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Weighted Moser–Onofri–Beckner and Logarithmic Sobolev Inequalities

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Abstract Motivated by recent results on the best constants and extremal functions for a family of the Caffarelli–Kohn–Nirenberg inequalities in [17,23], we will study weighted Moser–Onofri–Beckner inequalities on the Euclidean space $\mathbb{R}^N$. We also set up sharp weighted versions of the logarithmic Sobolev inequalities together with their best constants and optimizers.

Keywords Best constants · Extremal functions · Moser–Onofri–Beckner inequalities · Logarithmic Sobolev inequalities

Mathematics Subject Classification 26D10 · 46E35

1 Introduction

To study the prescribed Gauss curvature problem on two-dimensional sphere $\mathbb{S}^2$, in [25], Moser established the following exponential type inequality on the 2-dimensional sphere $\mathbb{S}^2$ with an optimal constant: For every $\beta \leq 4\pi$, there exists a constant $C(\beta) > 0$ such that for all $u \in H^1(\mathbb{S}^2)$

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$$\int_{\mathbb{S}^2} \exp\left(\frac{\beta\,(u-\overline{u})^2}{4\pi \int_{\mathbb{S}^2} |\nabla u|^2 \, \mathrm{d}\omega}\right) \mathrm{d}\omega \le C(\beta).$$

Here $\mathrm{d}\omega$ denotes the standard surface measure on $\mathbb{S}^2$, normalized so that $\int_{\mathbb{S}^2} \mathrm{d}\omega = 1$, and $\overline{u} = \int_{\mathbb{S}^2} u \mathrm{d}\omega$. Moreover, the constant 4π is sharp in the sense that if $\beta > 4\pi$, then

$$\sup_{u\in H^1(\mathbb{S}^2)} \int_{\mathbb{S}^2} \exp\left(\frac{\beta\,(u-\overline{u})^2}{4\pi \int_{\mathbb{S}^2} |\nabla u|^2 \, \mathrm{d}\omega}\right) \mathrm{d}\omega = \infty.$$

As a consequence of this result, we have that the functional

$$J(u) = \frac{1}{4}\int_{\mathbb{S}^2} |\nabla u|^2 \, \mathrm{d}\omega + \int_{\mathbb{S}^2} u \mathrm{d}\omega - \ln\left(\int_{\mathbb{S}^2} e^u \mathrm{d}\omega\right),$$

is bounded from below on $H^1(\mathbb{S}^2)$.

Moser used the following classical Schwarz rearrangement argument: every smooth function u is associated to a function u^* such that u^* is constant on the parallel circles and such that for any continuous function f:

$$\int_{\mathbb{S}^2} f(u)\mathrm{d}\omega = \int_{\mathbb{S}^2} f(u^*)\mathrm{d}\omega.$$

Moreover, the well-known Pólya–Szegö inequality

$$\int_{\mathbb{S}^2} \left|\nabla u^*\right|^p \mathrm{d}\omega \le \int_{\mathbb{S}^2} |\nabla u|^p \, \mathrm{d}\omega \tag{1.1}$$

plays a crucial role in the approach of Moser and enables him to reduce the consideration to a one-dimensional problem. Onofri, using conformal invariance and results in a paper of Aubin [4], showed in [26] that actually $J(u)$ is bounded from below by 0, and that modulo conformal transformations, $u = 0$ is the optimizer: For $\beta \ge 1$,

$$\inf_{u\in H^1(\mathbb{S}^2)} J_\beta(u) = 0,$$

while if $0 \le \beta < 1$

$$\inf_{u\in H^1(\mathbb{S}^2)} J_\beta(u) = -\infty.$$

Here

$$J_\beta(u) = \frac{\beta}{4}\int_{\mathbb{S}^2} |\nabla u|^2 \, \mathrm{d}\omega + \int_{\mathbb{S}^2} u \mathrm{d}\omega - \ln\left(\int_{\mathbb{S}^2} e^u \mathrm{d}\omega\right)$$

Other proofs for this result were provided by Hong in [22] and by Osgood–Phillips–Sarnak in [27]. Also, in 2005, Ghigi made use of the convex analysis, in particular, the well-known Prékopa–Leindler inequality, to give a new proof of the Moser–Onofri inequality in [19]. See Chaps. 16–18 in the book [20] of Ghoussoub and Moradifam for more details.

The Moser–Onofri inequality was also extended into higher-dimensional N-spheres $\mathbb{S}^N$. For instance, Beckner in [5] proved that for a real-valued function F defined on the sphere $\mathbb{S}^N$ with an expansion in spherical harmonics $F = \sum_{k=0}^{\infty} Y_k$,

$$\ln\left(\int_{\mathbb{S}^N} e^F \mathrm{d}\xi\right) \le \int_{\mathbb{S}^N} F \mathrm{d}\xi + \frac{1}{2N}\sum_{k=1}^{\infty} \frac{\Gamma(N+k)}{\Gamma(N)\Gamma(k)} \int_{\mathbb{S}^N} |Y_k|^2 \, \mathrm{d}\xi.$$

Here $\mathrm{d}\xi$ denotes the normalized surface measure. Moreover, equality happens if and only if e^F is given by the Jacobian determinant of a conformal transformation of the N-sphere. Also, in [9], Carlen and Loss used the method of competing symmetries to derive the sharp version of the logarithmic Hardy–Littlewood–Sobolev inequality and deduce Beckner's result as an application.

Sharp Moser inequalities and a weak form of the Moser–Onofri type inequalities were also derived on spheres in complex space $\mathbb{C}^N$ by Cohn and the second author in [11] using a sharp representation formula for functions on complex spheres in terms of complex tangential gradients (we refer to [10] for more background of complex tangential gradients). In these results, the smaller complex tangential gradient $|\nabla_{\mathbb{C}} u|$ replaces the real tangential gradient $|\nabla_t u|$, and also the critical exponent is $\frac{2N}{2N-1}$ rather than $\frac{2N-1}{2N-2}$, which differ from the real Moser inequality on the sphere in $\mathbb{R}^{2N}$.

It is worth noting that if we use the stereographic projection from $\mathbb{S}^2$ to $\mathbb{R}^2$, then we could obtain the following Euclidean version of the Moser–Onofri inequality: for all $u \in L^1(\mathbb{R}^2, \mathrm{d}\mu)$ such that $\nabla u \in L^2(\mathbb{R}^2, \mathrm{d}x)$ with $\mathrm{d}\mu = \frac{1}{\pi}\frac{1}{(1+|x|^2)^2}\mathrm{d}x$,

$$\ln\left(\int_{\mathbb{R}^2} e^u \mathrm{d}\mu\right) - \int_{\mathbb{R}^2} u \mathrm{d}\mu \le \frac{1}{16\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x.$$

This Euclidean Moser–Onofri inequality can also be deduced as a limiting procedure based on other functional inequalities [16] or from optimal mass transport [3].

Effort has also been made in [15] in order to get the Euclidean Moser–Onofri–Beckner inequality on $\mathbb{R}^N$. More precisely, Del Pino and Dolbeault proved that for any smooth compactly supported function u

$$\ln\left[\int_{\mathbb{R}^N} e^u \mathrm{d}\mu_N\right] - \int_{\mathbb{R}^N} u \mathrm{d}\mu_N \le \alpha_N \int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)\mathrm{d}x.$$

The best constant α_N is given by

$$\alpha_N = \frac{N^{1-N}\Gamma\left(\frac{N}{2}\right)}{2(N-1)\pi^{\frac{N}{2}}}.$$

Here we denote for $X, Y \in \mathbb{R}^N$

$$\mathcal{R}_N(X, Y) = |X+Y|^N - |X|^N - N|X|^{N-2}X \cdot Y$$

and

$$\mathcal{H}_N(X,Y)=\mathcal{R}_N\left(-\frac{N\,|X|^{-\frac{N-2}{N-1}}}{1+|X|^{\frac{N}{N-1}}}X,\ \frac{N-1}{N}Y\right).$$

The Euclidean Moser–Onofri inequality has also been studied in the presence of weights. For example, the following result has been set up recently in [16,17] using the weighted Caffarelli–Kohn–Nirenberg inequalities: Let $0\le s<2$. Then for any $u: u\in L^1\left(\mathbb{R}^2,\mathrm{d}\mu_s\right)$ and $|\nabla u|\in L^2\left(\mathbb{R}^2,\mathrm{d}x\right)$,

$$\ln\left(\int_{\mathbb{R}^2}e^u\mathrm{d}\mu_s\right)-\int_{\mathbb{R}^2}u\mathrm{d}\mu_s\le\frac{1}{8\left(2-s\right)\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x,$$

where

$$\mathrm{d}\mu_s=\frac{2-s}{2\pi}\frac{1}{\left(1+|x|^{2-s}\right)^2}\frac{\mathrm{d}x}{|x|^s}.$$

Motivated by the above discussions, in this article, we will set up the following version of the weighted Euclidean Moser–Onofri–Beckner inequality:

Theorem 1.1 *Let $0\le s<N$ and assume that u is any smooth compactly supported function. Then we have*

$$\ln\left[\int_{\mathbb{R}^N}e^u\mathrm{d}\mu_{N,s}\right]-\int_{\mathbb{R}^N}u\mathrm{d}\mu_{N,s}\le\alpha_{N,s}\int_{\mathbb{R}^N}\mathcal{H}_N\left(x,\nabla u\right)\mathrm{d}x. \tag{1.2}$$

Here

$$\mathrm{d}\mu_{N,s}=\frac{N-s}{N}\frac{\Gamma\left(\frac{N}{2}+1\right)}{\pi^{\frac{N}{2}}}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-N}\frac{\mathrm{d}x}{|x|^s},$$

and

$$\alpha_{N,s}=\left(\frac{N}{N-s}\right)^{N-1}\frac{\Gamma\left(\frac{N}{2}\right)}{2(N-1)N^{N-1}\pi^{\frac{N}{2}}}.$$

It is easy to check that the equality in (1.2) is attained by constants. However, we do not know whether or not (1.2) can be achieved by nonconstant optimizers. This question is still left open.

