{u \in D^{N,q}(\mathbb{R}^N):\ \|\nabla u\|_N \leq 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx$$

$$= \sup_{u \in D^{N,q}(\mathbb{R}^N):\ \|\nabla u\|_N \leq 1; \|u\|_q = 1} \int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u^{\frac{N}{N-1}}\right)}{\left(1+\lambda\,|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}} dx$$

Then $TME_{p,q,N,\lambda,\alpha,\beta}$ *can be attained in any of the following cases*

(a) $\beta > 0$ *and all* $0 < \alpha \leq \alpha_N$*,*

(b) $\beta = 0$*,* $\frac{q(N-1)}{N} \notin \mathbb{N}$ *and all* $0 < \alpha \leq \alpha_N$*,*

(c) $\beta = 0$*,* $\frac{q(N-1)}{N} \in \mathbb{N}$*,* $p = N$*,* $\lambda \leq \frac{\alpha}{\frac{q(N-1)}{N}+1}$ *and all* $0 < \alpha \leq \alpha_N$*,*

(d) $\beta = 0$*,* $\frac{q(N-1)}{N} \in \mathbb{N}$*,* $p > N$ *and all* $0 < \alpha \leq \alpha_N$*,*

(d) $\beta = 0$*,* $\frac{q(N-1)}{N} \in \mathbb{N}$*,* $p \leq N$*,* $q < p < \frac{N-1}{N-2}q$ *and* $\alpha = \alpha_N$*.*

Moreover, if $TME_{p,q,N,\lambda,\alpha,\beta}$ *cannot be attained, then* $\beta = 0$*,* $\frac{q(N-1)}{N} \in \mathbb{N}$ *and*

$$TME_{p,q,N,\lambda,\alpha,0} = \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

We remark that our main results have been announced in [11] (see Theorems C, D. E there). We note that techniques and ideas discovered and developed in this current paper have been used in [18,16,17]. The organization of the paper is as follows. In Section 2, we will do some preparations and prove some necessary lemmas which are needed to

prove our main theorems. Section 3 will provide the proofs of the maximizers of the subcritical Trudinger-Moser inequalities. Namely, we will prove Theorems 1.1 and 1.2. In Section 4, we give the proof of the existence of maximizers of the critical Trudinger-Moser inequalities and prove Theorems 1.3 and 1.4. Section 5 gives the proof of Theorem 1.5.

2. Some useful and preliminary results

In this section, we introduce some useful results that will be used in our proofs. We first recall the definition of rearrangement and some useful inequalities. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a measurable set. We denote by $\Omega^\#$ the open ball $B_R \subset \mathbb{R}^N$ centered at 0 of radius $R > 0$ such that $|B_R| = |\Omega|$.

Let $u : \Omega \to \mathbb{R}$ be a real-valued measurable function that vanishes at infinity, that is $|\{x : |u(x)| > t\}|$ is finite for all $t > 0$. The distribution function of u is the function

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|$$

and the decreasing rearrangement of u is the right-continuous, non-increasing function u^* that is equi-measurable with u:

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\}.$$

It is clear that $suppu^* \subseteq [0, |\Omega|]$. We also define

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(t)dt \geq u^*(s).$$

Moreover, we define the spherically symmetric decreasing rearrangement of u:

$$u^\# : \Omega^\# \to [0, \infty]$$
$$u^\#(x) = u^*\left(\sigma_N |x|^N\right).$$

Then we have the following important result that can be found in [32]:

Lemma 2.1 *(Pólya-Szegö inequality). Let $u \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$. Then $f^\# \in W^{1,p}(\mathbb{R}^n)$ and*

$$\left\|\nabla f^\#\right\|_p \leq \|\nabla f\|_p.$$

Lemma 2.2. *Let f and g be nonnegative functions on $\mathbb{R}^N$, vanishing at infinity. Then*

$$\int_{\mathbb{R}^N} f(x)\, g(x)\, dx \leq \int_{\mathbb{R}^N} f^\#(x)\, g^\#(x)\, dx$$

in the sense that when the left side is infinite so is the right. Moreover, if f is strictly symmetric-decreasing, then there is equality if and only if $g = g^{\#}$.

We will next prove a lemma that will be used several times in our paper.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$, $|\Omega| < \infty$. Suppose that*

$$f_n \to f \text{ a.e. in } \Omega$$

and there exists $q > 1$ such that f_n is uniformly bounded in $L^q(\Omega)$ and $f \in L^q(\Omega)$. Then

$$f_n \to f \text{ in } L^1(\Omega).$$

Proof. For arbitrary $\varepsilon > 0$, by Egorov's theorem, we can find a measurable $D \subset \Omega$ such that

$$\begin{aligned} &f_n \to f \text{ uniformly in } D, \\ &|\Omega \setminus D| < \varepsilon. \end{aligned}$$

Thus

$$\int\limits_D |f_n - f|\, dx \to 0.$$

Also, by Holder's inequality

$$\begin{aligned} \int\limits_{\Omega\setminus D} |f_n - f|\, dx &\le \left(\int\limits_{\Omega\setminus D} |f_n - f|^q\, dx \right)^{1/q} \left(\int\limits_{\Omega\setminus D} 1^{q^*} dx \right)^{1/q^*} \\ &\le C\varepsilon^{1/q^*}. \end{aligned}$$

Hence $f_n \to f$ in $L^1(\Omega)$. □

Next, we will show that the supremums of both subcritical and critical Trudinger-Moser inequalities can be normalized.

Lemma 2.4.

$$TMSC_{p,q,N,\alpha,\beta} = \sup_{\|\nabla u\|_N = \|u\|_q = 1} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta};$$

$$TMSC_{N,\alpha,\beta} = \sup_{\|\nabla u\|_N = \|u\|_N = 1} \int\limits_{\mathbb{R}^N} \Phi_{N,N,0}\left(\alpha\left(1 - \frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta};$$

$$TMSC_{q,N,\alpha,\beta} = \sup_{\|\nabla u\|_N = \|u\|_q = 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx;$$

$$TMC_{a,b}(q,N,\beta) = \sup_{\|\nabla u\|_N^a + \|u\|_q^b = 1} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx$$

Proof. *We will just show the following identity:*

$$TMSC_{p,q,N,\alpha,\beta} = \sup_{\|\nabla u\|_N = \|u\|_q = 1} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta};$$

The other identities can be showed similarly.

First, for any $u \in D^{N,q}(\mathbb{R}^N) : \|\nabla u\|_N \le 1$, *we set*

$$v(x) = \frac{u(\lambda x)}{\|\nabla u\|_N}; \ \lambda = \left(\frac{\|u\|_q}{\|\nabla u\|_N}\right)^{\frac{q}{N}}.$$

Then, it is easy to verify that

$$\|\nabla v\|_N = 1;$$
$$\|v\|_q = \frac{1}{\|\nabla u\|_N} \frac{1}{\lambda^{N/q}} \|u\|_q = 1$$

and

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-1}}\right)|v|^p \frac{dx}{|x|^\beta} \\
&= \frac{1}{\|\nabla u\|_N^p} \int\limits_{\mathbb{R}^N} \exp\left(\frac{\alpha\left(1-\frac{\beta}{N}\right)}{\|\nabla u\|_N^{\frac{N}{N-1}}}|u(\lambda x)|^{\frac{N}{N-1}}\right)|u(\lambda x)|^p \frac{dx}{|x|^\beta} \\
&= \frac{1}{\|\nabla u\|_N^p} \frac{1}{\lambda^{N-\beta}} \int\limits_{\mathbb{R}^N} \exp\left(\frac{\alpha\left(1-\frac{\beta}{N}\right)}{\|\nabla u\|_N^{\frac{N}{N-1}}}|u(x)|^{\frac{N}{N-1}}\right)|u(x)|^p \frac{dx}{|x|^\beta} \\
&= \frac{1}{\|\nabla u\|_N^{p-q\left(1-\frac{\beta}{N}\right)}} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\frac{\alpha\left(1-\frac{\beta}{N}\right)}{\|\nabla u\|_N^{\frac{N}{N-1}}}|u(x)|^{\frac{N}{N-1}}\right)|u(x)|^p \frac{dx}{|x|^\beta} \\
&\ge \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u(x)|^{\frac{N}{N-1}}\right)|u(x)|^p \frac{dx}{|x|^\beta}. \qquad \square
\end{aligned}$$

3. The existence of maximizers for the subcritical Trudinger-Moser supremums

3.1. The existence of maximizers for $TME_{p,q,N,\beta}$

We will prove in this subsection that the subcritical Trudinger-Moser inequalities have extremal functions for the supremums $TME_{p,q,N,\beta}$. Namely, we will prove Theorem 1.1.

Proof of Theorem 1.1. Let (w_n) be a maximizing sequence of

$$TMSC_{p,q,N,\alpha,\beta} = \sup_{\|\nabla u\|_N \le 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta}.$$

By symmetrization argument, the Hardy-Littlewood inequality and Lemma 2.4, we may assume further that w_n is radial and

$$\int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p \frac{dx}{|x|^\beta} \to TMSC_{p,q,N,\alpha,\beta},$$

$$\|\nabla w_n\|_N = \|w_n\|_q = 1.$$

Without loss of generality, we assume that

$$w_n \rightharpoonup w \text{ weakly in } D_{rad}^{1,N}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) \text{ as } n \to \infty.$$

It's clear that

$$\|\nabla w\|_N \le 1; \quad \|w\|_q \le 1. \tag{3.1}$$

Case $\beta = 0,\ p > q$

For arbitrary $\varepsilon > 0$, by Radial Lemma, we could choose $R > 0$ sufficiently large such that $|w_n(x)| < \varepsilon$ when $|x| \ge R$. Then, by Theorem A, we have

$$\int\limits_{|x|\ge R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,dx \le \varepsilon^{p-q} \int\limits_{|x|\ge R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^q\,dx$$

$$\le C(N,q,\alpha,\beta)\,\varepsilon^{p-q}.$$

Also, we have

$$\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,dx \to \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx \text{ a.e. in } B(0,R).$$

Next, noting that there exists $\delta > 0$ sufficiently small such that by Theorem A, we have $\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,dx$ is uniformly bounded in $L^{1+\delta}(B(0,R))$. Hence, by Lemma 2.3, we get

$$\int\limits_{|x|<R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,dx \to \int\limits_{|x|<R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx$$

Thus,

$$\begin{aligned} TMSC_{p,q,N,\alpha,\beta} &\longleftarrow \frac{1}{\|w_n\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,dx \\ &= \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,dx \\ &\leq \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx + C\left(N,q\right)\varepsilon^{p-q}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$TMSC_{p,q,N,\alpha,\beta} \leq \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx.$$

As a consequence, $w \neq 0$. Also, by (3.1):

$$\begin{aligned} TMSC_{p,q,N,\alpha,\beta} &\leq \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx \\ &\leq \frac{1}{\|w\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx. \end{aligned}$$

Thus,

$$\|w\|_q = 1$$

and

$$TMSC_{p,q,N,\alpha,\beta} = \frac{1}{\|w\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,dx.$$

In particular, w is a maximizer of $TMSC_{p,q,N,\alpha,\beta}$.

