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Sharp Affine and Improved Moser–Trudinger–Adams Type Inequalities on Unbounded Domains in the Spirit of Lions

Nguyen Lam[1] · **Guozhen Lu**[2] · **Hanli Tang**[3]

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Abstract The purpose of this paper is threefold. First, we prove sharp singular affine Moser–Trudinger inequalities on both bounded and unbounded domains in $\mathbb{R}^n$. In particular, we will prove the following much sharper affine Moser–Trudinger inequality in the spirit of Lions (Rev Mat Iberoamericana 1(2):45–121, 1985) (see our Theorem 1.4): Let $\alpha_n = n\left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\right)^{\frac{1}{n-1}}$, $0 \le \beta < n$ and $\tau > 0$. Then there exists a constant $C = C(n,\beta) > 0$ such that for all $0 \le \alpha \le \left(1-\frac{\beta}{n}\right)\alpha_n$ and $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ with the affine energy $\mathcal{E}_n(u) < 1$, we have

$$\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\alpha}{(1+\mathcal{E}_n(u)^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx \le C(n,\beta)\frac{\|u\|_n^{n-\beta}}{\left|1-\mathcal{E}_n(u)^n\right|^{1-\frac{\beta}{n}}}.$$

Moreover, the constant $\left(1-\frac{\beta}{n}\right)\alpha_n$ is the best possible in the sense that there is no uniform constant $C(n,\beta)$ independent of u in the above inequality when $\alpha >$

✉ Guozhen Lu
gzlu@wayne.edu

Nguyen Lam
nhlam@pitt.edu

Hanli Tang
rainthl@163.com

1 Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

2 Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

3 School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

$\left(1-\frac{\beta}{n}\right)\alpha_n$. Second, we establish the following improved Adams type inequality in the spirit of Lions (Theorem 1.8): Let $0 \le \beta < 2m$ and $\tau > 0$. Then there exists a constant $C = C(m, \beta, \tau) > 0$ such that

$$\sup_{u \in W^{2,m}(\mathbb{R}^{2m}),\int_{\mathbb{R}^{2m}}|\Delta u|^m+\tau|u|^m\le 1}\int_{\mathbb{R}^{2m}}\frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\alpha}{(1+\|\Delta u\|_m^m)^{\frac{1}{m-1}}}|u|^{\frac{m}{m-1}}\right)}{|x|^{\beta}}dx \le C(m,\beta,\tau),$$

for all $0 \le \alpha \le \left(1-\frac{\beta}{2m}\right)\beta(2m,2)$. When $\alpha > \left(1-\frac{\beta}{2m}\right)\beta(2m,2)$, the supremum is infinite. In the above, we use

$$\phi_{p,q}(t) = e^t - \sum_{j=0}^{j_{\frac{p}{q}}-2}\frac{t^j}{j!},\quad j_{\frac{p}{q}} = \min\left\{j \in \mathbb{N} : j \ge \frac{p}{q}\right\} \ge \frac{p}{q}.$$

The main difficulties of proving the above results are that the symmetrization method does not work. Therefore, our main ideas are to develop a rearrangement-free argument in the spirit of Lam and Lu (J Differ Equ 255(3):298–325, 2013; Adv Math 231(6): 3259–3287, 2012), Lam et al. (Nonlinear Anal 95: 77–92, 2014) to establish such theorems. Third, as an application, we will study the existence of weak solutions to the biharmonic equation

$$\begin{cases} \Delta^2 u + V(x)u = f(x,u) \text{ in } \mathbb{R}^4 \\ u \in H^2\left(\mathbb{R}^4\right),\ u \ge 0 \end{cases},$$

where the nonlinearity f has the critical exponential growth.

Keywords Affine Moser–Trudinger inequalities · Best constants for Moser–Trudinger and Adams inequalities · Unbounded domains · Lions type

Mathematics Subject Classification Primary: 46E35 · 35J30 · Secondary: 46E30

1 Introduction

Sobolev spaces and geometric inequalities can be considered as one of the central tools in many areas such as analysis, differential geometry, mathematical physics, partial differential equations, calculus of variations, etc. The main aim of this paper is to study such inequalities. More precisely, we will prove many versions of Moser–Trudinger type inequalities and Adams type inequalities, which are the borderline cases of the Sobolev embeddings. Basically, the Sobolev embeddings assert that $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for $1 \le q \le \frac{np}{n-kp}$, $kp < n$, $n \ge 2$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. However, in the limiting case, $n = kp$, we can show by many examples that $W_0^{k,\frac{n}{k}}(\Omega) \not\subseteq L^\infty(\Omega)$. In this case, the Moser–Trudinger and Adams inequalities are the perfect replacement. In fact, Yudovich [41], Pohozaev [33] and Trudinger [39] worked independently and proved that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(\beta |t|^{n/(n-1)}\right) - 1$ for some $\beta > 0$. More precisely, they proved that there exist constants $\beta > 0$ and $C_n > 0$ depending only on n such that

$$\sup_{u \in W_0^{1,n}(\Omega),\ \int_\Omega |\nabla u|^n dx \le 1} \int_\Omega \exp\left(\beta |u|^{\frac{n}{n-1}}\right) dx \le C_n |\Omega| .$$

Nevertheless, the best possible constant β was not exhibited until the 1971 paper [30] of Moser. In fact, using the symmetrization argument to reduce to the one dimensional case, Moser established the following result:

Theorem (Moser [30], 1971). *Let Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$, $n \ge 2$. Then there exists sharp constant $\alpha_n = n\left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\right)^{\frac{1}{n-1}}$ such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \le c_0$$

for any $\alpha \le \alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \le 1$. This constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality can no longer hold with some c_0 independent of u.

The existence of extremal functions for Moser's inequality was first established by Carleson and Chang on balls in Euclidean spaces [5] and then extended to more general domains in [10] and [25]. Moser's inequalities have played important roles and have been widely used in geometric analysis and PDEs; see for example [13,14,36,38], the expository articles [6] and [18] and references therein.

Recently, using the L^p affine energy $\mathcal{E}_p(f)$ of f instead of the standard L^p energy of gradient $\|\nabla f\|_p$, where

$$\mathcal{E}_p(f) = c_{n,p}\left(\int_{S^{n-1}} \|D_v f\|_p^{-n} dv\right)^{-1/n},$$

$$c_{n,p} = \left(\frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}}\right)^{1/p}(n\omega_n)^{1/n},$$

$$\|D_v f\|_p = \left(\int_{\mathbb{R}^n} |v \cdot \nabla f(x)|^p dx\right)^{1/p},$$

the authors of [8] proved a sharp version of affine Moser–Trudinger inequality, namely,

Theorem ([8], 2009). *Let Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$, $n \ge 2$. Then there exists a constant $m_n > 0$ such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \le m_n$$

for any $\alpha \le \alpha_n$, *any* $u \in W_0^{1,n}(\Omega)$ *with* $\mathcal{E}_n(u) \le 1$. *The constant* α_n *is sharp in the sense that if* $\alpha > \alpha_n$, *then the above inequality can no longer hold with some* m_n *independent of* u.

It is worth noting that by the Hölder inequality and Fubini's theorem, we have that

$$\mathcal{E}_p(f) \le \|\nabla f\|_p$$

for every $f \in W^{1,p}(\mathbb{R}^n)$ and $p \ge 1$. Moreover, since the ratio $\frac{\|\nabla f\|_p}{\mathcal{E}_p(f)}$ is not uniformly bounded from above by any constant (see [8] and [42]), this affine Moser–Trudinger inequality is actually stronger than the standard Moser–Trudinger inequality.

Our first result is a sharp version of the singular affine Moser–Trudinger inequality on bounded domains which extends the result of [8]:

Theorem 1.1 *Let* Ω *be a domain with finite measure in Euclidean n-space* $\mathbb{R}^n$, $n \ge 2$ *and* $0 \le \beta < n$. *Then there exists a constant* $m_{n,\beta} > 0$ *such that*

$$\frac{1}{|\Omega|^{1-\frac{\beta}{n}}} \int_\Omega \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \le m_{n,\beta}$$

for any $\alpha \le \left(1 - \frac{\beta}{n}\right)\alpha_n$, *any* $u \in W_0^{1,n}(\Omega)$ *with* $\mathcal{E}_n(u) \le 1$. *The constant* $\left(1 - \frac{\beta}{n}\right)\alpha_n$ *is sharp in the sense that if* $\alpha > \left(1 - \frac{\beta}{n}\right)\alpha_n$, *then the above inequality can no longer hold with some* $m_{n,\beta}$ *independent of* u.

When Ω has infinite volume, the above results become meaningless. In this case, the subcritical Moser–Trudinger type inequalities for unbounded domains were considered in [7] when $n = 2$ and [9] and [1] for the general case $n \ge 2$. More precisely, they proved that for any $u \in W^{1,n}(\mathbb{R}^n)$ with $\|\nabla u\|_n \le m < 1$ and $\|u\|_n \le M < \infty$, there exists a constant $C(m, M) > 0$ and $\alpha > 0$ independent of u such that

$$\int_{\mathbb{R}^n} \phi_{n,1}\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \le C(m, M)$$

where

$$\phi_{n,1}(t) = e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}.$$

The sharp form is given as follows:

Theorem *(*[9] and [1]*) For any* $\alpha \in (0, \alpha_n)$, *there exists a constant* $C_\alpha > 0$ *such that*

$$\int_{\mathbb{R}^n} \phi_{n,1}\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \le C_\alpha \|u\|_n^n, \ \forall u \in W^{1,n}\left(\mathbb{R}^n\right), \ \|\nabla u\|_n \le 1,$$

This inequality is false for $\alpha \geq \alpha_n$.

It can be noted that unlike in the case of the bounded domains, the best constant α_n cannot be achieved. Thus, the above result can be considered as the subcritical Moser–Trudinger type inequality on unbounded domains. We also note that the proofs given in [1] and [9] use the symmetrization argument in Euclidean spaces. On the Heisenberg group where such a symmetrization argument fails, the subcritical Moser–Trudinger inequality has been established in [20].

As our next result, we will study the singular affine Moser–Trudinger type inequality on unbounded domains:

Theorem 1.2 *Let* $0 \leq \beta < n$. *For any* $\alpha \in \left(0, \left(1-\frac{\beta}{n}\right)\alpha_n\right)$, *there exists a constant* $C_{\alpha,\beta} > 0$ *such that*

$$\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx \leq C_{\alpha,\beta}\,\|u\|_n^{n-\beta},$$

for any $u \in W^{1,n}(\mathbb{R}^n)$ *with* $\mathcal{E}_n(u) \leq 1$. *This inequality is false for* $\alpha \geq \left(1-\frac{\beta}{n}\right)\alpha_n$ *in the sense that if* $\alpha \geq \left(1-\frac{\beta}{n}\right)\alpha_n$, *then the above inequality can no longer hold with some* $C_{\alpha,\beta}$ *independent of* u.

