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Sharp subcritical Moser–Trudinger inequalities on Heisenberg groups and subelliptic PDEs[☆]

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ABSTRACT

The aim of this paper is to prove a sharp subcritical Moser–Trudinger inequality on the whole Heisenberg group. Let $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ be the $n-$dimensional Heisenberg group, $Q = 2n+2$ be the homogeneous dimension of $\mathbb{H}$, $Q' = \frac{Q}{Q-1}$, and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ be the homogeneous norm of $\xi = (z, t) \in \mathbb{H}$. Then we establish the following inequality on $\mathbb{H}$ (Theorem 1.1): there exists a positive constant $\alpha_Q = Q\left(2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1}\right)^{Q'-1}$ such that for any pair $\beta,\ \alpha$ satisfying $0 \leq \beta < Q$, $0 < \alpha < \alpha_Q(1 - \frac{\beta}{Q})$ there exists a constant $0 < C_{\alpha,\beta} = C(\alpha, \beta) < \infty$ such that the following inequality holds

$$\sup_{\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \leq 1} \frac{1}{\|u\|^{Q-\beta}_{L^Q(\mathbb{H})}} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha |u|^{Q/(Q-1)}\right) - \sum_{k=0}^{Q-2} \frac{\alpha^k}{k!} |u|^{kQ/(Q-1)} \right\} \leq C_{\alpha,\beta}.$$

The above result is the best possible in the sense when $\alpha \geq \alpha_Q(1 - \frac{\beta}{Q})$, the integral is still finite for any $u \in W^{1,Q}(\mathbb{H})$, but the supremum is infinite.

In contrast to the analogous inequality in Euclidean spaces proved in Adachi and Tanaka (1999) [6] using symmetrization, our argument is completely different and avoids the symmetrization method which is not available on the Heisenberg group in an optimal way. Moreover, our restriction on the norm $\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \leq 1$ of the function u is much weaker than $\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} + \|u\|_{L^Q(\mathbb{H})} \leq 1$ which was assumed in Lam and Lu (2012) [16]. As a consequence, our inequality fails at $\alpha = \alpha_Q(1 - \frac{\beta}{Q})$ in contrast to the one in [16].

As an application of this inequality, we will prove that the following nonlinear subelliptic equation of Q-Laplacian type without perturbation:

$$-\Delta_Q u + V(\xi)|u|^{Q-2} u = \frac{f(\xi, u)}{\rho(\xi)^\beta} \text{ in } \mathbb{H} \tag{0.1}$$

has a nontrivial weak solution, where the nonlinear term f has the critical exponential growth $e^{\alpha|u|^{\frac{Q}{Q-1}}}$ as $u \to \infty$, but does not satisfy the Ambrosetti–Rabinowitz condition.

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1. Introduction

Geometric inequalities are very important tools in the study of geometric analysis, a mathematical discipline at the interface of differential geometry and partial differential equations. Sobolev embedding can be considered as one of such inequalities. Basically, the Sobolev inequality asserts that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ when $kp < n$, where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain, $1 \leq q \leq \frac{np}{n-kp}$ and that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q < \infty$ when $kp = n$. However, it can be showed by many examples that $W_0^{k,\frac{n}{k}}(\Omega) \not\subseteq L^\infty(\Omega)$. In fact, Yudovich [1], Pohozaev [2] and Trudinger [3] proved independently that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(\beta |t|^{n/(n-1)}\right) - 1$ for some $\beta > 0$. It was established in his 1971 paper [4] by J. Moser the following inequality using the symmetrization argument:

Theorem A (*Moser–Trudinger Inequality*). *Let Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$, $n \geq 2$. Then there exists a sharp constant $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit n-ball, such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq c_0$$

for any $\alpha \leq \alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$. This constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality can no longer hold with some c_0 independent of u.

There have been many generalizations related to the Moser–Trudinger inequality. For instance, Adimurthi and Sandeep in [5] established an interpolation of Hardy inequality and Moser–Trudinger inequality and proved that with $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $|\Omega| < \infty$, there exists a constant $C_0 = C_0(n) > 0$ such that

$$\frac{1}{|\Omega|^{1-\frac{\alpha}{n}}}\int_\Omega \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \leq C_0$$

for any $\beta \in [0, n)$, $0 \leq \alpha \leq \left(1 - \frac{\beta}{n}\right)\alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$. Moreover, this constant $\left(1 - \frac{\beta}{n}\right)\alpha_n$ is sharp in the sense that if $\alpha > \left(1 - \frac{\beta}{n}\right)\alpha_n$, then the above inequality can no longer hold with some C_0 independent of u.

Another interesting extension is to study the Moser–Trudinger inequality on unbounded domains. In fact, when Ω has infinite volume, the above results are trivial. Adachi and Tanaka [6], using Moser's approach of symmetrization, proved that

Theorem B. *For any $\alpha \in (0, \alpha_n)$, there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^n} \phi\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq C_\alpha \|u\|_n^n, \quad \forall u \in W^{1,n}\left(\mathbb{R}^n\right), \quad \|\nabla u\|_n \leq 1,$$

where

$$\phi(t) = e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}.$$

This inequality is false for $\alpha \geq \alpha_n$ in the sense that there is no finite C_α such that the inequality holds uniformly for all u.

It is particularly interesting to note that in this situation, the constant α_n cannot be achieved. Namely, the inequality fails when $\alpha = \alpha_n$.

In order to obtain a Moser–Trudinger type inequality including the critical case α_n, Ruf [7] and Li–Ruf [8] used the full norm of the Sobolev space $W^{1,n}(\mathbb{R}^n)$, namely $\left[\|u\|_n^n + \|\nabla u\|_n^n\right]^{1/n}$, instead of $\|\nabla u\|_n$, to set up the result in the critical case $\alpha = \alpha_n$. These results were generalized recently in [9] where they proved that for all $\alpha \leq \left(1 - \frac{\beta}{n}\right)\alpha_n$ and $\tau > 0$,

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^n} \frac{\phi\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx < \infty$$

where

$$\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^n} \left(|\nabla u|^n + \tau |u|^n\right) dx\right)^{1/n}.$$

Moreover, this constant $\left(1 - \frac{\beta}{n}\right)\alpha_n$ is sharp in the sense that if $\alpha > \left(1 - \frac{\beta}{n}\right)\alpha_n$, then the supremum is infinity.

All the proofs of the above theorems use symmetrization argument in Euclidean spaces. More precisely, for a given function f we denote $\lambda_f(t) = |\{x : |f(x)| > t\}$ and let $f^\sharp$ be defined by

$$f^\sharp(s) = \inf\{t : \lambda_f(t) \le s\}.$$

Then we can define the non-increasing rearrangement f^* of f by $f^*(x) = f^\sharp(c_n|x|^n)$, where c_n is the volume of the unit ball in $\mathbb{R}^n$. Thus, we have (see e.g. [10])

$$\|f^*\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{and} \quad \|\nabla f^*\|_{L^p(\mathbb{R}^n)} \le \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

for $1 \le p < \infty$.

These properties of the rearrangement functions allow them to reduce the proofs of Theorems A and B to radial functions [4,6]. Nevertheless, such rearrangement inequalities are not true on the Heisenberg group. Therefore, analogous theorems on the Heisenberg group to Theorems A and B become more difficult to prove.

The first main purpose of this paper is to establish the sharp subcritical Moser–Trudinger type inequalities on Heisenberg groups. To state our theorems, we shall begin with some preliminaries.

Let $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ be the n-dimensional Heisenberg group whose group structure is given by

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\mathrm{Im}(z \cdot \overline{z'})),$$

for any two points (z, t) and (z', t') in $\mathbb{H}$. The Lie algebra of $\mathbb{H}$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \qquad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

for $i = 1, \dots n$. These generators satisfy the non-commutative relationship

$$[X_i, Y_j] = -4\delta_{ij} T.$$

Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with Heisenberg group structure which is usually denoted as

$$\delta_r(z, t) = (rz, r^2 t).$$

However, for simplicity we will write ru to denote $\delta_r u$. The Jacobian determinant of δ_r is r^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}$.

We use $\xi = (z, t)$ to denote any point $(z, t) \in \mathbb{H}$ and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ to denote the homogeneous norm of $\xi \in \mathbb{H}$. With this norm, we can define a Heisenberg ball centered at $\xi = (z, t)$ with radius R : $B(\xi, R) = \{v \in \mathbb{H} : |\xi^{-1}v| < R\}$. The volume of such a ball is $\sigma_Q = C_Q R^Q$ for some constant depending on Q.