Case $\beta = 0,\ p = q$

Similarly as in the case $\beta = 0,\ p > q$, we could show that for any $R > 0$:

$$\int\limits_{|x|<R} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right) - 1\right]|w_n|^p\,dx$$

$$\overset{n\to\infty}{\to} \int\limits_{|x|<R} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)-1\right]|w|^p\,dx.$$

Now, for $R>0$ sufficiently large, such that $|w_n(x)|<\varepsilon$ when $|x|\geq R$, we have

$$\begin{aligned}
&\int\limits_{|x|\geq R} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)-1\right]|w_n|^p\,dx\\
&\leq \alpha \int\limits_{|x|\geq R} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)\right]|w_n|^{p+\frac{N}{N-1}}\,dx\\
&\leq \alpha\varepsilon^{\frac{N}{N-1}} \int\limits_{|x|\geq R} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)\right]|w_n|^p\,dx\\
&\leq C(q,N)\,\varepsilon^{\frac{N}{N-1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
TMSC_{p,q,N,\alpha,\beta} &\longleftarrow \frac{1}{\|w_n\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^q\,dx\\
&= \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^q\,dx\\
&= \int\limits_{\mathbb{R}^N} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)-1\right]|w_n|^q\,dx + \|w_n\|_q^q\\
&\leq \int\limits_{\mathbb{R}^N} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)-1\right]|w|^q\,dx + 1 + C(q,N)\,\varepsilon^{\frac{N}{N-1}}.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we can conclude that

$$TMSC_{p,q,N,\alpha,\beta} \leq \int\limits_{\mathbb{R}^N} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)-1\right]|w|^q\,dx + 1. \tag{3.2}$$

As a consequence, $w\neq 0$.

Also, from (3.1) and (3.2), one gets

$$TMSC_{p,q,N,\alpha,\beta} \leq \frac{\int\limits_{\mathbb{R}^N} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)-1\right]|w|^q\,dx}{\|w\|_q^q} + 1$$

$$= \frac{\int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^q\,dx}{\|w\|_q^q}.$$

Hence we can conclude that

$$\|w\|_q = 1$$

and

$$TMSC_{p,q,N,\alpha,\beta} = \frac{\int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^q\,dx}{\|w\|_q^q}.$$

Hence w is a maximizer of $TMSC_{p,q,N,\alpha,\beta}$.

Case $\beta > 0$

Similarly as in the above two cases:

$$\int\limits_{|x|<R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,\frac{dx}{|x|^\beta}$$
$$\overset{n\to\infty}{\to} \int\limits_{|x|<R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,\frac{dx}{|x|^\beta}.$$

Also, by Theorem A, we get for $\gamma(\beta) \gtrapprox 0$:

$$\int\limits_{|x|\geq R} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p\,\frac{dx}{|x|^\beta}$$
$$\leq \frac{1}{R^{\gamma(\beta)}} \int\limits_{|x|\geq R} \frac{\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_n|^{\frac{N}{N-1}}\right)|w_n|^p}{|x|^{\beta-\gamma(\beta)}}dx \leq C(N,p,q,\alpha,\beta)\,\frac{1}{R^{\gamma(\beta)}}.$$

Hence similarly as in the two above cases, we have $w \neq 0$,

$$\|w\|_q = 1$$

and

$$TMSC_{p,q,N,\alpha,\beta} = \frac{1}{\|w\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p\,\frac{dx}{|x|^\beta}.$$

In other words, w is a maximizer of $TMSC_{p,q,N,\alpha,\beta}$.

Finally, we will show that $\|\nabla w\|_N = 1$. Indeed, assume $\|\nabla w\|_N = \frac{1}{c},\ c \geq 1$. Setting

$$w_0(x) = cw(\lambda_0 x),\ \lambda_0 = c^{q/N},$$

by direct calculations, we get

$$\|\nabla w_0\|_N = 1;$$
$$\|w_0\|_q = 1$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w_0|^{\frac{N}{N-1}}\right)|w_0|^p \frac{dx}{|x|^\beta} \\
&= c^{p-q\left(1-\frac{\beta}{N}\right)} \int_{\mathbb{R}^N} \exp\left(c^{\frac{N}{N-1}}\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p \frac{dx}{|x|^\beta} \\
&\geq \frac{1}{\|w\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|w|^{\frac{N}{N-1}}\right)|w|^p \frac{dx}{|x|^\beta} \\
&= TMSC_{p,q,N,\alpha,\beta}.
\end{aligned}$$

Hence $c = 1$.

The attainabilities of $TMSC_{N,\alpha,\beta}$, $TMSC_{q,N,\alpha,\beta}$ can be showed similarly. □

3.2. Maximizers are symmetric-Proof of Theorem 1.2

In this subsection, we will prove that maximizers for the singular subcritical Trudinger-Moser inequalities are radially symmetric. Namely, we will establish Theorem 1.2.

Proof of Theorem 1.2. Assume that u is a maximizer of

$$TMSC_{p,q,N,\alpha,\beta} = \sup_{u \in D^{N,q}(\mathbb{R}^N):\ \|\nabla u\|_N \leq 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta}.$$

Then, it is clear that $|u|$ is also a maximizer. Now, let $v = |u|^\#$, then $v \in D^{N,q}(\mathbb{R}^N)$: $\|\nabla v\|_N \leq 1$, $\|v\|_q^{q\left(1-\frac{\beta}{N}\right)} = \|u\|_q^{q\left(1-\frac{\beta}{N}\right)}$. Moreover, using the Hardy-Littlewood inequality and the fact that $|u|$ is a maximizer of $TMSC_{p,q,N,\alpha,\beta}$, we get

$$\frac{1}{\|v\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-1}}\right)|v|^p \frac{dx}{|x|^\beta}$$

$$\leq \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta}$$
$$\leq \frac{1}{\|v\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-1}}\right)|v|^p \frac{dx}{|x|^\beta}.$$

It means that the equality in the Hardy-Littlewood inequality happens:

$$\int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p \frac{dx}{|x|^\beta} = \int\limits_{\mathbb{R}^N} \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-1}}\right)|v|^p \frac{dx}{|x|^\beta}$$
$$= \int\limits_{\mathbb{R}^N} \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p\right]^\# \frac{dx}{|x|^\beta}.$$

Since, $\frac{1}{|x|^\beta}$ is strictly symmetric-decreasing, we get that

$$\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p = \left[\exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)|u|^p\right]^\#$$
$$= \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|v|^{\frac{N}{N-1}}\right)|v|^p.$$

As a consequence:

$$|u| = v = |u|^\#.$$

Similarly for $TMSC_{N,\alpha,\beta}$, $TMSC_{q,N,\alpha,\beta}$.

4. Proofs of Theorem 1.3 and Theorem 1.4

From now, we fix $a > 0, b > 0, q > 1$ and $0 \leq \beta < N$. Set

$$f(\alpha) = TMSC_{q,N,\alpha,\beta}$$

and

$$g_{a,b}(\alpha) = TMC_{a,b,q,N,\beta}(\alpha) = \sup_{\|\nabla u\|_N^a + \|u\|_q^b \leq 1} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx.$$

Similarly as in [25, Theorem 1.2], we can show that

$$g_{a,b}(\alpha) = \sup_{s\in(0,\alpha)} \left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s).$$

Lemma 4.1. *f is a continuous function on $(0, \alpha_N)$.*

Proof. Indeed, let $\varepsilon_k \to 0^+$ and $u_k \in D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right) : \|\nabla u_k\|_N \leq 1, \ \|u_k\|_q = 1$ such that

$$f\left(\alpha + \varepsilon_k\right) = \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\left(\alpha + \varepsilon_k\right)\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx.$$

Then

$$\begin{aligned} 0 &\leq f\left(\alpha + \varepsilon_k\right) - f\left(\alpha\right) \\ &\leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\left(\alpha + \varepsilon_k\right)\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx. \end{aligned}$$

WLOG, assume that

$$\begin{aligned} &u_k \rightharpoonup u \text{ weakly in } D^{1,N}_{rad}\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right); \\ &u_k \to u \text{ pointwise a.e. in } \mathbb{R}^N. \end{aligned}$$

By Radial lemma, we can choose $R_0 > 0$ such that

$$|u_k(x)| \leq 1 \text{ for every } k \in \mathbb{N}, \ |x| \geq R_0.$$

Then by Lemma 2.3, we can deduce

$$\int\limits_{\{|x|<R_0\}} \frac{\left[\Phi_{N,q,\beta}\left(\left(\alpha + \varepsilon_k\right)\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)\right]}{|x|^\beta} dx \to 0 \text{ as } k \to \infty,$$

since

$$\frac{\left[\Phi_{N,q,\beta}\left(\left(\alpha + \varepsilon_k\right)\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)\right]}{|x|^\beta} \to 0 \text{ a.e.}$$