Corollary 1.1 *When $N=2$, $\mathcal{R}_2\left(X,Y\right)=|Y|^2$ and $\mathcal{H}_2\left(X,Y\right)=\frac{1}{4}|Y|^2$. So we get the following weighted Euclidean Moser–Onofri inequality:*

$$\ln\left[\int_{\mathbb{R}^2}e^u\mathrm{d}d\mu_{2,s}\right]-\int_{\mathbb{R}^2}u\mathrm{d}\mu_{2,s}\le\frac{1}{8\pi\left(2-s\right)}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x,$$

where

$$\mathrm{d}\mu_{2,s} = \frac{2-s}{2\pi}\left(1+|x|^{2-s}\right)^{-2}\frac{\mathrm{d}x}{|x|^s}.$$

Corollary 1.2 *When $N = 4$,*

$$\mathcal{R}_4(X,Y) = |Y|^4 + 2|X|^2|Y|^2 + 4|Y|^2 X\cdot Y + 4(X\cdot Y)^2$$

and

$$\begin{aligned}\mathcal{H}_4(X,Y) &= \mathcal{R}_N\left(-\frac{4|X|^{-\frac{2}{3}}}{1+|X|^{\frac{4}{3}}}X, \frac{3}{4}Y\right)\\ &= \left(\frac{3}{4}\right)^4|Y|^4 + 18\frac{|X|^{\frac{2}{3}}|Y|^2}{\left(1+|X|^{\frac{4}{3}}\right)^2} + \frac{27}{4}\frac{|X|^{-\frac{2}{3}}|Y|^2}{1+|X|^{\frac{4}{3}}}X\cdot Y\\ &\quad + 36\frac{|X|^{-\frac{4}{3}}}{\left(1+|X|^{\frac{4}{3}}\right)^2}(X\cdot Y)^2.\end{aligned}$$

Hence, we get the following weighted Euclidean Moser–Onofri–Beckner inequality:

$$\begin{aligned}&\ln\left[\int_{\mathbb{R}^4} e^u \mathrm{d}\mu_{4,s}\right] - \int_{\mathbb{R}^4} u\,\mathrm{d}\mu_{4,s}\\ &\leq \frac{1}{6(4-s)^3\pi^2}\int_{\mathbb{R}^4}\left(\left[\frac{9}{16}|\nabla u|^2 + 6\frac{|x|^{\frac{1}{3}}}{1+|x|^{\frac{4}{3}}}\left(\frac{x}{|x|}\cdot\nabla u\right)\right]^2 + 18\frac{|x|^{\frac{2}{3}}|\nabla u|^2}{\left(1+|x|^{\frac{4}{3}}\right)^2}\right)\mathrm{d}x.\end{aligned}$$

Here

$$\mathrm{d}\mu_{4,s} = \frac{4-s}{2\pi^2}\left(1+|x|^{\frac{4-s}{3}}\right)^{-4}\frac{\mathrm{d}x}{|x|^s}.$$

As we can see, this inequality contains inhomogeneous Sobolev–Orlicz norms which are quite different (and unexpected) from the results of Onofri [26] and Beckner [5].

From (1.2), for smooth compactly supported function u, we get that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(\ln\left[\int_{\mathbb{R}^N} e^{\varepsilon(u-\overline{u})}\mathrm{d}\mu_{N,s}\right]\right) \leq \alpha_{N,s}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N}\mathcal{H}_N\left(x,\varepsilon\nabla\left(u-\overline{u}\right)\right)\mathrm{d}x,$$

with

$$\overline{u} = \int_{\mathbb{R}^N} u\,\mathrm{d}\mu_{N,s}.$$

We also note that

$$\ln\left[\int_{\mathbb{R}^N} e^{\varepsilon(u-\overline{u})}\mathrm{d}\mu_{N,s}\right] = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^N} |u-\overline{u}|^2\,\mathrm{d}\mu_{N,s} + o(\varepsilon^2),$$

and

$$\int_{\mathbb{R}^N} \mathcal{H}_N\left(x, \varepsilon\nabla(u-\overline{u})\right)\mathrm{d}x = \varepsilon^2\int_{\mathbb{R}^N} \mathcal{G}_N\left(x, \nabla\left(u-\overline{u}\right)\right)\mathrm{d}x + o(\varepsilon^2).$$

Hence, as a byproduct, we obtain a Poincaré type inequality:

Corollary 1.3 *For $0 \leq s < N$ and u being a smooth compactly supported function, we have*

$$\frac{1}{2}\int_{\mathbb{R}^N} |u-\overline{u}|^2\, d\mu_{N,s} \leq \alpha_{N,s}\int_{\mathbb{R}^N} \mathcal{G}_N(x, \nabla u)\mathrm{d}x.$$

Here

$$\begin{aligned}\mathcal{G}_N\left(X, Y\right) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathcal{H}_N\left(X, \varepsilon Y\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathcal{R}_N\left(-\frac{N|X|^{-\frac{N-2}{N-1}}}{1+|X|^{\frac{N}{N-1}}}X, \varepsilon\frac{N-1}{N}Y\right)\\ &= \mathcal{L}_N\left(-\frac{N|X|^{-\frac{N-2}{N-1}}}{1+|X|^{\frac{N}{N-1}}}X, \frac{N-1}{N}Y\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_N\left(X, Y\right) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathcal{R}_N\left(X, \varepsilon Y\right) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left|X+tY\right|^N|_{t=0}\\ &= \frac{1}{2}N|X|^{N-4}\left[(N-2)\left(X\cdot Y\right)^2 + |X|^2\left|Y\right|^2\right].\end{aligned}$$

As in [17], we will obtain Theorem 1.1 as a result from the limiting process of a particular family of the Caffarelli–Kohn–Nirenberg (CKN) inequalities. More specifically, the proof of Theorem 1.1 will reply on the following weighted CKN inequalities established recently in [17,23]:

Theorem A *Let $0 \leq s < N$. When $r = N\frac{q-1}{N-1}$ and $q > N$, we have*

$$\begin{aligned}\mathrm{CKN}(N, s, q, r) = &\left(\frac{N}{N-s}\right)^{\frac{1}{N'}\left(1-\frac{q}{r}\right)}\left(\frac{q-N}{N\sqrt{\pi}}\right)^{1-\frac{q}{r}}\\ &\times\left(\frac{q}{q-N}\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}\left(\frac{N}{q}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{N-1}{q-N}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)\Gamma(N)}\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}.\end{aligned}$$

Moreover,

$$V_0(x) = \alpha\left(1 + \beta|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}} \text{ for some } \alpha \in \mathbb{R} \text{ and } \beta > 0,$$

are optimizers of CKN(N, s, q, r). *Here*

$$\mathrm{CKN}(N, s, q, r) = \sup_{u \in D_{0,s}^{N,q}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla u|^N dx\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^s}\right)^{\frac{1}{r}}}$$

and $D_{0,s}^{N,q}\left(\mathbb{R}^N\right)$ *is the completion of the space of smooth compactly supported functions under the norm* $\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{1/N} + \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^s}\right)^{1/q}$.

Theorem B *Assume that*

$$\begin{aligned} &1 < p \le p + \mu < N,\ 0 \le s = \frac{N\mu}{N-p} < N, \\ &p < q < r = p\frac{q-1}{p-1} < \frac{Np}{N-p};\ a = \frac{Np(q-p)}{(p-1)[pq - N(q-p)]r}. \end{aligned} \tag{C1}$$

Denote by $D_{\mu,s}^{p,q}\left(\mathbb{R}^N\right)$ *the completion of the space of smooth compactly supported functions with the norm* $\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{1/p} + \left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{1/q}$, *and set*

$$\mathrm{CKN}(N, \mu, s, p, q, r) = \sup_{u \in D_{\mu,\theta}^{p,q}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1-a}{q}}}.$$

Then

$$\begin{aligned} &\mathrm{CKN}(N, \mu, s, p, q, r) \\ &= \left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r} + \frac{p-1}{p} - \frac{1-a}{q} - \frac{p-1}{p}(1-a)} \\ &\quad \times \left(\frac{q-p}{p\sqrt{\pi}}\right)^a \left(\frac{pq}{N(q-p)}\right)^{\frac{a}{p}} \left(\frac{Np - q(N-p)}{pq}\right)^{\frac{1}{r}} \left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{p-1}{p}\frac{Np-q(N-p)}{q-p}\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}, \end{aligned}$$

and is achieved when

$$u(x) = A\left(1 + B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}} \text{ for some } A \in \mathbb{R},\ B > 0.$$

The proof of Theorem A will be provided in Sect. 2 (see Lemma 2.1) for the completeness, while the proof of Theorem B could be found in [23].

The CKN inequalities were introduced by Caffarelli, Kohn, and Nirenberg in their 1984 paper [7]. They play an important role in geometric analysis, partial differential equations, and other branches of modern mathematics. They also generalize many well-known inequalities such as Gagliardo–Nirenberg inequalities, Sobolev inequalities, Hardy–Sobolev inequalities, and Nash's inequalities.

There is another well-known inequality that has a close connection to the CKN inequalities and has been studied extensively in the literature. That is the following sharp L^p-logarithmic Sobolev inequality:

Theorem C *Let $p \geq 1$. For any smooth function f such that $\int_{\mathbb{R}^N} |f|^p \, \mathrm{d}x = 1$, we have*

$$\int_{\mathbb{R}^N} |f|^p \ln |f|^p \, \mathrm{d}x \leq \frac{N}{p} \ln \left[L_p \int_{\mathbb{R}^N} |\nabla f|^p \, \mathrm{d}x \right],$$

where

$$L_p = \frac{p}{N} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{p-1}{p}+1\right)} \right)^{\frac{p}{N}}.$$

The L^1-logarithmic Sobolev inequalities were studied by Ledoux in [24]. Their optimizers, which are the characteristic functions of the balls, were found by Beckner in [6]. The fact that the sharp L^2-logarithmic Sobolev inequality is equivalent to the sharp Gross logarithmic Sobolev inequality for the Gaussian measure [21] was pointed out, for example by Carlen in [8]. The optimizers in this case were also proved in [8] and are exactly the Gaussians $u(x) = (\pi\tau)^{-\frac{N}{2}} \exp\left(-\frac{1}{4}\tau |x|^2\right)$. The optimal L^p-logarithmic Sobolev inequalities together with their extremal functions with $1 \leq p \leq N$ were investigated by Del Pino and Dolbeault in [14,15]. Also, Gentil set up in [18] the general sharp L^p-logarithmic Sobolev inequalities for all $p \geq 1$ and under arbitrary norm on $\mathbb{R}^N$, using the Prékopa–Leindler inequality and a particular Hamilton–Jacobi equation. It is also worth noting that one can apply the optimal mass transport to provide other proofs for the sharp L^p-logarithmic Sobolev inequalities. See [2,12] for example.

