SHARP CONSTANTS AND OPTIMIZERS FOR A CLASS OF THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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Abstract. In this paper, we will use a suitable tranform to investigate the sharp constants and optimizers for the following Caffarelli-Kohn-Nirenberg inequalities for a wide range of parameters $(r, p, q, s, \mu, \sigma)$ and $0 \leq a \leq 1$:

$$\left(\int |u|^r \frac{dx}{|x|^s}\right)^{1/r} \leq C \left(\int |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{a/p} \left(\int |u|^q \frac{dx}{|x|^\sigma}\right)^{(1-a)/q}. \tag{CKN}$$

We are able to compute the best constants and the explicit forms of the extremal functions in numerous cases. When $0 < a < 1$, we can deduce the existence and symmetry of optimizers for a wide range of parameters. Moreover, in the particular classes $r = p\frac{q-1}{p-1}$ and $q = p\frac{r-1}{p-1}$, the forms of maximizers will also be provided in the spirit of Del Pino and Dolbeault [12, 13]. In the case $a = 1$, that is the Caffarelli-Kohn-Nirenberg inequality without the interpolation term, we will provide the exact maximizers for all the range of $\mu \geq 0$. The Caffarelli-Kohn-Nirenberg inequalities with arbitrary norms on the Euclidean spaces will also be considered in the spirit of Cordero-Erausquin, Nazaret and Villani [10].

1. Introduction

Geometric and Functional inequalities have a wide range of applications and play a crucial role in geometric analysis, partial differential equations and other branches of modern mathematics. In many situations, the validity of the inequality and some explicit bounds for its best constant are enough to run the process. However, there are numerous circumstances that it is required to know the exact sharp constant and information on extremal functions.

Among those inequalities, the Caffarelli-Kohn-Nirenberg (CKN) inequality is one of the most important and interesting ones. It is worth noting that many well-known and important inequalities such as Gagliardo-Nirenberg inequalities, Sobolev inequalities, Hardy-Sobolev inequalities, Nash's inequalities, etc are just the special cases of the CKN inequalities.

The CKN inequalities were first introduced in 1984 by Caffarelli, Kohn and Nirenberg in their celebrated work [8]:

Theorem A. *There exists a positive constant* $C = C(N, r, p, q, \gamma, \alpha, \beta)$ *such that for all* $u \in C_0^\infty(\mathbb{R}^N)$:

$$\left\| |x|^\gamma u \right\|_r \leq C \left\| |x|^\alpha |\nabla u| \right\|_p^a \left\| |x|^\beta u \right\|_q^{1-a} \tag{1.1}$$

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where

$$p,q\geq 1,\ r>0,\ 0\leq a\leq 1$$

$$\frac{1}{p}+\frac{\alpha}{N},\ \frac{1}{q}+\frac{\beta}{N},\ \frac{1}{r}+\frac{\gamma}{N}>0 \text{ where}$$

$$\gamma=a\sigma+\left(1-a\right)\beta$$

$$\frac{1}{r}+\frac{\gamma}{N}=a\left(\frac{1}{p}+\frac{\alpha-1}{N}\right)+\left(1-a\right)\left(\frac{1}{q}+\frac{\beta}{N}\right),$$

and

$$0\leq\alpha-\sigma \text{ if } a>0 \text{ and}$$

$$\alpha-\sigma\leq 1 \text{ if } a>0 \text{ and } \frac{1}{p}+\frac{\alpha-1}{N}=\frac{1}{r}+\frac{\gamma}{N}.$$

In this paper, if we perform the following change of exponents as in [40]:

$$\alpha=-\frac{\mu}{p},\ \beta=-\frac{\theta}{q},\ \gamma=-\frac{s}{r}.$$

Then (1.1) will become the following equivalent form:

$$\left(\int_{\mathbb{R}^N}\left|u\right|^r\frac{dx}{\left|x\right|^s}\right)^{1/r}\leq C\left(\int_{\mathbb{R}^N}\left|\nabla u\right|^p\frac{dx}{\left|x\right|^{\mu}}\right)^{a/p}\left(\int_{\mathbb{R}^N}\left|u\right|^q\frac{dx}{\left|x\right|^{\theta}}\right)^{(1-a)/q} \tag{CKN}$$

where

$$a=\frac{\left[\left(N-\theta\right)r-\left(N-s\right)q\right]p}{\left[\left(N-\theta\right)p-\left(N-\mu-p\right)q\right]r}.$$

We will restrict our consideration in this paper to the case $1<p<N$.

When $s=\mu=\theta=0$ and $a=1$, we recover the well-known Sobolev inequality:

$$\left(\int_{\mathbb{R}^N}\left|u\right|^{p^*}dx\right)^{1/p^*}\leq S\left(N,p\right)\left(\int_{\mathbb{R}^N}\left|\nabla u\right|^p dx\right)^{1/p} \tag{1.2}$$

where $p^*=\frac{Np}{N-p}$. This inequality has important applications in many areas of mathematics and there is a vast literature. We just mention here that when $p>1$, the best constant $S\left(N,p\right)$ was found in the works of Aubin [3] and Talenti [37] using rather classical tools such as Schwarz rearrangement, and solution of a particular one-dimensional problem, the Bliss inequality. The case $p=2$ was explored more by Beckner in [4] due to its conformal invariance. For $p=1$, it has been known that the Sobolev inequality is equivalent to the classical Euclidean isoperimetric inequality.