Also, we can find some $b \gtrapprox 1$ such that

$$\int\limits_{\{|x|<R_0\}} \left[\frac{\Phi_{N,q,\beta}\left(\left(\alpha + \varepsilon_k\right)\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1 - \frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)}{|x|^\beta}\right]^b dx$$

$$\leq \int\limits_{\mathbb{R}^N} \left[\frac{\Phi_{N,q,\beta}\left((\alpha+\varepsilon_k)\left(1-\frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)}{|x|^{\beta}} \right]^b dx$$

$$\leq C\left(N,q,\beta,\alpha\right).$$

Concerning

$$\int\limits_{\{|x|\geq R_0\}} \left[\frac{\Phi_{N,q,\beta}\left((\alpha+\varepsilon_k)\left(1-\frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)}{|x|^{\beta}} \right] dx,$$

we use the following estimate

$$\int\limits_{\{|x|\geq R_0\}} \left[\frac{\Phi_{N,q,\beta}\left((\alpha+\varepsilon_k)\left(1-\frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)|u_k|^{\frac{N}{N-1}}\right)}{|x|^{\beta}} \right] dx$$

$$\leq \left[\Phi_{N,q,\beta}\left((\alpha+\varepsilon_k)\left(1-\frac{\beta}{N}\right)\right) - \Phi_{N,q,\beta}\left(s\left(1-\frac{\beta}{N}\right)\right)\right]$$

$$\times \int\limits_{\{|x|\geq R_0\}} \frac{|u_k|^{j_{N,q,\beta}\frac{N}{N-1}}}{|x|^{\beta}} dx \to 0 \text{ as } k\to\infty$$

where $j_{N,q,\beta}$ is the smallest natural number such that

$$\begin{cases} j_{N,q,\beta} > \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right) \text{ if } \beta > 0. \\ j_{N,q,\beta} \geq \frac{q(N-1)}{N} \text{ if } \beta = 0. \end{cases}$$

Similarly, we can also have that

$$0 \leq f(\alpha) - f(\alpha-\varepsilon_k) \to 0 \text{ as } k\to\infty.$$

Hence $f(\alpha)$ is continuous on $(0,\alpha_N)$. □

4.1. Proof of Theorem 1.3

For the existence of maximizers for $TMC_{a,b,q,N,\beta}(\alpha)$ when $0<\alpha<\alpha_N$, it is now enough to show that

$$\overline{\lim}_{s\to 0}\left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s) < g_{a,b}(\alpha). \tag{4.1}$$

$$\overline{\lim}_{s\to\alpha}\left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s) < g_{a,b}(\alpha). \tag{4.2}$$

Indeed, if we have (4.1) and (4.2), then there exists $s_\alpha \in (0,\alpha)$ such that

$$g_{a,b}(\alpha) = \left[\frac{1-\left(\frac{s_\alpha}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s_\alpha}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s_\alpha).$$

Now, let U_α be the maximizer of $TMSC_{q,N,s_\alpha,\beta}$. Then, we could check easily that

$$V_\alpha(x) = \left(\frac{s_\alpha}{\alpha}\right)^{\frac{N-1}{N}} U_\alpha(\lambda x)$$

where

$$\lambda = \left(\frac{\left(\frac{s_\alpha}{\alpha}\right)^{b\frac{N-1}{N}} \|U_\alpha\|_q^b}{1-\left(\frac{s_\alpha}{\alpha}\right)^{\frac{N-1}{N}a}}\right)^{\frac{q}{Nb}}$$

is a maximizer of $TMC_{a,b,q,N,\beta}(\alpha)$.

Also, if V_α is an arbitrary extremal function of $TMC_{a,b,q,N,\beta}(\alpha)$, then similarly as in [25], we could verify that

$$U_\alpha(x) = \frac{V_\alpha(x)}{\theta},$$

where

$$\theta = \|\nabla V_\alpha\|_N$$

is a maximizer of $TMSC_{q,N,s,\beta}$ with $s = \alpha\theta^{\frac{N}{N-1}}$. By Theorem 1.2, when $\beta > 0$, $|U_\alpha| = (|U_\alpha|)^\#$. Hence $|V_\alpha| = (|V_\alpha|)^\#$.

Now, (4.2) is clear since

$$\overline{\lim}_{s\to\alpha}\left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s) = 0 < g_{a,b}(\alpha).$$

Hence, we just need to consider

$$\overline{\lim}_{s\to 0}\left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s).$$

Case 1: $\beta > 0$

Let (s_n) be an arbitrary sequence that $s_n \to 0$. Then by Theorem 1.1, we can find a sequence (u_n) of radial functions in $D^{N,q}\left(\mathbb{R}^N\right) : \|\nabla u_n\|_N \leq 1, \ \|u_n\|_q = 1$ such that

$$
\begin{aligned}
f\left(s_n\right) &= \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(s_n\left(1-\frac{\beta}{N}\right)|u_n|^{\frac{N}{N-1}}\right)}{|x|^\beta} dx \\
&= \sum_{j=\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1}^{\infty} \int\limits_{\mathbb{R}^N} \frac{s_n^j\left(1-\frac{\beta}{N}\right)^j |u_n|^{j\frac{N}{N-1}}}{j!\,|x|^\beta} \\
&= \int\limits_{\mathbb{R}^N} \frac{\left[s_n\left(1-\frac{\beta}{N}\right)\right]^{\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1} |u_n|^{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1\right)\frac{N}{N-1}}}{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1\right)!\,|x|^\beta} \\
&+ \sum_{j=\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+2}^{\infty} \int\limits_{\mathbb{R}^N} \frac{s_n^j\left(1-\frac{\beta}{N}\right)^j |u_n|^{j\frac{N}{N-1}}}{j!\,|x|^\beta} \\
&\leq \frac{s_n^{\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1}}{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1\right)!} \int\limits_{\mathbb{R}^N} \frac{|u_n|^{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1\right)\frac{N}{N-1}}}{|x|^\beta} \\
&+ s_n^{\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+2} \int\limits_{\mathbb{R}^N} \frac{\exp\left(s_n|u_n|^{\frac{N}{N-1}}\right)|u_n|^{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+2\right)\frac{N}{N-1}}}{|x|^\beta}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left[\frac{1-\left(\frac{s_n}{\alpha}\right)^{\frac{N-1}{N}}a}{\left(\frac{s_n}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f\left(s_n\right) \\
&\leq \left[\frac{1-\left(\frac{s_n}{\alpha}\right)^{\frac{N-1}{N}}a}{\left(\frac{1}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} \left[\frac{1}{(s_n)^{q\frac{N-1}{N}}}\right]^{\left(1-\frac{\beta}{N}\right)} \frac{s_n^{\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1}}{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1\right)!} \\
&\cdot \int\limits_{\mathbb{R}^N} \frac{|u_n|^{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+1\right)\frac{N}{N-1}}}{|x|^\beta} \\
&+ \left[\frac{1-\left(\frac{s_n}{\alpha}\right)^{\frac{N-1}{N}}a}{\left(\frac{1}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} \left[\frac{1}{(s_n)^{q\frac{N-1}{N}}}\right]^{\left(1-\frac{\beta}{N}\right)} s_n^{\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+2}
\end{aligned}
$$

$$\cdot \int_{\mathbb{R}^N} \frac{\exp\left(s_n |u_n|^{\frac{N}{N-1}}\right) |u_n|^{\left(\left\lfloor \frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)\right\rfloor+2\right)\frac{N}{N-1}}}{|x|^{\beta}}$$

$\to 0$ as $n \to \infty$.

Case 2: $\beta = 0$ and $\frac{q(N-1)}{N} \notin \mathbb{N}$
Similarly as in the previous case, we also get

$$\overline{\lim}_{s\to 0} \left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}\left(1-\frac{\beta}{N}\right)} f(s) = 0.$$

Case 3: $\beta = 0$ and $\frac{q(N-1)}{N} \in \mathbb{N}$
Then,

$$\begin{aligned}
&\left[\frac{1-\left(\frac{s_n}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s_n}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}} f(s_n) \\
&\leq \left[\frac{1-\left(\frac{s_n}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{1}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}} \left[\frac{1}{(s_n)^{\frac{q(N-1)}{N}}}\right] \frac{s_n^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \int_{\mathbb{R}^N} e^{s_n|u_n|^{\frac{N}{N-1}}} |u_n|^q \\
&\leq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!},
\end{aligned}$$

since we could show that

$$\int_{\mathbb{R}^N} e^{s_n|u_n|^{\frac{N}{N-1}}} |u_n|^q = \int_{\mathbb{R}^N} \left(e^{s_n|u_n|^{\frac{N}{N-1}}} - 1\right) |u_n|^q \, dx + 1 \to 1.$$

Hence

$$\overline{\lim}_{s\to 0} \left[\frac{1-\left(\frac{s}{\alpha}\right)^{\frac{N-1}{N}a}}{\left(\frac{s}{\alpha}\right)^{b\frac{N-1}{N}}}\right]^{\frac{q}{b}} f(s) \leq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

It is clear that (4.1) holds when $(a, b, \alpha) \in X$, i.e.,

$$g_{a,b}(\alpha) > \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}. \tag{4.3}$$

We note here that (4.3) was showed to be true in [15] when $a = b = q = N$ with $0 < \alpha \leq \alpha_N$ when $N \geq 3$ and for $\frac{2}{B_2} < \alpha \leq 4\pi$ when $N = 2$ where $B_2 = \sup_{\phi \neq 0,\ \phi \in H^1} \frac{\|\phi\|_4^4}{\|\phi\|_2^2 \|\nabla \phi\|_2^2}$.