As a consequence of Theorem 1.2, we have the following singular subcritical Moser–Trudinger type inequality which extends the result of [1] for $\beta = 0$ to the singular case $0 \leq \beta < n$:

Theorem 1.3 *Let* $0 \leq \beta < n$. *For any* $\alpha \in \left(0, \left(1-\frac{\beta}{n}\right)\alpha_n\right)$, *there exists a constant* $C_{\alpha,\beta} > 0$ *such that*

$$\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx \leq C_{\alpha,\beta}\,\|u\|_n^{n-\beta},$$

for any $u \in W^{1,n}(\mathbb{R}^n)$ *with* $\|\nabla u\|_n \leq 1$. *This inequality is false for* $\alpha \geq \left(1-\frac{\beta}{n}\right)\alpha_n$ *in the sense that if* $\alpha \geq \left(1-\frac{\beta}{n}\right)\alpha_n$, *then the above inequality can no longer hold with some* $C_{\alpha,\beta}$ *independent of* u.

We notice that if we replace the norm $\|\nabla u\|_n$ by the full norm $\|\nabla u\|_n + \|u\|_n$ in the Sobolev space $W^{1,n}(\mathbb{R}^n)$, the best constants in the Moser–Trudinger inequalities in unbounded domains can be attained. Thus, they can be considered as the critical Moser–Trudinger inequalities on unbounded domains. In fact, these results are studied in the work of [34] and [24] using symmetrization argument. We also note that on the Heisenberg group where symmetrization does not work, such a sharp critical Moser–Trudinger inequality was proved in [17].

We recall that in the paper [3], the authors used the blow-up technique to study an improvement of the Moser–Trudinger inequality in the spirit of Lions [26]. In fact, they proved that

$$C_\alpha(\Omega) := \sup_{u\in W_0^{1,2}(\Omega),\ \|\nabla u\|_2\le 1} \int_\Omega e^{4\pi u^2\left(1+\alpha\|u\|_2^2\right)} dx < \infty \text{ iff } 0 \le \alpha < \lambda_1(\Omega),$$
$$\lambda_1(\Omega) = \inf_{u\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

We note that $\lambda_1(\Omega)$ is the first eigenvalue for the Dirichlet problem of the Laplace operator on $\Omega \subset \mathbb{R}^2$. It is easy to see that this inequality is stronger than the original one of Moser where 4π is the best constant, while this inequality of [3] has the constant $4\pi\left(1+\alpha\|u\|_2^2\right)$ which is larger than 4π for $u \neq 0$. This result is extended to L^p norms in the two-dimensional case in [27] and to the high dimensional case in [40] and [43]. Such a blow-up analysis technique was used by Y. X. Li in his works in proving the existence of extremal functions for Moser–Trudinger inequalities on compact Riemannian manifolds (see [21] and [22]), and has also been used to establish the existence of extremal functions of the Adams inequality for Paneitz operator on compact Riemannian manifolds of dimension four in [23] and for bi-Laplacian operator on domains in $\mathbb{R}^4$ in [28].

Our next main theorem is to establish an even sharper affine Moser–Trudinger inequality in the entire space in the spirit of P. L. Lions [26] in which he proved a sharpened version of Moser's result on domains of finite measure. More precisely, we will prove that

Theorem 1.4 *Let $0 \le \beta < n$ and $\tau > 0$. Then there exists a constant $C = C(n,\beta) > 0$ such that for all $\alpha \le \left(1-\frac{\beta}{n}\right)\alpha_n$ and $u \in C_0^\infty(\mathbb{R}^n)\setminus\{0\}$, $\mathcal{E}_n(u) < 1$, we have*

$$\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\alpha}{\left(1+\mathcal{E}_n(u)^n\right)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \le C(n,\beta)\frac{\|u\|_n^{n-\beta}}{\left|1-\mathcal{E}_n(u)^n\right|^{1-\frac{\beta}{n}}}.$$

Moreover, the constant $\left(1-\frac{\beta}{n}\right)\alpha_n$ is the best possible in the sense that if $\alpha > \left(1-\frac{\beta}{n}\right)\alpha_n$, then there is no uniform finite constant $C(n,\beta)$ independent of u such that the above inequality holds. As a consequence, we have that there exists a constant $C = C(n,\beta,\tau) > 0$ such that

$$M_{4,\alpha} \le M_{3,\alpha} \le M_{1,\alpha} \le C(n,\beta,\tau),$$
$$M_{4,\alpha} \le M_{2,\alpha} \le M_{1,\alpha} \le C(n,\beta,\tau),$$

for all $0 \le \alpha \le \left(1-\frac{\beta}{n}\right)\alpha_n$, where

$$M_{1,\alpha}=\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \mathcal{E}_n(u)^n+\tau\|u\|_n^n\le 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\alpha}{(1+\mathcal{E}_n(u)^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}}dx$$

$$M_{2,\alpha}=\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \mathcal{E}_n(u)^n+\tau\|u\|_n^n\le 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\alpha|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}}dx$$

$$M_{3,\alpha}=\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \|\nabla u\|_n^n+\tau\|u\|_n^n\le 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\alpha}{(1+\|\nabla u\|_n^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}}dx$$

$$M_{4,\alpha}=\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \|\nabla u\|_n^n+\tau\|u\|_n^n\le 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\alpha|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}}dx.$$

Moreover, the constant $\left(1-\frac{\beta}{n}\right)\alpha_n$ *in the above supremums is sharp in the sense that when* $\alpha>\left(1-\frac{\beta}{n}\right)\alpha_n$, $M_{1,\alpha}=M_{2,\alpha}=M_{3,\alpha}=M_{4,\alpha}=\infty$.

Again, since the ratio $\frac{\|\nabla f\|_p}{\mathcal{E}_p(f)}$ is not uniformly bounded from above by any constant, our affine Moser–Trudinger type inequalities (Theorems 1.1, 1.2 and 1.4) are truly stronger than the standard Moser–Trudinger type inequalities. Moreover, as a consequence of Theorem 1.4, we have the following sharp Moser–Trudinger type inequality in the whole space in the spirit of P. L. Lions:

Theorem 1.5 *Let* $0\le\beta<n$ *and* $\tau>0$. *Then there exists a constant* $C=C(n,\beta,\tau)>0$ *such that*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \|\nabla u\|_n^n+\tau\|u\|_n^n\le 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\alpha}{(1+\|\nabla u\|_n^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}}dx\le C(n,\beta,\tau)<+\infty,$$

for all $0\le\alpha\le\left(1-\frac{\beta}{n}\right)\alpha_n$. *The constant* $\left(1-\frac{\beta}{n}\right)\alpha_n$ *is the best possible in the sense that if* $\alpha>\left(1-\frac{\beta}{n}\right)\alpha_n$, *then the integral is still finite but the supremum is infinite.*

We note here that since

$$\frac{2^{\frac{1}{n-1}}\left(1-\frac{\beta}{n}\right)\alpha_n}{\left(1+\|\nabla u\|_n^n\right)^{\frac{1}{n-1}}}\ge\left(1-\frac{\beta}{n}\right)\alpha_n,$$

Theorem 1.5 is stronger than the Moser–Trudinger type inequality in [24,34].

We now turn to the discussion of high order Adams inequalities. Regarding the case of higher order derivatives, since the symmetrization is not available, D. Adams

[2] proposed a new idea to find the sharp constants for higher order Moser's type inequality, namely, to express u as the Riesz potential of its gradient of order m, and then apply O'Neil's result on the rearrangement of convolution functions and use techniques of symmetric decreasing rearrangements. To state Adams's result, we use the symbol $\nabla^m u$, m is a positive integer, to denote the m-th order gradient for $u \in C^m$, the class of m-th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{for} \quad m \quad \text{even} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{for} \quad m \quad \text{odd} \end{cases},$$

where ∇ is the usual gradient operator and $\triangle$ is the Laplacian. We use $||\nabla^m u||_p$ to denote the L^p norm $(1 \le p \le \infty)$ of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $\left[||u||^p_{L^p(\Omega)} + \sum_{j=1}^{k} ||\nabla^j u||^p_{L^p(\Omega)} \right]^{1/p}$. Then Adams proved the following:

Theorem *(*Adams [2], 1988*). Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \le 1$, then*

$$\frac{1}{|\Omega|} \int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \le C_0$$

for all $\beta \le \beta(n,m)$ where

$$\beta(n, m) = \begin{cases} \frac{n}{w_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\ \frac{n}{w_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, the constant $\beta(n,m)$ is best possible in the sense that for any $\beta > \beta(n,m)$, the integral can be made as large as possible.

It's easy to check that $\beta(n,1)$ coincides with Moser's value of α_n and $\beta(2m,m) = 2^{2m}\pi^m \Gamma(m+1)$ for both odd and even m. In fact, Adams's result was extended recently by Tarsi [37] to a larger space, namely, the Sobolev space with homogeneous Navier boundary conditions $W_N^{m,\frac{n}{m}}(\Omega)$:

$$W_N^{m,\frac{n}{m}}(\Omega) := \left\{ u \in W^{m,\frac{n}{m}} : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \le j \le \left[\frac{m-1}{2} \right] \right\}.$$

We note that the Moser–Trudinger–Adams type inequality was extended to spheres in $\mathbb{R}^n$ by Beckner in [4].

Concerning the Adams inequality for unbounded domains, in the spirit of Adachi–Tanaka [1], Ogawa and Ozawa [31] in the case $\frac{n}{m} = 2$ and Ozawa [32] in the general case proved that there exist positive constants α and C_α such that

$$\int_{\mathbb{R}^n} \phi_{n,m}\left(\alpha\, |u|^{\frac{n}{n-m}}\right) dx \le C_\alpha \, \|u\|_{\frac{n}{m}}^{\frac{n}{m}}, \ \forall u \in W^{m,\frac{n}{m}}\left(\mathbb{R}^n\right), \ \ \left\|\nabla^m u\right\|_{\frac{n}{m}} \le 1,$$

where

$$\phi_{n,m}(t) = e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!}$$

$$j_{\frac{n}{m}} = \min\left\{ j \in \mathbb{N} : j \ge \frac{n}{m} \right\} \ge \frac{n}{m}.$$

Their approach of proving the above result is similar to the idea of Yudovich [41], Pohozaev [33] and Trudinger [39] and thus, the problem of determining the best constant cannot be investigated in this way. It seems that it is still left as an open problem to determine the best constant for the above inequality. Thus, it is very interesting to identify the best constants in such inequalities.

The next aim is to study the sharp subcritical Adams type inequalities in some special cases. More precisely, we will prove that

Theorem 1.6 *For any* $\alpha \in (0, \beta(n,2))$*, there exists a constant* $C_\alpha > 0$ *such that*

$$\int_{\mathbb{R}^n} \phi_{n,2}\left(\alpha\, |u|^{\frac{n}{n-2}}\right) dx \le C_\alpha \, \|u\|_{\frac{n}{2}}^{\frac{n}{2}}, \ \ \forall u \in W^{2,\frac{n}{2}}\left(\mathbb{R}^n\right), \ \ \|\Delta u\|_{\frac{n}{2}} \le 1. \tag{1.1}$$

Theorem 1.7 *For any* $\alpha \in (0, \beta(2m,m))$*, there exists a constant* $C_\alpha > 0$ *such that*

$$\int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\, |u|^2\right) dx \le C_\alpha \, \|u\|_2^2, \ \ \forall u \in W^{m,2}\left(\mathbb{R}^{2m}\right), \ \ \left\|\nabla^m u\right\|_2 \le 1. \tag{1.2}$$

It was proved in [12] that the inequality (1.1) in Theorem 1.6 does not hold when $\alpha > \beta(n,2)$, neither does inequality (1.2) in Theorem 1.7 when $\alpha > \beta(2m,m)$.