When $a=1,\ \mu=0,\ 0\leq s\leq p<N$ and $r=p^*\left(s\right)=\frac{N-s}{N-p}p$, the CKN inequality becomes the Hardy-Sobolev (HS) inequality that is the interpolation of the Sobolev inequality and the Hardy inequality:

$$\left(\int_{\mathbb{R}^N}\left|u\right|^{p^*(s)}\frac{dx}{\left|x\right|^s}\right)^{1/p^*(s)}\leq HS\left(N,p,s\right)\left(\int_{\mathbb{R}^N}\left|\nabla u\right|^p dx\right)^{1/p}. \tag{1.3}$$

In this situation, in [30] Lieb applied the symmetrization arguments to study (1.3) in the case $p=2$ and gave the best constants and explicit optimizers. The study of the best constant $HS\left(N,p,s\right)$ and extremal functions for the inequalities (1.3) in the general range goes back to

Ghoussoub and Yuan in [27] and maybe even earlier (see references in [27]): The maximizers for the HS inequality when $0 \leq s < p < N$ are the functions

$$u_{c,\lambda}(x) = c\left(\lambda + |x|^{\frac{p-s}{p-1}}\right)^{-\frac{N-p}{p-s}} \text{ for some } c \neq 0,\ \lambda > 0. \tag{1.4}$$

Actually, $u_{c,\lambda}$ (after rescaling) is the only positive radial solutions of

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \frac{u^{p^*(s)-1}}{|x|^s} \text{ on } \mathbb{R}^N.$$

When $a = 1$ and $0 < \mu, s < N$, the CKN inequality does not contain the interpolation term. There are great efforts to investigate the sharp constants, existence/nonexistence and symmetry/symmetry breaking of maximizers in this situation, especially when $p = 2$. See [6, 9, 11, 14, 25, 38], among others. For instance, Chou and Chu considered the case $p = 2$ and $\frac{\mu}{2} \leq \frac{s}{r} \leq \frac{\mu}{2} + 1$ and provided the best constants and explicit optimizers. In [38], Wang and Willem studied the compactness of all maximizing sequences up to dilations in the spirit of Lions [32, 33, 34, 35]. In [9], Catrina and Wang investigated the class of $p = 2$ and $\mu < 0$ and established the attainability/inattainability and symmetry breaking of extremal functions. Caldiroli and Musina studied the symmetry breaking of extremals for the CKN inequalities in a non-Hilbertian setting in [7]. In a recent paper [19], Dolbeault, Esteban and Loss studied the characterization of the optimal symmetry breaking region in HS inequalities with $p = 2$. As a consequence, maximizers and best constants are calculated in the symmetry region. Their result solves a longstanding conjecture on the optimal symmetry range.

In the case $0 < a < 1$, the CKN inequality includes the interpolation term. This situation is much harder to study. When there is no singular term, i.e. $s = \theta = \mu = 0$, the non-weighted CKN inequality, namely the Gagliardo-Nirenberg inequality, has been studied at length by many authors, see e.g., [5, 18, 21, 28], to mention just a few. Especially, for very particular classes, the best constant and the maximizers for the Gagliardo-Nirenberg inequality are provided explicitly by Del Pino and Dolbeault in [12, 13]. Indeed, in the special class $r = p\frac{q-1}{p-1}$, Del Pino and Dolbeault proved that the maximizers for the Gagliardo-Nirenberg inequality have the form $A\left(1 + B|x-\overline{x}|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}$ while in the case $q = p\frac{r-1}{p-1}$, the optimizers are $A\left(1 - B|x-\overline{x}|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}}$, for some $A \in \mathbb{R}$, $B > 0$ and $\overline{x} \in \mathbb{R}^N$. See also [1, 2] where Agueh gives a proof by studying a $p-$Laplacian type equation and by transforming the unknown of the equation via some change of functions. We also cite [10] where Cordero-Erausquin, Nazaret, and Villani set up a beautiful link between optimal transportation and certain Sobolev inequalities and Gagliardo-Nirenberg inequalities.

However, as far as we know, there are only a few papers concerning the full weighted CKN inequalities (i.e., $0 < a < 1$ and at least one of s, μ, θ is nonzero), see the review paper by Dolbeault and Esteban [15]. Compared with the special cases of the Gagliardo-Nirenberg inequalities without the interpolation term (i.e., $a = 1$), dealing with such CKN inequalities encounters considerably more difficulty. For instance, the Fourier analysis techniques cannot be applied in this setting. Moreover, the classical Schwarz rearrangement, which is based on an isoperimetric inequality, is unavailable due to the presence of singular terms (i.e., the weights $\frac{1}{|x|^s}$, $\frac{1}{|x|^\theta}$ and $\frac{1}{|x|^\mu}$). It is worth noting that symmetrization has been a very useful and efficient (and almost inevitable) method when dealing with the sharp geometric inequalities. Hence in general we are not able to reduce to work our problem on CKN inequalities to the case of radial setting. Actually, the problem of symmetry and symmetry breaking of optimizers for the CKN inequalities has been investigated by many researchers, see [15, 16, 20] for instance.