It also means that $TMC_{a,b}(q, N, 0, \alpha)$ is not attained only if

$$g_{a,b}(\alpha) \leq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

Lemma 4.2. *When $\beta = 0$ and $\frac{q(N-1)}{N} \in \mathbb{N}$, then $g_{a,b}(\alpha) \geq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$. Hence $X_{a,b} \cup Y_{a,b} = (0, \alpha_N)$.*

Proof. We choose a function $v \in D^{N,q}(\mathbb{R}^N) \setminus \{0\}$ such that $\|v\|_q = 1$, and set

$$v_t(x) = t^{1/q} v\left(t^{1/N} x\right),$$

then it is easy to check that

$$\|\nabla v_t\|_N = t^{1/q} \|\nabla v\|_N;$$
$$\|v_t\|_p = t^{\frac{p-q}{pq}} \|v\|_p.$$

Also, for each $t \gtrapprox 0$, there exists a unique $c_t > 0$ such that with $u_t = c_t v_t$:

$$\|\nabla u_t\|_N^a + \|u_t\|_q^b = c_t^a t^{a/q} \|\nabla v\|_N^a + c_t^b \|v\|_q^b = 1.$$

We note here that $c_t \uparrow 1$ as $t \downarrow 0$. Then

$$\begin{aligned}
g_{a,b}(\alpha) &\geq \int_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha |u_t|^{\frac{N}{N-1}}\right) dx \\
&\geq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \int_{\mathbb{R}^N} |u_t|^{\frac{N}{N-1}\frac{q(N-1)}{N}} + \frac{\alpha^{\frac{q(N-1)}{N}+1}}{\left(\frac{q(N-1)}{N}+1\right)!} \int_{\mathbb{R}^N} |u_t|^{\frac{N}{N-1}\left[\frac{q(N-1)}{N}+1\right]} \\
&= \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} c_t^q \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}\right) \\
&\to \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \text{ as } t \downarrow 0. \qquad \square
\end{aligned}$$

Hence if $TMC_{a,b}(q,N,0,\alpha)$ is not attained, then

$$g_{a,b}(\alpha)=\frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

It means that $B_{a,b}\subset Y_{a,b}$.

We note here that if u is a maximizer of $TMC_{a,b}(q,N,0,\alpha)$. Then $\|\nabla u\|_N^a+\|u\|_q^b=1$. Indeed, if $\|\nabla u\|_N^a+\|u\|_q^b<1$, we can choose some $c>1$ such that $c^a\|\nabla u\|_N^a+c^b\|u\|_q^b=1$ and hence

$$\begin{aligned}TMC_{a,b}(q,N,0,\alpha)&\geq\int\limits_{\mathbb{R}^N}\Phi_{N,q,0}\left(\alpha|cu|^{\frac{N}{N-1}}\right)dx\\&>\int\limits_{\mathbb{R}^N}\Phi_{N,q,0}\left(\alpha|u|^{\frac{N}{N-1}}\right)dx\end{aligned}$$

which is impossible.

Lemma 4.3. *$g_{a,b}(\alpha)$ is continuous on $(0,\alpha_N)$. As a consequence, $Y_{a,b}$ is closed and thus, $X_{a,b}$ is open.*

Proof. Let $\varepsilon_k\to 0^+$ and $u_k\in D_{rad}^{1,N}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N):\|\nabla u_k\|_N^a+\|u_k\|_q^b\leq 1$ such that

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right)dx\\&\leq g_{a,b}(\alpha+\varepsilon_k)\leq\int\limits_{\mathbb{R}^N}\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right)dx+\varepsilon_k\end{aligned}$$

Then

$$\begin{aligned}0&\leq g_{a,b}(\alpha+\varepsilon_k)-g_{a,b}(\alpha)\\&\leq\int\limits_{\mathbb{R}^N}\left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right)-\Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right]dx+\varepsilon_k.\end{aligned}$$

WLOG, assume that

$$\begin{aligned}&u_k\rightharpoonup u\text{ weakly in }D_{rad}^{1,N}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N);\\&u_k\to u\text{ pointwise a.e. in }\mathbb{R}^N.\end{aligned}$$

By Radial lemma, we can choose $R_0>0$ such that

$$|u_k(x)| \leq 1 \text{ for every } k \in \mathbb{N}, \ |x| \geq R_0.$$

Then by Lemma 2.3, we can deduce

$$\int\limits_{\{|x|<R_0\}} \left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right] dx \to 0 \text{ as } k \to \infty,$$

since

$$\left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right] \to 0 \text{ a.e.}$$

Also, we can find some $b \gtrapprox 1$ such that

$$\begin{aligned}
&\int\limits_{\{|x|<R_0\}} \left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right]^b dx \\
&\leq \int\limits_{\mathbb{R}^N} \left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right]^b dx \\
&\leq C(N,q,\alpha).
\end{aligned}$$

Concerning

$$\int\limits_{\{|x|\geq R_0\}} \left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right] dx,$$

we use the following estimate with a quick remind that $u_k \leq 1$ on $\{|x| \geq R_0\}$:

$$\begin{aligned}
&\int\limits_{\{|x|\geq R_0\}} \left[\Phi_{N,q,0}\left((\alpha+\varepsilon_k)|u_k|^{\frac{N}{N-1}}\right) - \Phi_{N,q,0}\left(\alpha|u_k|^{\frac{N}{N-1}}\right)\right] dx \\
&= \int\limits_{\{|x|\geq R_0\}} \sum_{j=q\frac{N}{N-1}}^{\infty} \frac{(\alpha+\varepsilon_k)^j - \alpha^j}{j!} |u_k|^{j\frac{N}{N-1}} \\
&\leq \left[\Phi_{N,q,0}((\alpha+\varepsilon_k)) - \Phi_{N,q,0}(\alpha)\right] \int\limits_{\{|x|\geq R_0\}} |u_k|^q dx \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Similarly, we can also have that

$$0 \leq g_{a,b}(\alpha) - g_{a,b}(\alpha-\varepsilon_k) \to 0 \text{ as } k \to \infty.$$

Hence $g_{a,b}(\alpha)$ is continuous on $(0, \alpha_N)$. □

Now, let $\alpha_0 \in X_{a,b}$. We will show that $\alpha \in X_{a,b}$ for every $\alpha > \alpha_0$. Indeed, since $\alpha_0 \in X_{a,b}$, there exists some $u : \|\nabla u\|_N^a + \|u\|_q^b \leq 1$ such that

$$TMC_{a,b,q,N,0}(\alpha_0) = \int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha_0 |u|^{\frac{N}{N-1}}\right) dx > \frac{\alpha_0^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

It means

$$\sum_{j=0}^{\infty} \frac{\alpha_0^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!} \|u\|^{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}_{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}} > \frac{\alpha_0^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}$$

i.e.

$$\sum_{j=0}^{\infty} \frac{\alpha_0^{j}}{\left(\frac{q(N-1)}{N}+j\right)!} \|u\|^{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}_{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}} > \frac{1}{\frac{q(N-1)}{N}!}.$$

Hence

$$\begin{aligned} TMC_{a,b,q,N,0}(\alpha) &\geq \int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \\ &= \sum_{j=0}^{\infty} \frac{\alpha^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!} \|u\|^{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}_{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}} \\ &\geq \alpha^{\frac{q(N-1)}{N}} \sum_{j=0}^{\infty} \frac{\alpha_0^{j}}{\left(\frac{q(N-1)}{N}+j\right)!} \|u\|^{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}_{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}} \\ &> \frac{1}{\frac{q(N-1)}{N}!}. \end{aligned}$$

Hence $\alpha \in X_{a,b}$ for every $\alpha > \alpha_0$.

This fact, together with the fact that $X_{a,b}$ is open, gives us that $X_{a,b}$ must have the form

$$X_{a,b} = (\alpha_{a,b}, \alpha_N)$$

for some $\alpha_{a,b} \in [0, \alpha_N]$ (when $\alpha_{a,b} = \alpha_N$, we understand that $X_{a,b} = \emptyset$).

It also gives us

$$Y_{a,b} = (0, \alpha_{a,b}].$$

Assume $\alpha_{a,b} \in (0, \alpha_N)$, we claim that

$$A_{a,b} = (\alpha_{a,b}, \alpha_N) \text{ or } A_{a,b} = [\alpha_{a,b}, \alpha_N).$$

Indeed, it is clear that $(\alpha_{a,b}, \alpha_N) \subset A_{a,b}$. We prove that $(0, \alpha_{a,b}) \in B_{a,b}$. Indeed, assume that there is some $\alpha < \alpha_{a,b}$ such that $TMC_{a,b,q,N,0}(\alpha)$ can be attained. Then, there exists some $u \neq 0 : \|\nabla u\|_N^a + \|u\|_q^b \leq 1$ such that

$$TMC_{a,b,q,N,0}(\alpha) = \int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha |u|^{\frac{N}{N-1}}\right) dx = \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

Equivalently,

$$\sum_{j=0}^{\infty} \frac{\alpha^j}{\left(\frac{q(N-1)}{N} + j\right)!} \|u\|^{\frac{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}} = \frac{1}{\frac{q(N-1)}{N}!}.$$

However, since $\alpha_{a,b} \in Y_{a,b} : TMC_{a,b,q,N,0}(\alpha_{a,b}) = \frac{\alpha_{a,b}^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}$. Hence

$$\int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha_{a,b} |u|^{\frac{N}{N-1}}\right) dx \leq \frac{\alpha_{a,b}^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

It means

$$\sum_{j=0}^{\infty} \frac{\alpha_{a,b}^j}{\left(\frac{q(N-1)}{N} + j\right)!} \|u\|^{\frac{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}} \leq \frac{1}{\frac{q(N-1)}{N}!}.$$

But it is impossible since $\forall j \geq 0$:

$$\frac{\alpha_{a,b}^j}{\left(\frac{q(N-1)}{N} + j\right)!} \|u\|^{\frac{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}} > \frac{\alpha^j}{\left(\frac{q(N-1)}{N} + j\right)!} \|u\|^{\frac{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}}.$$

Suppose $X_{a,b} = (0, \alpha_N)$. Then it is clear that $X_{a,b} = A_{a,b} = (0, \alpha_N)$; $Y_{a,b} = B_{a,b} = \emptyset$.

If $X_{a,b} = \emptyset$, then we prove that $A_{a,b} = \emptyset$. Indeed, suppose $\alpha \in (0, \alpha_N) \cap A_{a,b} = Y_{a,b} \cap A_{a,b}$. We can find $u \neq 0 : \|\nabla u\|_N^a + \|u\|_q^b \leq 1$ such that

$$TMC_{a,b,q,N,0}(\alpha) = \int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha |u|^{\frac{N}{N-1}}\right) dx = \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

Hence

$$\sum_{j=0}^{\infty} \frac{\alpha^j}{\left(\frac{q(N-1)}{N} + j\right)!} \|u\|^{\frac{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}} = \frac{1}{\frac{q(N-1)}{N}!}.$$

Thus, for every $\beta \in (\alpha, \alpha_N)$:

$$\sum_{j=0}^{\infty} \frac{\beta^j}{\left(\frac{q(N-1)}{N}+j\right)!} \|u\|^{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}}_{\left(\frac{q(N-1)}{N}+j\right)\frac{N}{N-1}} > \frac{1}{\frac{q(N-1)}{N}!}.$$

It means

$$TMC_{a,b,q,N,0}(\beta) \geq \int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\beta |u|^{\frac{N}{N-1}}\right) dx > \frac{\beta^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

Thus $(\alpha, \alpha_N) \subset X$ which is a contradiction.