The critical Adams type inequality was also studied using the full norm in order to get the best constant. Indeed, it was investigated in [35] when m is even and in [16] when m is odd. It was established in [15] for the fractional derivative case in Sobolev spaces of fractional orders. Moreover, the sharp singular Adams inequalities were also proved in [19]. We now state the sharp critical Adams inequality in fractional order Sobolev spaces proved by Lam and Lu in [15] as follows:

Theorem *Let* $0 < \alpha < n$ *be an arbitrary real positive number,* $p = \frac{n}{\alpha}$ *and* $\tau > 0$*. Then it holds that*

$$\sup_{u \in W^{\alpha,p}(\mathbb{R}^n),\, \left\|(\tau I - \Delta)^{\frac{\alpha}{2}} u\right\|_p \le 1} \int_{\mathbb{R}^n} \phi_{n,\alpha}\left(\beta_0(n,\alpha)\, |u|^{p'}\right) dx < \infty$$

where

$$\beta_0(n,\alpha) = \frac{n}{\omega_{n-1}}\left[\frac{\pi^{n/2}2^{\alpha}\Gamma(\alpha/2)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\right]^{p'}.$$

Furthermore, this inequality is sharp, i.e., if $\beta_0(n,\alpha)$ is replaced by any $\gamma > \beta_0(n,\alpha)$, then the supremum is infinite.

Our last main result in this paper is an improved version of the Adams type inequality in the Sobolev space $W^{2,m}\left(\mathbb{R}^{2m}\right)$. In this special case, it has been proved in [15] that: Let $0 \le \alpha < 2m$ and $\tau > 0$. Then for all $0 \le \beta \le \left(1-\frac{\alpha}{2m}\right)\beta(2m,2)$, we have

$$\sup_{u\in W^{2,m}(\mathbb{R}^{2m}),\ \int_{\mathbb{R}^{2m}}|\Delta u|^m+\tau|u|^m\le 1}\int_{\mathbb{R}^{2m}}\frac{\phi_{2m,2}\left(\beta|u|^{\frac{m}{m-1}}\right)}{|x|^{\alpha}}dx < \infty.$$

Moreover, the constant $\left(1-\frac{\alpha}{2m}\right)\beta(2m,2)$ is sharp in the sense that if $\beta > \left(1-\frac{\alpha}{2m}\right)\beta(2m,2)$, then the supremum is infinite.

We should note this result does not require the restriction on the full standard norm and hence, it extends the results in [19]. Indeed, the results there are for the special case $m=2$ and they require that the full standard norm $\int_{\mathbb{R}^4}\left(|\Delta u|^2+\sigma|\nabla u|^2+\tau|u|^2\right)dx$ is less than 1.

We are now ready to state our last main result which is an improved version of the sharp Adams inequality in the whole space in the spirit of P. L. Lions [3,26,27]:

Theorem 1.8 *Let $0 \le \beta < 2m$ and $\tau > 0$. Then there exists a constant $C = C(m,\beta) > 0$ such that for all $u \in C_0^{\infty}\left(\mathbb{R}^{2m}\right)\setminus\{0\}$, $\|\Delta u\|_m < 1$, we have for all $0 \le \alpha \le \left(1-\frac{\beta}{2m}\right)\beta(2m,2)$ the following inequality:*

$$\int_{\mathbb{R}^{2m}}\frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\alpha}{(1+\|\Delta u\|_m^m)^{\frac{1}{m-1}}}|u|^{\frac{m}{m-1}}\right)}{|x|^{\beta}}dx \le C(m,\beta)\frac{\|u\|_m^{m-\frac{\beta}{2}}}{\left|1-\|\Delta u\|_m^m\right|^{1-\frac{\beta}{2m}}}.$$

Moreover, the constant $\left(1-\frac{\beta}{2m}\right)\beta(2m,2)$ is the best possible in the sense that if $\alpha > \left(1-\frac{\beta}{2m}\right)\beta(2m,2)$, then there is no uniform finite constant $C(m,\beta)$ independent of u such that the above inequality holds.

Consequently, we have that there exists a constant $C = C(m,\beta,\tau) > 0$ such that

$$\sup_{u\in W^{2,m}(\mathbb{R}^{2m}),\ \int_{\mathbb{R}^{2m}}|\Delta u|^m+\tau|u|^m\le 1}\int_{\mathbb{R}^{2m}}\frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\alpha}{(1+\|\Delta u\|_m^m)^{\frac{1}{m-1}}}|u|^{\frac{m}{m-1}}\right)}{|x|^{\beta}}dx \le C(m,\beta,\tau),$$

for all $0 \leq \alpha \leq \left(1-\frac{\beta}{2m}\right)\beta(2m,2)$. *Moreover, the constant* $\left(1-\frac{\beta}{2m}\right)\beta(2m,2)$ *is the best possible in the sense that if* $\alpha > \left(1-\frac{\beta}{2m}\right)\beta(2m,2)$, *then the above supremum is infinite.*

As an application of our result, we will investigate the existence of nontrivial weak solutions of the following biharmonic equation:

$$\begin{cases} \Delta^2 u + V(x)u = f(x,u) \text{ in } \mathbb{R}^4 \\ u \in H^2\left(\mathbb{R}^4\right) \end{cases}, \tag{1.3}$$

where V is a continuous positive potential bounded away from zero and the nonlinearity $f(x,u)$ behaves like $\exp\left(\alpha u^2\right)$ at infinity for some $\alpha > 0$. We refer to Sect. 6 for more details on the assumptions on the potential V and the nonlinear term f.

The organization of the paper is as follows. In Sect. 2, we give the proofs of Theorem 1.1 and 1.2, i.e., the sharp affine singular Moser–Trudinger inequalities both on bounded and unbounded domains. Section 3 deals with the proof of Theorem 1.4, namely, the affine Moser–Trudinger inequality on the entire spaces in the spirit of P. L. Lions. This is one of the main theorems of this paper. Section 4 offers the proofs of the sharp subcritical Adams type inequalities on the entire spaces when the restrictions are only on the norms of the highest order derivatives in the case of the second order derivatives $m = 2$ and when the dimension $n = 2m$, i.e., Theorems 1.6 and 1.7. These are the second main results of this paper. The proof of the sharp Adams type inequality on the entire space $\mathbb{R}^n$ in the spirit of Lions, namely, Theorem 1.8, is given in Sect. 5. This is another main theorem of the paper. Section 6 includes the last main result of the paper, namely, the application of our sharp inequalities to nonlinear PDEs of bi-harmonic type equations with nonlinear term of exponential growth, i.e., the existence of nonnegative solutions to (1.3).

2 Proof of Theorems 1.1 and 1.2

2.1 Proof of Theorem 1.1

First, we note that for every $f \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1, \mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f)$ and $\mathcal{E}_p(f^\star) = \|\nabla f^\star\|_p$, where $f^\star$ is the nonincreasing spherically symmetric rearrangement of f. This can be found in [29] and [42]. Then we can assume that u is a positive smooth decreasing symmetric function and Ω is a ball $B_R = B(0,R)$. The proof now is similar to the standard Moser–Trudinger inequality using symmetrization. Indeed, we have

$$\begin{aligned} \mathcal{E}_n(u)^n &= \|\nabla u\|_n^n \\ &= \int_{B_R} |\nabla u|^n \\ &= \omega_{n-1}\int_0^R \left(u'(r)\right)^n r^{n-1}dr. \end{aligned}$$

Letting $t = \frac{\alpha}{\alpha_n} = 1 - \frac{\beta}{n}$, then we have

$$\int_{\Omega} \frac{\exp\left(\alpha_n \left(1 - \frac{\beta}{n}\right) |u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx = \int_{B_R} \frac{\exp\left(t\alpha_n |u|^{\frac{n}{n-1}}\right)}{|x|^{(1-t)n}} dx$$
$$= \omega_{n-1} \int_0^R \exp\left(t\alpha_n |u|^{\frac{n}{n-1}}\right) r^{tn-1} dr.$$

Now, we define a function v as follows:

$$v(s) = t^{\frac{n-1}{n}} u\left(s^{\frac{1}{t}}\right) \text{ for } s \in \left[0, R^t\right].$$

Then, we can check that

$$\omega_{n-1} \int_0^{R^t} \left(v'(r)\right)^n r^{n-1} dr = \omega_{n-1} \int_0^R \left(u'(r)\right)^n r^{n-1} dr$$
$$= 1$$

and

$$\frac{1}{t} \int_0^{R^t} \exp\left(\alpha_n |v(r)|^{\frac{n}{n-1}}\right) r^{n-1} dr = \int_0^R \exp\left(t\alpha_n |u|^{\frac{n}{n-1}}\right) r^{tn-1} dr.$$

Hence, we get

$$\sup_{\mathcal{E}_n(u) \le 1} \int_{\Omega} \frac{\exp\left(\alpha_n \left(1 - \frac{\beta}{n}\right) |u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx \le \frac{1}{t} \sup_{\|\nabla v\|_n \le 1} \int_{B_{R^t}} \exp\left(\alpha_n |u|^{\frac{n}{n-1}}\right) dx$$
$$= C_{n,\beta} \left|R^t\right|^n$$
$$= C_{n,\beta} |\Omega|^{1-\frac{\beta}{n}}.$$

Now, noting that

$$\sup_{\mathcal{E}_n(u) \le 1} \frac{1}{|\Omega|^{1-\frac{\beta}{n}}} \int_{\Omega} \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx \ge \sup_{\|\nabla u\|_n \le 1} \frac{1}{|\Omega|^{1-\frac{\beta}{n}}} \int_{\Omega} \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx,$$

we can conclude that $\left(1 - \frac{\beta}{n}\right) \alpha_n$ is sharp in the sense of Theorem 1.1. Namely, the supremum $\sup_{\mathcal{E}_n(u) \le 1} \frac{1}{|\Omega|^{1-\frac{\beta}{n}}} \int_{\Omega} \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx$ is infinite if $\alpha > \left(1 - \frac{\beta}{n}\right) \alpha_n$. This completes the proof of Theorem 1.1.

2.2 Proof of Theorem 1.2

Fix $\alpha \in \left(0, \left(1-\frac{\beta}{n}\right)\alpha_n\right)$, we want to prove that there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx \le C_{\alpha,\beta}\,\|u\|_n^{n-\beta}\,, \tag{2.1}$$

for any $u \in W^{1,n}(\mathbb{R}^n)$ with $\mathcal{E}_n(u) \le 1$. We will present here a new method, a rearrangement-free argument, to study the inequality (2.1). In fact, using a new idea of splitting the domain, we can prove Theorem 1.2 without using the symmetrization.

By a standard density argument, we can suppose that $u \in C_0^{\infty}(\mathbb{R}^n) \setminus \{0\}$, $u \ge 0$ and $\mathcal{E}_n(u) \le 1$.