We use $|\nabla_{\mathbb{H}} f|$ to express the norm of the subelliptic gradient of the function $f : \mathbb{H} \to R$:

$$|\nabla_{\mathbb{H}} f| = \left(\sum_{i=1}^{n} ((X_i f)^2 + (Y_i f)^2) \right)^{1/2}.$$

Let Ω be an open set in $\mathbb{H}$. We use $W_0^{1,p}(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ under the norm $\|f\|_{W^{1,p}(\Omega)} = (\int_\Omega (|\nabla_{\mathbb{H}^n} f|^p + |f|^p) du)^{1/p}$.

As pointed out earlier, it is not known whether or not the L^p norm of the subelliptic gradient of the rearrangement of a function is dominated by the L^p norm of the subelliptic gradient of the function on the Heisenberg group $\mathbb{H}$. In other words, an inequality like

$$\| \nabla_{\mathbb{H}} u^* \|_{L^p} \le \| \nabla_{\mathbb{H}} u \|_{L^p}$$

is not available on $\mathbb{H}$. Thus, in order to establish the Moser–Trudinger inequality on bounded domains on the Heisenberg group, we also have to avoid the rearrangement argument. Nevertheless, Cohn and Lu [11], using a sharp representation formula on the Heisenberg group, adapted D. Adams' idea (see [12]) to avoid considering the subelliptic gradient of the rearrangement function. Instead, they considered the rearrangement of the convolution of the subelliptic gradient with an optimal kernel (see also [13] in the case of complex spheres).

The sharp constant for the Moser–Trudinger inequality on domains of finite measure in the Heisenberg group established in [11] is stated as follows:

Theorem C. *Let $\alpha_Q = Q\left(2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1}\right)^{Q'-1}$. Then there exists a uniform constant C_0 depending only on Q such that for all $\Omega \subset \mathbb{H}$, $|\Omega| < \infty$ and $\alpha \le \alpha_Q$*

$$\sup_{u \in W_0^{1,Q}(\Omega),\ \|\nabla_{\mathbb{H}} u\|_{L^Q} \le 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha\, |u(\xi)|^{Q'}) d\xi \le C_0 < \infty. \tag{1.1}$$

The constant α_Q is the best possible in the sense that if $\alpha > \alpha_Q$, then the supremum in the inequality (1.1) is infinite.

Using similar ideas of [11], we considered in [14] the sharp singular Moser–Trudinger inequality on bounded domains in the Heisenberg group $\mathbb{H}$. This is stated as follows:

Theorem D. *There exists a constant C_0 depending only on $Q, \ \beta$ such that for all $\Omega \in \mathbb{H}$, $|\Omega| < \infty$, and for all $u \in W_0^{1,Q}(\Omega)$,*

$$\frac{1}{|\Omega|^{1-\frac{\beta}{Q}}} \int_{\Omega} \frac{\exp\left(\alpha_Q \left(1-\frac{\beta}{Q}\right) |u(\xi)|^{Q'}\right)}{\rho(\xi)^{\beta}} d\xi \leq C_0,$$

provided $\|\nabla_{\mathbb{H}} u\|_{L^Q} \leq 1$. Furthermore, if $\alpha_Q(1-\frac{\beta}{Q})$ is replaced by any larger number, then the above statement is false.

The situation is more complicated when concerning the Moser–Trudinger type inequalities for unbounded domains on Heisenberg group since the Adams' approach does not work. In this case, the authors of [15] established a non-optimal Moser–Trudinger inequality by using a symmetrization argument. Recently, the first two authors of this paper developed a new idea of establishing sharp constants for Moser–Trudinger inequalities on unbounded domains of Heisenberg groups and for Adams inequalities on high order Sobolev spaces on unbounded domains without using rearrangement argument [16, 17]. With this new method, we can set up the following sharp Moser–Trudinger type inequality on unbounded domain in [16]:

Theorem E. *Let τ be any positive real number. Then for any pair $\beta, \ \alpha$ satisfying $0 \leq \beta < Q$ and $0 < \alpha \leq \alpha_Q(1-\frac{\beta}{Q})$, there holds*

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{\exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u)\right\} < \infty. \tag{1.2}$$

When $\alpha > \alpha_Q(1-\frac{\beta}{Q})$, the integral in (1.2) is still finite for any $u \in W^{1,Q}(\mathbb{H})$, but the supremum is infinite. Here

$$\|u\|_{1,\tau} = \left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q\right]^{1/Q},$$

$$S_{Q-2}(\alpha, u) = \sum_{k=0}^{Q-2} \frac{\alpha^k}{k!} |u|^{kQ/(Q-1)}.$$

We notice that in the above result, we used the restriction of the full norm of the Sobolev space $W^{1,Q}(\mathbb{H}) : \left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q\right]^{1/Q}$.

These results on the Heisenberg group raised a very interesting question: Is the analogous theorem to Theorem B on the Heisenberg group true? Namely, can we only impose the restriction on the norm $\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q$ without restricting the full norm

$$\left[\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q + \tau \int_{\mathbb{H}} |u|^Q\right]^{1/Q} \leq 1?$$

The proof of Theorem B by Adachi–Tanaka [6] in the Euclidean spaces requires a symmetrization argument which is not available on the Heisenberg group. Therefore, it is nontrivial to know if such an inequality holds on the Heisenberg group.

In this paper, we will use a rearrangement-free argument to prove the singular Moser–Trudinger type inequality in the spirit of Adachi–Tanaka. More precisely, we will prove that

Theorem 1.1. *For any pair $\beta, \ \alpha$ satisfying $0 \leq \beta < Q$ and $0 < \alpha < \alpha_Q(1-\frac{\beta}{Q})$ there exists a constant $0 < C_{\alpha,\beta} = C(\alpha, \beta) < \infty$ such that the following inequality holds*

$$\sup_{\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \leq 1} \frac{1}{\|u\|_{L^Q(\mathbb{H})}^{Q-\beta}} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{\exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u)\right\} \leq C_{\alpha,\beta}. \tag{1.3}$$

The above result is sharp in the sense when $\alpha \geq \alpha_Q(1-\frac{\beta}{Q})$, the integral in (1.3) is still finite for any $u \in W^{1,Q}(\mathbb{H})$, but the supremum is infinite.

In fact, to prove the first part of Theorem 1.1, we will prove the following more general result:

Theorem 1.2. *Let β be a nonnegative real number satisfying $0 \leq \beta < Q$ and $\{\alpha_k\}_{k=0}^{\infty}$ be a positive sequence satisfying the following: there exist constants $0 < \alpha < \alpha_Q(1-\frac{\beta}{Q})$ and $C = C(\alpha) > 0$ such that $\sum_{k=Q-1}^{\infty} \alpha_k |x|^{kQ/(Q-1)} \leq C(\alpha) e^{\alpha|x|^{Q/(Q-1)}}$, $\forall |x| \geq 1$. Then there exists a constant $0 < C_{\alpha,\beta} = C(\alpha, \beta) < \infty$ such that the following inequality holds*

$$\sup_{\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \leq 1} \frac{1}{\|u\|_{L^Q(\mathbb{H})}^{Q-\beta}} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^{\beta}} \left\{\sum_{k=Q-1}^{\infty} \alpha_k |u|^{kQ/(Q-1)}\right\} \leq C_{\alpha,\beta}.$$

It is clear that if we choose $\alpha_k = \frac{\alpha^{kQ/(Q-1)}}{k!}$ in Theorem 1.2, we will have Theorem 1.1.

We note here that our proof of Theorem 1.2 (and hence Theorem 1.1) does not rely on the method of symmetrization which was used by Adachi–Tanaka in [6] on the Euclidean space. As a matter of fact, such a symmetrization is not available on the Heisenberg group $\mathbb{H}$ as we pointed out earlier. Therefore, the argument in [6] does not work on $\mathbb{H}$.

It is important to observe that there is a sharp difference between Theorems 1.1 and E. The inequality (1.2) in Theorem E holds for all $\alpha \le (1-\frac{\beta}{Q})\alpha_Q$, while the inequality (1.3) in Theorem 1.1 only holds for $\alpha < (1-\frac{\beta}{Q})\alpha_Q$. This indicates the restriction of Sobolev norms on the functions under consideration has a substantial impact on the sharp constants for the geometric inequalities.

As an application of our results proved in this paper, we will study and investigate some properties of the solutions to the Q-sub-Laplacian equation

$$-\Delta_Q u + V(\xi)\,|u|^{Q-2}\,u = \frac{f(\xi,u)}{\rho(\xi)^{\beta}} \quad \text{in } \mathbb{H}, \tag{1.4}$$

where $\Delta_Q u = \operatorname{div}_{\mathbb{H}}\left(|\nabla_{\mathbb{H}} u|^{Q-2}\,\nabla_{\mathbb{H}} u\right)$. When the nonlinear term f satisfies the Ambrosetti–Rabinowitz condition (see [18,19]), the existence of a nonnegative solution has been established in [16]. We will deal with the case when f does not satisfy the Ambrosetti–Rabinowitz condition in this paper.