So when $X_{a,b} = \emptyset : A_{a,b} = \emptyset$ and $Y_{a,b} = B_{a,b} = (0, \alpha_N)$. In this case, we have since $Y_{a,b} = B_{a,b} = (0, \alpha_N)$:

$$TMC_{a,b,q,N,0}(\alpha) = \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!} \ \forall 0 < \alpha < \alpha_N.$$

Let $\alpha \uparrow \alpha_N$, we get

$$TMC_{a,b,q,N,0}(\alpha_N) = \frac{\alpha_N^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}.$$

4.2. Proof of Theorem 1.4

As in the above proof, $\alpha \in X_{a,b}$ (and hence, $TMC_{a,b}(q, N, 0, \alpha)$ can be achieved) if we can show that

$$L(t) = c_t^q \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|^{q+\frac{N}{N-1}}_{q+\frac{N}{N-1}}\right) > 1 \text{ for some } t \gtrapprox 0. \tag{4.4}$$

We recall that

$$c_t^a t^{a/q} \|\nabla v\|_N^a + c_t^b = 1.$$

By the implicit theorem, we get

$$bc_t^{b-1} c_t' = -ac_t^{a-1} c_t' t^{a/q} \|\nabla v\|_N^a - \frac{a}{q} c_t^a \|\nabla v\|_N^a t^{a/q-1}.$$

Hence

$$c_t' = \frac{-\frac{a}{q} c_t^a \|\nabla v\|_N^a t^{a/q-1}}{bc_t^{b-1} + ac_t^{a-1} t^{a/q} \|\nabla v\|_N^a}.$$

Now,

$$
\begin{aligned}
L'(t) &= q c_t^{q-1} c_t' \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}\right) \\
&\quad + c_t^q \begin{pmatrix} \frac{N}{N-1} \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{1}{N-1}} c_t' t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}} \\ + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} \frac{N}{(N-1)q} t^{\frac{N}{(N-1)q}-1} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}} \end{pmatrix} \\
&= c_t' \begin{bmatrix} q c_t^{q-1} \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}\right) \\ + \frac{N}{N-1} \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^q c_t^{\frac{1}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}} \end{bmatrix} \\
&\quad + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^q c_t^{\frac{N}{N-1}} \frac{N}{(N-1)q} t^{\frac{N}{(N-1)q}-1} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}} \\
&= \frac{-\frac{a}{q} c_t^a \|\nabla v\|_N^a t^{a/q-1}}{b c_t^{b-1} + a c_t^{a-1} t^{a/q} \|\nabla v\|_N^a} \begin{bmatrix} q c_t^{q-1} \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}\right) \\ + \frac{N}{N-1} \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^q c_t^{\frac{1}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}} \end{bmatrix} \\
&\quad + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^q c_t^{\frac{N}{N-1}} \frac{N}{(N-1)q} t^{\frac{N}{(N-1)q}-1} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&t^{1-a/q} \frac{b c_t^{b-1} + a c_t^{a-1} t^{a/q} \|\nabla v\|_N^a}{\frac{a}{q} c_t^a \|\nabla v\|_N^a} L'(t) \qquad (4.5) \\
&= - \begin{bmatrix} q c_t^{q-1} \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^{\frac{N}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}\right) \\ + \frac{N}{N-1} \frac{\alpha}{\frac{q(N-1)}{N}+1} c_t^q c_t^{\frac{1}{N-1}} t^{\frac{N}{(N-1)q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}} \end{bmatrix} \\
&\quad + \frac{\alpha}{\frac{q(N-1)}{N}+1} \frac{b c_t^{b-1} + a c_t^{a-1} t^{a/q} \|\nabla v\|_N^a}{\frac{a}{q} c_t^a \|\nabla v\|_N^a} c_t^q c_t^{\frac{N}{N-1}} \frac{N}{(N-1)q} t^{\frac{N}{(N-1)q}-\frac{a}{q}} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}.
\end{aligned}
$$

If $a > \frac{N}{N-1}$, then $\frac{N}{(N-1)q} - \frac{a}{q} < 0$. Hence when letting $t \downarrow 0$, the RHS of (4.5) goes to $+\infty$. Combining with $c_t \uparrow 1$ as $t \downarrow 0$, we will have that $L'(t) > 0$ for some $t \gtrsim 0$ which means (4.4).

The same thing happens when $\frac{N}{(N-1)q} - \frac{a}{q} = 0$ and

$$
q < \frac{\alpha}{\frac{q(N-1)}{N}+1} \frac{b}{\frac{a}{q}\|\nabla v\|_N^a} \frac{N}{(N-1)q} \|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}
$$

or equivalently

$$\alpha > q\left(\frac{q\,(N-1)}{N}+1\right)\frac{a}{b}\frac{(N-1)}{N}\frac{\|\nabla v\|_N^a}{\|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}}$$

$$= q\left(\frac{q\,(N-1)}{N}+1\right)\frac{1}{b}\frac{\|\nabla v\|_N^{\frac{N}{N-1}}}{\|v\|_{q+\frac{N}{N-1}}^{q+\frac{N}{N-1}}}.$$

Now, if $a \leq \frac{N}{N-1}$, we will show that $X_{a,b} \neq \emptyset$. Indeed, it is enough to show that for $\alpha \gtrsim 0$:

$$\int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha\,|u|^{\frac{N}{N-1}}\right)dx < \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\ \forall u \neq 0 : \|\nabla u\|_N^a + \|u\|_q^b = 1.$$

We first note that by definition: for any $\gamma \in (0, \alpha_N)$:

$$\int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\gamma\left|\frac{u}{\|\nabla u\|_N}\right|^{\frac{N}{N-1}}\right)dx \leq TMSC_{q,N,\gamma,0}\frac{\|u\|_q^q}{\|\nabla u\|_N^q}.$$

Hence

$$\|u\|_{q+\frac{N}{N-1}k}^{q+\frac{N}{N-1}k} \leq TMSC_{q,N,\gamma,0}\frac{\left(\frac{q(N-1)}{N}+k\right)!}{\gamma^{\frac{q(N-1)}{N}+k}}\|\nabla u\|_N^{\frac{N}{N-1}k}\|u\|_q^q\ \ \forall k \geq 0$$

$$= TMSC_{q,N,\gamma,0}\frac{\left(\frac{q(N-1)}{N}+k\right)!}{\gamma^{\frac{q(N-1)}{N}+k}}\theta^{\frac{N}{N-1}k}\left(1-\theta^a\right)^{\frac{q}{b}}$$

where $\theta = \|\nabla u\|_N \in (0,1)$. Hence

$$\int\limits_{\mathbb{R}^N} \Phi_{N,q,0}\left(\alpha\,|u|^{\frac{N}{N-1}}\right)dx$$

$$= \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\|u\|_q^q + \sum_1^\infty \frac{\alpha^{\frac{q(N-1)}{N}+k}}{\left(\frac{q(N-1)}{N}+k\right)!}\|u\|_{q+\frac{N}{N-1}k}^{q+\frac{N}{N-1}k}$$

$$\leq \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\left(1-\theta^a\right)^{\frac{q}{b}} + TMSC_{q,N,\gamma,0}\sum_1^\infty \frac{\alpha^{\frac{q(N-1)}{N}+k}}{\left(\frac{q(N-1)}{N}+k\right)!}\frac{\left(\frac{q(N-1)}{N}+k\right)!}{\gamma^{\frac{q(N-1)}{N}+k}}\theta^{\frac{N}{N-1}k}\left(1-\theta^a\right)^{\frac{q}{b}}$$

$$= \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\left(1-\theta^a\right)^{\frac{q}{b}} + TMSC_{q,N,\gamma,0}\sum_1^\infty \frac{\alpha^{\frac{q(N-1)}{N}+k}}{\gamma^{\frac{q(N-1)}{N}+k}}\theta^{\frac{N}{N-1}k}\left(1-\theta^a\right)^{\frac{q}{b}}$$

$$= \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\left(1-\theta^a\right)^{\frac{q}{b}} + TMSC_{q,N,\gamma,0}\frac{\alpha^{\frac{q(N-1)}{N}}}{\gamma^{\frac{q(N-1)}{N}}}\left(1-\theta^a\right)^{\frac{q}{b}}\sum_1^\infty \frac{\alpha^k}{\gamma^k}\theta^{\frac{N}{N-1}k}$$

$$= \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\left(1-\theta^a\right)^{\frac{q}{b}} + TMSC_{q,N,\gamma,0}\frac{\alpha^{\frac{q(N-1)}{N}}}{\gamma^{\frac{q(N-1)}{N}}}\left(1-\theta^a\right)^{\frac{q}{b}}\left[\frac{1}{1-\frac{\alpha}{\gamma}\theta^{\frac{N}{N-1}}}-1\right]$$
$$= \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\left(1-\theta^a\right)^{\frac{q}{b}} + TMSC_{q,N,\gamma,0}\frac{\alpha^{\frac{q(N-1)}{N}}}{\gamma^{\frac{q(N-1)}{N}}}\left(1-\theta^a\right)^{\frac{q}{b}}\left[\frac{\alpha\theta^{\frac{N}{N-1}}}{\gamma-\alpha\theta^{\frac{N}{N-1}}}\right].$$

Hence it is enough to choose

$$TMSC_{q,N,\gamma,0}\frac{\alpha^{\frac{q(N-1)}{N}}}{\gamma^{\frac{q(N-1)}{N}}}\left(1-\theta^a\right)^{\frac{q}{b}}\left[\frac{\alpha\theta^{\frac{N}{N-1}}}{\gamma-\alpha\theta^{\frac{N}{N-1}}}\right] \leq \frac{\alpha^{\frac{q(N-1)}{N}}}{\frac{q(N-1)}{N}!}\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]$$

i.e.