Denote

$$\begin{aligned}
\Omega(u) &= \{x \in \mathbb{R}^n\colon\ u(x) > 1\},\\
I_1 &= \int_{\Omega(u)} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx,\\
I_2 &= \int_{\mathbb{R}^n\setminus\Omega(u)} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx.
\end{aligned}$$

First, we will estimate I_1. First, it can be noted that since $u \in C_0^{\infty}(\mathbb{R}^n)$, $\Omega(u)$ is a bounded domain. Moreover, the volume of $\Omega(u)$ satisfies

$$\int_{\mathbb{R}^n} |u|^n \ge \int_{\Omega(u)} |u|^n \ge |\Omega(u)|.$$

Second, if we set $v(x) = u(x) - 1$ in $\Omega(u)$, then it is clear that $v(x) \in W_0^{1,n}(\Omega(u))$, and $\mathcal{E}_n(v) = \mathcal{E}_n(u)$. Put $\varepsilon = \frac{\alpha_n}{\alpha}(1-\frac{\beta}{n}) - 1 > 0$. Then using the following elementary inequality:

$$(a+b)^p - b^p \le \varepsilon b^p + (1-(1+\varepsilon)^{-\frac{1}{p-1}})^{1-p} a^p,$$

for all $a, b \ge 1$ and $p > 1$, we have in $\Omega(u)$ that

$$|u(x)|^{\frac{n}{n-1}} = (v(x)+1)^{\frac{n}{n-1}} \le (1+\varepsilon)|v(x)|^{\frac{n}{n-1}} + (1-\frac{1}{(1+\varepsilon)^{n-1}})^{\frac{1}{1-n}}.$$

Hence, by Theorem 1.1,

$$\begin{aligned}
I_1 &= \int_{\Omega(u)} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx\\
&\le \int_{\Omega(u)} \frac{\exp\left(\alpha|u|^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx\\
&= \int_{\Omega(u)} \frac{\exp\left(\alpha(v+1)^{\frac{n}{n-1}}\right)}{|x|^{\beta}} dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\Omega(u)} \frac{\exp\left(\alpha(1+\varepsilon)|v|^{\frac{n}{n-1}} + \alpha C_\varepsilon\right)}{|x|^\beta} dx \\
&\leq \int_{\Omega(u)} \frac{\exp\left(\alpha_n(1-\frac{\beta}{n})|v|^{\frac{n}{n-1}} + \alpha C_\varepsilon\right)}{|x|^\beta} dx \\
&\leq e^{\alpha C_\varepsilon}|\Omega(u)|^{1-\frac{\beta}{n}} \\
&\leq C_{\alpha,\beta}\|u\|_n^{n-\beta}.
\end{aligned}
$$

To estimate I_2, we first note that $u \leq 1$ in $\mathbb{R}^n \setminus \Omega(u)$. As a consequence, we have

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^n\setminus\Omega(u)} \frac{\phi_{n,1}\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \\
&\leq \int_{\{u\leq 1\}} \frac{1}{|x|^\beta} \sum_{j=n-1}^{\infty} \frac{\alpha^k}{k!}|u|^{k\frac{n}{n-1}} \\
&\leq \int_{\{u\leq 1\}} \frac{1}{|x|^\beta} \sum_{j=n-1}^{\infty} \frac{\alpha^k}{k!}|u|^n \\
&\leq e^\alpha \int_{\{u\leq 1,\ |x|\leq\|u\|_n\}} \frac{1}{|x|^\beta}|u|^n + e^\alpha \int_{\{u\leq 1,\ |x|>\|u\|_n\}} \frac{1}{|x|^\beta}|u|^n.
\end{aligned}
$$

Now, since $0 \leq \beta < n$, we can fix $\gamma > 0$ such that $\beta < \gamma < n$ (say $\gamma = \frac{n+\beta}{2}$), then

$$
\begin{aligned}
&\int_{\{u\leq 1,\ |x|\leq\|u\|_n\}} \frac{1}{|x|^\beta}|u|^n dx \\
&\quad\leq \int_{\{u\leq 1,\ |x|\leq\|u\|_n\}} \frac{1}{|x|^\beta}|u|^\gamma dx \\
&\quad\leq \left(\int_{|x|\leq\|u\|_n} \left(\frac{1}{|x|^\beta}\right)^{\frac{1}{1-\gamma/n}} dx\right)^{1-\gamma/n} \left(\int_{\mathbb{R}^n} (|u|^\gamma)^{\frac{n}{\gamma}} dx\right)^{\frac{\gamma}{n}} \\
&\quad= C_{\alpha,\beta}\|u\|_n^\gamma\|u\|_n^{n-\beta-\gamma} \\
&\quad= C_{\alpha,\beta}\|u\|_n^{n-\beta},
\end{aligned}
$$

where in the second inequality, we used the Hölder inequality.

We also have that

$$
\begin{aligned}
&\int_{\{u\leq 1,\ |x|>\|u\|_n\}} \frac{1}{|x|^\beta}|u|^n dx \\
&\quad\leq \frac{1}{\|u\|_n^\beta} \int_{\mathbb{R}^n} |u|^n \\
&\quad= \|u\|_n^{n-\beta}.
\end{aligned}
$$

Therefore we get the following inequality:

$$I_2 \leq C_{\alpha,\beta} \|u\|_n^{n-\beta}.$$

Finally, noting that $\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx = I_1 + I_2$, we have the inequality (2.1).

Now, it remains to show that there exists a sequence $\{u_k\}$ in $W^{1,n}(\mathbb{R}^n) \setminus \{0\}$ with $\mathcal{E}_n(u_k) \leq 1$ such that

$$\frac{1}{\|u_k\|_n^{n-\beta}} \int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\alpha |u_k|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \to \infty \text{ as } k \to \infty.$$

In fact, such a sequence can be showed explicitly. More precisely, we set

$$u_k(x) = \frac{1}{\alpha_n^{\frac{n-1}{n}}} \begin{cases} k^{\frac{n-1}{n}} & \text{if } 0 \leq |x| \leq e^{-k/n} \\ k^{-\frac{1}{n}} n \ln \frac{1}{|x|}, & \text{if } e^{-k/n} \leq |x| \leq 1 \\ 0, & \text{if } 1 < |x| \end{cases}.$$

By calculation, we have $|\nabla u_k| = \frac{1}{\alpha_n^{\frac{n-1}{n}}} k^{-\frac{1}{n}} n \frac{1}{|x|} \chi_{B(0,1) \setminus B(0,e^{-k/n})}$ and then

$$\begin{aligned} \mathcal{E}_n(u_k)^n &\leq \int_{\mathbb{R}^n} |\nabla u_k|^n dx \\ &= \int_{e^{-k/n} \leq |x| \leq 1} \frac{1}{\alpha_n^{n-1}} k^{-1} n^n \frac{1}{|x|^n} dx \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} \int_{\mathbb{R}^n} |u_k|^n dx &= \frac{1}{\alpha_n^{n-1}} \int_{e^{-k/n} \leq |x| \leq 1} k^{-1} n^n \left(\ln \frac{1}{|x|}\right)^n dx \\ &\quad + \frac{1}{\alpha_n^{n-1}} \int_{0 \leq |x| \leq e^{-k/n}} k^{n-1} dx \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{\exp\left(\alpha_n (1 - \frac{\beta}{n}) |u_k|^{\frac{n}{n-1}}\right) - \sum_{j=0}^{n-2} \frac{\left(\alpha_n (1-\frac{\beta}{n})\right)^j}{j!} |u_k|^{j \frac{n}{n-1}}}{|x|^\beta} dx \\ &= \int_{e^{-k/n} \leq |x| \leq 1} \frac{\exp\left((1 - \frac{\beta}{n}) n^{\frac{n}{n-1}} k^{-\frac{1}{n-1}} \left|\ln |x|\right|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{e^{-k/n}\le|x|\le 1}\frac{\sum_{j=0}^{n-2}\frac{\left(1-\frac{\beta}{n}\right)^j n^{j\frac{n}{n-1}}}{j!}k^{-\frac{j}{n-1}}\left|\ln|x|\right|^{j\frac{n}{n-1}}}{|x|^\beta}dx\\
&+\int_{0\le|x|\le e^{-k/n}}\frac{\exp\left(k(1-\frac{\beta}{n})\right)-\sum_{j=0}^{n-2}\frac{\left(k\left(1-\frac{\beta}{n}\right)\right)^j}{j!}}{|x|^\beta}dx\\
&\ge -\sum_{j=1}^{n-2}k^{-\frac{j}{n-1}}\int_{0\le|x|\le 1}\frac{\left(1-\frac{\beta}{n}\right)^j n^{j\frac{n}{n-1}}}{j!\,|x|^\beta}\left|\ln|x|\right|^{j\frac{Q}{Q-1}}d\xi\\
&+\frac{\omega_{n-1}\left(e^{(1-\frac{\beta}{n})k}-\sum_{j=0}^{n-2}\frac{((1-\frac{\beta}{n})k)^j}{j!}\right)}{(n-\beta)\,e^{\left(1-\frac{\beta}{n}\right)k}}\\
&\to\frac{\omega_{n-1}}{n-\beta}>0 \text{ as } k\to\infty
\end{aligned}
$$

since

$$
\int_{0\le|x|\le 1}\frac{\left|\ln|x|\right|^{j\frac{n}{n-1}}}{|x|^\beta}dx<+\infty \text{ for any } j\in\{1,\ldots,n-2\}.
$$

The proof of Theorem 1.2 is now completed.

3 Proof of Theorem 1.4

First, we need to prove that there exists a constant $C=C(n,\beta)>0$ such that for $u\in C_0^\infty(\mathbb{R}^n)\setminus\{0\}$, $\mathcal{E}_n(u)<1$, we have

$$
\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\left(1-\frac{\beta}{n}\right)\alpha_n}{\left(1+\mathcal{E}_n(u)^n\right)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx\le C(n,\beta)\frac{\|u\|_n^{n-\beta}}{\left|1-\mathcal{E}_n(u)^n\right|^{1-\frac{\beta}{n}}}.
$$

Indeed, let $u\in C_0^\infty(\mathbb{R}^n)\setminus\{0\}$, $\mathcal{E}_n(u)<1,\ u\ge 0$. We fix the following notation:

$$
\begin{aligned}
A(u)&=\left(1-\mathcal{E}_n(u)^n\right)^{\frac{1}{n}},\\
\alpha(u)&=\frac{1+\mathcal{E}_n(u)^n}{2},\\
\varepsilon(u)&=\frac{\alpha(u)^{\frac{1}{n-1}}}{\mathcal{E}_n(u)^{n/(n-1)}}-1,\\
C(u)&=\left(1-\frac{1}{(1+\varepsilon(u))^{n-1}}\right)^{\frac{1}{1-n}},
\end{aligned}
$$

$$\beta(u) = \alpha\,(u)^{\frac{1}{n-1}}\,,$$
$$\Omega\,(u) = \left\{x \in \mathbb{R}^n : u(x) > A(u)\right\}.$$

We have that

$$C(u) = \left[\frac{1+\mathcal{E}_n\,(u)^n}{1-\mathcal{E}_n\,(u)^n}\right]^{\frac{1}{n-1}},$$
$$\frac{C\,(u)\,|A(u)|^{\frac{n}{n-1}}}{\beta(u)} = 2^{\frac{1}{n-1}}.$$

We note that since $u \in C_0^\infty(\mathbb{R}^n)$, $\Omega\,(u)$ is a bounded domain. Moreover, since

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^n\,dx &\geq \int_{\Omega(u)} |u|^n\,dx \\ &\geq \int_{\Omega(u)} |A(u)|^n\,dx \\ &= |A(u)|^n\,|\Omega\,(u)|\end{aligned}$$

we get

$$\begin{aligned}|\Omega\,(u)| &\leq \frac{\|u\|_n^n}{|A(u)|^n} \\ &= \frac{\|u\|_n^n}{1-\mathcal{E}_n\,(u)^n}.\end{aligned}$$

On $\Omega\,(u)$, we define functions

$$v(x) = u(x) - A(u),$$
$$w(x) = (1+\varepsilon(u))^{\frac{n-1}{n}}\,v(x).$$

Then, it's clear that $v, w \in W_0^{1,n}(\Omega\,(u))$.