The second purpose of this paper is to use Theorem A to derive a version of the sharp weighted L^N-logarithmic Sobolev inequalities. More precisely, we will prove the following:

Theorem 1.2 *Let $0 \leq s < N$. For any $u \in D_{0,s}^{N,N}\left(\mathbb{R}^N\right)$ such that $\int_{\mathbb{R}^N} |u|^N \frac{\mathrm{d}x}{|x|^s} = 1$, we have*

$$\ln \left(\mathrm{SL}(N,s) \int_{\mathbb{R}^N} |\nabla u|^N \, \mathrm{d}x \right) \geq \int_{\mathbb{R}^N} |u|^N \ln |u|^N \frac{\mathrm{d}x}{|x|^s}, \tag{1.3}$$

with

$$SL(N,s)=\left(\frac{N}{N-s}\right)^{N-1}\frac{(N-1)^{N-1}}{\left(\sqrt{\pi}\right)^{N}}\frac{1}{e^{N-1}}\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma(N)}.$$

The equality happens in (1.3) when

$$u(x)=\exp\left(-b\,|x|^{\frac{N-s}{N-1}}\right),$$

where

$$b>0 \text{ and } b^{N-1}=\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\frac{N-1}{N-s}\frac{\Gamma(N-1)}{N^{N-1}}.$$

We will also apply Theorem B to deduce the following sharp weighted L^p-logarithmic Sobolev inequalities:

Theorem 1.3 *Assume (C1). For any $u\in D^{p,p}_{\mu,s}\left(\mathbb{R}^N\right)$ such that $\int_{\mathbb{R}^N}|u|^p\frac{\mathrm{d}x}{|x|^s}=1$, we have*

$$\frac{N}{p}\ln\left(\mathrm{LS}(N,p,\mu)\int_{\mathbb{R}^N}|\nabla u|^p\frac{\mathrm{d}x}{|x|^{\mu}}\right)\geq\int_{\mathbb{R}^N}|u|^p\ln|u|^p\frac{\mathrm{d}x}{|x|^s}, \tag{1.4}$$

with

$$\mathrm{LS}(N,p,\mu)=\frac{p}{N}\left(\frac{N-p}{N-p-\mu}\right)^{\frac{N-1}{N}p}\left(\frac{p-1}{e}\right)^{p-1}\left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{p}{N}}\left(\frac{1}{\sqrt{\pi}}\right)^{p}.$$

The equality happens in (1.4) when

$$u\left(x\right)=\exp\left(-b\,|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right),$$

where

$$b>0 \text{ and } b^{N\frac{p-1}{p}}=\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\frac{(N-p)\,(p-1)}{(N-p-\mu)\,p}\left(\frac{1}{p}\right)^{N\frac{p-1}{p}}\Gamma\left(N\frac{p-1}{p}\right).$$

The rest of the paper is organized as follows: in Sect. 2, we will show several key computations that will be used in the proofs of our main results. Weighted Moser–Onofri–Beckner inequality will then be studied in Sect. 3. Finally in Sects. 4 and 5, we will establish sharp versions of the weighted L^N and L^p-logarithmic Sobolev inequalities correspondingly.

2 Some Important Lemmas

In this section, we will show some useful estimations and results that will be used in the next sections.

Lemma 2.1 *Let* $0 \leq s < N$*. When* $r = N\frac{q-1}{N-1}$ *and* $q > N$*, we have*

$$\mathrm{CKN}\,(N,s,q,r) = \left(\frac{N}{N-s}\right)^{\frac{1}{N'}\left(1-\frac{q}{r}\right)}\left(\frac{q-N}{N\sqrt{\pi}}\right)^{1-\frac{q}{r}} \times \left(\frac{q}{q-N}\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}\left(\frac{N}{q}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{N-1}{q-N}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)\Gamma(N)}\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}.$$

Moreover,

$$V_0\,(x) = \alpha\left(1+\beta\,|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}} \text{ for some } \alpha \in \mathbb{R} \text{ and } \beta > 0,$$

are optimizers of $\mathrm{CKN}\,(N,s,q,r)$ *. Here*

$$\mathrm{CKN}\,(N,s,q,r) = \sup_{u\in D_{0,s}^{N,q}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N}|u|^r\,\frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N}|\nabla u|^N\,\mathrm{d}x\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}\left(\int_{\mathbb{R}^N}|u|^q\,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}},$$

and $D_{0,s}^{N,q}\left(\mathbb{R}^N\right)$ *is the completion of the space of smooth compactly supported functions under the norm* $\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{1/N} + \left(\int_{\mathbb{R}^N}|u|^q\,\frac{dx}{|x|^s}\right)^{1/q}$.

Proof Set

$$\mathrm{GN}\,(N,q,r) = \sup_{u\in D_{0,0}^{N,q}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N}|u|^r\,\mathrm{d}x\right)^{1/r}}{\left(\int_{\mathbb{R}^N}|\nabla u|^N\,\mathrm{d}x\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}\left(\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x\right)^{\frac{1}{r}}}.$$

Then from the results in [1,13,14], we get that $GN\,(N,q,r)$ is equal to

$$\left(\frac{q-N}{N\sqrt{\pi}}\right)^{1-\frac{q}{r}}\left(\frac{q}{q-N}\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)}\left(\frac{N}{q}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{N-1}{q-N}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)\Gamma(N)}\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)},$$

and can be achieved by $U_0(x) = \alpha \left(1 + \beta |x|^{\frac{N}{N-1}}\right)^{-\frac{N-1}{q-N}}$ for some $\alpha \in \mathbb{R}$ and $\beta > 0$. We now set $V_0 = D_{N,s}^{-1} U_0$, that is $U_0 = D_{N,s} V_0$ where

$$D_{N,s}u(x) := \left(\frac{N-s}{N}\right)^{\frac{1}{N'}} u(F_{N,s}(x)),$$
$$D_{N,s}^{-1}u(x) := \left(\frac{N}{N-s}\right)^{\frac{1}{N'}} u(F_{N,s}^{-1}(x)),$$

and

$$F_{N,s}(x) = |x|^{\frac{s}{N-s}} x,$$
$$F_{N,s}^{-1}(x) = |x|^{-\frac{s}{N}} x.$$

We will show that V_0 is a maximizer of CKN (N, s, q, r). Indeed, for any v, we need to show

$$\frac{\left(\int_{\mathbb{R}^N} |v|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla v|^N \mathrm{d}x\right)^{\frac{1}{N}(1-\frac{q}{r})} \left(\int_{\mathbb{R}^N} |v|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}} \le \frac{\left(\int_{\mathbb{R}^N} |V_0|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla V_0|^N \mathrm{d}x\right)^{\frac{1}{N}(1-\frac{q}{r})} \left(\int_{\mathbb{R}^N} |V_0|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}.$$

By Lemma 2.2 in [23], we get

$$\int_{\mathbb{R}^N} |v|^r \frac{\mathrm{d}x}{|x|^s} = \frac{N}{N-s} \left(\frac{N}{N-s}\right)^{\frac{r}{N'}} \int_{\mathbb{R}^N} \left(D_{N,s} v(x)\right)^r \mathrm{d}x,$$
$$\int_{\mathbb{R}^N} |v|^q \frac{\mathrm{d}x}{|x|^s} = \frac{N}{N-s} \left(\frac{N}{N-s}\right)^{\frac{q}{N'}} \int_{\mathbb{R}^N} \left(D_{N,s} v(x)\right)^q \mathrm{d}x,$$
$$\int_{\mathbb{R}^N} |\nabla v|^N \mathrm{d}x \ge \int_{\mathbb{R}^N} \left|\nabla D_{N,s} v\right|^N \mathrm{d}x.$$

Hence

$$\frac{\left(\int_{\mathbb{R}^N} |v|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla v|^N \mathrm{d}x\right)^{\frac{1}{N}(1-\frac{q}{r})} \left(\int_{\mathbb{R}^N} |v|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}$$
$$\le \left(\frac{N}{N-s}\right)^{\frac{1}{N'}(1-\frac{q}{r})} \frac{\left(\int_{\mathbb{R}^N} \left|D_{N,s} v(x)\right|^r \mathrm{d}x\right)^{1/r}}{\left(\int_{\mathbb{R}^N} \left|\nabla D_{N,s} v\right|^N \mathrm{d}x\right)^{\frac{1}{N}(1-\frac{q}{r})} \left(\int_{\mathbb{R}^N} \left|D_{N,s} v(x)\right|^q \mathrm{d}x\right)^{\frac{1}{r}}}$$

$$\leq \left(\frac{N}{N-s}\right)^{\frac{1}{N'}\left(1-\frac{q}{r}\right)} \frac{\left(\int |U_0(x)|^r \,\mathrm{d}x\right)^{1/r}}{\left(\int |\nabla U_0|^N \,\mathrm{d}x\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)} \left(\int |U_0(x)|^q \,\mathrm{d}x\right)^{\frac{1}{r}}}$$
$$= \frac{\left(\int |V_0|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int |\nabla V_0|^N \,\mathrm{d}x\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)} \left(\int |V_0|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}.$$

We note that we have the equality in the last row because U_0 is radially symmetric. Moreover, we also obtain

$$\mathrm{CKN}\,(N,s,q,r) = \left(\frac{N}{N-s}\right)^{\frac{1}{N'}\left(1-\frac{q}{r}\right)} \mathrm{GN}(N,q,r).$$