$$\alpha < \frac{\gamma\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]}{\frac{q(N-1)}{N}!}.\frac{1}{TMSC_{q,N,\gamma,0}\frac{\left(1-\theta^a\right)^{\frac{q}{b}}}{\gamma^{\frac{q(N-1)}{N}}}\theta^{\frac{N}{N-1}} + \frac{\theta^{\frac{N}{N-1}}\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]}{\frac{q(N-1)}{N}!}} \quad \forall\theta\in(0,1).$$

Noting that

$$\lim_{\theta\to 1}\frac{\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]}{TMSC_{q,N,\gamma,0}\frac{\left(1-\theta^a\right)^{\frac{q}{b}}}{\gamma^{\frac{q(N-1)}{N}}}\theta^{\frac{N}{N-1}} + \frac{\theta^{\frac{N}{N-1}}\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]}{\frac{q(N-1)}{N}!}} > 0,$$

it is enough to show

$$\lim_{\theta\to 0}\frac{\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]}{TMSC_{q,N,\gamma,0}\frac{\left(1-\theta^a\right)^{\frac{q}{b}}}{\gamma^{\frac{q(N-1)}{N}}}\theta^{\frac{N}{N-1}} + \frac{\theta^{\frac{N}{N-1}}\left[1-\left(1-\theta^a\right)^{\frac{q}{b}}\right]}{\frac{q(N-1)}{N}!}} > 0.$$

But it is clear by L'Hôpital's rule since $a \leq \frac{N}{N-1}$.

5. Proof of Theorem 1.5

Before proving Theorem 1.5, we will study a lower estimate for $TME_{p,q,N,\lambda,\alpha,\beta}$ when $\beta = 0$; $\frac{q(N-1)}{N} \in \mathbb{N}$.

Lemma 5.1. *Let $\beta = 0$, $\frac{q(N-1)}{N} \in \mathbb{N}$ and $0 < \alpha \leq \alpha_N$, the following estimates hold*

(a) $TME_{p,q,N,\lambda,\alpha,0} \geq \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$.

(b) $TME_{p,q,N,\lambda,\alpha,0} > \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$ *if* $p = N$ *and* $\lambda \leq \frac{\alpha}{\frac{q(N-1)}{N}+1}$.

(c) if $p > N$, *then* $TME_{p,q,N,\lambda,\alpha,0} > \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$.

(d) if $q < p \leq N$ *and* $p < \frac{N-1}{N-2}q$, *then* $TME_{p,q,N,\lambda,\alpha_N,0} > \frac{\alpha_N^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$.

Proof. Fix $v \in C_0^\infty\left(\mathbb{R}^N\right) \setminus \{0\} : \|\nabla v\|_N \leq 1;\ \|v\|_q = 1$ and let

$$v_t(x) = t^{1/q} v\left(t^{1/N} x\right).$$

Then it is easy to check that

$$\|\nabla v_t\|_N = t^{1/q} \|\nabla v\|_N;$$
$$\|v_t\|_p = t^{\frac{p-q}{pq}} \|v\|_p.$$

Hence by Fatou's lemma:

$$\begin{aligned} TME_{p,q,N,\lambda,\alpha,0} &\geq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,0}(\alpha v_t^{\frac{N}{N-1}})}{1+\lambda |v_t|^{\frac{p}{N-1}}} dx \\ &> \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \int\limits_{\mathbb{R}^N} \frac{v_t^q}{1+\lambda |v_t|^{\frac{p}{N-1}}} dx + \frac{\alpha^{\frac{q(N-1)}{N}+1}}{\left(\frac{q(N-1)}{N}+1\right)!} \int\limits_{\mathbb{R}^N} \frac{v_t^{q+\frac{N}{N-1}}}{1+\lambda |v_t|^{\frac{p}{N-1}}} dx \\ &= \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \left[\int\limits_{\mathbb{R}^N} \frac{v^q \left(1 + \frac{\alpha}{\frac{q(N-1)}{N}+1} t^{\frac{N}{q(N-1)}} v^{\frac{N}{N-1}}\right)}{1+\lambda t^{\frac{p}{q(N-1)}} |v|^{\frac{p}{N-1}}} dx \right] \\ &\to \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \text{ as } t \downarrow 0. \end{aligned}$$

It is clear to see that if $p = N$ and $\lambda \leq \frac{\alpha}{\frac{q(N-1)}{N}+1}$ then

$$TME_{p,q,N,\lambda,\alpha,0} > \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

Now, we consider the sequence

$$u_n(x) = \begin{cases} \left(\frac{1}{\omega_{N-1}}\right)^{1/N} \left(\frac{n}{N}\right)^{\frac{N-1}{N}}, & 0 \leq |x| \leq e^{-\frac{n}{N}}, \\ \left(\frac{N}{\omega_{N-1} n}\right)^{1/N} \log\left(\frac{1}{|x|}\right), & e^{-\frac{n}{N-\beta}} < |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{5.1}$$

We note that $\|\nabla u_n\|_N = 1$ and

$$\|u_n\|_q^q = \omega_{N-1}\int_0^{e^{-\frac{n}{N}}}(\frac{1}{\omega_{N-1}})^{q/N}(\frac{n}{N})^{q\frac{N-1}{N}}r^{N-1}dr + \omega_{N-1}\int_{e^{-\frac{n}{N}}}^1(\frac{N}{\omega_{N-1}n})^{q/N}\left(\log\frac{1}{r}\right)^q r^{N-1}dr$$

$$= \omega_{N-1}^{1-\frac{q}{N}}(\frac{n}{N})^{q\frac{N-1}{N}}\frac{e^{-n}}{N} + \omega_{N-1}^{1-\frac{q}{N}}\left(\frac{N}{n}\right)^{\frac{q}{N}}\int_0^{\frac{n}{N}}y^q e^{-Ny}dy$$

$$:= A + B.$$

We recall that in this case $\Phi_{N,q,0}(x) = \sum_{j=q\frac{N-1}{N}}^{\infty}\frac{x^j}{j!}$, and

$$\int_{\mathbb{R}^N}\frac{\Phi_{N,q,0}(\alpha u_n^{\frac{N}{N-1}})}{1+\lambda|u_n|^{\frac{p}{N-1}}}dx$$

$$= \omega_{N-1}\int_0^{e^{-\frac{n}{N}}}\frac{\Phi_{N,q,0}\left(\alpha(\frac{1}{\omega_{N-1}})^{\frac{1}{N-1}}\frac{n}{N}\right)}{1+\lambda(\frac{1}{\omega_{N-1}})^{\frac{p}{N(N-1)}}(\frac{n}{N})^{\frac{p}{N}}}r^{N-1}dr$$

$$+ \omega_{N-1}\int_{e^{-\frac{n}{N}}}^1\frac{\Phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}(\log\frac{1}{r})^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}(\log\frac{1}{r})^{\frac{p}{N-1}}}r^{N-1}dr$$

$$:= I + II.$$

For sufficiently large n, we have

$$I = \omega_{N-1}\frac{\Phi_{N,q,0}\left(\alpha(\frac{1}{\omega_{N-1}})^{\frac{1}{N-1}}\frac{n}{N}\right)}{1+\lambda(\frac{1}{\omega_{N-1}})^{\frac{p}{N(N-1)}}(\frac{n}{N})^{\frac{p}{N}}}\frac{e^{-n}}{N}$$

$$> \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A \text{ for all } 0<\alpha\le N\omega_{N-1}^{\frac{1}{N-1}},$$

and in particular, $I - \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A \approx \frac{1}{n^{\frac{p}{N}}}$ when $\alpha = \alpha_N$. Also,

$$II = \omega_{N-1}\int_{e^{-\frac{n}{N}}}^1\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}(\log\frac{1}{r})^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}(\log\frac{1}{r})^{\frac{p}{N-1}}}r^{N-1}dr$$

$$= \omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}} - \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q\right)e^{-Ny}dy$$

$$
\begin{aligned}
&+\omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q\right)e^{-Ny}dy\\
&=\omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q\right)e^{-Ny}dy\\
&+\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}\omega_{N-1}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}\int_0^{\frac{n}{N}}y^qe^{-Ny}dy\\
&:=III+\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}B
\end{aligned}
$$

where

$$
\begin{aligned}
III:=\omega_{N-1}\int_0^{\frac{n}{N}}&\left(\frac{\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}}y^{\frac{N}{N-1}}\right)-\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}}y^q}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}\right.\\
&\left.\frac{-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}y^{q+\frac{p}{N-1}}}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}\right)\\
&\cdot e^{-Ny}dy\\
:=\omega_{N-1}&\int_0^{\frac{n}{N}}S(y)e^{-Ny}dy.
\end{aligned}
$$

If $\alpha=\alpha_N$ and $p\le N$, then

$$
\begin{aligned}
III&\gtrsim-\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}}\cdot\omega_{N-1}\int_0^{\frac{n}{N}}\left(\frac{y^{q+\frac{p}{N-1}}}{1+\lambda(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}y^{\frac{p}{N-1}}}\right)e^{-Ny}dy\\
&\approx-\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{p}{N(N-1)}}
\end{aligned}
$$

Taking advantage of the assumption $\frac{p}{q}<\frac{N-1}{N-2}$, we have $\frac{p}{N}<\frac{q}{N}+\frac{p}{N(N-1)}$. Hence

$$
\begin{aligned}
\frac{1}{\|u_n\|_q^q}\int_{\mathbb{R}^N}\frac{\phi_{N,q,0}(\alpha u_n^{\frac{N}{N-1}})}{1+u^{\frac{p}{N-1}}}dx&=\frac{I+II}{A+B}\\
&>\frac{\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A+O\left(\frac{1}{n}\right)^{\frac{p}{N}}-O\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{p}{N(N-1)}}+\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}B}{A+B}
\end{aligned}
$$

$$> \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!} \text{ when } n \text{ is large enough.}$$