In the particular case $p = q = 2$, $-\frac{N-2}{2} < \alpha$, $\beta = \alpha - 1$, $\alpha - 1 \leq \gamma < \alpha$, $r = \frac{2N}{N+2(\gamma-\alpha)}$, together with other conditions, the best constant and maximizers for CKN inequality (1.1) in Theorem A together with the weighted logarithmic inequality are investigated by Dolbeault and Esteban [17], Dolbeault, Esteban, Tarantello and Tertikas [20]. The authors can also give the exact sharp constants and the form of maximizers, however, only in the radial setting.

Concerning the inequality (CKN), for the special class $q = \frac{p(r-1)}{p-1}$, $1 < p < r$, $N - \theta < \left(1 + \frac{\mu}{p} - \frac{\theta}{p}\right)\frac{(r-1)p}{r-p}$ and

$$s = \frac{\mu}{p} + 1 + \frac{p-1}{p}\theta,$$

Xia could guess and then verify in [39] that $\left(\lambda + |x|^{1+\frac{\mu}{p}-\frac{\theta}{p}}\right)^{-\frac{p-1}{r-p}}$, $\lambda > 0$, are extremal functions. But he could not prove that these are the all possible optimizers. Moreover, this case does not cover the interesting situations in [12, 13].

In [40], Zhong and Zou study the existence of extremal functions for the CKN inequality under a wider region, and use it to set up the continuity and compactness of embeddings on weighted Sobolev spaces. However, there is no information about the maximizers provided there.

In a very recent paper [22], the authors studied the CKN inequality in the regime $s = \theta > 0$, $p = 2$ and $r = 2(q-1) > 2$. In this case, they were able to show that for $s = \theta > 0$ small enough, then the CKN inequality can be achieved by the optimizers of the form $\left(1 + |x|^{2-s}\right)^{-\frac{1}{q-2}}$, up to multiplications by a constant and scalings.

In [23, 24, 29], when dealing with the sharp singular Trudinger-Moser inequalities, which can be considered as the limiting Sobolev embeddings, where again the classical Schwarz rearrangement could not be used, the authors propose a new approach. Namely, we define a new Kelvin-type transform to convert those sharp singular inequalities to the nonweighted ones. Moreover, in [23], we treat successfully the CKN inequalities in the special case $p = N$, $\mu = 0$, $0 \leq s = \theta < N$, $1 \leq q < r$ and $a = 1 - \frac{q}{r}$ using this new transform. Especially, for a 1-parameter family of inequalities, the best constants and the maximizers for the CKN inequality are calculated explicitly there.

Motivated by results in [23] and [1, 2, 10, 13, 22], in this paper, we will use convenient vector fields to investigate the CKN inequality in some special regions. Our main idea is that under our suitable transforms, the CKN inequalities can be converted to the simpler versions, namely, the Hardy-Sobolev inequalities and Gagliardo-Nirenberg inequalities. Since the sharp constants and optimizers of those inequalities are easier to study, and are known in some particular classes, we can get the best constants and maximizers for CKN inequalities in the corresponding regions.

More precisely, we study the extremal functions for CKN inequality involving the interpolation term (i..e., $0 < a < 1$). We will consider the following class:

$$\begin{aligned} &1 < p < \ p + \mu < N,\, \theta \leq \frac{N\mu}{N-p} \leq s < N, \\ &1 \leq q < r < \frac{Np}{N-p};\ a = \frac{\left[(N-\theta)\, r - (N-s)\, q\right] p}{\left[(N-\theta)\, p - (N-\mu-p)\, q\right] r}. \end{aligned} \tag{C1}$$

Denote $D_{\mu,\theta}^{p,q}\left(\mathbb{R}^N\right)$ the completion of the space of smooth compactly supported functions with the norm $\left(\int\limits_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{1/p} + \left(\int\limits_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{1/q}$, and set

$$CKN\left(N,\mu,\theta,s,p,q,r\right) = \sup_{u\in D_{\mu,\theta}^{p,q}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}.$$

Then we have the following result:

Theorem 1.1. *Assume (C1). Then $CKN\left(N,\mu,\theta,s,p,q,r\right)$ can be achieved. Moreover, all the extremal functions of $CKN\left(N,\mu,\theta,s,p,q,r\right)$ are radially symmetric.*

Moreover, we will also give the explicit forms for all maximizers and the exact best constant for $CKN\left(N,\mu,\theta,s,p,q,r\right)$ in the following special cases:

Theorem 1.2. *Assume (C1) with $\theta = s = \frac{N\mu}{N-p}$. If $r = p\frac{q-1}{p-1}$, then with $\delta = Np - q\left(N-p\right)$:*

$$\begin{aligned} & CKN\left(N,\mu,\theta,s,p,q,r\right) \\ &= \left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \\ &\times \left(\frac{q-p}{p\sqrt{\pi}}\right)^a \left(\frac{pq}{N\left(q-p\right)}\right)^{\frac{a}{p}} \left(\frac{\delta}{pq}\right)^{\frac{1}{r}} \left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}} \end{aligned}$$

and all the maximizers have the form

$$V_0\left(x\right) = A\left(1 + B\left|x\right|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}} \text{ for some } A \in \mathbb{R},\ B > 0.$$

Theorem 1.3. *Assume (C1) with $\theta = s = \frac{N\mu}{N-p}$. If $q = p\frac{r-1}{p-1}$, then with $\delta = Np - r\left(N-p\right)$:*

$$\begin{aligned} & CKN\left(N,\mu,\theta,s,p,q,r\right) \\ &= \left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \\ &\times \left(\frac{p-r}{p\sqrt{\pi}}\right)^a \left(\frac{pr}{N\left(p-r\right)}\right)^{\frac{a}{p}} \left(\frac{pr}{\delta}\right)^{\frac{1-a}{q}} \left(\frac{\Gamma\left(\frac{p-1}{p}\frac{\delta}{p-r}+1\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(r\frac{p-1}{p-r}+1\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}. \end{aligned}$$

If $r > 2 - \frac{1}{p}$, then all the maximizers have the form

$$V_0\left(x\right) = A\left(1 - B\left|x\right|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}} \text{ for some } A \in \mathbb{R},\ B > 0.$$

We also provide the explicit optimizers for the CKN inequalities in the following regime:

$$\begin{aligned} & p = 2 < 2+\mu < N,\ 2 < r = 2(q-1) < \frac{2N}{N-2} \\ & \mu + 2 > s = \theta > \frac{N\mu}{N-2};\ a = \frac{\left(N-s\right)\left[q-2\right]}{\left[\left(N-s\right)2 - \left(N-\mu-2\right)q\right]\left(q-1\right)} \end{aligned} \tag{C2}$$

Again, we denote

$$CKN(N,\mu,s,q)=\sup_{u\in D^{2;q}_{\mu,s}(\mathbb{R}^N)}\frac{\left(\int\limits_{\mathbb{R}^N}|u|^{2(q-1)}\frac{dx}{|x|^s}\right)^{\frac{1}{2(q-1)}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\frac{dx}{|x|^\mu}\right)^{\frac{a}{2}}\left(\int\limits_{\mathbb{R}^N}|u|^q\frac{dx}{|x|^s}\right)^{\frac{1-a}{q}}}.$$

Then, we will prove that

Theorem 1.4. *There exists $s^*=s^*(N,q,\mu)\in(0,N-(q-1)(N-2-\mu))$ such that for all $\frac{N\mu}{N-2}<s<s^*$, $CKN(N,\mu,s,q)$ is attained by the optimizers of the form*

$$V_0(x)=A\left(1+B|x|^{\mu+2-s}\right)^{-\frac{1}{q-2}}\text{ for some }A\in\mathbb{R},\ B>0.$$

2. Preliminaries and some important lemmas

To carry through our argument, it is necessary to show our new Kelvin type transform can indeed be used to reduce the CKN inequalities with more complicated weights to simpler ones and vice versa. This interchange is nicely done through the following lemmas which are of independent interests and can be found useful in other settings as well.

Lemma 2.1. *We have that $|x\cdot\nabla u(x)|=|x||\nabla u(x)|$ for a.e. $x\in\mathbb{R}^N$ if and only if u is radially symmetric, that is $u(x)=u(y)$ when $|x|=|y|$.*

Proof. If u is radial, then we have

$$\frac{\partial u}{\partial x_j}(x)=u'(|x|)\frac{x_j}{|x|}.$$

Hence,

$$|\nabla u(x)|=|u'(|x|)|.$$

Also,

$$\left|\sum_{j=1}^N x_j\frac{\partial u}{\partial x_j}(x)\right|=|u'(|x|)|\left|\sum_{j=1}^N x_j\frac{x_j}{|x|}\right|=|u'(|x|)||x|=|x||\nabla u(x)|.$$

Now, assume that for all x :

$$|x\cdot\nabla u(x)|=|x||\nabla u(x)|.$$

It means that $\nabla u(x)$ has the same direction with x. That is we can find a scalar function $g(x)$ such that

$$\nabla u(x)=g(x)x.$$

Now, let a and b be two points on the sphere with radius $r>0$ (that is $|a|=|b|=r$). We connect x and y by a piecewise smooth curve $r(t)$ on this sphere, i.e. $|r(t)|=r$, $r(0)=a$ and $r(1)=b$. Then we have

$$\nabla u(r(t))=g(r(t))r(t).$$

Noting that from

$$|r(t)|=r\text{ for all }t$$

we can get that

$$\nabla r(t)\cdot r(t)=0.$$

Thus

$$\nabla u(r(t))\cdot\nabla r(t)=g(r(t))r(t)\cdot\nabla r(t)=0.$$

So

$$u(b)-u(a)=u(r(1))-u(r(0))=\int\limits_0^1\nabla u(r(t))\cdot\nabla r(t)\,dt=0.$$