We assume that $f : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(\xi,u) = 0$ for all $(\xi,u) \in \mathbb{H} \times (-\infty,0]$ and f behaves like $\exp\left(\alpha\,|u|^{Q/(Q-1)}\right)$ as $|u| \to \infty$. More precisely, we assume the following growth conditions on the nonlinearity $f(\xi,u)$:

$(f1)$ There exist constants $\alpha_0,\ b_1,\ b_2 > 0$ such that for all $(\xi,u) \in \mathbb{H} \times \mathbb{R}$,

$$|f(\xi,u)| \le b_1\,|u|^{Q-1} + b_2\left[\exp\left(\alpha_0\,|u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0,u)\right],$$

$(f2)$ $L(\xi,u) \le L(\xi,v)$ for all $\xi \in \mathbb{H}$ and $0 < u < v$, where

$$F(\xi,s) = \int_0^s f(\xi,\tau)d\tau,$$
$$L(\xi,\tau) = uf(\xi,\tau) - QF(\xi,\tau).$$

$(f3)$ $\lim_{u\to\infty} \frac{F(\xi,u)}{|u|^Q} = \infty$ uniformly on $\xi \in \mathbb{H}$.

$(f4)$ There exists $c > 0$ such that for all $(\xi,s) \in \mathbb{H} \times \mathbb{R}^+ : F(\xi,s) \le c\,|s|^Q + cf(\xi,s)$.

$(f5)$ $\limsup_{s\to 0+} \frac{QF(\xi,s)}{|s|^Q} < \lambda_1(Q)$ uniformly in $\xi \in \mathbb{H}$ where

$$\lambda_1(Q) = \inf\left\{\frac{\|u\|_X^Q}{\int_{\mathbb{H}} \frac{|u|^Q}{\rho(\xi)^{\beta}}d\xi} : u \in X \setminus \{0\}\right\}.$$

$(f6)$ $\lim_{s\to\infty} sf(\xi,s)\exp\left(-\alpha_0\,|s|^{Q/(Q-1)}\right) = \infty$ uniformly on compact subsets of $\mathbb{H}$.

We also assume that the potential satisfies

$(V1)$ $V : \mathbb{H} \to \mathbb{R}$ is a continuous function bounded from below by a positive constant V_0; and one of the two following conditions:

$(V2)$ For every $M > 0$, $\mu\left(\{\xi \in \mathbb{H} : V(\xi) \le M\}\right) < \infty$.

$(V3)$ The function $[V(\xi)]^{-1}$ is in $L^1(\mathbb{H})$.

The main features of our equation are that it is defined in the whole Heisenberg group $\mathbb{H}$ (therefore it has the non-compact nature for the problem) and that the singular nonlinearity is with the critical growth, but does not satisfy the classical Ambrosetti–Rabinowitz condition. The failure of the Ambrosetti–Rabinowitz condition for the nonlinear term f adds extra difficulty (we refer the reader to [20–22]) where nonlinear equations and systems in Euclidean spaces have been considered and existence theorems have been proved when the nonlinear terms do not satisfy the Ambrosetti–Rabinowitz condition. In spite of a possible failure of the Palais–Smale compactness condition, in this paper, we still use a version of the Mountain-pass approach due to Cerami [23,24] for the critical growth to derive a nontrivial weak solution. More precisely, we will prove in this paper that:

Theorem 1.3. *Suppose that $(V1)$ and $(V2)$ (or $(V3)$) and $(f1)$–$(f6)$ are satisfied. Then Eq. (1.4) has a nontrivial weak solution.*

We are now ready to make the following remarks. First of all, all the results, including the sharp critical and subcritical Moser–Trudinger inequalities (Theorems 1.1 and E) and existence of nontrivial solutions of the subelliptic PDEs of exponential growth on the Heisenberg group (Theorem 1.3), hold true for more general groups such as stratified (also know as Carnot) groups. Using the same rearrangement-free argument, we can first extend Theorems 1.1 and E to the arbitrary Carnot (stratified) groups. Second, for functions which are restricted to be in the class of first-layer symmetric on the groups of Heisenberg type, our Theorem 1.1 has been extended to weighted Moser–Trudinger inequalities on unbounded domains with sharp constants by the first and third authors in [25] (we note that our Theorem 1.1 does not restrict to this class of functions). Third, best constants for critical and subcritical Moser–Trudinger inequalities on hyperbolic spaces of any dimension have been established by the second and third authors in [26] using a rearrangement-free argument. It is worthwhile to note that the symmetrization method on the hyperbolic space does not work to establish such sharp singular Moser–Trudinger inequalities on the entire hyperbolic space.

Following our first remark, we now state that the sharp critical Moser–Trudinger inequality on the Heisenberg group (Theorem E) can be extended to the following:

Theorem 1.4. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q and τ be any positive real number. Let $\nabla_{\mathbb{G}}u$ be the subelliptic gradient on $\mathbb{G}$. Then for any pair $\beta,\ \alpha$ satisfying $0 \le \beta < Q,\ 0 < \alpha \le \alpha_Q \left(1 - \frac{\beta}{Q}\right)$, there holds*

$$\sup_{u \in W^{1,Q}(\mathbb{G}), \|u\|_{1,\tau} \le 1} \int_{\mathbb{G}} \frac{\phi\left(\alpha\, |u(\xi)|^{\frac{Q}{Q-1}}\right)}{N\,(\xi)^{\beta}} d\xi < \infty.$$

Moreover, the constant α_Q is sharp in the sense that if $\alpha > \alpha_Q \left(1 - \frac{\beta}{Q}\right)$, then the supremum is infinite. Here

$N$is the homogeneous norm on $\mathbb{G}$,

$$\alpha_Q = Q\left[\int_{\mathbb{S}} |\nabla_{\mathbb{G}} N(\xi)|^{Q}\, d\sigma(\xi)\right]^{\frac{1}{Q-1}},$$

$$\mathbb{S} = \{N = 1\},$$

$$\|u\|_{1,\tau} = \left[\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u(\xi)|^{Q}\, d\xi + \tau \int_{\mathbb{G}} |u(\xi)|^{Q}\, d\xi\right]^{1/Q}$$

$$\phi(t) = e^{t} - \sum_{j=0}^{Q-2} \frac{t^{j}}{j!}.$$

Next, the sharp subcritical Moser–Trudinger inequality on the Heisenberg group (Theorem 1.1) can be generalized to the following:

Theorem 1.5. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q. Let $\nabla_{\mathbb{G}}u$ be the subelliptic gradient on $\mathbb{G}$. Then for any pair $\beta,\ \alpha$ satisfying $0 \le \beta < Q,\ 0 < \alpha < \alpha_Q \left(1 - \frac{\beta}{Q}\right)$, there holds*

$$\sup_{u \in W^{1,Q}(\mathbb{G}), \|\nabla_{\mathbb{G}} u\|_{L^Q} \le 1} \int_{\mathbb{G}} \frac{\phi\left(\alpha\, |u(\xi)|^{\frac{Q}{Q-1}}\right)}{N(\xi)^{\beta}} d\xi < \infty.$$

Moreover, the constant α_Q is sharp in the sense that if $\alpha \ge \alpha_Q \left(1 - \frac{\beta}{Q}\right)$, then the supremum is infinite.

The proofs of Theorems 1.4 and 1.5 on the Carnot group are identical to those of Theorems 1.1 and E with very minimal modifications. We have chosen in this paper to present our results and their proofs on the Heisenberg group only for the purpose of clarity and simplicity. We note the Moser–Trudinger inequality on domains of finite measure in the Carnot group was given in [27] which extends the results on the Heisenberg group and groups of Heisenberg type in [11,28]. For analysis on Carnot (stratified) groups, we refer to [29,30].

The paper is organized as follows. We will prove the sharp subcritical Moser–Trudinger inequality Theorems 1.1 and 1.2 in Section 2. The existence of a nontrivial weak solution to Eq. (1.4) when the nonlinear term f does not satisfy the well-known Ambrosetti–Rabinowitz condition will be studied in Section 3.

2. Proof of Theorems 1.1 and 1.2

We will begin with the proof of Theorem 1.2 from which the first part of Theorem 1.1 follows.