Note that if $p > N$, we can conclude that when

$$y < \left(\frac{\alpha}{\lambda(\frac{q(N-1)}{N}+1)}\right)^{\frac{N-1}{p-N}} \left(\frac{\omega_{N-1}n}{N}\right)^{\frac{1}{N}} := c(n),$$

we have

$$\begin{aligned}
&\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}} y^{\frac{N}{N-1}}\right) - \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}} y^q \\
&\qquad - \lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}} y^{q+\frac{p}{N-1}} \\
&= \left(\frac{\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}} y^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+1}}{\left(\frac{q(N-1)}{N}+1\right)!} - \lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}} y^{q+\frac{p}{N-1}}\right) \\
&\qquad + \sum_{j=\frac{q(N-1)}{N}+2}^{\infty} \frac{\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}} y^{\frac{N}{N-1}}\right)^j}{j!} \\
&\geq \sum_{j=\frac{q(N-1)}{N}+2}^{\infty} \frac{\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}} y^{\frac{N}{N-1}}\right)^j}{j!}.
\end{aligned}$$

Then we get the following estimates

$$\begin{aligned}
&\int_0^{c(n)} S(y)e^{-Ny}dy \\
&\geq \int_0^{c(n)} \left(\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}} y^{\frac{N}{N-1}}\right) - \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}} y^q \right. \\
&\qquad \left. -\lambda\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}} y^{q+\frac{p}{N-1}}\right) \\
&\qquad \cdot e^{-Ny}dy \cdot \frac{\omega_{N-1}}{1+(\frac{N}{\omega_{N-1}n})^{\frac{p}{N(N-1)}}(\frac{n}{N})^{\frac{p}{N-1}}} \\
&\gtrsim \left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{2}{N-1}} \int_0^1 y^{q+\frac{2N}{N-1}} e^{-Ny}dy \cdot \frac{1}{n^{\frac{p}{N}}} \gtrsim \left(\frac{1}{n}\right)^{\frac{p}{N}+\frac{q}{N}+\frac{2}{N-1}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{c(n)}^{\frac{n}{N}} S(y)e^{-Ny}dy \\
&\geq \int_{c(n)}^{\frac{n}{N}} \left(\phi_{N,q,0}\left(\alpha(\frac{N}{\omega_{N-1}n})^{\frac{1}{N-1}} y^{\frac{N}{N-1}}\right) - \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}} y^q \right. \\
&\qquad \left. -\lambda \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}(\frac{N}{\omega_{N-1}n})^{\frac{q}{N}+\frac{p}{N(N-1)}} y^{q+\frac{p}{N-1}} \right) \\
&\qquad \cdot e^{-Ny}dy \\
&\gtrsim -\left(\frac{1}{n}\right)^{\frac{q}{N}+\frac{p}{N(N-1)}} \int_{c(n)}^{\frac{n}{N}} y^{q+\frac{p}{N-1}} e^{-Ny}dy \geq -n^{\frac{q(N-1)}{N}+\frac{p}{N}} \int_{c(n)}^{\frac{n}{N}} e^{-Ny}dy \\
&\gtrsim -n^{\frac{q(N-1)}{N}+\frac{p}{N}} e^{-Nc(n)}.
\end{aligned}$$

Since $c(n) \sim n^{\frac{1}{N}}$ and n is sufficiently large, we conclude

$$III \gtrsim \left(\frac{1}{n}\right)^{\frac{p}{N}+\frac{q}{N}+\frac{2}{N-1}} - n^{\frac{q(N-1)}{N}+\frac{p}{N}} e^{-Nc(n)} \geq 0$$

Thus

$$\begin{aligned}
\frac{1}{\|u_n\|_q^q} \int_{\mathbb{R}^N} \frac{\phi_{N,q,0}(\alpha u_n^{\frac{N}{N-1}})}{1+u^{\frac{p}{N-1}}}dx = \frac{I+II}{A+B} &> \frac{\frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}A + III + \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}B}{A+B} \\
&\geq \frac{\alpha^{\frac{q(N-1)}{N}}}{(\frac{q(N-1)}{N})!}, \qquad \square
\end{aligned}$$

Proof of Theorem 1.5.. We recall that

$$TME_{p,q,N,\lambda,\alpha,\beta} = \sup_{u \in D^{N,q}(\mathbb{R}^N):\ \|\nabla u\|_N \leq 1} \frac{1}{\|u\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u^{\frac{N}{N-1}}\right)}{\left(1+\lambda|u|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

where $0 \leq \beta < N$, $0 < \lambda$, $q > 1$, $p > 0$ ($p > q$ if $\alpha = \alpha_N$), and

$$\Phi_{N,q,\beta}(t)=\begin{cases}\displaystyle\sum_{j\in\mathbb{N},\ j>\frac{q(N-1)}{N}\left(1-\frac{\beta}{N}\right)}\frac{t^j}{j!} \text{ if } \beta>0.\\ \displaystyle\sum_{j\in\mathbb{N},\ j\geq\frac{q(N-1)}{N}}\frac{t^j}{j!} \text{ if } \beta=0.\end{cases}.$$

Let (u_k) be a maximizing sequence of $TME_{p,q,N,\lambda,\alpha,\beta}$ in $D^{N,q}\left(\mathbb{R}^N\right)$ such that $\|\nabla u_k\|_N\leq 1$, i.e.,

$$\frac{1}{\|u_k\|_q^{q\left(1-\frac{\beta}{N}\right)}}\int\limits_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|u_k\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}}dx\to TME_{p,q,N,\lambda,\alpha,\beta}.$$

By symmetrization arguments, we can also assume that each u_k is radially nonnegative nonincreasing function. Now, setting

$$v_k(x)=u_k(\lambda_k x)\ \text{ where } \lambda_k=\|u_k\|_q^{q/N},$$

then we have that v_k is radially nonnegative nonincreasing function in $D^{N,q}\left(\mathbb{R}^N\right)$; $\|\nabla v_k\|_N\leq 1$; $\|v_k\|_q=1$ and

$$\begin{aligned}&\frac{1}{\|u_k\|_q^{q\left(1-\frac{\beta}{N}\right)}}\int\limits_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)u_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|u_k\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}}dx\\ &=\frac{1}{\|v_k\|_q^{q\left(1-\frac{\beta}{N}\right)}}\int\limits_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}}dx\\ &=\int\limits_{\mathbb{R}^N}\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^{\beta}}dx\\ &\to TME_{p,q,N,\lambda,\alpha,\beta}.\end{aligned}$$

Hence, we can assume without loss of generality that

$$\begin{aligned}&v_k\rightharpoonup v \text{ weakly in } D^{N,q}\left(\mathbb{R}^N\right);\\ &v_k\to v \text{ a.e. in } \mathbb{R}^N;\|\nabla v\|_N\leq 1;\|v\|_q\leq 1.\end{aligned}$$

Case 1: $\beta>0$

We note here that for all $R > 0$:

$$1 = \|v_k\|_q^q = \int\limits_{\mathbb{R}^N} |v_k|^q \, dx \geq \left|S^{N-1}\right| \int\limits_0^R |v_k(r)|^q \, r^{N-1} dr$$

$$\geq \left|S^{N-1}\right| |v_k(R)|^q \int\limits_0^R r^{N-1} dr = \left|S^{N-1}\right| |v_k(R)|^q \frac{R^N}{N}.$$

Hence

$$v_k(R) \leq \left(\frac{N}{|S^{N-1}|}\right)^{1/q} \frac{1}{R^{N/q}}.$$

We now fix $\varepsilon > 0$ and set $R_\varepsilon = \left(\frac{N}{|S^{N-1}|\varepsilon}\right)^{1/N}$. Then for every $R \geq R_\varepsilon : v_k(R) \leq \varepsilon$.

We denote $j_{N,q,\beta} \geq 1$ to be the smallest natural number such that $j_{N,q,\beta} > \frac{q(N-1)}{N}\left(1 - \frac{\beta}{N}\right)$. Then for $\gamma(\beta) \gtrapprox 0$:

$$\int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right) v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda |v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx$$

$$\leq \frac{1}{R_\varepsilon^{\gamma(\beta)}} \int\limits_{|x|\geq R_\varepsilon} \sum_{j\geq j_{N,q,\beta}} \frac{\left(\alpha\left(1-\frac{\beta}{N}\right) v_k^{\frac{N}{N-1}}\right)^j}{j!} \frac{dx}{|x|^{\beta-\gamma(\beta)}}$$

$$\leq \frac{1}{R_\varepsilon^{\gamma(\beta)}} \alpha_N^{j_{N,q,\beta}} \int\limits_{|x|\geq R_\varepsilon} \exp\left(\alpha\left(1-\frac{\beta}{N}\right) v_k^{\frac{N}{N-1}}\right) v_k^{\frac{N}{N-1} j_{N,q,\beta}} \frac{dx}{|x|^{\beta-\gamma(\beta)}}$$

$$\leq \frac{1}{R_\varepsilon^{\gamma(\beta)}} \alpha_N^{j_{N,q,\beta}} \int\limits_{|x|\geq R_\varepsilon} \exp\left(\alpha\left(1-\frac{\beta}{N}\right) v_k^{\frac{N}{N-1}}\right) v_k^{\frac{N}{N-1} j_{N,q,\beta}} \frac{dx}{|x|^{\beta-\gamma(\beta)}}$$

$$\leq \frac{1}{R_\varepsilon^{\gamma(\beta)}} \alpha_N^{j_{N,q,\beta}} C_{N,q,\beta} \|v_k\|_q^{q\left(1-\frac{\beta-\gamma(\beta)}{N}\right)} \text{ (by Theorem B)}$$

$$\leq \frac{1}{R_\varepsilon^{\gamma(\beta)}} \alpha_N^{j_{N,q,\beta}} C_{N,q,\beta} \ (\to 0 \text{ as } \varepsilon \to 0).$$

Now, consider

$$\int\limits_{|x|<R_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right) v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda |v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} dx.$$

If $\alpha = \alpha_N$: Since $\frac{1+\lambda|s|^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}}{1+\lambda|s|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}} \to 0$ as $|s| \to \infty$, we can find L_ε (that goes to ∞ as $\varepsilon \to 0$) such that $\frac{1+\lambda|s|^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}}{1+\lambda|s|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}} \leq \varepsilon$ for every $|s| \geq L_\varepsilon$. Then

$$\int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

$$\leq \varepsilon \int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{q}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

$$\leq \varepsilon C_{p,q,N,\lambda,\beta}\ (\to 0 \text{ as } \varepsilon \to 0).$$