A direct calculation shows that

$$V_0(x) = \alpha\left(1+\beta\,|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}} \text{ for some } \alpha\in\mathbb{R} \text{ and } \beta>0.$$

$\square$

Lemma 2.2 *Let $q > N > s \geq 0$ and*

$$V_q(x) = \left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}}.$$

Then

$$\int_{\mathbb{R}^N} \left(1+|x|^{\frac{N-s}{N-1}}\right)^{-N} \frac{\mathrm{d}x}{|x|^s} = \frac{1}{N-s}\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)},$$
$$\frac{1}{q}\int_{\mathbb{R}^N} \left|V_q\right|^q \frac{\mathrm{d}x}{|x|^s} \to \frac{\omega_{N-1}}{N(N-s)} \text{ as } q\to\infty,$$
$$\int_{\mathbb{R}^N} \left|\nabla V_q\right|^N \mathrm{d}x = \left(\frac{N-s}{N}\right)^{N-1} \frac{2N^{N-2}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} q^{1-N} + o\left(\frac{1}{q^{N-1}}\right) \text{ as } q\to\infty.$$

Moreover,

$$\Delta_N V_q = (N-1)\left(\frac{N-s}{q-N}\right)^{N-2} \frac{N-s}{q-N}\frac{N-s}{N-1} \times \left[\begin{array}{l} \frac{q-1}{q-N}\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}(N-2)-\frac{2q-N-1}{q-N}} r^{\frac{1-s}{N-1}(N-2)+2\frac{1-s}{N-1}} \\ -\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}(N-2)-\frac{q-1}{q-N}} r^{\frac{1-s}{N-1}(N-2)+\frac{2-s-N}{N-1}} \end{array}\right].$$

Proof If we perform the change of variable:

$$t=\frac{1}{1+r^{\frac{N-s}{N-1}}}\ \text{that is}\ r=\left(\frac{1-t}{t}\right)^{\frac{N-1}{N-s}}\ \text{and}\ \mathrm{d}r=\frac{N-1}{N-s}\left(\frac{1-t}{t}\right)^{\frac{-1+s}{N-s}}\cdot\frac{-1}{t^2}\mathrm{d}t,$$

then

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-N}\frac{\mathrm{d}x}{|x|^s}\\
&=\omega_{N-1}\int_0^\infty\left(1+r^{\frac{N-s}{N-1}}\right)^{-N}r^{N-1-s}\mathrm{d}r\\
&=\frac{N-1}{N-s}\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\int_0^1 t^N\left(\frac{1-t}{t}\right)^{\frac{N-1}{N-s}N-1-s}\left(\frac{1-t}{t}\right)^{\frac{-1+s}{N-s}}\cdot\frac{1}{t^2}\mathrm{d}t\\
&=\frac{N-1}{N-s}\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\int_0^1(1-t)^{N-2}\,\mathrm{d}t=\frac{1}{N-s}\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}.
\end{aligned}\tag{2.1}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^N}\left|V_q\right|^q\frac{\mathrm{d}x}{|x|^s}&=\int_{\mathbb{R}^N}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-q\frac{N-1}{q-N}}\frac{\mathrm{d}x}{|x|^s}\\
&=\omega_{N-1}\int_0^\infty\left(1+r^{\frac{N-s}{N-1}}\right)^{-q\frac{N-1}{q-N}}r^{N-1-s}\mathrm{d}r.
\end{aligned}$$

Again, we perform the same change of variables, then

$$\begin{aligned}
&\int_0^\infty\left(1+r^{\frac{N-s}{N-1}}\right)^{-q\frac{N-1}{q-N}}r^{N-1-s}\mathrm{d}r\\
&=\frac{N-1}{N-s}\int_0^1 t^{q\frac{N-1}{q-N}}\left(\frac{1-t}{t}\right)^{\frac{N-1}{N-s}N-1-s}\left(\frac{1-t}{t}\right)^{\frac{-1+s}{N-s}}\cdot\frac{1}{t^2}\mathrm{d}t\\
&=\frac{N-1}{N-s}\int_0^1 t^{\frac{N^2-q}{q-N}}(1-t)^{N-2}\,\mathrm{d}t=\frac{N-1}{N-s}\mathcal{B}\left(\frac{N^2-N}{q-N},N-1\right)\\
&=\frac{N-1}{N-s}\frac{\Gamma(N-1)}{\prod_{k=1}^{N-1}\left(\frac{q(N-1)}{q-N}-k\right)}\\
&=\frac{N-1}{N-s}\frac{\Gamma(N-1)}{\prod_{k=1}^{N-1}\left[q(N-1-k)+kN\right]}(q-N)^{N-1},
\end{aligned}$$

where $\mathcal{B}$ is the Euler beta function. Hence, as $q\to\infty$

$$\frac{1}{q}\int_{\mathbb{R}^N}\left|V_q\right|^q\frac{\mathrm{d}x}{|x|^s}\to\omega_{N-1}\frac{N-1}{N-s}\frac{1}{N(N-1)}=\frac{\omega_{N-1}}{N(N-s)}.\tag{2.2}$$

Now, noting that V_q is a radial function: $V_q(x) = V_q(r)$ with $r = |x|$, we get

$$\nabla V_q = V_q'(r)\,\frac{x}{r},$$

and so

$$\left|\nabla V_q\right|^{N-2} \nabla V_q = \left|V_q'(r)\right|^{N-2} V_q'(r)\,\frac{x}{r}.$$

Hence

$$\begin{aligned}
\Delta_N V_q &= \nabla \cdot \left(\left|\nabla V_q\right|^{N-2} \nabla V_q \right) \\
&= \sum \frac{\partial}{\partial x_i} \left(\left|V_q'(r)\right|^{N-2} V_q'(r)\,\frac{x_i}{r} \right) \\
&= \sum \left|V_q'(r)\right|^{N-2} V_q'(r)\,\frac{r - x_i \frac{x_i}{r}}{r^2} + \sum \left|V_q'(r)\right|^{N-2} \frac{x_i}{r} V_q''(r)\,\frac{x_i}{r} \\
&\quad + \sum V_q'(r)\,\frac{x_i}{r}\,\frac{N-2}{2} \left|V_q'(r)\right|^{2\left(\frac{N-2}{2}-1\right)} 2V_q'(r)\,V_q''(r)\frac{x_i}{r} \\
&= \left|V_q'(r)\right|^{N-2} V_q'(r)\,\frac{N-1}{r} + (N-1) \left|V_q'(r)\right|^{N-2} V_q''(r) \\
&= (N-1) \left|V_q'(r)\right|^{N-2} \left[\frac{1}{r} V_q'(r) + V_q''(r) \right].
\end{aligned}$$

Noting that

$$V_q(r) = -\frac{N-s}{q-N} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}} r^{\frac{1-s}{N-1}},$$

and

$$\begin{aligned}
V_q(r) = \frac{N-s}{q-N} \Bigg[&\frac{q-1}{q-N}\,\frac{N-s}{N-1} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{2q-N-1}{q-N}} r^{2\frac{1-s}{N-1}} \\
&- \frac{1-s}{N-1} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}} r^{\frac{2-s-N}{N-1}} \Bigg],
\end{aligned}$$

we obtain

$$\begin{aligned}
\Delta_N V_q = (N-1) &\left| \frac{N-s}{q-N} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}} r^{\frac{1-s}{N-1}} \right|^{N-2} \\
\times \Bigg[&-\tfrac{N-s}{q-N} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}} r^{\frac{1-s}{N-1}} \tfrac{1}{r} + \tfrac{N-s}{q-N}\,\tfrac{q-1}{q-N}\,\tfrac{N-s}{N-1} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{2q-N-1}{q-N}} r^{2\frac{1-s}{N-1}} \\
&- \tfrac{N-s}{q-N}\,\tfrac{1-s}{N-1} \left(1 + r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}} r^{\frac{2-s-N}{N-1}} \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= (N-1)\left(\frac{N-s}{q-N}\right)^{N-2}\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}(N-2)} r^{\frac{1-s}{N-1}(N-2)} \\
&\quad\times \frac{N-s}{q-N}\frac{N-s}{N-1}\left[\frac{q-1}{q-N}\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{2q-N-1}{q-N}} r^{2\frac{1-s}{N-1}} - \left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}} r^{\frac{2-s-N}{N-1}}\right] \\
&= (N-1)\left(\frac{N-s}{q-N}\right)^{N-2}\frac{N-s}{q-N}\frac{N-s}{N-1} \\
&\quad\times\left[\begin{array}{l} \frac{q-1}{q-N}\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}(N-2)-\frac{2q-N-1}{q-N}} r^{\frac{1-s}{N-1}(N-2)+2\frac{1-s}{N-1}} \\ \quad -\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}(N-2)-\frac{q-1}{q-N}} r^{\frac{1-s}{N-1}(N-2)+\frac{2-s-N}{N-1}} \end{array}\right]. \qquad (2.3)
\end{aligned}
$$

Next, since

$$
\nabla V_q = \nabla\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}} = -\frac{N-1}{q-N}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}-1}\frac{N-s}{N-1}|x|^{\frac{N-s}{N-1}-1}\frac{x}{|x|},
$$

we get that

$$
\begin{aligned}
\int_{\mathbb{R}^N}\left|\nabla V_q\right|^N \mathrm{d}x &= \left(\frac{N-s}{q-N}\right)^N \int_{\mathbb{R}^N}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}N}|x|^{\frac{1-s}{N-1}N}\mathrm{d}x \\
&= \left(\frac{N-s}{q-N}\right)^N \omega_{N-1}\int_0^\infty\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{q-1}{q-N}N} r^{\frac{1-s}{N-1}N} r^{N-1}\mathrm{d}r.
\end{aligned}
$$