Using the following elementary inequality:

$$(a+b)^p - b^p \leq \varepsilon b^p + \left(1-(1+\varepsilon)^{-\frac{1}{p-1}}\right)^{1-p} a^p,\ \forall p > 1,\ a, b, \varepsilon > 0,$$

we can deduce that

$$\begin{aligned}|u(x)|^{\frac{n}{n-1}} &= (v(x)+A(u))^{\frac{n}{n-1}} \\ &\leq (1+\varepsilon\,(u)\,)|v(x)|^{\frac{n}{n-1}} + \left(1-\frac{1}{(1+\varepsilon\,(u))^{n-1}}\right)^{\frac{1}{1-n}} |A(u)|^{\frac{n}{n-1}} \\ &= |w(x)|^{\frac{n}{n-1}} + C\,(u)\,|A(u)|^{\frac{n}{n-1}}.\end{aligned}$$

Moreover, since on $\Omega(u)$,

$$\nabla w = (1+\varepsilon(u))^{\frac{n-1}{n}} \nabla v = (1+\varepsilon(u))^{\frac{n-1}{n}} \nabla u,$$

we obtain

$$\begin{aligned}\mathcal{E}_n(w)^n &\le (1+\varepsilon(u))^{n-1}\mathcal{E}_n(u)^n\\ &= \alpha(u)\\ &= \frac{1+\mathcal{E}_n(u)^n}{2}.\end{aligned}$$

Hence, by Theorem 1.1, we have

$$\begin{aligned}&\int_{\Omega(u)} \frac{\exp\left(\frac{2^{\frac{1}{n-1}}\left(1-\frac{\beta}{n}\right)\alpha_n}{(1+\mathcal{E}_n(u)^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx\\
&\le \int_{\Omega(u)} \frac{\exp\left(\frac{\alpha_n\left(1-\frac{\beta}{n}\right)}{\beta(u)}|w(x)|^{\frac{n}{n-1}}\right)\exp\left(\frac{\alpha_n\left(1-\frac{\beta}{n}\right)C(u)|A(u)|^{\frac{n}{n-1}}}{\beta(u)}\right)}{|x|^\beta}dx\\
&\le \exp\left(\frac{\alpha_n\left(1-\frac{\beta}{n}\right)C(u)|A(u)|^{\frac{n}{n-1}}}{\beta(u)}\right)\int_{\Omega(u)}\frac{\exp\left(\frac{\alpha_n\left(1-\frac{\beta}{n}\right)}{\beta(u)}|w(x)|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx\\
&\le \exp\left(2^{\frac{1}{n-1}}\alpha_n\left(1-\frac{\beta}{n}\right)\right)\int_{\Omega(u)}\frac{\exp\left(\alpha_n\left(1-\frac{\beta}{n}\right)\left|\frac{w(x)}{\mathcal{E}_n(w)}\right|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx\\
&\le C(n,\beta)\,|\Omega(u)|^{1-\frac{\beta}{n}}\\
&= C(n,\beta)\frac{\|u\|_n^{n-\beta}}{\left|1-\mathcal{E}_n(u)^n\right|^{1-\frac{\beta}{n}}}.\end{aligned}$$

Now, noting that on the domain $\mathbb{R}^n\setminus\Omega(u)$, we have $|u(x)|\le 1$, we can deduce easily that

$$\int_{\mathbb{R}^n\setminus\Omega(u)} \frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\left(1-\frac{\beta}{n}\right)\alpha_n}{(1+\mathcal{E}_n(u)^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx \le C(n,\beta)\,\|u\|_n^{n-\beta}.$$

Hence, we can conclude that

$$\int_{\mathbb{R}^n} \frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\left(1-\frac{\beta}{n}\right)\alpha_n}{(1+\mathcal{E}_n(u)^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx \leq C(n,\beta)\frac{\|u\|_n^{n-\beta}}{\left|1-\mathcal{E}_n(u)^n\right|^{1-\frac{\beta}{n}}}.$$

Now, if we have $\mathcal{E}_n(u)^n + \tau\|u\|_n^n \leq 1$, then

$$\frac{\|u\|_n^{n-\beta}}{\left|1-\mathcal{E}_n(u)^n\right|^{1-\frac{\beta}{n}}} \leq \frac{1}{\tau^{1-\frac{\beta}{n}}}.$$

Thus

$$M_{1,\alpha} = \sup_{u\in W^{1,n}(\mathbb{R}^n),\ \mathcal{E}_n(u)^n+\tau\|u\|_n^n\leq 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\frac{2^{\frac{1}{n-1}}\alpha}{(1+\mathcal{E}_n(u)^n)^{\frac{1}{n-1}}}|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx \leq C(n,\beta,\tau).$$

Finally, when $\alpha > \left(1-\frac{\beta}{n}\right)\alpha_n$, we have that

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \|\nabla u\|_n^n+\tau\|u\|_n^n\leq 1}\int_{\mathbb{R}^n}\frac{\phi_{n,1}\left(\alpha|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx = +\infty,$$

and hence we can conclude that $\left(1-\frac{\beta}{n}\right)\alpha_n$ is sharp.

The proof is now completed.

4 Proof of Theorems 1.6 and 1.7

4.1 Proof of Theorem 1.6

Fix $\alpha \in (0, \beta(n,2))$, we will prove that there exists a constant $C_\alpha > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)\setminus\{0\}$, $u \geq 0$ and $\|\Delta u\|_{\frac{n}{2}} \leq 1$, we have

$$\int_{\mathbb{R}^n}\phi_{n,2}\left(\alpha|u|^{\frac{n}{n-2}}\right)dx \leq C_\alpha\|u\|_{\frac{n}{2}}^{\frac{n}{2}}.$$

Indeed, set

$$\Omega(u) = \left\{x\in\mathbb{R}^n : u(x) > 1\right\}.$$

Since $u \in C_0^\infty(\mathbb{R}^n)$, we have that $\Omega(u)$ is a bounded set. Moreover, we have

$$|\Omega(u)| \leq \int_{\Omega(u)} |u|^{\frac{n}{2}} \leq \|u\|_{\frac{n}{2}}^{\frac{n}{2}} < \infty.$$

Thus, on $\mathbb{R}^n \setminus \Omega(u)$, we have $|u(x)| \leq 1$. Thus,

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n \setminus \Omega(u)} \phi_{n,2}\left(\alpha\, |u|^{\frac{n}{n-2}}\right) dx \\ &\leq \int_{\mathbb{R}^n \setminus \Omega(u)} \sum_{j=j_{\frac{n}{2}}-1}^{\infty} \frac{\left(\alpha\, |u|^{\frac{2m}{2m-2}}\right)^j}{j!} dx \\ &\leq \int_{\mathbb{R}^n \setminus \Omega(u)} e^{\alpha}\, |u|^{\frac{n}{2}}\, dx \\ &\leq C_\alpha\, \|u\|_{\frac{n}{2}}^{\frac{n}{2}}. \end{aligned}$$

Next, set

$$I_2 = \int_{\Omega(u)} \phi_{n,2}\left(\alpha\, |u|^{\frac{n}{n-2}}\right) dx.$$

Since $0 < \alpha < \beta(n,2)$, we can fix $\varepsilon = \frac{\beta(n,2)}{\alpha} - 1 > 0$. On $\Omega(u)$, we define $v(x) = u(x) - 1$. Thus $v \in W_N^{2,\frac{n}{2}}(\Omega(u))$. Also, $\|\Delta v\|_{\frac{n}{2}} \leq 1$ and

$$|u(x)|^{\frac{n}{n-2}} = (v(x)+1)^{\frac{n}{n-2}} \leq (1+\varepsilon)|v(x)|^{\frac{n}{n-2}} + C_\varepsilon.$$

Here C_ε is a constant depending only on α and n.

Using the Adams inequality for bounded domains on the Sobolev space with homogeneous Navier boundary conditions $W_N^{2,\frac{n}{2}}(\Omega(u))$ (see [37]), we get

$$\begin{aligned} I_2 &= \int_{\Omega(u)} \phi_{n,2}\left(\alpha\, |u|^{\frac{n}{n-2}}\right) dx \\ &= \int_{\Omega(u)} \phi_{n,2}\left(\alpha\, |v(x)+1|^{\frac{n}{n-2}}\right) dx \\ &\leq \int_{\Omega(u)} \exp\left(\alpha(1+\varepsilon)|v(x)|^{\frac{n}{n-2}} + \alpha C_\varepsilon\right) dx \\ &\leq C_\alpha \int_{\Omega(u)} \exp\left(\beta(n,2)\, |v(x)|^{\frac{n}{n-2}}\right) dx \\ &\leq C_\alpha |\Omega(u)| \\ &\leq C_\alpha\, \|u\|_{\frac{n}{2}}^{\frac{n}{2}}. \end{aligned}$$

Finally, noting that

$$\int_{\mathbb{R}^n} \phi_{n,2}\left(\alpha\left|u\right|^{\frac{n}{n-2}}\right) dx = I_1 + I_2,$$

we get our desired result.

If $\alpha > \beta(n, 2)$, it was showed in [12] that the inequality is false.

4.2 Proof of Theorem 1.7

Lemma 4.1 *For any $\alpha \in (0, \beta(2m, m))$, there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left|u\right|^2\right) dx \le C_\alpha, \ \ \forall u \in W^{m,2}\left(\mathbb{R}^{2m}\right), \ \ \left\|\nabla^m u\right\|_2 \le 1, \ \ \|u\|_2 = 1.$$

Proof Fix $\alpha \in (0, \beta(2m, m))$. We first note that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for every $u \in W^{m,2}\left(\mathbb{R}^{2m}\right)$ and $0 < \tau < 1$:

$$\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2^2 \le (1+\varepsilon)\left\|\nabla^m u\right\|_2^2 + C_\varepsilon \tau \|u\|_2^2. \tag{4.1}$$

Indeed, by Fourier transform, we get that

$$\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2^2 = \sum_{j=0}^{m} \binom{m}{j} \tau^{m-j} \left\|\nabla^j u\right\|_2^2.$$

Now, by interpolation inequalities, noting that $0 < \tau < 1$, we can conclude (4.1).