If $0 < \alpha < \alpha_N$: then again since $\frac{1}{1+\lambda|s|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}} \to 0$ as $|s| \to \infty$, we can find L_ε (that goes to ∞ as $\varepsilon \to 0$) such that $\frac{1}{1+\lambda|s|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}} \leq \varepsilon$ for every $|s| \geq L_\varepsilon$. Then

$$\int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx$$

$$\leq \varepsilon \int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{|x|^\beta}dx \leq \varepsilon C_{p,q,N,\lambda,\alpha,\beta}(\to 0 \text{ as } \varepsilon \to 0).$$

It remains to consider

$$I_{k,\varepsilon} = \int\limits_{|x|<R_\varepsilon;\ |v_k|<L_\varepsilon} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}dx.$$

But by Dominated Convergence Theorem, it is easy to deduce that (since $\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} \to \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta}$ a.e. and $\frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)|x|^\beta} \leq \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)L_\varepsilon^{\frac{N}{N-1}}\right)}{|x|^\beta} \in L^1(B_{R_\varepsilon})$)

$$\overline{\lim}_{k\to\infty} I_{k,\varepsilon} \leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)\left|x\right|^{\beta}}dx$$

Hence, when we let $\varepsilon \to 0$, we have

$$TME_{p,q,N,\lambda,\alpha,\beta} \leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)\left|x\right|^{\beta}}dx.$$

Thus $v \neq 0$ and then

$$\begin{aligned} TME_{p,q,N,\lambda,\alpha,\beta} &\leq \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v^{\frac{N}{N-1}}\right)}{\left(1+\left|v\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)\left|x\right|^{\beta}}dx \\ &\leq \frac{1}{\left\|v\right\|_q^{q\left(1-\frac{\beta}{N}\right)}} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,\beta}\left(\alpha\left(1-\frac{\beta}{N}\right)v^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v\right|^{\frac{p}{N-1}\left(1-\frac{\beta}{N}\right)}\right)\left|x\right|^{\beta}}dx. \end{aligned}$$

As a consequence, $\left\|v\right\|_q = 1$ and v is a maximizer for $TME_{p,q,N,\lambda,\alpha,\beta}$.

Case 2: $\beta = 0$; $\frac{q(N-1)}{N} \notin \mathbb{N}$

We can denote $j_{N,q} \geq 1$ to be the smallest natural number such that $j_{N,q} \geq \frac{q(N-1)}{N}$ (actually, $j_{N,q} > \frac{q(N-1)}{N}$ since $\frac{q(N-1)}{N} \notin \mathbb{N}$). In this case, we have

$$\begin{aligned} \int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}dx &\leq \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=0}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{j_{N,q}+j}}{(j_{N,q}+j)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}dx \\ &\leq \int\limits_{|x|\geq R_\varepsilon} v_k^{\frac{N}{N-1}j_{N,q}-q} \frac{\sum\limits_{j=0}^{\infty}\frac{\alpha_N^{j_{N,q}+j} v_k^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}dx \\ &\leq \varepsilon^{\frac{N}{N-1}j_{N,q}-q} \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=0}^{\infty}\frac{\alpha_N^{j_{N,q}+j} v_k^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}dx \\ &\leq \varepsilon^{\frac{N}{N-1}j_{N,q}-q} C_{p,q,N,\lambda}\left\|v_k\right\|_q^q \\ &= \varepsilon^{\frac{N}{N-1}j_{N,q}-q} C_{p,q,N,\lambda} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Here, the last inequality comes from Remark 1, since the function

$$\Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)=\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}$$

satisfying

$$\Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)\leq C_{N,q}\exp\left(\alpha|u|^{\frac{N}{N-1}}\right);$$
$$\Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)\leq C_{N,q}|u|^{q}\ \text{ for every }\ |u|\leq 1.$$

(the second one is clear since the smallest power in $\Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)$ is $|u|^q$; To explain the first one, we note that $j_{N,q}>\frac{q(N-1)}{N}>j_{N,q}-1\geq 0$:

$$\Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)=\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{q+\frac{N}{N-1}j}}{(j_{N,q}+j)!}=\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{\frac{N}{N-1}(j_{N,q}-1+j)}u^{\frac{q(N-1)}{N}+1-j_{N,q}}}{(j_{N,q}+j)!}.$$

Since $0<\frac{q(N-1)}{N}+1-j_{N,q}<1$, we can find two positive numbers A and B such that $u^{\frac{q(N-1)}{N}+1-j_{N,q}}\leq Au+B$. Hence

$$\begin{aligned}\Phi\left(\alpha|u|^{\frac{N}{N-1}}\right)&\leq\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{\frac{N}{N-1}(j_{N,q}-1+j)}(Au+B)}{(j_{N,q}+j)!}\\&\leq A\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j}u^{\frac{N}{N-1}(j_{N,q}+j)}}{(j_{N,q}+j)!}+B\alpha_N\sum_{j=0}^{\infty}\frac{\alpha^{j_{N,q}+j-1}u^{\frac{N}{N-1}(j_{N,q}+j-1)}}{(j_{N,q}+j-1)!}\\&\leq C_{N,q}\exp\left(\alpha|u|^{\frac{N}{N-1}}\right)).\end{aligned}$$

The integral $\int\limits_{|x|<R_\varepsilon}\frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v_k|^{\frac{p}{N-1}}\right)}dx$ can be dealed similarly as in Case 1. Hence, again we have that v is a maximizer for $TME_{p,q,N,\lambda,\alpha,\beta}$.

Case 3: $\beta=0$; $\frac{q(N-1)}{N}\in\mathbb{N}$

We set

$$F(u)=\frac{\Phi_{N,q,0}\left(\alpha u^{\frac{N}{N-1}}\right)}{\left(1+\lambda|u|^{\frac{p}{N-1}}\right)}-\frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}u^q.$$

Hence

$$TME_{p,q,N,\lambda,\alpha,0} = \lim_{k\to\infty} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}dx$$
$$= \lim_{k\to\infty} \int\limits_{\mathbb{R}^N} F(v_k) + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

Again, we will first consider here $\int\limits_{|x|\geq R_\varepsilon} F(v_k)dx$. We have

$$\int\limits_{|x|\geq R_\varepsilon} F(v_k)dx \leq \int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right) - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}v_k^q}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}$$
$$= \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=0}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}v_k^q}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}$$
$$= \int\limits_{|x|\geq R_\varepsilon} \frac{\sum\limits_{j=1}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}$$
$$\leq \int\limits_{|x|\geq R_\varepsilon} \alpha v_k^{\frac{N}{N-1}} \frac{\sum\limits_{j=0}^{\infty}\frac{\left(\alpha v_k^{\frac{N}{N-1}}\right)^{\frac{q(N-1)}{N}+j}}{\left(\frac{q(N-1)}{N}+j\right)!}}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}$$
$$\leq \alpha\varepsilon^{\frac{N}{N-1}} \int\limits_{|x|\geq R_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{p}{N-1}}\right)}$$
$$\leq \alpha\varepsilon^{\frac{N}{N-1}} C_{N,q,p,\lambda} \ (\to 0 \text{ as } \varepsilon\to 0).$$

Also, as in the above cases:

$$\int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} F(v_k) \leq \varepsilon \int\limits_{|x|<R_\varepsilon;\ |v_k|\geq L_\varepsilon} \frac{\Phi_{N,q,0}\left(\alpha v_k^{\frac{N}{N-1}}\right)}{\left(1+\lambda\left|v_k\right|^{\frac{q}{N-1}}\right)} \leq \varepsilon C_{N,q,p,\lambda} \ (\to 0 \text{ as } \varepsilon\to 0).$$

Considering $\int\limits_{|x|<R_\varepsilon;\ |v_k|<L_\varepsilon} F(v_k)$, again we can use the Dominated Convergence Theorem to conclude that

$$\lim_{k\to\infty} \int\limits_{|x|<R_\varepsilon;\ |v_k|<L_\varepsilon} F(v_k) = \int\limits_{\{|x|<R_\varepsilon\}\cap A_{v,\varepsilon}} F(v)$$

$$\chi_{A_{v,\varepsilon}} = \lim_{k\to\infty} \chi_{\{|v_k|<L_\varepsilon\}}.$$

Hence, we have

$$TME_{p,q,N,\lambda,\alpha,0} \leq \alpha\varepsilon^{\frac{N}{N-1}} C_{N,q,p,\lambda} + \varepsilon C_{N,q,p,\lambda} + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} + \int\limits_{\{|x|<R_\varepsilon\}\cap A_{v,\varepsilon}} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx.$$

Letting $\varepsilon \to 0$, and noting that $R_\varepsilon \to \infty;\ L_\varepsilon \to \infty$, again by the Dominated Convergence Theorem we get

$$TME_{p,q,N,\lambda,\alpha,0} \leq \int\limits_{\mathbb{R}^N} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}.$$

If $TME_{p,q,N,\lambda,\alpha,0} > \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$, we have that $v \neq 0$ and

$$\begin{aligned} TME_{p,q,N,\lambda,\alpha,0} &\leq \int\limits_{\mathbb{R}^N} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \\ &\leq \frac{1}{\|v\|_q^q} \int\limits_{\mathbb{R}^N} \left(\frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}}\right)} - \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} v^q \right) dx + \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!} \\ &= \frac{1}{\|v\|_q^q} \int\limits_{\mathbb{R}^N} \frac{\Phi_{N,q,0}\left(\alpha v^{\frac{N}{N-1}}\right)}{\left(1+\lambda|v|^{\frac{p}{N-1}}\right)} dx. \end{aligned}$$

In other words, v is a maximizer for $TME_{p,q,N,\lambda,\alpha,0}$.

Hence $TME_{p,q,N,\lambda,\alpha,0}$ is not attained only if $TME_{p,q,N,\lambda,\alpha,0} = \frac{\alpha^{\frac{q(N-1)}{N}}}{\left(\frac{q(N-1)}{N}\right)!}$.

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