We perform the same change of variables again:

$$
t = \frac{1}{1+r^{\frac{N-s}{N-1}}},\ \text{that is } r = \left(\frac{1-t}{t}\right)^{\frac{N-1}{N-s}} \text{ and } \mathrm{d}r = \frac{N-1}{N-s}\left(\frac{1-t}{t}\right)^{\frac{-1+s}{N-s}}\cdot\frac{-1}{t^2}\mathrm{d}t,
$$

and obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left|\nabla V_q\right|^N \mathrm{d}x \\
&= \frac{N-1}{N-s}\left(\frac{N-s}{q-N}\right)^N \omega_{N-1}\int_0^1 t^{\frac{q-1}{q-N}N}\left(\frac{1-t}{t}\right)^{\frac{N-1}{N-s}\frac{1-s}{N-1}N}\left(\frac{1-t}{t}\right)^{\frac{N-1}{N-s}(N-1)} \\
&\quad\times\left(\frac{1-t}{t}\right)^{\frac{-1+s}{N-s}}\frac{1}{t^2}\mathrm{d}t \\
&= \frac{N-1}{N-s}\left(\frac{N-s}{q-N}\right)^N \omega_{N-1}\int_0^1 t^{\frac{q-1}{q-N}N-N-1}(1-t)^{N-1} \\
&= \frac{N-1}{N-s}\left(\frac{N-s}{q-N}\right)^N \omega_{N-1}\mathcal{B}\left(\frac{q-1}{q-N}N-N, N\right) \\
&= \frac{N-1}{N-s}\left(\frac{N-s}{q-N}\right)^N \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\mathcal{B}\left(\frac{N^2-N}{q-N}, N\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{N-1}{N-s}\left(\frac{N-s}{q-N}\right)^{N} \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \frac{\Gamma\left(\frac{N^2-N}{q-N}\right)\Gamma(N)}{\Gamma\left(\frac{N(q-1)}{q-N}\right)} \\
&= \left(\frac{N-s}{N}\right)^{N-1} \frac{N-1}{N}\left(\frac{N}{q-N}\right)^{N} \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \frac{\Gamma\left(\frac{N^2-N}{q-N}\right)\Gamma(N)}{\Gamma\left(\frac{N(q-1)}{q-N}\right)} \\
&= \left(\frac{N-s}{N}\right)^{N-1} \frac{N-1}{N}\left(\frac{N}{q-N}\right)^{N} \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \frac{\Gamma(N)}{\prod_{k=1}^{N}\left(\frac{N(q-1)}{q-N}-k.\right)}.
\end{aligned}
$$

Hence, as $q \to \infty$

$$
\begin{aligned}
&q^{N-1}\int_{\mathbb{R}^N}\left|\nabla V_q\right|^N \mathrm{d}x \to (N-1)(N-s)^{N-1} \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \frac{\Gamma(N)}{\prod_{k=1}^{N-1}(N-k)} \frac{1}{N(N-1)} \\
&\quad = \frac{(N-s)^{N-1}}{N} \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
\int_{\mathbb{R}^N}\left|\nabla V_q\right|^N \mathrm{d}x &= \frac{(N-s)^{N-1}}{N} \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} q^{1-N} + o\left(\frac{1}{q^{N-1}}\right) \\
&= \left(\frac{N-s}{N}\right)^{N-1} \frac{2N^{N-2}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} q^{1-N} + o\left(\frac{1}{q^{N-1}}\right). \qquad (2.4)
\end{aligned}
$$

□

Now, we consider the limit when $q \downarrow N$ of

$$
\begin{aligned}
&N\ln\left(\frac{q-N}{N\sqrt{\pi}}\right) + \ln\left(\frac{q}{q-N}\right) + \ln\left(\frac{\Gamma\left(q\frac{N-1}{q-N}\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)}\right) \\
&\quad = \ln\left[\left(\frac{q-N}{N\sqrt{\pi}}\right)^{N}\left(\frac{q}{q-N}\right)\frac{\Gamma\left(q\frac{N-1}{q-N}\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)}\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
\Gamma\left(q\frac{N-1}{q-N}\right) &= \Gamma\left(q\frac{N-1}{q-N}-(N-1)\right)\prod_{k=1}^{N-1}\left(q\frac{N-1}{q-N}-k\right) \\
&= \Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)\prod_{k=1}^{N-1}\left(\frac{q(N-1-k)}{q-N}+\frac{Nk}{q-N}\right).
\end{aligned}
$$

Hence

$$\begin{aligned}
&\left(\frac{q-N}{N\sqrt{\pi}}\right)^N \left(\frac{q}{q-N}\right) \frac{\Gamma\left(q\frac{N-1}{q-N}\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)} \\
&= \frac{q}{\left(N\sqrt{\pi}\right)^N} (q-N)^{N-1} \prod_{k=1}^{N-1} \left(\frac{q\,(N-1-k)+Nk}{q-N}\right) \\
&= \frac{q}{\left(N\sqrt{\pi}\right)^N} \prod_{k=1}^{N-1} (q\,(N-1-k)+Nk) \\
&\to \frac{N}{\left(N\sqrt{\pi}\right)^N} \prod_{k=1}^{N-1} (N\,(N-1-k)+Nk) \\
&= \frac{N}{\left(N\sqrt{\pi}\right)^N} [N\,(N-1)]^{N-1} = \frac{(N-1)^{N-1}}{\left(\sqrt{\pi}\right)^N}.
\end{aligned}$$

So as $q \downarrow N$,

$$N\ln\left(\frac{q-N}{N\sqrt{\pi}}\right) + \ln\left(\frac{q}{q-N}\right) + \ln\left(\frac{\Gamma\left(q\frac{N-1}{q-N}\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)}\right) \to \ln\left(\frac{(N-1)^{N-1}}{\left(\sqrt{\pi}\right)^N}\right). \tag{2.5}$$

Lemma 2.3 *For* $0 \le s < N$ *and* $b > 0$, *we have*

$$\begin{aligned}
\int_{\mathbb{R}^N} \exp\left(-Nb\,|x|^{\frac{N-s}{N-1}}\right) \frac{\mathrm{d}x}{|x|^s} &= \omega_{N-1} \frac{N-1}{N-s} \left(\frac{1}{Nb}\right)^{N-1} \Gamma(N-1) \\
\int_{\mathbb{R}^N} \left|\exp\left(-b\,|x|^{\frac{N-s}{N-1}}\right)\right|^N \ln\left|\exp\left(-b\,|x|^{\frac{N-s}{N-1}}\right)\right| \frac{\mathrm{d}x}{|x|^s} &= -b\omega_{N-1} \frac{N-1}{N-s} \left(\frac{1}{Nb}\right)^N \Gamma(N) \\
\int_{\mathbb{R}^N} \left|\nabla \exp\left(-b\,|x|^{\frac{N-s}{N-1}}\right)\right|^N \mathrm{d}x &= \left(\frac{N-s}{N-1}\right)^N \omega_{N-1} \frac{N-1}{N-s} \left(\frac{1}{N}\right)^N \Gamma(N).
\end{aligned}$$

Proof We first consider the integral

$$\int_{\mathbb{R}^N} \exp\left(-k\,|x|^n\right) \frac{\mathrm{d}x}{|x|^s} = \omega_{N-1} \int_0^\infty \exp\left(-kr^n\right) r^{N-1-s} \mathrm{d}r.$$

By the change of variable $t = kr^n$, that is $r = \left(\frac{1}{k}\right)^{\frac{1}{n}} t^{\frac{1}{n}}$ and $\mathrm{d}r = \left(\frac{1}{k}\right)^{\frac{1}{n}} \frac{1}{n} t^{\frac{1}{n}-1} \mathrm{d}t$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \exp\left(-k|x|^n\right) \frac{\mathrm{d}x}{|x|^s} &= \omega_{N-1} \int_0^\infty \exp(-t) \left(\frac{1}{k}\right)^{\frac{N-1-s}{n}} t^{\frac{N-1-s}{n}} \left(\frac{1}{k}\right)^{\frac{1}{n}} \frac{1}{n} t^{\frac{1}{n}-1} \mathrm{d}t \\
&= \omega_{N-1} \frac{1}{n} \left(\frac{1}{k}\right)^{\frac{N-s}{n}} \int_0^\infty \exp(-t)\, t^{\frac{N-s}{n}-1} \mathrm{d}t \\
&= \omega_{N-1} \frac{1}{n} \left(\frac{1}{k}\right)^{\frac{N-s}{n}} \Gamma\left(\frac{N-s}{n}\right). \qquad (2.6)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_{\mathbb{R}^N} \exp\left(-k|x|^n\right) |x|^n \frac{\mathrm{d}x}{|x|^s} &= \omega_{N-1} \int_0^\infty \exp\left(-kr^n\right) r^{n+N-1-s} \mathrm{d}r \\
&= \omega_{N-1} \int_0^\infty \exp(-t) \left(\frac{1}{k}\right)^{\frac{n+N-1-s}{n}} t^{\frac{n+N-1-s}{n}} \left(\frac{1}{k}\right)^{\frac{1}{n}} \frac{1}{n} t^{\frac{1}{n}-1} \mathrm{d}t \\
&= \omega_{N-1} \frac{1}{n} \left(\frac{1}{k}\right)^{1+\frac{N-s}{n}} \int_0^\infty \exp(-t)\, t^{\frac{N-s}{n}} \mathrm{d}t \\
&= \omega_{N-1} \frac{1}{n} \left(\frac{1}{k}\right)^{1+\frac{N-s}{n}} \Gamma\left(\frac{N-s}{n}+1\right). \qquad (2.7)
\end{aligned}
$$

Hence

$$
\int_{\mathbb{R}^N} \exp\left(-Nb|x|^{\frac{N-s}{N-1}}\right) \frac{\mathrm{d}x}{|x|^s} = \omega_{N-1} \frac{N-1}{N-s} \left(\frac{1}{Nb}\right)^{N-1} \Gamma(N-1), \qquad (2.8)
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left|\exp\left(-b|x|^{\frac{N-s}{N-1}}\right)\right|^N \ln\left|\exp\left(-b|x|^{\frac{N-s}{N-1}}\right)\right| \frac{\mathrm{d}x}{|x|^s} \\
&= -b \int_{\mathbb{R}^N} \exp\left(-Nb|x|^{\frac{N-s}{N-1}}\right) |x|^{\frac{N-s}{N-1}} \frac{\mathrm{d}x}{|x|^s} \\
&= -b\omega_{N-1} \frac{N-1}{N-s} \left(\frac{1}{Nb}\right)^N \Gamma(N). \qquad (2.9)
\end{aligned}
$$