Using (4.1), we can fix $0 < \varepsilon, \tau < 1$ such that for all $u \in W^{m,2}\left(\mathbb{R}^{2m}\right), \ \|\nabla^m u\|_2 \le 1, \ \ \|u\|_2 = 1$, we have

$$\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2^2 \le \frac{\beta(2m, m)}{\alpha}.$$

Thus, using Theorem D, we get

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left|u\right|^2\right) dx &= \int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2^2 \left|\frac{u}{\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2}\right|^2\right) dx \\
&\le \int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\beta(2m, m)\left|\frac{u}{\left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_2}\right|^2\right) dx \\
&\le C_\alpha.
\end{aligned}$$

□

Next, we will use our Lemma 4.1 to prove Theorem 1.7. Indeed, for every $u \in W^{m,2}\left(\mathbb{R}^{2m}\right)$, $\|\nabla^m u\|_2 \leq 1$, we let

$$u_\lambda(x) = u\left(\lambda x\right),$$

where

$$\lambda = \|u\|_2^{\frac{1}{m}}$$

then we have

$$\begin{gathered}
\|u_\lambda\|_2^2 = \lambda^{-2m} \ \|u\|_2^2 = 1. \\
\int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left|u_\lambda\right|^2\right) dx = \lambda^{-2m} \int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left|u\right|^2\right) dx, \\
\left\|\nabla^m u_\lambda\right\|_2 = \left\|\nabla^m u\right\|_2 \leq 1
\end{gathered}$$

Applying Lemma 4.1, we have

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left|u\right|^2\right) dx &= \lambda^{2m} \int_{\mathbb{R}^{2m}} \phi_{2m,m}\left(\alpha\left|u_\lambda\right|^2\right) dx \\
&\leq \lambda^{2m}.C_\alpha \\
&= C_\alpha \left\|u\right\|_2^2.
\end{aligned}$$

5 Proof of Theorem 1.8

The method here is similar to the proof of Theorem 1.4. Indeed, fix $u \in C_0^\infty\left(\mathbb{R}^{2m}\right) \setminus \{0\}$, $\|\Delta u\|_m < 1$, $u \geq 0$. We will use the following notation:

$$\begin{aligned}
A(u) &= \left(1 - \|\Delta u\|_m^m\right)^{\frac{1}{m}}, \\
\alpha\left(u\right) &= \frac{1 + \|\Delta u\|_m^m}{2}, \\
\varepsilon(u) &= \frac{\left(1 + \|\Delta u\|_m^m\right)^{\frac{1}{m-1}}}{2^{\frac{1}{m-1}} \ \|\Delta u\|_m^{\frac{m}{m-1}}} - 1, \\
C(u) &= \left(\frac{1 - \|\Delta u\|_m^m}{1 + \|\Delta u\|_m^m}\right)^{\frac{1}{1-m}}, \\
\beta(u) &= \left(\frac{1 + \|\Delta u\|_m^m}{2}\right)^{\frac{1}{m-1}}, \\
\Omega\left(u\right) &= \left\{x \in \mathbb{R}^{2m} : u(x) > A(u)\right\}.
\end{aligned}$$

We note here that

$$\frac{C(u)\,|A(u)|^{\frac{m}{m-1}}}{\beta(u)} = 2^{\frac{1}{m-1}}.$$

Now since $u \in C_0^\infty\left(\mathbb{R}^{2m}\right)$, we have that $\Omega(u)$ is a bounded domain and the volume of $\Omega(u)$ satisfies

$$\begin{aligned} |\Omega(u)| &\leq \frac{\|u\|_m^m}{|A(u)|^m} \\ &= \frac{\|u\|_m^m}{1 - \|\Delta u\|_m^m}, \end{aligned}$$

since

$$\begin{aligned} \int_{\mathbb{R}^{2m}} |u|^m\, dx &\geq \int_{\Omega(u)} |u|^m\, dx \\ &\geq \int_{\Omega(u)} |A(u)|^m\, dx \\ &= |A(u)|^m\, |\Omega(u)|. \end{aligned}$$

On the domain $\Omega(u)$, we define two functions

$$\begin{aligned} v(x) &= u(x) - A(u), \\ w(x) &= (1+\varepsilon(u))^{\frac{m-1}{m}}\, v(x). \end{aligned}$$

Then, it's clear that $v, w \in W_N^{2,m}(\Omega(u))$.

Now, if we make use of the following elementary inequality:

$$(a+b)^p - b^p \leq \varepsilon b^p + \left(1 - (1+\varepsilon)^{-\frac{1}{p-1}}\right)^{1-p} a^p,\ \forall p > 1,\ a, b, \varepsilon > 0,$$

we can get the following inequalities

$$\begin{aligned} |u(x)|^{\frac{m}{m-1}} &= (v(x) + A(u))^{\frac{m}{m-1}} \\ &\leq (1+\varepsilon(u))|v(x)|^{\frac{m}{m-1}} + \left(1 - \frac{1}{(1+\varepsilon(u))^{m-1}}\right)^{\frac{1}{1-m}} |A(u)|^{\frac{m}{m-1}} \\ &= |w(x)|^{\frac{m}{m-1}} + C(u)\,|A(u)|^{\frac{m}{m-1}}. \end{aligned}$$

Moreover, since on $\Omega(u)$,

$$\Delta w = (1+\varepsilon(u))^{\frac{m-1}{m}}\, \Delta v = (1+\varepsilon(u))^{\frac{m-1}{m}}\, \Delta u,$$

we obtain

$$\begin{aligned}\|\Delta w\|_m^m &\le (1+\varepsilon(u))^{m-1}\|\Delta u\|_m^m \\ &= \alpha(u) \\ &= \frac{1+\|\Delta u\|_m^m}{2}.\end{aligned}$$

Hence, by the singular Adams type inequality (Theorem 1.2 in [19]), we have

$$\begin{aligned}&\int_{\Omega(u)} \frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\left(1-\frac{\beta}{2m}\right)\beta(2m,2)}{\left(1+\|\Delta u\|_m^m\right)^{\frac{1}{m-1}}}|u|^{\frac{m}{m-1}}\right)}{|x|^\beta}dx \\ &\le \int_{\Omega(u)} \frac{\exp\left(\frac{\left(1-\frac{\beta}{2m}\right)\beta(2m,2)}{\beta(u)}|w(x)|^{\frac{m}{m-1}}\right)\exp\left(\frac{\left(1-\frac{\beta}{2m}\right)\beta(2m,2)C(u)|A(u)|^{\frac{m}{m-1}}}{\beta(u)}\right)}{|x|^\beta}dx \\ &= \exp\left(\frac{\left(1-\frac{\beta}{2m}\right)\beta(2m,2)C(u)\,|A(u)|^{\frac{m}{m-1}}}{\beta(u)}\right)\int_{\Omega(u)} \frac{\exp\left(\frac{\left(1-\frac{\beta}{2m}\right)\beta(2m,2)}{\beta(u)}|w(x)|^{\frac{m}{m-1}}\right)}{|x|^\beta}dx \\ &\le \exp\left(2^{\frac{1}{m-1}}\left(1-\frac{\beta}{2m}\right)\beta(2m,2)\right)\int_{\Omega(u)} \frac{\exp\left(\left(1-\frac{\beta}{2m}\right)\beta(2m,2)\left|\frac{w(x)}{\|\Delta w\|_m}\right|^{\frac{m}{m-1}}\right)}{|x|^\beta}dx \\ &\le C(m,\beta)\,|\Omega(u)|^{1-\frac{\beta}{2m}} \\ &= C(m,\beta)\,\frac{\|u\|_m^{m-\frac{\beta}{2}}}{\left|1-\|\Delta w\|_m^m\right|^{1-\frac{\beta}{2n}}}.\end{aligned}$$

Also, on the exterior domain $\mathbb{R}^{2m}\setminus\Omega(u)$, we have $|u(x)|\le 1$. Hence we can deduce easily that

$$\int_{\mathbb{R}^{2m}\setminus\Omega(u)} \frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\left(1-\frac{\beta}{2m}\right)\beta(2m,2)}{\left(1+\|\Delta u\|_m^m\right)^{\frac{1}{m-1}}}|u|^{\frac{m}{m-1}}\right)}{|x|^\beta}dx \le C(m,\beta)\,\|u\|_m^{m-\frac{\beta}{2}}.$$

Thus, we finally can conclude that

$$\int_{\mathbb{R}^n} \frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\left(1-\frac{\beta}{2m}\right)\beta(2m,2)}{\left(1+\|\Delta u\|_m^m\right)^{\frac{1}{m-1}}}|u|^{\frac{m}{m-1}}\right)}{|x|^\beta}dx \le C(m,\beta)\,\frac{\|u\|_m^{m-\frac{\beta}{2}}}{\left|1-\|\Delta w\|_m^m\right|^{1-\frac{\beta}{2n}}}.$$

Now, if $\int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m \leq 1$, then

$$\frac{\|u\|_m^{m-\frac{\beta}{2}}}{\left|1 - \|\Delta w\|_m^m\right|^{1-\frac{\beta}{2n}}} \leq \left(\frac{1}{\tau}\right)^{1-\frac{\beta}{2n}}.$$

As a consequence,

$$\sup_{u \in W^{2,m}(\mathbb{R}^{2m}),\ \int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m \leq 1} \int_{\mathbb{R}^{2m}} \frac{\phi_{2m,2}\left(\frac{2^{\frac{1}{m-1}}\alpha}{(1+\|\Delta u\|_m^m)^{\frac{1}{m-1}}} |u|^{\frac{m}{m-1}}\right)}{|x|^\beta} dx \leq C(m, \beta, \tau),$$

for all $0 \leq \alpha \leq \left(1 - \frac{\beta}{2m}\right) \beta(2m, 2)$.

When $\alpha > \left(1 - \frac{\beta}{2m}\right) \beta(2m, 2)$, the supremum is infinite since

$$\sup_{u \in W^{2,m}(\mathbb{R}^{2m}),\ \int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m \leq 1} \int_{\mathbb{R}^{2m}} \frac{\phi_{2m,2}\left(\left(1 - \frac{\beta}{2m}\right) \beta(2m, 2) |u|^{\frac{m}{m-1}}\right)}{|x|^\beta} dx = +\infty$$

by a result in [15].

The proof now is completed.