2.1. Proof of Theorem 1.2

It is enough to prove that for all $u \in C_0^{\infty}(\mathbb{H}) \setminus \{0\}$, $u \ge 0$ and $\|\nabla_{\mathbb{H}} u\|_Q = 1$, there holds

$$\int_{\mathbb{H}} \frac{\Phi_Q(|u|^{\frac{Q}{Q-1}})}{\rho(\xi)^{\beta}} d\xi \le C_{\alpha,\beta} \|u\|_Q^{Q-\beta},$$

where

$$\Phi_Q(t) = \sum_{k=Q-1}^{\infty} \alpha_k |t|^k .$$

Set $\Omega(u) = \{\xi \in \mathbb{H} : u > 1\}$. Since $u \in C_0^\infty(\mathbb{H})$, $\Omega(u)$ is a bounded domain. Moreover, we have $\int_{\mathbb{H}} |u|^Q \geq \int_{\Omega(u)} |u|^Q \geq |\Omega(u)|$.

Now, we split the integral as follows:

$$\int_{\mathbb{H}} \frac{\Phi_Q\left(|u|^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = I_1 + I_2,$$

where

$$I_1 = \int_{\mathbb{H}\setminus\Omega(u)} \frac{\Phi_Q\left(|u|^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi,$$

$$I_2 = \int_{\Omega(u)} \frac{\Phi_Q\left(|u|^{\frac{Q}{Q-1}}\right)}{\rho\,(\xi)^\beta} d\xi.$$

We first estimate I_1. Since $u \leq 1$ in $\mathbb{H} \setminus \Omega(u)$, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{H}\setminus\Omega(u)} \frac{\Phi_Q\left(|u|^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\ &\leq \int_{\{u\leq 1\}} \frac{1}{\rho(\xi)^\beta} \sum_{k=Q-1}^{\infty} \alpha_k |u|^{k\frac{Q}{Q-1}} \\ &\leq \int_{\{u\leq 1\}} \frac{1}{\rho(\xi)^\beta} \sum_{k=Q-1}^{\infty} \alpha_k |u|^{Q} \\ &\leq C\,(\alpha)\,e^\alpha \int_{\{u\leq 1,\ \rho(\xi)\leq \|u\|_Q\}} \frac{1}{\rho(\xi)^\beta} |u|^Q + C\,(\alpha)\,e^\alpha \int_{\{u\leq 1,\ \rho(\xi)> \|u\|_Q\}} \frac{1}{|\rho(\xi)^\beta} |u|^Q. \end{aligned}$$

Now, since $\beta < Q$, we can fix $\gamma > 0$ such that $0 < \gamma < Q - \beta$ (say $\gamma = \frac{Q-\beta}{2}$), then

$$\begin{aligned} \int_{\{u\leq 1,\ \rho(\xi)\leq \|u\|_Q\}} \frac{1}{\rho(\xi)^\beta} |u|^Q &\leq \int_{\{u\leq 1,\ \rho(\xi)\leq \|u\|_Q\}} \frac{1}{\rho(\xi)^\beta} |u|^\gamma \\ &\leq \left(\int_{\rho(\xi)\leq \|u\|_Q} \left(\frac{1}{\rho(\xi)^\beta}\right)^{\frac{1}{1-\gamma/Q}} \right)^{1-\gamma/Q} \left(\int_{\mathbb{H}} (|u|^\gamma)^{\frac{Q}{\gamma}} \right)^{\frac{\gamma}{Q}} \\ &= \|u\|_Q^\gamma \left(\int_\Sigma \int_0^{\|u\|_Q} r^{Q-1-\frac{\beta Q}{Q-\gamma}} dr d\mu(\xi^*) \right)^{\frac{Q-\gamma}{Q}} \\ &= C_{\alpha,\beta} \|u\|_Q^\gamma \|u\|_Q^{Q-\beta-\gamma} \\ &= C_{\alpha,\beta} \|u\|_Q^{Q-\beta}, \end{aligned}$$

where in the second inequality, we used the Hölder inequality.

We also have that

$$\begin{aligned} \int_{\{u\leq 1,\ \rho(\xi)> \|u\|_Q\}} \frac{1}{\rho(\xi)^\beta} |u|^Q &\leq \frac{1}{\|u\|_Q^\beta} \int_{\mathbb{H}} |u|^Q \\ &= \|u\|_Q^{Q-\beta}. \end{aligned}$$

Therefore we get the following inequality:

$$I_1 \leq C_{\alpha,\beta} \|u\|_Q^{Q-\beta}.$$

To estimate I_2, we first notice that if we set $v(\xi) = u(\xi) - 1$ in $\Omega(u)$, then $v(\xi) \in W_0^{1,Q}(\Omega(u))$, and $\|\nabla_{\mathbb{H}} v\|_Q = \|\nabla_{\mathbb{H}} u\|_Q = 1$. Moreover in $\Omega(u)$,

$$|u(\xi)|^{\frac{Q}{Q-1}} = (v(\xi) + 1)^{\frac{Q}{Q-1}} \leq (1+\varepsilon)|v(\xi)|^{\frac{Q}{Q-1}} + \left(1 - \frac{1}{(1+\varepsilon)^{Q-1}}\right)^{\frac{1}{1-Q}}$$

for any small $\varepsilon > 0$, where we use the following elementary inequality:

$$(a+b)^p - b^p \le \varepsilon b^p + \left(1 - (1+\varepsilon)^{-\frac{1}{p-1}}\right)^{1-p},$$

for all $a, b > 1$ and $p > 1$.

Now since $0 < \alpha < \alpha_Q(1 - \frac{\beta}{Q})$, we can fix $\varepsilon = \frac{\alpha_Q}{\alpha}(1 - \frac{\beta}{Q}) - 1 > 0$, and set $C_\varepsilon = (1 - (1+\varepsilon)^{-\frac{1}{p-1}})^{1-p}$. Then C_ε is a constant only depending on α, β.

Since $\Omega(u)$ is bounded, using Theorem D, we get

$$\begin{aligned}
I_2 &= \int_{\Omega(u)} \frac{\Phi_Q\left(|u|^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi \\
&\le C_\alpha \int_{\Omega(u)} \frac{\exp\left(\alpha |u|^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} \\
&= C_\alpha \int_{\Omega(u)} \frac{\exp\left(\alpha (v+1)^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} \\
&\le C_\alpha \int_{\Omega(u)} \frac{\exp\left(\alpha (1+\varepsilon)|v|^{\frac{Q}{Q-1}} + \alpha C_\varepsilon\right)}{\rho(\xi)^\beta} \\
&\le C_\alpha \int_{\Omega(u)} \frac{\exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |v|^{\frac{Q}{Q-1}} + \alpha C_\varepsilon\right)}{\rho(\xi)^\beta} \\
&\le C_\alpha e^{\alpha C_\varepsilon} |\Omega(u)|^{1-\frac{\beta}{Q}} \\
&\le C_{\alpha,\beta} \|u\|_Q^{Q-\beta}.
\end{aligned}$$

Thus

$$\int_{\mathbb{H}} \frac{\Phi_Q\left(|u|^{\frac{Q}{Q-1}}\right)}{\rho(\xi)^\beta} d\xi = I_1 + I_2 \le C_{\alpha,\beta} \|u\|_Q^{Q-\beta}.$$

This completes the proof of Theorem 1.2.

2.2. Proof of Theorem 1.1

We first introduce some notations needed in the proof. Given any $\xi = (z, t)$ set $z^* = z/\rho(\xi)$, $t^* = t/\rho(\xi)^2$ and $\xi^* = (z^*, t^*)$. Thus for any $u \in \mathbb{H}$ and $\xi \neq 0$ we have $\xi^* \in \Sigma = \{\xi \in \mathbb{H} : \rho(\xi) = 1\}$.

It is clear that if we choose $\alpha_k = \frac{\alpha^{kQ/(Q-1)}}{k!}$, we have

$$\sup_{\|\nabla_{\mathbb{H}} u\|_{L^Q(\mathbb{H})} \le 1} \frac{1}{\|u\|_{L^Q(\mathbb{H})}^{Q-\beta}} \int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{\exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u)\right\} < \infty$$

for any pair β, α satisfying $0 \le \beta < Q$ and $0 < \alpha < \alpha_Q(1 - \frac{\beta}{Q})$. Also, the fact that $\frac{1}{\rho(\xi)^\beta} \left\{\exp\left(\alpha |u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, u)\right\} \in L^1(\mathbb{H})$ for all $u \in W^{1,Q}(\mathbb{H})$ can be found in [15] or [16].