Also, since $\nabla \exp\left(-b|x|^{\frac{N-s}{N-1}}\right) = -b\frac{N-s}{N-1} \exp\left(-b|x|^{\frac{N-s}{N-1}}\right) |x|^{\frac{N-s}{N-1}-1} \frac{x}{|x|}$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \left|\nabla \exp\left(-b|x|^{\frac{N-s}{N-1}}\right)\right|^N \mathrm{d}x &= \left(b\frac{N-s}{N-1}\right)^N \int_{\mathbb{R}^N} \exp\left(-Nb|x|^{\frac{N-s}{N-1}}\right) |x|^{N\frac{1-s}{N-1}} \mathrm{d}x \\
&= \left(b\frac{N-s}{N-1}\right)^N \int_{\mathbb{R}^N} \exp\left(-Nb|x|^{\frac{N-s}{N-1}}\right) |x|^{N\frac{1-s}{N-1}+s} \frac{\mathrm{d}x}{|x|^s}
\end{aligned}
$$

$$= \left(b\frac{N-s}{N-1}\right)^N \int_{\mathbb{R}^N} \exp\left(-Nb\,|x|^{\frac{N-s}{N-1}}\right) |x|^{\frac{N-s}{N-1}} \frac{\mathrm{d}x}{|x|^s}$$
$$= \left(b\frac{N-s}{N-1}\right)^N \omega_{N-1}\frac{N-1}{N-s}\left(\frac{1}{Nb}\right)^N \Gamma(N). \qquad (2.10)$$

3 Weighted Moser–Onofri–Beckner Inequality: Proof of Theorem 1.1

Proof of Theorem 1.1 By Lemma 2.1, we have

$$\frac{\left(\int_{\mathbb{R}^N} |\nabla v|^N \,\mathrm{d}x\right)^{\frac{1}{N}\theta} \left(\int_{\mathbb{R}^N} |v|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1-\theta}{q}}}{\left(\int_{\mathbb{R}^N} |v|^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{N-1}{N(q-1)}}} \geq \frac{\left(\int_{\mathbb{R}^N} \left|\nabla V_q\right|^N \mathrm{d}x\right)^{\frac{1}{N}\theta} \left(\int_{\mathbb{R}^N} \left|V_q\right|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1-\theta}{q}}}{\left(\int_{\mathbb{R}^N} \left|V_q\right|^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{N-1}{N(q-1)}}},$$

where

$$V_q(x) = \left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}}.$$

Let u be a smooth compactly supported function such that $\int_{\mathbb{R}^N} u \,\mathrm{d}\mu_{N,s} = 0$ and set

$$v_q = V_q\left(1+\frac{N-1}{Nq}u\right).$$

We have that

$$\left(\frac{\int_{\mathbb{R}^N} \left|\nabla v_q\right|^N \mathrm{d}x}{\int_{\mathbb{R}^N} \left|\nabla V_q\right|^N \mathrm{d}x}\right)^{\frac{q-N}{N(N-1)}} \left(\frac{\int_{\mathbb{R}^N} \left|v_q\right|^q \frac{\mathrm{d}x}{|x|^s}}{\int_{\mathbb{R}^N} \left|V_q\right|^q \frac{\mathrm{d}x}{|x|^s}}\right) \geq \left(\frac{\int_{\mathbb{R}^N} \left|v_q\right|^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s}}{\int_{\mathbb{R}^N} \left|V_q\right|^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s}}\right).$$

First, we have by (2.1)

$$\lim_{p\to\infty}\int_{\mathbb{R}^N} \left|V_q\right|^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s} = \lim_{p\to\infty}\int_{\mathbb{R}^N} \left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s}$$
$$= \int_{\mathbb{R}^N} \left(1+|x|^{\frac{N-s}{N-1}}\right)^{-N} \frac{\mathrm{d}x}{|x|^s} = \frac{1}{N-s}\omega_{N-1},$$

and

$$\lim_{p\to\infty}\int_{\mathbb{R}^N} \left|v_q\right|^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s} = \lim_{p\to\infty}\int_{\mathbb{R}^N} \left|V_q\right|^{\frac{N(q-1)}{N-1}} \left(1+\frac{N-1}{Nq}u\right)^{\frac{N(q-1)}{N-1}} \frac{\mathrm{d}x}{|x|^s}$$
$$= \int_{\mathbb{R}^N} \left(1+|x|^{\frac{N-s}{N-1}}\right)^{-N} e^u \frac{\mathrm{d}x}{|x|^s},$$

so

$$\frac{\int_{\mathbb{R}^N}\left|v_q\right|^{N\alpha}\frac{\mathrm{d}x}{|x|^s}}{\int_{\mathbb{R}^N}\left|V_q\right|^{N\alpha}\frac{\mathrm{d}x}{|x|^s}}\to\int_{\mathbb{R}^N}e^u\frac{1}{\frac{1}{N-s}\omega_{N-1}}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-N}\frac{\mathrm{d}x}{|x|^s}=\int_{\mathbb{R}^N}e^u\mathrm{d}\mu_{N,s}.$$

Also, by (2.2), when $q\to\infty$

$$\begin{aligned}\int_{\mathbb{R}^N}\left|V_q\right|^q\frac{\mathrm{d}x}{|x|^s}&\approx\frac{\omega_{N-1}}{N\left(N-s\right)}\frac{1}{q},\\ \lim_{q\to\infty}\int_{\mathbb{R}^N}\left|v_q\right|^q\frac{\mathrm{d}x}{|x|^s}&=\infty,\end{aligned}$$

but

$$\lim_{q\to\infty}\frac{\int_{\mathbb{R}^N}\left|v_q\right|^q\frac{\mathrm{d}x}{|x|^s}}{\int_{\mathbb{R}^N}\left|V_q\right|^q\frac{\mathrm{d}x}{|x|^s}}=\lim_{q\to\infty}\frac{\int_{\mathbb{R}^N}\left|V_q\right|^q\left(1+\frac{N-1}{Nq}u\right)^q\frac{\mathrm{d}x}{|x|^s}}{\int_{\mathbb{R}^N}\left|V_q\right|^q\frac{\mathrm{d}x}{|x|^s}}=1.$$

Now, with

$$\begin{aligned}X_q&=\left(1+\frac{N-1}{Nq}u\right)\nabla V_q\\ Y_q&=\frac{N-1}{Nq}V_q\nabla u,\end{aligned}$$

then

$$\begin{aligned}\left|\nabla v_q\right|^N=&\left|\nabla V_q\right|^N\left(1+\frac{N-1}{Nq}u\right)^N+V_q\left|\nabla V_q\right|^{N-2}\nabla V_q\cdot\nabla\left(1+\frac{N-1}{Nq}u\right)^N\\ &+\mathcal{R}_N\left(X_q,Y_q\right).\end{aligned}$$

We estimate the second term as follows:

$$\begin{aligned}&\int_{\mathbb{R}^N}V_q\left|\nabla V_q\right|^{N-2}\nabla V_q\cdot\nabla\left(1+\frac{N-1}{Nq}u\right)^N\\ &\quad=-\int_{\mathbb{R}^N}\left(1+\frac{N-1}{Nq}u\right)^N\nabla\left(V_q\left|\nabla V_q\right|^{N-2}\nabla V_q\right)\\ &\quad=-\int_{\mathbb{R}^N}\left(1+\frac{N-1}{Nq}u\right)^N\left|\nabla V_q\right|^N-\int_{\mathbb{R}^N}\left(1+\frac{N-1}{Nq}u\right)^N V_q\Delta_N V_q.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^N}\left|\nabla v_q\right|^N=-\int_{\mathbb{R}^N}\left(1+\frac{N-1}{Nq}u\right)^N V_q\Delta_N V_q+\int_{\mathbb{R}^N}\mathcal{R}_N\left(X_q,Y_q\right).$$

Now, we note that

$$
\begin{aligned}
qX_q &= \left(1+\frac{N-1}{Nq}u\right)q\nabla V_q \\
&= -\left(1+\frac{N-1}{Nq}u\right)q\frac{N-1}{q-N}\left(1+|x|^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}-1}\frac{N-s}{N-1}|x|^{\frac{N-s}{N-1}-1}\frac{x}{|x|} \\
&\to -(N-s)\,|x|^{\frac{N-s}{N-1}-1}\frac{x}{|x|}\frac{1}{1+|x|^{\frac{N-s}{N-1}}} \text{ a.e. as } q\to\infty
\end{aligned}
$$

and

$$
qY_q = q\frac{N-1}{Nq}V_q\nabla u \to \frac{N-1}{N}\nabla u.
$$

As a consequence, both X_q and Y_q in $\mathcal{R}_N\left(X_q,Y_q\right)$ are of the order of $\frac{1}{q}$. Hence, when $q\to\infty$

$$
\begin{aligned}
q^N\mathcal{R}_N\left(X_q,Y_q\right) &\to \mathcal{R}_N\left(-(N-s)\,|x|^{\frac{N-s}{N-1}-1}\frac{x}{|x|}\frac{1}{1+|x|^{\frac{N-s}{N-1}}},\frac{N-1}{N}\nabla u\right) \\
&= \mathcal{H}_N\left(x,\nabla u\right).
\end{aligned}
$$