This completes the proof of the lemma. $\square$

Let $d > 1$. We define the vector-valued function $L_{N,d} : \mathbb{R}^N \to \mathbb{R}^N$ by

$$L_{N,d}(x) = |x|^{d-1}x.$$

The Jacobian matrix of this function $L_{N,d}$ is

$$\mathbf{J_{L_{N,d}}} = \begin{pmatrix} |x|^{d-1} + (d-1)\,|x|^{d-3}x_1^2 & (d-1)\,|x|^{d-3}x_1x_2 & \dots & (d-1)\,|x|^{d-3}x_1x_N \\ (d-1)\,|x|^{d-3}x_2x_1 & |x|^{d-1} + (d-1)\,|x|^{d-3}x_2^2 & \dots & (d-1)\,|x|^{d-3}x_2x_N \\ \vdots & \vdots & \ddots & \vdots \\ (d-1)\,|x|^{d-3}x_Nx_1 & (d-1)\,|x|^{d-3}x_Nx_2 & \dots & |x|^{d-1} + (d-1)\,|x|^{d-3}x_N^2 \end{pmatrix}.$$

We will now show that

$$\det(J_{L_{N,d}}) = d|x|^{N(d-1)}. \tag{2.1}$$

Indeed, consider the matrix

$$\mathbf{A} = \begin{pmatrix} (d-1)\,|x|^{d-3}x_1^2 & (d-1)\,|x|^{d-3}x_1x_2 & \dots & (d-1)\,|x|^{d-3}x_1x_N \\ (d-1)\,|x|^{d-3}x_2x_1 & (d-1)\,|x|^{d-3}x_2^2 & \dots & (d-1)\,|x|^{d-3}x_2x_N \\ \vdots & \vdots & \ddots & \vdots \\ (d-1)\,|x|^{d-3}x_Nx_1 & (d-1)\,|x|^{d-3}x_Nx_2 & \dots & (d-1)\,|x|^{d-3}x_N^2 \end{pmatrix}.$$

It is easy to check that

$$rank\,(A) = 1 \text{ and } tr\,(A) = (d-1)\,|x|^{d-1}.$$

Hence, its characteristic polynomial is

$$\det\left(\lambda \mathbb{I}_N - A\right) = \lambda^N - (d-1)\,|x|^{d-1}\lambda^{N-1}.$$

Choose $\lambda = -|x|^{d-1}$, we get

$$\det(J_{L_{N,d}}) = (-1)^N \det\left(-|x|^{d-1}\mathbb{I}_N - A\right) = d|x|^{N(d-1)}.$$

We now define mappings $D_{N,d,p}$ with $p > 1$ by

$$D_{N,d,p}u(x) := \left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(L_{N,d}(x)) = \left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(|x|^{d-1}x). \tag{2.2}$$

We also define $D_{N,d,p}^{-1}$

$$D_{N,d,p}^{-1}u = v \text{ if } u = D_{N,d,p}v.$$

Under the transform $D_{N,d,p}$, we have the following result that will play an important part in our paper:

Lemma 2.2. *(1) For continuous function f, we have*

$$\int\limits_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{d}\right)^{\frac{p-1}{p}} u\,(x)\right)}{|x|^t} dx = d \int\limits_{\mathbb{R}^N} \frac{f\,(D_{N,d,p}u\,(x))}{|x|^{N+td-Nd}} dx.$$

In particular, we obtain that $u \in L^s\left(\frac{dx}{|x|^t}\right)$ if and only if $D_{N,d,p}u \in L^s\left(\frac{dx}{|x|^{N+td-Nd}}\right)$.

(2) If $\nabla u \in L^p\left(\frac{dx}{|x|^\mu}\right)$, then $\nabla D_{N,d,p}u \in L^p\left(\frac{dx}{|x|^{d(p+\mu-N)+N-p}}\right)$. Moreover,

$$\int\limits_{\mathbb{R}^N} \frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{d(p+\mu-N)+N-p}} dx \le \int\limits_{\mathbb{R}^N} \frac{|\nabla u(x)|^p}{|x|^\mu} dx.$$

The equality occurs if and only if u is radially symmetric.