Now, we will verify that the constant $\alpha_Q(1 - \frac{\beta}{Q})$ is our best possible. Indeed, we choose the sequence $\{u_k\}$ as follows

$$u_k(\xi) = \frac{1}{\alpha_Q^{\frac{Q-1}{Q}}} \begin{cases} k^{\frac{Q-1}{Q}} & \text{if } 0 \le \rho(\xi) \le e^{-k/Q}, \\ -k^{-\frac{1}{Q}} Q \ln \rho(\xi), & \text{if } e^{-k/Q} \le \rho(\xi) \le 1, \\ 0, & \text{if } 1 < \rho(\xi). \end{cases}$$

We can verify that

$$|\nabla_{\mathbb{H}} u_k| = \frac{1}{\alpha_Q^{\frac{Q-1}{Q}}} k^{-1/Q} Q \frac{|z|}{\rho(\xi)^2} \chi_{B(0,1)\setminus B(0,e^{-k})} \quad \text{where } \xi = (z, t) \in \mathbb{H}$$

since $|\nabla_{\mathbb{H}} \rho(\xi)| = \frac{|z|}{\rho(\xi)}$.

By [11], $\alpha_Q = Qc_Q^{1/(Q-1)}$, where $c_Q = \int_\Sigma |z^*|^Q d\mu(\xi^*)$. Then we get

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u_k|^Q = \frac{Q^Q}{\alpha_Q^{Q-1}} \int_\Sigma \int_{e^{-k/Q}}^1 k^{-1} |z^*|^Q \frac{1}{r} dr d\mu(\xi^*) = \frac{Q^Q}{\alpha_Q^{Q-1}} c_Q k^{-1} \int_{e^{-k/Q}}^1 \frac{1}{r} dr = 1.$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{H}} |u_k|^Q &= \frac{1}{\alpha_Q^{Q-1}} \int_\Sigma \int_{e^{-k/Q}}^1 \left(k^{\frac{Q-1}{Q}} Q \frac{\ln r}{k} \right)^Q r^{Q-1} dr d\mu(\xi^*) \\
&\quad + \frac{1}{\alpha_Q^{Q-1}} \int_\Sigma \int_0^{e^{-k/Q}} \left(k^{\frac{Q-1}{Q}} \right)^Q r^{Q-1} dr d\mu(\xi^*) \\
&\le \frac{1}{k} \frac{\int_\Sigma d\mu(\xi^*)}{\alpha_Q^{Q-1}} \int_\Sigma d\mu(\xi^*) \int_0^1 r^{Q-1} (\ln r)^Q dr + \frac{k^{Q-1}}{e^k} \frac{\int_\Sigma d\mu(\xi^*)}{Q\alpha_Q^{Q-1}} \\
&\overset{k\to\infty}{\to} 0.
\end{aligned}$$

Thus, we can conclude that $\{u_k(\xi)\}_{k=1}^\infty \subset W^{1,Q}(\mathbb{H})$.

Moreover, we have

$$\begin{aligned}
&\int_{\mathbb{H}} \frac{1}{\rho(\xi)^\beta} \left\{ \exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u_k|^{Q/(Q-1)} \right) - S_{Q-2}\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right), u_k \right) \right\} d\xi \\
&= \int_{\mathbb{H}} \frac{\exp\left(\alpha_Q \left(1 - \frac{\beta}{Q}\right) |u_k|^{\frac{Q}{Q-1}} \right) - \sum_{j=0}^{Q-2} \frac{\left(\alpha_Q\left(1-\frac{\beta}{Q}\right)\right)^j}{j!} |u_k|^{j\frac{Q}{Q-1}}}{\rho(\xi)^\beta} d\xi \\
&= \int_{e^{-k/Q} \le \rho(\xi) \le 1} \frac{\exp\left(\left(1 - \frac{\beta}{Q}\right) Q^{\frac{Q}{Q-1}} k^{-\frac{1}{Q-1}} |\ln \rho(\xi)|^{\frac{Q}{Q-1}} \right)}{\rho(\xi)^\beta} d\xi \\
&\quad - \int_{e^{-k/Q} \le \rho(\xi) \le 1} \frac{\sum_{j=0}^{Q-2} \frac{\left(1-\frac{\beta}{Q}\right)^j Q^{j\frac{Q}{Q-1}}}{j!} k^{-\frac{j}{Q-1}} |\ln \rho(\xi)|^{j\frac{Q}{Q-1}}}{\rho(\xi)^\beta} d\xi + \int_{0 \le \rho(\xi) \le e^{-k/Q}} \frac{\Phi_Q\left(\left(1 - \frac{\beta}{Q}\right) k \right)}{\rho(\xi)^\beta} d\xi \\
&\ge -\sum_{j=1}^{Q-2} k^{-j\frac{1}{Q-1}} \int_{0 \le \rho(\xi) \le 1} \frac{\left(1 - \frac{\beta}{Q}\right)^j Q^{j\frac{Q}{Q-1}}}{j! \rho(\xi)^\beta} |\ln \rho(\xi)|^{j\frac{Q}{Q-1}} d\xi \\
&\quad + \frac{\int_\Sigma d\mu(\xi^*)}{Q - \beta} \left(e^{\left(1-\frac{\beta}{Q}\right)k} - \sum_{j=0}^{Q-2} \frac{\left(\left(1 - \frac{\beta}{Q}\right) k\right)^j}{j!} \right) e^{-\left(1-\frac{\beta}{Q}\right)k} \\
&\to \frac{\int_\Sigma d\mu(\xi^*)}{Q - \beta} > 0 \quad \text{as } k \to \infty
\end{aligned}$$

since

$$\int_{0 \le \rho(\xi) \le 1} \frac{|\ln \rho(\xi)|^{j\frac{Q}{Q-1}}}{\rho(\xi)^\beta} d\xi < +\infty \quad \text{for any } j \in \{1, \dots, Q-2\}.$$

Therefore the chosen sequence satisfies that $\{u_k(\xi)\}_{k=1}^\infty \subset W^{1,Q}(\mathbb{H})$, $\|\nabla_{\mathbb{H}} u_k\|_{L^Q} = 1$ and $\frac{1}{\|u_k\|_{L^Q}^{Q-\beta}} \int_{\mathbb{H}} \frac{\exp\left(\alpha_Q\left(1-\frac{\beta}{Q}\right)|u_k|^{Q/(Q-1)}\right) - S_{Q-2}\left(\alpha_Q\left(1-\frac{\beta}{Q}\right), u_k\right)}{\rho(\xi)^\beta} d\xi \to \infty$. That completes the proof of Theorem 1.1, namely, the supremum in Theorem 1.1 is infinite when $\alpha = \alpha_Q$.

3. Q-sub-Laplace equation

3.1. Variational framework

We define the function space:

$$X = \left\{ u \in W^{1,Q}(\mathbb{H}) : \int_{\mathbb{H}} V(\xi) |u|^Q d\xi < \infty \right\}.$$

By the assumptions of the potential V, we see that X with the norm

$$\|u\|_X := \left[\int_{\mathbb{H}} \left(|\nabla_{\mathbb{H}} u|^Q + V(\xi)\,|u|^Q\right) d\xi\right]^{1/Q}$$

is a reflexive Banach space. Moreover, we also get the continuous embedding

$$X \hookrightarrow W^{1,Q}\left(\mathbb{R}^Q\right) \hookrightarrow L^q\left(\mathbb{R}^Q\right)$$

for all $Q \le q < \infty$, and the compactness of the embedding

$$X \hookrightarrow L^p\left(\mathbb{R}^Q\right) \quad \text{for all } p \ge Q.$$

Thus, we can conclude that

$$\lambda_1(Q) = \inf\left\{\frac{\|u\|_X^Q}{\int_{\mathbb{H}} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi} : u \in X \setminus \{0\}\right\} > 0 \quad \text{for any } 0 \le \beta < Q.$$

Now, by assumptions on the nonlinear term, we obtain for all $(\xi, u) \in \mathbb{H} \times \mathbb{R}$,

$$|F(\xi, u)| \le b_3\left[\exp\left(\alpha_1\,|u|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_1, u)\right]$$

for some constants $\alpha_1, b_3 > 0$. Thus, by our Theorem 1.1, $F(\xi, u) \in L^1(\mathbb{H})$ for all $u \in W^{1,Q}\left(\mathbb{R}^Q\right)$. Thus, we can define the following functional $J : X \to \mathbb{R}$ by

$$J(u) = \frac{1}{Q}\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^Q\, d\xi + \frac{1}{Q}\int_{\mathbb{H}} V(\xi)\,|u|^Q\, d\xi - \int_{\mathbb{H}} \frac{F(\xi, u)}{\rho(\xi)^\beta} d\xi.$$

Moreover, by standard arguments, J is a C^1 functional on X and $\forall u, v \in X$,

$$DJ(u)v = \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^{Q-2}\,\nabla_{\mathbb{H}} u \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi)\,|u|^{Q-2}\, v d\xi - \int_{\mathbb{H}} \frac{f(\xi, u)v}{\rho(\xi)^\beta} d\xi.$$