Next, we have from (2.4) that

$$
-\int_{\mathbb{R}^N}V_q\Delta_N V_q = \int_{\mathbb{R}^N}\left|\nabla V_q\right|^N \mathrm{d}x = \left(\frac{N-s}{N}\right)^{N-1}\frac{2N^{N-2}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}q^{1-N},
$$

and from (2.3) that

$$
\begin{aligned}
&q^{N-1}V_q\Delta_N V_q \\
&= q^{N-1}(N-1)\left(\frac{N-s}{q-N}\right)^{N-2}\frac{N-s}{q-N}\frac{q-1}{q-N}\frac{N-s}{N-1} \\
&\quad\times\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}-\frac{q-1}{q-N}(N-2)-\frac{2q-N-1}{q-N}}r^{\frac{1-s}{N-1}(N-2)+2\frac{1-s}{N-1}} \\
&\quad - q^{N-1}(N-1)\left(\frac{N-s}{q-N}\right)^{N-2}\frac{N-s}{q-N}\frac{N-s}{N-1} \\
&\quad\times\left(1+r^{\frac{N-s}{N-1}}\right)^{-\frac{N-1}{q-N}-\frac{q-1}{q-N}(N-2)-\frac{q-1}{q-N}}r^{\frac{1-s}{N-1}(N-2)+\frac{2-s-N}{N-1}} \\
&\quad\to (N-s)^N\left[\left(1+r^{\frac{N-s}{N-1}}\right)^{-N}r^{\frac{1-s}{N-1}(N-2)+2\frac{1-s}{N-1}} - \left(1+r^{\frac{N-s}{N-1}}\right)^{-N+1}r^{\frac{1-s}{N-1}(N-2)+\frac{2-s-N}{N-1}}\right] \\
&= (N-s)^N\left(1+r^{\frac{N-s}{N-1}}\right)^{-N}r^{\frac{1-s}{N-1}(N-2)}\left[r^{2\frac{1-s}{N-1}}-\left(1+r^{\frac{N-s}{N-1}}\right)r^{\frac{2-s-N}{N-1}}\right]
\end{aligned}
$$

$$= -(N-s)^N \left(1 + r^{\frac{N-s}{N-1}}\right)^{-N} \frac{1}{r^s}$$
$$= -(N-s)^{N-1}\, \omega_{N-1}\mu_{N,s} \text{ as } q \to \infty.$$

Hence

$$-\int_{\mathbb{R}^N} V_q \Delta_N V_q u \mathrm{d}x = \frac{1}{q^{N-1}} (N-s)^{N-1}\, \omega_{N-1} \int_{\mathbb{R}^N} u \mathrm{d}\mu_{N,s} + o\left(\frac{1}{q^{N-1}}\right)$$
$$= o\left(\frac{1}{q^{N-1}}\right).$$

So

$$\int_{\mathbb{R}^N} \left|\nabla v_q\right|^N = -\int_{\mathbb{R}^N} \left(1 + N\frac{N-1}{Nq} u + o\left(\frac{1}{q}\right)\right) V_q \Delta_N V_q + \frac{1}{q^N} \int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)$$
$$= \int_{\mathbb{R}^N} \left|\nabla V_q\right|^N \mathrm{d}x + \frac{1}{q^N} \int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)\, \mathrm{d}x + o\left(\frac{1}{q^N}\right).$$

Letting $q \to \infty$, we can deduce that

$$\left(\frac{\int_{\mathbb{R}^N} \left|\nabla v_q\right|^N \mathrm{d}x}{\int_{\mathbb{R}^N} \left|\nabla V_q\right|^N \mathrm{d}x}\right)^{\frac{q-N}{N(N-1)}} \approx \left(1 + \frac{\int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)\, \mathrm{d}x}{q\left(\frac{N-s}{N}\right)^{N-1} \frac{2N^{N-2}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}}\right)^{\frac{q-N}{N(N-1)}}$$
$$\approx \left(1 + \frac{N(N-1)}{q} \alpha_{N,s} \int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)\, \mathrm{d}x\right)^{\frac{q-N}{N(N-1)}}$$
$$\approx \exp\left[\alpha_{N,s} \int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)\, \mathrm{d}x\right].$$

Here

$$\alpha_{N,s} = \frac{1}{N(N-1)} \frac{1}{\left(\frac{N-s}{N}\right)^{N-1} \frac{2N^{N-2}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}} = \left(\frac{N}{N-s}\right)^{N-1} \frac{\Gamma\left(\frac{N}{2}\right)}{2(N-1)N^{N-1}\pi^{\frac{N}{2}}}.$$

Finally, putting everything together, we obtain that with $\int_{\mathbb{R}^N} u \mathrm{d}\mu_{N,s} = 0$, then

$$\int_{\mathbb{R}^N} e^u \mathrm{d}\mu_{N,s} \leq \exp\left[\alpha_{N,s} \int_{\mathbb{R}^N} \mathcal{H}_N(x, \nabla u)\, \mathrm{d}x\right].$$

□

4 Weighted L^N-Logarithmic Sobolev Inequality: Proof of Theorem 1.2

Proof of Theorem 1.2 We first recall that from Theorem A, we obtain

$$\left(\int_{\mathbb{R}^N} |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}\left(1-\frac{q}{r}\right)} \left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}} CKN\,(N,s,q,r) \geq \left(\int_{\mathbb{R}^N} |u|^r \,\frac{\mathrm{d}x}{|x|^s}\right)^{1/r}.$$

Hence

$$\begin{aligned}
&\left(1-\frac{q}{r}\right)\ln\left(\int_{\mathbb{R}^N} |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}} + \frac{q}{r}\ln\left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}} + \ln\left(CKN\,(N,s,q,r)\right) \\
&\quad \geq \ln\left(\int_{\mathbb{R}^N} |u|^r \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}.
\end{aligned}$$

Equivalently,

$$\ln \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}}}{\left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} \geq \frac{1}{1-\frac{q}{r}} \ln \frac{\left(\int_{\mathbb{R}^N} |u|^r \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} - \frac{1}{1-\frac{q}{r}} \ln\left(CKN\,(N,s,q,r)\right).$$

Noting that $r = N\frac{q-1}{N-1}$, we have $1-\frac{q}{r} = \frac{q-N}{N(q-1)}$ and

$$\begin{aligned}
&\ln \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}}}{\left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} \\
&\quad \geq \frac{N\,(q-1)}{q-N} \ln \frac{\left(\int_{\mathbb{R}^N} |u|^r \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} - \frac{N\,(q-1)}{q-N} \ln\left(CKN\,(N,s,q,r)\right).
\end{aligned}$$

It is clear that as $q \downarrow N$

$$\ln \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}}}{\left(\int_{\mathbb{R}^N} |u|^q \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} \to \ln \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^N \,\mathrm{d}x\right)^{\frac{1}{N}}}{\left(\int_{\mathbb{R}^N} |u|^N \,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{N}}}$$

and by (2.5)

$$\begin{aligned}
&\frac{N\,(q-1)}{q-N} \ln\left(CKN\,(N,s,q,r)\right) \\
&\quad = \frac{1}{N'} \ln\left(\frac{N}{N-s}\right) + \ln\left(\frac{q-N}{N\sqrt{\pi}}\right) + \frac{1}{N}\ln\left(\frac{q}{q-N}\right) + \frac{N\,(q-1)}{q-N}\frac{1}{r}\ln\left(\frac{N}{q}\right) \\
&\qquad + \frac{1}{N}\ln\left(\frac{\Gamma\left(q\frac{N-1}{q-N}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N-1}{N}\frac{N^2}{q-N}\right)\Gamma(N)}\right)
\end{aligned}$$

$$\to \frac{1}{N'}\ln\left(\frac{N}{N-s}\right)+\frac{1}{N}\ln\left(\frac{(N-1)^{N-1}}{\left(\sqrt{\pi}\right)^{N}}\right)-\frac{N-1}{N}+\frac{1}{N}\ln\left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma(N)}\right)$$
$$= SL\,(N,s)\,.$$

Now, we consider the limit when $q\downarrow N$ of

$$\frac{1}{q-N}\ln\frac{\left(\int_{\mathbb{R}^N}|u|^r\,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N}|u|^q\,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}}=\frac{\frac{N-1}{N(q-1)}\ln\left(\int_{\mathbb{R}^N}|u|^{N\frac{q-1}{N-1}}\,\frac{\mathrm{d}x}{|x|^s}\right)-\frac{1}{q}\ln\left(\int_{\mathbb{R}^N}|u|^q\,\frac{\mathrm{d}x}{|x|^s}\right)}{q-N}.$$

By L'Hospital's Rule, it is equal to

$$\frac{\frac{N-1}{N(q-1)}\int_{\mathbb{R}^N}|u|^{N\frac{q-1}{N-1}}\ln|u|\,\frac{N}{N-1}\frac{\mathrm{d}x}{|x|^s}}{\left(\int_{\mathbb{R}^N}|u|^{N\frac{q-1}{N-1}}\,\frac{\mathrm{d}x}{|x|^s}\right)}-\frac{N-1}{N\,(q-1)^2}\ln\left(\int_{\mathbb{R}^N}|u|^{N\frac{q-1}{N-1}}\,\frac{\mathrm{d}x}{|x|^s}\right)$$
$$-\frac{\frac{1}{q}\int_{\mathbb{R}^N}|u|^q\ln|u|\,\frac{\mathrm{d}x}{|x|^s}}{\left(\int_{\mathbb{R}^N}|u|^q\,\frac{\mathrm{d}x}{|x|^s}\right)}+\frac{1}{q^2}\ln\left(\int_{\mathbb{R}^N}|u|^q\,\frac{\mathrm{d}x}{|x|^s}\right).$$

Noting that $\int_{\mathbb{R}^N}|u|^N\,\frac{\mathrm{d}x}{|x|^s}=1$, we get

$$\frac{1}{q-N}\ln\frac{\left(\int_{\mathbb{R}^N}|u|^r\,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{r}}}{\left(\int_{\mathbb{R}^N}|u|^q\,\frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}}\to\frac{1}{N-1}\int_{\mathbb{R}^N}|u|^N\ln|u|\,\frac{\mathrm{d}x}{|x|^s}-\frac{1}{N}\int_{\mathbb{R}^N}|u|^N\ln|u|\,\frac{\mathrm{d}x}{|x|^s}$$
$$=\frac{1}{N\,(N-1)}\int_{\mathbb{R}^N}|u|^N\ln|u|\,\frac{\mathrm{d}x}{|x|^s}.$$