Proof. (1)

$$\int_{\mathbb{R}^N} \frac{f\left(D_{N,d,p}u\left(x\right)\right)}{\left|x\right|^{N+td-Nd}}dx = \int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(|x|^{d-1}x)\right)}{\left|x\right|^{N+td-Nd}}dx.$$

Using change of variables $y_i = |x|^{d-1}x_i$, $i = 1, 2, ..., N$, we have

$$dy = \det(J_{L_{N,d}})dx = d|x|^{N(d-1)}dx, \tag{2.3}$$

and

$$dx = \frac{1}{d}\frac{dy}{|y|^{N\frac{d-1}{d}}}. \tag{2.4}$$

Hence

$$\int_{\mathbb{R}^N} \frac{f\left(D_{N,d,p}u\left(x\right)\right)}{\left|x\right|^{N+td-Nd}}dx = \frac{1}{d}\int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{d}\right)^{\frac{p-1}{p}} u\left(y\right)\right)}{|y|^{N\frac{d-1}{d}}\left|y\right|^{\frac{N+td-Nd}{d}}}dy = \frac{1}{d}\int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{d}\right)^{\frac{p-1}{p}} u\left(y\right)\right)}{\left|y\right|^{t}}dy.$$

(2) Now we begin to consider the gradient of $D_{N,d,p}u$. After calculations, we have

$$\begin{pmatrix} \frac{\partial D_{N,d,p}u}{\partial x_1}(x) \\ \frac{\partial D_{N,d,p}u}{\partial x_2}(x) \\ \vdots \\ \frac{\partial D_{N,d,p}u}{\partial x_N}(x) \end{pmatrix} = \nabla D_{N,d,p}u(x) = \left(\frac{1}{d}\right)^{\frac{p-1}{p}} \nabla(u(|x|^{d-1}x))$$

$$= \left(\frac{1}{d}\right)^{\frac{p-1}{p}} J^T_{L_{N,d}} \begin{pmatrix} \frac{\partial u}{\partial x_1}(|x|^{d-1}x) \\ \frac{\partial u}{\partial x_2}(|x|^{d-1}x) \\ \vdots \\ \frac{\partial u}{\partial x_N}(|x|^{d-1}x) \end{pmatrix}.$$

Hence we have

$$\frac{\partial D_{N,d,p}u}{\partial x_i}(x) = \left(\frac{1}{d}\right)^{\frac{p-1}{p}} \left(|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + A_i\right),$$

for $i = 1, 2, ...N$, where

$$A_i := \sum_{j=1}^{N} \left(d-1\right)|x|^{d-3}x_i x_j \frac{\partial u}{\partial x_j}(|x|^{d-1}x).$$

Hence, we obtain

$$\begin{aligned} |\nabla D_{N,d,p}u(x)|^2 &= \sum_{i=1}^{N}\left(\frac{\partial D_{N,d,p}u}{\partial x_i}(x)\right)^2 \\ &= d^{-2\frac{p-1}{p}}\sum_{i=1}^{N}\left(|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + A_i\right)^2 \\ &= d^{-2\frac{p-1}{p}}\left[\sum_{i=1}^{N}|x|^{2(d-1)}\left(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\right)^2 + \sum_{i=1}^{N}2A_i|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) + \sum_{i=1}^{N}A_i^2\right] \\ &:= d^{-2\frac{p-1}{p}}\left(I_1 + I_2 + I_3\right). \end{aligned}$$

Direct computations show

$$I_1 = \sum_{i=1}^{N}|x|^{2(d-1)}\left(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\right)^2 = |x|^{2(d-1)}\left|\nabla u(|x|^{\frac{t}{N-t}}x)\right|^2.$$

Applying the Cauchy-Schwarz inequality to estimate the second term, we get

$$\begin{aligned} I_2 &= \sum_{i=1}^{N} 2A_i|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x) \\ &= \sum_{i=1}^{N} 2|x|^{d-1}\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\sum_{j=1}^{N}(d-1)\,|x|^{d-3}x_ix_j\frac{\partial u}{\partial x_j}(|x|^{d-1}x) \\ &= 2\,(d-1)\,|x|^{2d-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{x_ix_j}{|x|^2}\frac{\partial u}{\partial x_j}(|x|^{d-1}x)\frac{\partial u}{\partial x_i}(|x|^{d-1}x) \\ &= 2\,(d-1)\,|x|^{2d-2}\left(\sum_{i=1}^{N}\frac{x_i}{|x|}\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\right)^2 \\ &\le 2\,(d-1)\,|x|^{2d-2}\left[\sum_{i=1}^{N}\left(\frac{x_i}{|x|}\right)^2\right]\left[\sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}(|x|^{d-1}x)\right)^2\right] \\ &= 2\,(d-1)\,|x|^{2d-2}\left|\nabla u(|x|^{d-1}x)\right|^2 . \end{aligned} \tag{2.5}$$

Similarly for the last term, we have

$$\begin{aligned} I_3 = \sum_{i=1}^{N} A_i^2 &= \sum_{i=1}^{N}\left(\sum_{j=1}^{N}(d-1)\,|x|^{d-3}x_ix_j\frac{\partial u}{\partial x_j}(|x|^{d-1}x)\right)^2 \\ &\le (d-1)^2\,|x|^{2d-6}\sum_{i=1}^{N}\left[\sum_{j=1}^{N}(x_ix_j)^2\right]\left[\sum_{j=1}^{N}\left(\frac{\partial u}{\partial x_j}(|x|^{\frac{t}{N-t}}x)\right)^2\right] \\ &= (d-1)^2\,|x|^{2d-6}\sum_{i=1}^{N}|x|^2\,x_i^2\left|\nabla u(|x|^{d-1}x)\right|^2 \\ &= (d-1)^2\,|x|^{2d-2}\left|\nabla u(|x|^{d-1}x)\right|^2 . \end{aligned}$$