As a consequence, critical points of J are weak solutions of Eq. (1.4). We will search such critical points by the Mountain Pass Theorems. We stress that to use the Mountain-pass Theorem, we need to verify some types of compactness for the associated Lagrange–Euler functional, namely the Palais–Smale condition. Or at least, we must prove the boundedness of the Palais–Smale sequence. In almost all of works, we can easily establish this condition thanks to the Ambrosetti–Rabinowitz (AR) condition which is not assumed in our work. Nevertheless, we will use the following version of Mountain Pass Theorem with Cerami sequence [23,24]:

Lemma 3.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies $I(0) = 0$ and*

(i) *There are constants $\rho, \ \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$.*
(ii) *There is an $x \in X \setminus B_\rho$ such that $I(x) \le 0$.*

Let C_M be characterized by

$$C_M = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t))$$

where

$$\Gamma = \left\{\gamma \in C^0([0, 1], X),\ \gamma(0) = 0,\ \gamma(1) = x\right\}.$$

Then I possesses a $(C)_{C_M}$ sequence, i.e., there exists a sequence $\{x_n\} \subset X$ with

$$I(x_n) \to C_M, \quad \|DI(x_n)\|_{X^*}(1 + \|x_n\|_X) \to 0.$$

3.2. Basic lemmas

In this subsection, we recall some lemmas in [16].

Lemma 3.2. *For $\kappa > 0$ and $\|u\|_X \le M$ with M sufficiently small and $q > Q$, we have*

$$\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa\,|u|^{Q/(Q-1)}\right) - S_{Q-2}(\kappa, u)\right]|u|^q}{\rho(\xi)^\beta} d\xi \le C(Q, \kappa)\,\|u\|^q.$$

Lemma 3.3. *If $\kappa > 0, 0 \le \beta < Q$, $u \in X$ and $\|u\|_X \le M$ with $\kappa M^{Q/(Q-1)} < \left(1-\frac{\beta}{Q}\right)\alpha_Q$, then*

$$\int_{\mathbb{H}} \frac{\left[\exp\left(\kappa |u|^{Q/(Q-1)}\right) - S_{Q-2}(\kappa, u)\right]|u|}{\rho(\xi)^\beta} d\xi \le C(Q, M, \kappa)\|u\|_s$$

for some $s > Q$.

Lemma 3.4. *Let $\{w_k\} \subset X$, $\|w_k\|_X = 1$. If $w_k \to w \ne 0$ weakly and almost everywhere, $\nabla_{\mathbb{H}} w_k \to \nabla_{\mathbb{H}} w$ almost everywhere, then $\frac{\left[\exp\left(\alpha|w_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha, w_k)\right]}{\rho(\xi)^\beta}$ is bounded in $L^1(\mathbb{H})$ for $0 < \alpha < \alpha_Q\left(1-\frac{\beta}{Q}\right)\left(1-\|w\|_X^Q\right)^{-1/(Q-1)}$.*

3.3. Mountain pass geometry

Lemma 3.5. *There exists $\rho > 0$ such that $J(u) > 0$ if $\|u\|_X = \rho$.*

Proof. By the assumptions $(f5)$ and $(f1)$, we see that there exist $\tau,\ \delta > 0$ such that $|u| \le \delta$ implies

$$F(\xi, u) \le k_0(\lambda_1(Q) - \tau)|u|^Q \quad \text{for all } \xi \in \mathbb{H} \tag{3.1}$$

and for each $q > Q$, we can find a constant $C = C(q, \delta)$ such that for some $\kappa > 0$:

$$F(\xi, u) \le C|u|^q\left[\exp\left(\kappa|u|^{Q/(Q-1)}\right) - S_{Q-2}(\kappa, u)\right] \quad \text{for } |u| \ge \delta \text{ and } \xi \in \mathbb{H}. \tag{3.2}$$

Hence, we have

$$F(\xi, u) \le k_0(\lambda_1(Q) - \tau)|u|^Q + C|u|^q\left[\exp\left(\kappa|u|^{Q/(Q-1)}\right) - S_{Q-2}(\kappa, u)\right]$$

for all $(\xi, u) \in \mathbb{H} \times \mathbb{R}$. As a consequence, we obtain

$$\begin{aligned} J(u) &\ge \frac{1}{Q}\|u\|_X^Q - \frac{1}{Q}(\lambda_1(Q) - \tau)\int_{\mathbb{H}} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi - C\|u\|_X^q \\ &\ge \frac{1}{Q}\left(1 - \frac{(\lambda_1(Q) - \tau)}{\lambda_1(Q)}\right)\|u\|_X^Q - C\|u\|_X^q \\ &\ge \|u\|_X\left[\frac{1}{Q}\left(1 - \frac{(\lambda_1(Q) - \tau)}{\lambda_1(Q)}\right)\|u\|_X^{Q-1} - C\|u\|_X^{q-1}\right] \end{aligned} \tag{3.3}$$

by using Lemma 3.2 and noting the continuous embedding $E \hookrightarrow L^Q(\mathbb{H})$. Since $\tau > 0$ and $q > Q$, we may choose $\rho > 0$ such that $\frac{1}{Q}\left(1 - \frac{(\lambda_1(Q)-\tau)}{\lambda_1(Q)}\right)\rho^{Q-1} - C\rho^{q-1} > 0$. $\square$

Lemma 3.6. *There exists $x \in X$ with $\|x\|_X > \rho$ such that $J(x) < \inf_{\|u\|=\rho} J(u)$.*

Proof. Let $u \in E \setminus \{0\}$, $u \ge 0$ with compact support $\Omega = \operatorname{supp}(u)$. By $(f3)$, for all $M > 0$, there exists a constant $C > 0$ such that

$$\forall s \ge 0, \quad \forall \xi \in \Omega\ F(\xi, s) \ge Ms^Q - C. \tag{3.4}$$

Thus,

$$J(tu) \le \frac{t^Q}{Q}\|u\|_X^Q - Mt^Q\int_\Omega \frac{|u|^Q}{\rho(\xi)^\beta} d\xi + C|\Omega|.$$

Now, choosing $M > \frac{\|u\|_X^Q}{Q\int_\Omega \frac{|u|^Q}{\rho(\xi)^\beta} d\xi}$ and letting $t \to \infty$, we have $J_\varepsilon(tu) \to -\infty$. Setting $x = tu$ with t sufficiently large, we get the conclusion. $\square$

By Lemmas 3.5 and 3.6, we now can find a Cerami sequence at minimax level

$$C_M = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} J(\gamma(t))$$

where

$$\Gamma = \left\{\gamma \in C^0([0, 1], X),\ \gamma(0) = 0,\ \gamma(1) = x\right\}.$$

It means that there exist sequences $\{u_k\}$ and $\{\varepsilon_k\}$ such that for all $v \in X$:

$$\frac{1}{Q}\|u_k\|_X^Q - \int_{\mathbb{H}} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi \to C_M\left(1 + \|u_k\|_X\right)$$
$$\times \left| \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u_k|^{Q-1} \nabla_{\mathbb{H}} u_k \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi) |u_k|^{Q-1} u_k v d\xi - \int_{\mathbb{H}} \frac{f(\xi, u_k) v}{\rho(\xi)^\beta} d\xi \right| \le \varepsilon_k \|v\|_X \, \varepsilon_k \to 0.$$

We will now prove that this Cerami sequence is bounded. To do that, we first need to find more information about the minimax level C_M. In fact, it was proved in [16] that (see Lemma 6.2):

$$0 < C_M < \frac{1}{Q}\left(\left(1 - \frac{\beta}{Q}\right)\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}. \tag{3.5}$$

With the help of inequality (3.5), we are now able to prove the boundedness of the Cerami sequence.

Lemma 3.7. *Let $\{u_k\}$ be an arbitrary Cerami sequence associated to the functional*

$$I(u) = \frac{1}{Q}\|u\|_X^Q - \int_{\mathbb{H}} \frac{F(\xi, u)}{\rho(\xi)^\beta} d\xi$$

such that

$$\frac{1}{Q}\|u_k\|_X^Q - \int_{\mathbb{H}} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi \to C_M\left(1 + \|u_k\|_X\right)$$
$$\times \left| \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u_k|^{Q-1} \nabla_{\mathbb{H}} u_k \nabla_{\mathbb{H}} v d\xi + \int_{\mathbb{H}} V(\xi) |u_k|^{Q-1} u_k v d\xi - \int_{\mathbb{H}} \frac{f(\xi, u_k) v}{\rho(\xi)^\beta} d\xi \right| \le \varepsilon_k \|v\|_X \, \varepsilon_k \to 0$$

where $C_M \in \left(0, \frac{1}{Q}\left(\left(1 - \frac{\beta}{Q}\right)\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}\right)$. Then $\{u_k\}$ is bounded up to a subsequence.