Putting everything together, we obtain

$$\ln\left(SL\,(N,s)\int_{\mathbb{R}^N}|\nabla u|^N\,\mathrm{d}x\right)\geq\int_{\mathbb{R}^N}|u|^N\ln|u|^N\,\frac{\mathrm{d}x}{|x|^s}.$$

Now, with

$$U\,(x)=\exp\left(-b\,|x|^{\frac{N-s}{N-1}}\right),$$

where

$$b>0\text{ and }b^{N-1}=\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\frac{N-1}{N-s}\frac{\Gamma\,(N-1)}{N^{N-1}},$$

we have by (2.8), (2.9), and (2.10) that

$$\ln\left(\int_{\mathbb{R}^N} |\nabla U|^N \,\mathrm{d}x\right)^{\frac{1}{N}} = \frac{1}{N}\ln\left[\left(\frac{N-s}{N-1}\right)^N \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \frac{N-1}{N-s}\left(\frac{1}{N}\right)^N \Gamma(N)\right]$$
$$= \frac{N-1}{N}\ln\frac{N-s}{N-1} + \frac{1}{N}\ln\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}+1\right)} + \frac{1}{N}\ln\frac{\pi^{\frac{N}{2}}}{N^{N-1}},$$

and

$$\int_{\mathbb{R}^N} |U|^N \ln|U| \frac{\mathrm{d}x}{|x|^s} = -\frac{1}{b^{N-1}}\omega_{N-1}\frac{N-1}{N-s}\left(\frac{1}{N}\right)^N \Gamma(N)$$
$$= -\frac{N-1}{N-s}\left(\frac{1}{N}\right)^N \Gamma(N)\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \frac{\Gamma\left(\frac{N}{2}+1\right)(N-s)\,N^{N-1}}{N\pi^{\frac{N}{2}}(N-1)\,\Gamma(N-1)}$$
$$= -\frac{N-1}{N}.$$

Hence

$$\ln\left(\mathrm{SL}\,(N,s)\int_{\mathbb{R}^N} |\nabla U|^N \,\mathrm{d}x\right) = \int_{\mathbb{R}^N} |U|^N \ln|U|^N \frac{\mathrm{d}x}{|x|^s}.$$

□

5 Weighted L^p-Logarithmic Sobolev Inequality for $1 < p < N$

Proof of Theorem 1.3 By Theorem B, we have

$$\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1-a}{q}} CKN\,(N,\mu,s,p,q,r)$$
$$\geq \left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}.$$

Hence

$$\ln\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{\frac{1}{p}} + \left(\frac{1-a}{a}\right)\ln\left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}} + \frac{1}{a}\ln CKN\,(N,\mu,s,p,q,r)$$
$$\geq \frac{1}{a}\ln\left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r},$$

and

$$\ln \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} + \frac{1}{a} \ln CKN\,(N, \mu, s, p, q, r) \geq \frac{1}{a} \ln \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}}. \tag{5.1}$$

Now, we will consider the limiting process when $q \downarrow p$ of (5.1). In this case, it is easy to see that

$$\ln \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} \to \ln \left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{\frac{1}{p}}.$$

Now, by L'Hospital's Rule, we get

$$\begin{aligned}
&\lim_{q \downarrow p} \frac{\frac{1}{r} \ln \left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right) - \frac{1}{q} \ln \left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)}{q-p} \\
&= \lim_{q \downarrow p} \left[\begin{array}{c} -\frac{p}{(p-1)r^2} \ln \left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right) + \frac{1}{r} \frac{p}{p-1} \frac{\int_{\mathbb{R}^N} |u|^r \ln|u| \frac{\mathrm{d}x}{|x|^s}}{\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}} \\ + \frac{1}{q^2} \ln \left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right) - \frac{1}{q} \frac{\int_{\mathbb{R}^N} |u|^q \ln|u| \frac{\mathrm{d}x}{|x|^s}}{\left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)} \end{array} \right] \\
&= \frac{1}{p^2\,(p-1)} \int_{\mathbb{R}^N} |u|^p \ln |u|^p \frac{\mathrm{d}x}{|x|^s}.
\end{aligned}$$

Hence, as $q \downarrow p$

$$\frac{1}{a} \ln \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^s}\right)^{\frac{1}{q}}} \to \frac{1}{N} \int_{\mathbb{R}^N} |u|^p \ln |u|^p \frac{\mathrm{d}x}{|x|^s}.$$

Now, we will consider the limit problem

$$\lim_{q \downarrow p} \frac{1}{a} \ln \mathrm{CKN}(N, \mu, s, p, q, r).$$

First,

$$\begin{aligned}
&\lim_{q \downarrow p} \frac{1}{a} \ln \left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r} + \frac{p-1}{p} - \frac{1-a}{q} - \frac{p-1}{p}(1-a)} \\
&= \lim_{q \downarrow p} \frac{p-1}{p} \ln \left(\frac{N-p}{N-p-\mu}\right) + \lim_{q \downarrow p} \frac{1}{a} \frac{1}{q-1} \frac{(q-p)\,(N-p)}{p\,[pq - N\,(q-p)]} \ln \left(\frac{N-p}{N-p-\mu}\right) \\
&= \frac{N-1}{N} \ln \left(\frac{N-p}{N-p-\mu}\right).
\end{aligned}$$

Next, by L'Hospital's Rule

$$\lim_{q\downarrow p}\frac{1}{a}\ln\left(\frac{Np-q\,(N-p)}{pq}\right)^{\frac{1}{r}}$$
$$=\lim_{q\downarrow p}\frac{(p-1)\,[pq-N\,(q-p)]\,r}{Np}\frac{1}{r}\frac{1}{(q-p)}\ln\left(\frac{Np-q\,(N-p)}{pq}\right)=-\frac{(p-1)}{p}.$$

Also

$$\lim_{q\downarrow p}\frac{1}{a}\ln\left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}=\frac{1}{N}\ln\left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{p-1}{p}+1\right)}\right).$$

Finally, let us consider the limit

$$\lim_{q\downarrow p}\frac{1}{a}\ln\left[\left(\frac{q-p}{p\sqrt{\pi}}\right)^{a}\left(\frac{pq}{N\,(q-p)}\right)^{\frac{a}{p}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)}\right)^{\frac{a}{N}}\right]$$
$$=\lim_{q\downarrow p}\ln\left[\left(\frac{q-p}{p\sqrt{\pi}}\right)\left(\frac{pq}{N\,(q-p)}\right)^{\frac{1}{p}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)}{\Gamma\left(\frac{p-1}{p}\frac{Np-q(N-p)}{q-p}\right)}\right)^{\frac{1}{N}}\right]$$
$$=\lim_{q\downarrow p}\ln\left[\left(\frac{1}{p\sqrt{\pi}}\right)\left(\frac{pq}{N}\right)^{\frac{1}{p}}(q-p)^{1-\frac{1}{p}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)}{\Gamma\left(\frac{p-1}{p}\frac{Np-q(N-p)}{q-p}\right)}\right)^{\frac{1}{N}}\right].$$

Noting that with $n=q\frac{p-1}{q-p}\to\infty$ as $q\downarrow p$, and $z=-N\frac{p-1}{p}$, we have by the asymptotic approximations of the gamma function and by the Stirling's formula that

$$\left(\frac{q-p}{q\,(p-1)}\right)^{N\left(1-\frac{1}{p}\right)}\frac{\Gamma\left(q\frac{p-1}{q-p}\right)}{\Gamma\left(\frac{p-1}{p}\frac{Np-q(N-p)}{q-p}\right)}=\frac{\Gamma(N)n^{z}}{\Gamma\,(n+z)}\to 1\text{ as }q\downarrow p\text{ and }n\to\infty.$$

Hence

$$\lim_{q\downarrow p}\frac{1}{a}\ln\left[\left(\frac{q-p}{p\sqrt{\pi}}\right)^{a}\left(\frac{pq}{N\,(q-p)}\right)^{\frac{a}{p}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)}\right)^{\frac{a}{N}}\right]$$
$$=\ln\left[\left(\frac{1}{p\sqrt{\pi}}\right)\left(\frac{p^{2}}{N}\right)^{\frac{1}{p}}[p\,(p-1)]^{1-\frac{1}{p}}\right].$$

Combining all the estimations, we obtain

$$\frac{1}{p}\ln\left(\int_{\mathbb{R}^N}|\nabla u|^p\frac{\mathrm{d}x}{|x|^{\mu}}\right)-\frac{1}{N}\int_{\mathbb{R}^N}|u|^p\ln|u|^p\frac{\mathrm{d}x}{|x|^s}$$
$$\geq\ln\left[\left(\frac{N-p}{N-p-\mu}\right)^{\frac{N-1}{N}}\left(\frac{1}{e}\right)^{\frac{(p-1)}{p}}\left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{1}{N}}\left(\frac{1}{p\sqrt{\pi}}\right)\left(\frac{p^2}{N}\right)^{\frac{1}{p}}[p(p-1)]^{1-\frac{1}{p}}\right].$$

Equivalently,

$$\frac{N}{p}\ln\left(\mathrm{LS}(N,p,\mu)\int_{\mathbb{R}^N}|\nabla u|^p\frac{\mathrm{d}x}{|x|^{\mu}}\right)\geq\int_{\mathbb{R}^N}|u|^p\ln|u|^p\frac{\mathrm{d}x}{|x|^s}.$$

Moreover, using the identities (2.6) and (2.7), we can check that

$$\frac{N}{p}\ln\left(\mathrm{LS}(N,p,\mu)\int_{\mathbb{R}^N}|\nabla U|^p\frac{\mathrm{d}x}{|x|^{\mu}}\right)=\int_{\mathbb{R}^N}|U|^p\ln|U|^p\frac{\mathrm{d}x}{|x|^s},$$

where

$$U(x)=\exp\left(-b|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right),$$

with

$$b>0\text{ and }b^{N\frac{p-1}{p}}=\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\frac{(N-p)(p-1)}{(N-p-\mu)p}\left(\frac{1}{p}\right)^{N\frac{p-1}{p}}\Gamma\left(N\frac{p-1}{p}\right).$$

□

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