Combining them together, we have

$$|\nabla D_{N,d,p}u(x)|^2 \le d^{-2\frac{p-1}{p}}d^2|x|^{2d-2}\left|\nabla u(|x|^{d-1}x)\right|^2 .$$

This leads to

$$|\nabla D_{N,d,p}u(x)| \le d^{\frac{1}{p}}|x|^{d-1}\left|\nabla u(|x|^{d-1}x)\right| .$$

Using the change of variables again, we get

$$\begin{aligned} \int_{\mathbb{R}^N}\frac{|\nabla u(y)|^p}{|y|^{\mu}}dy &= \int_{\mathbb{R}^N}\frac{|\nabla u(|x|^{d-1}x)|^p}{||x|^{d-1}x|^{\mu}}d|x|^{N(d-1)}dx \\ &\ge \frac{1}{d}\int_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{p(d-1)}\,||x|^{d-1}x|^{\mu}}d|x|^{N(d-1)}dx \\ &= \int_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}u(x)|^p}{|x|^{d(p+\mu-N)+N-p}}dx. \end{aligned}$$

Finally, by Lemma 2.1, it is easy to check that the equalities hold if and only if u is radial. $\square$

3. Caffarelli-Kohn-Nirenberg inequality when $0 < a < 1$ under the condition (C1)

Theorem 1.1, Theorem 1.2 and Theorem 1.3 will be proved via the following series of lemmas. For the convenience of the reader, we recall that the conditions on the parameters are

$$1 < p < p + \mu < N,\ \theta \leq \frac{N\mu}{N-p} \leq s < N, \tag{C1}$$
$$1 \leq q < r < \frac{Np}{N-p};\ a = \frac{[(N-\theta)r - (N-s)q]p}{[(N-\theta)p - (N-\mu-p)q]r}.$$

Also,

$$CKN(N,\mu,\theta,s,p,q,r) = \sup_{u\in D^{p;q}_{\mu,\theta}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}.$$

We now set

$$GN(N,p,q,r,\mu,\theta,s) = \sup_{u\in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}},$$

where

$$d = \frac{N-p}{N-p-\mu}.$$

It is important to note here that since $\theta \leq \frac{N\mu}{N-p} \leq s < N$, we have

$$N + \theta d - Nd \leq 0 \leq N + sd - Nd < N.$$

Lemma 3.1. *The variational problem*

$$A(N,p,q,r,\mu,\theta,s)$$
$$= \inf\left\{ I(u) = \frac{1}{p}\int\limits_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{q}\int\limits_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}}dx : u \in D^{p,q}_{0,N+\theta d-Nd}\left(\mathbb{R}^N\right)\ \text{and} \int\limits_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}dx = 1 \right\}$$

has a minimizer. Moreover, $A(N,p,q,r,\mu,\theta,s) > 0$.

Proof. By the classical Schwarz rearrangement, we can assume that there exists a sequence of radial functions (u_n) :

$$I(u_n) \downarrow A(N,p,q,r,\mu,\theta,s)$$
$$\int\limits_{\mathbb{R}^N} \frac{|u_n|^r}{|x|^{N+sd-Nd}}dx = 1.$$

We can assume WLOG that $u_n \rightharpoonup u$ in $u \in D^{p,q}_{0,N+\theta d-Nd}\left(\mathbb{R}^N\right)$. Since it is clear that $I(u) \leq A(N,p,q,r,\mu,\theta,s)$, it is now enough to show

$$\int\limits_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}}dx = 1.$$

But this is easy to observe since we can write for $R > 0$ sufficiently large:

$$\int\limits_{\mathbb{R}^N} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}} dx = \int\limits_{B_R} + \int\limits_{B_R^c} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}} dx.$$

Then by the Radial Lemma, we get

$$\int\limits_{B_R^c} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}} dx \to 0.$$

Also, by the compactness of Sobolev embeddings, we can deduce

$$\int\limits_{B_R} \frac{|u_n - u|^r}{|x|^{N+sd-Nd}} dx \to 0.$$

As a consequence, $u \neq 0$ and $A(N, p, q, r, \mu, \theta, s) > 0$. Moreover, noting that for $\lambda > 0$:

$$u_\lambda(x) = \lambda^{\frac{Nd-sd}{r}} u(\lambda x),$$

then

$$\|\nabla u_\lambda\|_p = \lambda^{\frac{Nd-sd}{r} + \frac{p-N}{p}} \|\nabla u\|_p ;$$
$$\|u_\lambda\|_k = \lambda^{\frac{Nd-sd}{r} - \frac{N}{k}} \|u\|_k ,$$