Proof. We could suppose for a contradiction that

$$\|u_k\|_X \to \infty. \tag{3.6}$$

Now, letting

$$v_k = \frac{u_k}{\|u_k\|_X}$$

then

$$\|v_k\|_X = 1,$$
$$v_k \rightharpoonup v \quad \text{in } X \text{ (up to a subsequence).}$$

Similarly, we have $v_k^+ \rightharpoonup v^+$ in X. Here we are using the standard notation $w^+ = \max\{w, 0\}$. By assumptions on the potential V, the embedding $E \hookrightarrow L^q(\mathbb{H})$ is compact for all $q \ge Q$, we get that

$$\begin{cases} v_k^+(\xi) \to v^+(\xi) & \text{a.e. in } \mathbb{H} \\ v_k^+ \to v^+ & \text{in } L^q(\mathbb{H}), \ \forall q \ge Q. \end{cases}$$

Noting that $\{u_k\}$ is a Cerami sequence at level C_M, we see that

$$\|u_k\|_X^Q = QC_M + Q\int_{\mathbb{H}} \frac{F(\xi, u_k^+(\xi))}{\rho(\xi)^\beta} d\xi + o(1).$$

As consequences,

$$\int_{\mathbb{H}} \frac{F(\xi, u_k^+(\xi))}{\rho(\xi)^\beta} d\xi \to +\infty$$

and

$$\begin{aligned} \liminf_{k\to\infty} \int_{\mathbb{H}} \frac{F\left(\xi, u_k^+(\xi)\right)}{\rho(\xi)^\beta \left|u_k^+(\xi)\right|^Q} \left|v_k^+(\xi)\right|^Q d\xi &= \liminf_{k\to\infty} \int_{\mathbb{H}} \frac{F\left(\xi, u_k^+(\xi)\right)}{\rho(\xi)^\beta \|u_k\|_X^Q} d\xi \\ &= \liminf_{k\to\infty} \frac{\int_{\mathbb{H}} \frac{F\left(\xi, u_k^+(\xi)\right)}{\rho(\xi)^\beta} d\xi}{QC_M + Q\int_{\mathbb{H}} \frac{F(\xi, u_k^+(\xi))}{\rho(\xi)^\beta} d\xi + o(1)} \\ &= \frac{1}{Q}. \end{aligned} \tag{3.7}$$

Now, we will prove that $v^+ = 0$ a.e. $\mathbb{H}$. Indeed, if $S^+ = \left\{\xi \in \mathbb{H} : v^+(\xi) > 0\right\}$ has a positive measure, then in S^+, we have by $(f3)$ that

$$\lim_{k\to\infty} u_k^+(\xi) = \lim_{k\to\infty} v_k^+(\xi)\,\|u_k\|_X = +\infty,$$
$$\lim_{k\to\infty} \frac{F\left(\xi, u_k^+(\xi)\right)}{\rho(\xi)^\beta \left|u_k^+(\xi)\right|^Q} = +\infty \quad \text{a.e. in } S^+.$$

Thus

$$\lim_{k\to\infty} \frac{F\left(\xi, u_k^+(\xi)\right)}{\rho(\xi)^\beta \left|u_k^+(\xi)\right|^Q} \left|v_k^+(\xi)\right|^Q = +\infty \quad \text{a.e. in } S^+, \tag{3.8}$$
$$\int_{\mathbb{H}} \liminf_{k\to\infty} \frac{F\left(\xi, u_k^+(\xi)\right)}{\rho(\xi)^\beta \left|u_k^+(\xi)\right|^Q} \left|v_k^+(\xi)\right|^Q d\xi = +\infty. \tag{3.9}$$

This is a contradiction by Fatou's lemma, (3.9), (3.7) and noting that $F(\xi, s) \geq 0$. Thus, we can conclude that $v_k^+ \rightharpoonup 0$ in X.

Next, we choose $t_k \in [0, 1]$ such that

$$J(t_k u_k) = \max_{t\in[0,1]} J(tu_k).$$

For any given $M \in \left(0, \left(\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{\alpha_0}\right)^{\frac{Q-1}{Q}}\right)$, let $\varepsilon = \frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{M^{Q/(Q-1)}} - \alpha_0 > 0$. Since f has critical growth $(f1)$ on $\mathbb{H}$, there exists $C = C(M) > 0$ such that

$$F(\xi, s) \leq C\,|s|^Q + \left|\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{M^{Q/(Q-1)}} - \alpha_0\right| R\left(\alpha_0 + \frac{\varepsilon}{2}, s\right), \quad \forall (\xi, s) \in \mathbb{H}\times\mathbb{R}. \tag{3.10}$$

Since $\|u_k\|_X \to \infty$, we have

$$J(t_k u_k) \geq J\left(\frac{M}{\|u_k\|_X} u_k\right) = J(Mv_k). \tag{3.11}$$

By (3.10), $\|v_k\|_X = 1$ and the fact that $\int_{\mathbb{H}} \frac{F(\xi, v_k)}{\rho(\xi)^\beta} d\xi = \int_{\mathbb{H}} \frac{F\left(\xi, v_k^+\right)}{\rho(\xi)^\beta} d\xi$, we get

$$\begin{aligned}
QJ(Mv_k) &\geq M^Q - QCM^Q \int_{\mathbb{H}} \frac{\left|v_k^+\right|^Q}{\rho(\xi)^\beta} d\xi - Q\left|\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{M^{\frac{Q}{Q-1}}} - \alpha_0\right| \int_{\mathbb{H}} \frac{R\left(\alpha_0 + \frac{\varepsilon}{2}, M\left|v_k^+\right|\right)}{\rho(\xi)^\beta} d\xi \\
&\geq M^Q - QCM^Q \int_{\mathbb{H}} \frac{\left|v_k^+\right|^Q}{\rho(\xi)^\beta} d\xi - Q\left|\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{M^{\frac{Q}{Q-1}}} - \alpha_0\right| \int_{\mathbb{H}} \frac{R\left(\left(\alpha_0 + \frac{\varepsilon}{2}\right) M^{\frac{Q}{Q-1}}, \left|v_k^+\right|\right)}{\rho(\xi)^\beta} d\xi \\
&\geq M^Q - QCM^Q \int_{\mathbb{H}} \frac{\left|v_k^+\right|^Q}{\rho(\xi)^\beta} d\xi - Q\left|\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{M^{\frac{Q}{Q-1}}} - \alpha_0\right| \int_{\mathbb{H}} \frac{R(\alpha_1, |v_k|)}{\rho(\xi)^\beta} d\xi.
\end{aligned} \tag{3.12}$$

Here

$$\alpha_1 = \left[\alpha_0 + \frac{1}{2}\left(\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{M^{Q/(Q-1)}} - \alpha_0\right)\right] M^{\frac{Q}{Q-1}} \in \left(0, \left(1-\frac{\beta}{Q}\right)\alpha_Q\right).$$

Since $v_k^+ \rightharpoonup 0$ in X, the fact that the embedding $X \hookrightarrow L^p(\mathbb{H})$ is compact for all $p \geq Q$, using the Holder inequality, we can show easily that $\int_{\mathbb{H}} \frac{\left|v_k^+(\xi)\right|^Q}{\rho(\xi)^\beta} d\xi \overset{k\to\infty}{\to} 0$. Also, noting that $0 < \alpha_1 < \left(1-\frac{\beta}{Q}\right)\alpha_Q$, by our Theorem 1.1, $\int_{\mathbb{H}} \frac{R(\alpha_1, |v_k(\xi)|)}{\rho(\xi)^\beta} d\xi$ is bounded by a universal C.