6 Biharmonic Equation in $\mathbb{R}^4$ with Exponential Critical Growth

This section is devoted to the study of biharmonic equation in $\mathbb{R}^4$ when the nonlinearity has the exponential critical growth in the sense of the Adams inequality. More precisely, we study the existence of nontrivial weak solutions of the critical periodic and asymptotic periodic problem:

$$\begin{cases} \Delta^2 u + V(x)u = f(x, u) \text{ in } \mathbb{R}^4 \\ u \in H^2\left(\mathbb{R}^4\right), \ u \geq 0 \end{cases}. \tag{6.1}$$

Here the potential $V : \mathbb{R}^4 \to \mathbb{R}$ is continuous and $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^4$. Moreover, we assume that

(V) There exists a continuous 1-periodic function $W : \mathbb{R}^4 \to \mathbb{R}$ (i.e., $W(x + p) = W(x)$ for all $x \in \mathbb{R}^4$ and $p \in \mathbb{Z}^4$) such that $W(x) \geq V(x)$ for all $x \in \mathbb{R}^4$ and $W(x) - V(x) \to 0$ as $|x| \to \infty$.
$(f1)$ There exists a continuous 1-periodic function $f_0 : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ (i.e., $f(x + p, u) = f(x, u)$ for all $x \in \mathbb{R}^4$, $p \in \mathbb{Z}^4$ and $u \in \mathbb{R}$) such that $0 \leq f_0(x, s) \leq f(x, s)$ for all $(x, s) \in \mathbb{R}^4 \times [0, \infty)$, and for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $s \geq 0$ and $|x| \geq \eta : |f(x, s) - f_0(x, s)| \leq \varepsilon e^{32\pi^2 s^2}$.
$(f2)$ $f(x, s) = o_1(s)$ near the origin uniformly with respect to $x \in \mathbb{R}^4$.

$(f3)$ $f(x,s) \leq Ce^{32\pi^2 s^2}$ for all $(x,s) \in \mathbb{R}^4 \times [0,\infty)$.
$(f4)$ There exists $\mu > 2$ such that $0 \leq \mu F(x,s) < sf(x,s)$ and $0 \leq \mu F_0(x,s) < sf_0(x,s)$, all $(x,s) \in \mathbb{R}^4 \times (0,\infty)$. Here

$$F(x,s) = \int_0^s f(x,t)dt.$$
$$F_0(x,s) = \int_0^s f_0(x,t)dt.$$

$(f5)$ for each fixed $x \in \mathbb{R}^4$, the functions $s \to f_0(x,s)/s$ and $s \to f(x,s)/s$ are increasing;
$(f6)$ there are constants $p > 2$ and $C_p > 0$ such that

$$f_0(x,s) \geq C_p s^{p-1}, \text{ for all } (x,s) \in \mathbb{R}^4 \times \mathbb{R}^+,$$

where

$$C_p > \left[\frac{\mu(p-2)}{p\,(\mu-2)}\right]^{(p-2)/2} S_p^p,$$
$$S_p = \inf_{u \in H^2(\mathbb{R}^4)\setminus\{0\}} \frac{\left[\int_{\mathbb{R}^4} \left(|\Delta u|^2 + W_1\,|u|^2\right) dx\right]^{1/2}}{\left(\int_{\mathbb{R}^4} |u|^p\, dx\right)^{1/p}}.$$

Here $W_1 = \max_{x \in \mathbb{R}^4} W(x)$.

$(f7)$ For any set A of positive measure, at least one of the nonnegative continuous functions $f(x,s) - f_0(x,s)$ and $W(x) - V(x)$ is positive on A.

It is easy to see from the Sobolev embedding that $S_p > 0$.
Since we are interested in nonnegative solutions, we also assume that

$$f_0(x,s) = f(x,s) = 0 \text{ for all } (x,s) \in \mathbb{R}^4 \times (-\infty, 0].$$

We denote by X_0 the Sobolev space $H^2\left(\mathbb{R}^4\right)$ endowed with the norm

$$\|u\|_0 = \left[\int_{\mathbb{R}^4} \left(|\Delta u|^2 + W(x)\,|u|^2\right) dx\right]^{1/2}$$

and by X the Sobolev space $H^2\left(\mathbb{R}^4\right)$ endowed with the norm

$$\|u\| = \left[\int_{\mathbb{R}^4} \left(|\Delta u|^2 + V(x)\,|u|^2\right) dx\right]^{1/2}$$

Our first concern is about the existence of solutions for the periodic critical problem:

$$\begin{cases} \Delta^2 u + W(x)u = f_0(x,u) \text{ in } \mathbb{R}^4 \\ u \in H^2\left(\mathbb{R}^4\right), \ u \geq 0 \end{cases}. \tag{P}$$

In view of Adams type inequalities (Theorems 1.6, 1.7, 1.8), we have that the functionals

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^4} \left(|\Delta u|^2 + W(x)|u|^2\right) dx - \int_{\mathbb{R}^4} F_0(x,u)dx$$
$$J(u) = \frac{1}{2}\int_{\mathbb{R}^4} \left(|\Delta u|^2 + V(x)|u|^2\right) dx - \int_{\mathbb{R}^4} F(x,u)dx$$

are well defined. Moreover, by standard arguments, they are in C^1 and $\forall \varphi \in C_0^\infty\left(\mathbb{R}^4\right)$:

$$DJ_0(u)\varphi = \int_{\mathbb{R}^4} (\Delta u \Delta\varphi + W(x)u\varphi)\, dx - \int_{\mathbb{R}^4} f_0(x,u)\varphi dx$$
$$DJ(u)\varphi = \int_{\mathbb{R}^4} (\Delta u \Delta\varphi + V(x)u\varphi)\, dx - \int_{\mathbb{R}^4} f(x,u)\varphi dx.$$

Thus, critical points of J_0, J are weak solutions of (P), (6.1) respectively.

First, we will prove that

Theorem 6.1 *The equation (P) has a nontrivial weak solution.*

Theorem 6.2 *The equation (6.1) has a nontrivial weak solution.*

6.1 The Periodic Equation (P)

From the conditions on the potential W and nonlinear term f_0, it's now standard to check that J_0 satisfies the mountain-pass geometry:

Lemma 6.1 *(i) There exist $\rho,\ \theta > 0$ such that if $\|u\|_0 = \rho$, then $J_0(u) \geq \theta$.*
(ii) For any $u \in H^2\left(\mathbb{R}^4\right) \setminus \{0\}$ with $u \geq 0$, we have $J_0(tu) \to \infty$ as $t \to \infty$.

Consequently, we can get a (bounded) Palais–Smale sequence of the functional J_0 at the minimax level $M_0 = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} J_0(\gamma(t))$ where

$$\Gamma_0 = \left\{\gamma \in C\left([0,1], H^2\left(\mathbb{R}^4\right)\right) : J_0(\gamma(0)) \leq 0,\ J_0(\gamma(1)) \leq 0\right\}.$$

That is

$$J_0(u_n) \to M_0,\ DJ_0(u_n) \to 0.$$

Then

Lemma 6.2 *We have $M_0 \in \left[\theta, \frac{\mu-2}{2\mu}\right]$. Also, the $(PS)_{M_0}$ sequence (u_n) is bounded and its weak limit u_0 satisfies $DJ_0(u_0) = 0$.*

Proof It's clear that $M_0 \geq \theta$. Now, we can fix $\varepsilon > 0$ small enough and a function $v_{p,\varepsilon} \in H^2(\mathbb{R}^4)$ such that

$$S_p \leq \frac{\left[\int_{\mathbb{R}^4}\left(\left|\Delta v_{p,\varepsilon}\right|^2 + W_1\left|v_{p,\varepsilon}\right|^2\right)dx\right]^{1/2}}{\left[\int_{\mathbb{R}^4}\left|v_{p,\varepsilon}\right|^p dx\right]^{1/p}} \leq S_p + \varepsilon,$$

$$\int_{\mathbb{R}^4}\left(\left|\Delta v_{p,\varepsilon}\right|^2 + W_1\left|v_{p,\varepsilon}\right|^2\right)dx \leq \frac{C_p}{p}\int_{\mathbb{R}^4}\left|v_{p,\varepsilon}\right|^p dx.$$

Then,

$$\begin{aligned} M_0 &\leq \max_{t\in[0,1]} J_0\left(tv_{p,\varepsilon}\right) \\ &\leq \max_{t\in[0,1]}\left[\frac{t^2}{2}\int_{\mathbb{R}^4}\left(\left|\Delta v_{p,\varepsilon}\right|^2 + W_1\left|v_{p,\varepsilon}\right|^2\right)dx - \frac{t^pC_p}{p}\int_{\mathbb{R}^4}\left|v_{p,\varepsilon}\right|^p dx\right] \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{\left(S_p+\varepsilon\right)^{p/(p-2)}}{C_p^{2/(p-2)}} \\ &< \frac{\mu-2}{2\mu}. \end{aligned}$$

Now, by the standard Ambrosetti–Rabinowitz condition, it's clear that (u_n) is bounded and then we can assume that

$$\begin{aligned} u_n &\rightharpoonup u_0 \in X_0 \\ u_n &\to u_0 \text{ in } L^s_{loc}\left(\mathbb{R}^4\right) \\ u_n(x) &\to u_0(x) \text{ in } \mathbb{R}^4. \end{aligned}$$

By $(f4)$, we have

$$\begin{aligned} M_0 &= \lim_{n\to\infty} J_0(u_n) \\ &= \lim_{n\to\infty}\left[J_0(u_n) - \frac{1}{\mu}DJ(u_n)u_n\right] \\ &\geq \frac{\mu-2}{2\mu}\limsup_{n\to\infty}\|u_n\|_0^2. \end{aligned}$$

Hence, $\limsup\limits_{n\to\infty}\|u_n\|_0^2 = m \leq \frac{2\mu M_0}{\mu-2} < 1$.

By Adams type inequalities, we can find positive numbers $\gamma, q \gtrapprox 1$ and $C > 0$ such that

$$h_n(x) = e^{32\pi^2 \gamma u_n^2(x)} - 1 \in L^q\left(\mathbb{R}^4\right),$$
$$\|h_n\|_q \leq C, \ \forall n \in \mathbb{N}.$$

Consequently,

$$\int_{\mathbb{R}^4} f_0\left(x, u_n(x)\right) \varphi(x) dx \to \int_{\mathbb{R}^4} f_0\left(x, u_0(x)\right) \varphi(x) dx, \ \forall \varphi \in C_0^\infty\left(\mathbb{R}^4\right).$$

Now, we can conclude that

$$DJ_0\left(u_0\right) = 0.$$

□

Lemma 6.3 *Let* $(u_n) \in H^2\left(\mathbb{R}^4\right)$ *be a sequence with* $u_n \rightharpoonup 0$ *and*

$$\limsup_{n\to\infty} \|u_n\|_0^2 \leq m < 1.$$

If there exists $R > 0$ *such that*

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{B_R(y)} |u_n|^2 \, dx = 0,$$

we then have

$$\int_{\mathbb{R}^4} F_0(x, u_n) dx, \ \int_{\mathbb{R}^4} f_0(x, u_n) u_n dx \to 0 \text{ as } n \to \infty.$$

Proof Using Lemma 8.4 in [11] (which claims that: Let $1 < p \leq \infty, \ 1 \leq q < \infty$ (if $p < 4$, we assume more that $q \neq p^* = \frac{4p}{4-p}$). If (u_n) is bounded in $L^q\left(\mathbb{R}^4\right)$ such that $(|\nabla u_n|)$ is bounded in $L^p\left(\mathbb{R}^4\right)$ and if there exists $R > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{B_R(y)} |u_n|^q \, dx = 0,$$

then $u_n \to 0$ in $L^r\left(\mathbb{R}^4\right)$ for $\min\left(q, p^*\right) < r < \max\left(q, p^*\right)$ (we denote that $p^* = \infty$ if $p \geq 4$)), we get with $p = 4$ and $q = 2$ that

$$u_n \to 0 \text{ in } L^r\left(\mathbb{R}^4\right) \text{ for all } r > 2.$$

(Note that since (u_n) is bounded in $H^2\left(\mathbb{R}^4\right)$, we have $|\nabla u_n|$ is bounded in $L^4\left(\mathbb{R}^4\right)$ because of $\nabla u_n \in H^1\left(\mathbb{R}^4\right)$ and the Sobolev imbedding $H^1\left(\mathbb{R}^4\right) \hookrightarrow L^4\left(\mathbb{R}^4\right)$).