Thus using (3.11) and letting $k \to \infty$ in (3.12), and then letting $M \to \left[\left(\frac{\left(1-\frac{\beta}{Q}\right)\alpha_Q}{\alpha_0}\right)^{\frac{Q-1}{Q}}\right]^-$, we get

$$\liminf_{k\to\infty} J\left(t_k u_k\right) \geq \frac{1}{Q}\left(\left(1-\frac{\beta}{Q}\right)\frac{\alpha_Q}{\alpha_0}\right)^{Q-1} > C_M. \tag{3.13}$$

Note that $J(0) = 0$ and $J(u_k) \to C_M$, we can now suppose that $t_k \in (0, 1)$. Thus since $DJ(t_k u_k)t_k u_k = 0$,

$$t_k^Q \left\|u_k\right\|_X^Q = \int_{\mathbb{H}} \frac{f\left(\xi, t_k u_k\right) t_k u_k}{\rho(\xi)^\beta} d\xi.$$

By $(f2)$:

$$\begin{aligned} QJ\left(t_k u_k\right) &= t_k^Q \left\|u_k\right\|_X^Q - Q\int_{\mathbb{H}} \frac{F\left(\xi, t_k u_k\right)}{\rho\left(\xi\right)^\beta} d\xi \\ &= \int_{\mathbb{H}} \frac{\left[f\left(\xi, t_k u_k\right) t_k u_k - QF\left(\xi, t_k u_k\right)\right]}{\rho(\xi)^\beta} d\xi \\ &\leq \int_{\mathbb{H}} \frac{\left[f\left(\xi, u_k\right) u_k - QF\left(\xi, u_k\right)\right]}{\rho(\xi)^\beta} d\xi \\ &= \left\|u_k\right\|_X^Q + QC_M - \left\|u_k\right\|_X^Q + o(1) \\ &= QC_M + o(1). \end{aligned}$$

This is a contraction to (3.13). This proves that $\{u_k\}$ is bounded in X. □

Now, by standard arguments (see Lemma 5.3 in [16]), noting that the sequence $\{u_k\}$ is bounded, we have

Lemma 3.8. *Let $(u_k) \subset X$ be an arbitrary Cerami sequence of J at the minimax level C_M. Then there exist a subsequence of (u_k) (still denoted by (u_k)) and $u \in X$ such that*

$$\begin{cases} \dfrac{f\left(\xi, u_k\right)}{\rho(\xi)^\beta} \to \dfrac{f\left(\xi, u\right)}{\rho(\xi)^\beta} \quad \text{strongly in } L^1_{\text{loc}}\left(\mathbb{H}\right) \\ \nabla_{\mathbb{H}} u_k(\xi) \to \nabla_{\mathbb{H}} u(\xi) \quad \text{almost everywhere in } \mathbb{H} \\ \left|\nabla_{\mathbb{H}} u_k\right|^{Q-2} \nabla_{\mathbb{H}} u_k \rightharpoonup \left|\nabla_{\mathbb{H}} u\right|^{Q-2} \nabla_{\mathbb{H}} u \quad \text{weakly in } \left(L^{Q/(Q-1)}_{\text{loc}}\left(\mathbb{H}\right)\right)^{Q-2} \\ u_k \rightharpoonup u \quad \text{weakly in } X. \end{cases}$$

Furthermore u is a weak solution of (1.4).

Thus, our work will be completed if we can prove that u is nontrivial.

3.4. *Proof of Theorem 1.3*

Suppose that $u = 0$. First, we will prove that

$$\frac{F(\xi, u_k)}{\rho(\xi)^\beta} \to 0 \quad \text{in } L^1\left(\mathbb{H}\right). \tag{3.14}$$

Indeed, by Lemma 3.8, we have by $(f4)$ and the generalized Lebesgue dominated convergence theorem that:

$$\frac{F(\xi, u_k)}{\rho(\xi)^\beta} \to 0 \quad \text{in } L^1\left(B_R\right) \text{ for all } R > 0.$$

Hence, it is enough to show that for arbitrary $\delta > 0$, we can find $R > 0$ such that

$$\int_{\rho(\xi)>R} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi \leq 3\delta.$$

First, we would like to recall the following facts: there exists $C > 0$ such that for all $(\xi, s) \in \mathbb{H} \times \mathbb{R}^+$:

$$\begin{aligned} &F(\xi, s) \leq C\left|s\right|^Q + Cf(\xi, s) \\ &F(\xi, s) \leq C\left|s\right|^Q + CR\left(\alpha_0, s\right) s \\ &\int_{\mathbb{H}} \frac{f(\xi, u_k)u_k}{\rho(\xi)^\beta} d\xi \leq C, \qquad \int_{\mathbb{H}} \frac{F(\xi, u_k)}{\rho(\xi)^\beta} d\xi \leq C. \end{aligned} \tag{3.15}$$

Using (3.15), the fact that $\|u_k\|_X$ is bounded, we can choose A and R large enough such that

$$\begin{aligned}\int_{\substack{\rho(\xi)>R\\|u_k|>A}} \frac{F(\xi,u_k)}{\rho(\xi)^\beta}d\xi &\le C\int_{\substack{\rho(\xi)>R\\|u_k|>A}} \frac{|u_k|^Q}{\rho(\xi)^\beta}d\xi + C\int_{\substack{\rho(\xi)>R\\|u_k|>A}} \frac{f(\xi,u_k)}{\rho(\xi)^\beta}d\xi\\ &\le \frac{C}{R^\beta A}\int_{\rho(\xi)>R}|u_k|^{Q+1}\,d\xi + C\frac{1}{A}\int_{\mathbb{H}}\frac{f(\xi,u_k)u_k}{\rho(\xi)^\beta}d\xi\\ &\le \frac{C}{R^\beta A}\|u_k\|_X^{Q+1} + C\frac{1}{A}\int_{\mathbb{H}}\frac{f(\xi,u_k)u_k}{\rho(\xi)^\beta}d\xi\\ &\le 2\ \delta.\end{aligned}$$

Also,

$$\begin{aligned}\int_{\substack{\rho(\xi)>R\\|u_k|\le A}} \frac{F(\xi,u_k)}{\rho(\xi)^\beta}d\xi &\le \frac{C(\alpha_0,A)}{R^\beta}\int_{\substack{\rho(\xi)>R\\|u_k|\le A}}|u_k|^Q\,d\xi\\ &\le \frac{2^{Q-1}C(\alpha_0,A)}{R^\beta}\left(\int_{\substack{\rho(\xi)>R\\|u_k|\le A}}|u_k-u_0|^Q\,d\xi + \int_{\substack{\rho(\xi)>R\\|u_k|\le A}}|u_0|^Q\,d\xi\right).\end{aligned}$$

Using the compactness of embedding $E \hookrightarrow L^q(\mathbb{H})$, $q \ge Q$ and noticing that $u_k \rightharpoonup u_0$, again we can choose R sufficiently large such that

$$\int_{\substack{\rho(\xi)>R\\|u_k|\le A}} \frac{F(\xi,u_k)}{\rho(\xi)^\beta}d\xi \le \delta.$$

Thus, we have

$$\int_{\rho(\xi)>R} \frac{F(\xi,u_k)}{\rho(\xi)^\beta}d\xi \le 3\delta.$$

As a consequence, we get (3.14) and then

$$\|u_k\|_X^Q \to Qc_M > 0. \tag{3.16}$$

Also, since $C_M \in \left(0, \frac{1}{Q}\left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0}\right)^{Q-1}\right)$, we can find $\delta > 0$ and $K \in \mathbb{N}$ such that

$$\|u_k\|_X^Q \le \left(\frac{Q-\beta}{Q}\frac{\alpha_Q}{\alpha_0} - \delta\right)^{Q-1} \quad \text{for all } k \ge K. \tag{3.17}$$

Now, if we choose $\tau > 1$ sufficiently close to 1, then by $(f1)$ we have

$$|f(\xi,u_k)u_k| \le b_1|u_k|^Q + b_2\left[\exp\left(\alpha_0|u_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0,u_k)\right]|u_k|.$$

Hence

$$\int_{\mathbb{H}}\frac{|f(\xi,u_k)u_k|}{\rho(\xi)^\beta} \le b_1\int_{\mathbb{H}}\frac{|u_k|^Q}{\rho(\xi)^\beta} + b_2\int_{\mathbb{H}}\frac{\left[\exp\left(\alpha_0|u_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0,u_k)\right]|u_k|}{\rho(\xi)^\beta}.$$

Using Hölder inequality, the compactness of the embedding $X \hookrightarrow L^q(\mathbb{H})$, we have $\int_{\mathbb{H}}\frac{|u_k|^Q}{\rho(\xi)^\beta} \to 0$ as $k \to \infty$.

Now, by Theorem 1.1, Lemma 3.3 and (3.17), we can conclude that

$$\int_{\mathbb{H}}\frac{\left[\exp\left(\alpha_0|u_k|^{Q/(Q-1)}\right) - S_{Q-2}(\alpha_0,u_k)\right]|u_k|}{\rho(\xi)^\beta} \to 0 \quad \text{as } k \to \infty.$$

Thus,

$$\int_{\mathbb{H}}\frac{|f(\xi,u_k)u_k|}{\rho(\xi)^\beta} \to 0 \quad \text{as } k \to \infty.$$

Now, since $DJ_0(u_k) \to 0$, we get $\|u_k\|_X \to 0$ and it is a contradiction.

The proof is now completed.

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