This fact together with the Adams-type inequalities imply that for $\kappa > 1$ sufficiently close to 1 :

$$\int_{\mathbb{R}^4} \left(\exp\left(32\pi^2\kappa u_n^2(x)\right) - 1\right) dx \leq C.$$

By our assumptions on the nonlinear term, given $\varepsilon > 0$ there exist positive constants C_ε and $q, \gamma > 1$ sufficiently close to 1 such that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^4} f_0(x, u_n)u_n dx\right| \\
&\leq \varepsilon \int_{\mathbb{R}^4} |u_n|^2\, dx + C_\varepsilon \int_{\mathbb{R}^4} |u_n| \left[\exp\left(32\pi^2\gamma u_n^2(x)\right) - 1\right] dx \\
&\leq C\left(\int_{\mathbb{R}^4} |u_n|^{q'}\, dx\right)^{1/q'} \left(\int_{\mathbb{R}^4} \left[\exp\left(32\pi^2\gamma u_n^2(x)\right) - 1\right]^q dx\right)^{1/q} + \varepsilon C \\
&\leq C\left(\int_{\mathbb{R}^4} |u_n|^{q'}\, dx\right)^{1/q'} \left(\int_{\mathbb{R}^4} \left[\exp\left(32\pi^2 q\gamma u_n^2(x)\right) - 1\right] dx\right)^{1/q} + \varepsilon C \\
&= C\, \|u_n\|_{q'} + \varepsilon C
\end{aligned}$$

where $q' = q/(q-1)$. Hence

$$\int_{\mathbb{R}^4} F_0(x, u_n)dx,\ \int_{\mathbb{R}^4} f_0(x, u_n)u_n dx \to 0 \text{ as } n \to \infty.$$

$\square$

Now, by Lemma 6.2, we have that the weak limit u_0 of the $(PS)_{M_0}$ sequence satisfies $DJ_0(u_0) = 0$. If $u_0 \neq 0$, then u_0 is the nontrivial weak solution of (P). Now, if $u_0 = 0$, then there is a sequence $(z_n) \subset \mathbb{R}^4$, and $R,\ A > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(z_n)} |u_n|^2\, dx > A.$$

(If not, then by Lemma 6.4, we get

$$\int_{\mathbb{R}^4} F(x, u_n)dx,\ \int_{\mathbb{R}^4} f(x, u_n)u_n dx \to 0 \text{ as } n \to \infty.$$

which implies that $u_n \to 0$ in $H^2\left(\mathbb{R}^4\right)$. As a consequence, $M_0 = 0$ which is impossible.)

Without loss of generality, we may assume that $(z_n) \subset \mathbb{Z}^4$. Setting $\widetilde{u}_n(x) = u_n(x - z_n)$, since $W,\ f_0,\ F_0$ are 1-periodic functions, we get

$$\begin{aligned} \|u_n\|_0 &= \|\widetilde{u}_n\|_0 \\ J_0(u_n) &= J_0(\widetilde{u}_n) \\ DJ_0(\widetilde{u}_0) &\to 0. \end{aligned}$$

As before, we can find a $\widetilde{u}_0$ in $H^2\left(\mathbb{R}^4\right)$ such that $\widetilde{u}_n \rightharpoonup \widetilde{u}_0$ weakly in $H^2\left(\mathbb{R}^4\right)$ and $DJ_0(\widetilde{u}_0) = 0$. Finally, we notice that by taking a subsequence and R sufficiently large, we can get

$$A^{1/2} \leq \|\widetilde{u}_n\|_{L^2(B_R(0))} \leq \|\widetilde{u}_0\|_{L^2(B_R(0))} + \|\widetilde{u}_n - \widetilde{u}_0\|_{L^2(B_R(0))}.$$

By the compact embedding $H^2\left(\mathbb{R}^4\right) \hookrightarrow L^2_{loc}\left(\mathbb{R}^4\right)$ we have that $\widetilde{u}_0$ is nontrivial.

Also, since $f_0(x,s) = 0$ for all $s \leq 0$, we have that the nontrivial weak solution here is nonnegative.

6.2 The Nonperiodic Equation (6.1)

Similarly as in the previous subsection, we may check that the functional energy J has the geometry of the mountain-pass theorem. Also, we can find a bounded Palais–Smale sequence (v_n) in $H^2\left(\mathbb{R}^4\right)$ at the minimax level M:

$$\begin{aligned} J\left(v_n\right) &\to M \text{ and } DJ\left(v_n\right) \to 0,\ \text{ as } n \to \infty, \\ M &= \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} J\left(\gamma\left(t\right)\right) \end{aligned}$$

where

$$\Gamma_0 = \left\{\gamma \in C\left(\left[0,1\right], H^2\left(\mathbb{R}^4\right)\right) : J\left(\gamma\left(0\right)\right) \leq 0,\ J\left(\gamma\left(1\right)\right) \leq 0\right\}.$$

Furthermore, we also have $M \in (\kappa, (\mu - 2)/2\mu]$ for some positive constant κ and $v_n \rightharpoonup v_0$ in $H^2\left(\mathbb{R}^4\right)$ and that v_0 is a critical point of functional J.

We also get that

$$\limsup_{n \to \infty} \|v_n\|^2 \leq m' < 1$$

which again implies that for $\gamma > 1$ sufficiently close to 1, we have

$$\int_{\mathbb{R}^4} \left(\exp\left(32\pi^2\gamma v_n^2(x)\right) - 1\right) dx \leq C$$

for some universal constant $C > 0$. So now, it's sufficient to prove that v_0 is nontrivial.

Suppose that $v_0 = 0$. First, we will prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^4} |F_0(x, v_n) - F(x, v_n)|\, dx = \lim_{n\to\infty}\int_{\mathbb{R}^4} |f_0(x, v_n)v_n - f(x, v_n)v_n|\, dx = 0.$$

Indeed, given $\varepsilon > 0$, there exists $\eta > 0$ such that by the Adams type inequalities and Sobolev embeddings:

$$\begin{aligned}
&\int_{|x|\geq\eta} |f_0(x, v_n)v_n - f(x, v_n)v_n|\, dx \\
&\quad\leq \varepsilon \int_{|x|\geq\eta} \left| \left(\exp\left(32\pi^2\gamma v_n^2(x)\right) - 1\right) v_n \right| \\
&\quad\leq \varepsilon \left(\int_{\mathbb{R}^4} \left|\left(\exp\left(32\pi^2\gamma v_n^2(x)\right) - 1\right)\right|^q dx\right)^{1/q} \left(\int_{\mathbb{R}^4} |v_k|^{q'}\, dx\right)^{1/q'} \\
&\quad\leq C\varepsilon.
\end{aligned}$$

On the other hand, by the compact embedding $H^2\left(\mathbb{R}^4\right) \hookrightarrow L^r_{loc}\left(\mathbb{R}^4\right)$, $r \geq 1$:

$$\begin{aligned}
&\int_{|x|\leq\eta} |f_0(x, v_n)v_n - f(x, v_n)v_n|\, dx \\
&\quad\leq \left(\int_{\mathbb{R}^4} \left|\left(\exp\left(32\pi^2\gamma v_n^2(x)\right) - 1\right)\right|^q dx\right)^{1/q} \left(\int_{|x|\leq\eta} |v_n|^{q'}\, dx\right)^{1/q'} \\
&\qquad + \varepsilon \int_{\mathbb{R}^4} |v_n|^2 .
\end{aligned}$$

Combining these two inequalities, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^4} |F_0(x, v_n) - F(x, v_n)|\, dx = \lim_{n\to\infty}\int_{\mathbb{R}^4} |f_0(x, v_n)v_n - f(x, v_n)v_n|\, dx = 0.$$

We can also check easily that

$$\int_{\mathbb{R}^4} [W(x) - V(x)]\, v_n^2(x)\, dx \to 0 \text{ as } n \to \infty.$$

From these equations, we get

$$\begin{gathered}
|J_0(v_n) - J(v_n)| \to 0 \\
\|DJ_0(v_n) - DJ(v_n)\| \to 0,
\end{gathered}$$

which implies

$$\begin{gathered}
J_0(v_n) \to M \\
DJ_0(v_n) \to 0.
\end{gathered}$$

As in the previous subsection, there is a sequence $(z_n) \subset \mathbb{Z}^4$, and $R,\ A > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(z_n)} |v_n|^2\, dx > A.$$

Now, letting $\widetilde{v}_n(x) = v_n(x - z_n)$, since $W,\ f_0,\ F_0$ are 1-periodic functions, we get

$$\begin{aligned} \|v_n\|_0 &= \|\widetilde{v}_n\|_0 \\ J_0(v_n) &= J_0(\widetilde{v}_n) \\ DJ_0(\widetilde{v}_n) &\to 0. \end{aligned}$$

Then we can find a $\widetilde{v}_0$ in $H^2\left(\mathbb{R}^4\right)$ such that $\widetilde{v}_n \rightharpoonup \widetilde{v}_0$ weakly in $H^2\left(\mathbb{R}^4\right)$ and $DJ_0(\widetilde{v}_0) = 0$.

Next, by Fatou's lemma we have:

$$\begin{aligned} J_0(\widetilde{v}_0) &= J_0(\widetilde{v}_0) - \frac{1}{2} DJ_0(\widetilde{v}_0)\widetilde{v}_0 \\ &= \frac{1}{2}\int_{\mathbb{R}^{2m}} [f_0(x, \widetilde{v}_0)\widetilde{v}_0 - 2F_0(x, \widetilde{v}_0)] \\ &\le \liminf_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^4} [f_0(x, \widetilde{v}_n)\widetilde{v}_n - 2F_0(x, \widetilde{v}_n)] \\ &= \lim_{n\to\infty} \left[J_0(\widetilde{v}_n) - \frac{1}{2} DJ_0(\widetilde{v}_n)\widetilde{v}_n \right] = M. \end{aligned}$$

Similarly as in the previous section, we have that $\widetilde{v}_0 \neq 0$ and by $(f5)$, we can get that

$$M \ge J_0(\widetilde{v}_0) = \max_{t\ge 0} J_0(t\widetilde{v}_0) \ge M_0.$$

On the other hand, by assumptions $(f1)$, $(f5)$ and $(f7)$:

$$M \le \max_{t\ge 0} J(tu_0) = J(t_1 u_0) < J_0(t_1 u_0) \le \max_{t\ge 0} J_0(tu_0) = J_0(u_0) = M_0$$

and we get a contradiction. Therefore, v_0 is nontrivial.

Since $f(x, u) = 0$ for all $(x, u) \in \mathbb{R}^4 \times (-\infty, 0]$, from standard arguments, it's easy to see that this weak solution is nonnegative.

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