and

$$\int\limits_{\mathbb{R}^N} \frac{|u_\lambda|^r}{|x|^{N+sd-Nd}} dx = 1.$$

Also,

$$\begin{aligned} I(u_\lambda) &= \frac{1}{p} \int\limits_{\mathbb{R}^N} |\nabla u_\lambda|^p dx + \frac{1}{q} \int\limits_{\mathbb{R}^N} \frac{|u_\lambda|^q}{|x|^{N+\theta d-Nd}} dx \\ &= \frac{1}{p} \lambda^{\frac{Nd-sd}{r} p + p - N} \|\nabla u\|_p^p + \frac{1}{q} \lambda^{q \frac{Nd-sd}{r} - N + N + \theta d - Nd} \int\limits_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}} dx \\ &= \lambda^m A + \lambda^{-n} B \end{aligned}$$

with

$$\begin{aligned} m &= \frac{Nd - sd}{r} p + p - N;\ n = Nd - \theta d - q \frac{Nd - sd}{r} \\ A &= \frac{1}{p} \|\nabla u\|_p^p ;\ \ B = \frac{1}{q} \int\limits_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}} dx. \end{aligned}$$

Hence

$$A(N, p, q, r, \mu, \theta, s) = \inf_{\lambda > 0} I(u_\lambda) = I(u_{\lambda_0})$$

where

$$\lambda_0 = \left(\frac{nB}{mA} \right)^{\frac{1}{m+n}}.$$

It means that

$$A(N, p, q, r, \mu, \theta, s) = \frac{m+n}{m} \left(\frac{n}{m} \right)^{-\frac{n}{m+n}} A^{\frac{n}{m+n}} B^{\frac{m}{m+n}}.$$

□

Lemma 3.2. *$GN(N,p,q,r)$ can be achieved and*

$$GN(N,p,q,r,\mu,\theta,s)=\left[\frac{\frac{m+n}{m}\left(\frac{n}{m}\right)^{-\frac{n}{m+n}}\left(\frac{1}{p}\right)^{\frac{n}{m+n}}\left(\frac{1}{q}\right)^{\frac{m}{m+n}}}{A(N,p,q,r,\mu,\theta,s)}\right]^{\frac{\frac{a}{p}}{\frac{n}{m+n}}}.$$

Proof. For any v with

$$\int_{\mathbb{R}^N}\frac{|v|^r}{|x|^{N+sd-Nd}}dx=1,$$

we use the above process and get

$$\begin{aligned}A(N,p,q,r,\mu,\theta,s)&\le\inf_{\lambda>0}I(v_\lambda)\\&=\frac{m+n}{m}\left(\frac{n}{m}\right)^{-\frac{n}{m+n}}\left(\frac{1}{p}\|\nabla v\|_p^p\right)^{\frac{n}{m+n}}\left(\frac{1}{q}\int_{\mathbb{R}^N}\frac{|u|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{m}{m+n}}.\end{aligned}$$

Noting that

$$\frac{\frac{n}{m+n}}{\frac{m}{m+n}}=\frac{\frac{a}{p}}{\frac{1-a}{q}},$$

we obtain

$$\frac{\left(\int_{\mathbb{R}^N}\frac{|v|^r}{|x|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int_{\mathbb{R}^N}|\nabla v|^p\,dx\right)^{\frac{a}{p}}\left(\int_{\mathbb{R}^N}\frac{|v|^q}{|x|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}}\le\left[\frac{\frac{m+n}{m}\left(\frac{n}{m}\right)^{-\frac{n}{m+n}}\left(\frac{1}{p}\right)^{\frac{n}{m+n}}\left(\frac{1}{q}\right)^{\frac{m}{m+n}}}{A(N,p,q,r,\mu,\theta,s)}\right]^{\frac{\frac{a}{p}}{\frac{n}{m+n}}}.$$

Combining with the previous lemma, we conclude that $GN(N,p,q,r,\mu,\theta,s)$ can be achieved and

$$GN(N,p,q,r,\mu,\theta,s)=\left[\frac{\frac{m+n}{m}\left(\frac{n}{m}\right)^{-\frac{n}{m+n}}\left(\frac{1}{p}\right)^{\frac{n}{m+n}}\left(\frac{1}{q}\right)^{\frac{m}{m+n}}}{A(N,p,q,r,\mu,\theta,s)}\right]^{\frac{\frac{a}{p}}{\frac{n}{m+n}}}.$$

□

Using Lemma 3.1 and Lemma 3.2, we will now show that $CKN(N,\mu,\theta,s,p,q,r)$ can be achieved:

Lemma 3.3. *Under (C1), $CKN(N,\mu,\theta,s,p,q,r)$ can be attained and*

$$CKN(N,\mu,\theta,s,p,q,r)=\left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}GN(N,p,q,r,\mu,\theta,s).$$

Proof. We begin by an observation that if $u\ge0$ is a maximizer for $GN(N,p,q,r,\mu,\theta,s)$, then we can assume that u is radial. Indeed, that fact is just a consequence of the Schwarz rearrangement. See for instance [31]. Now, let us assume that $U_0\ge0$ is a radial maximizer of $GN(N,p,q,r,\mu,\theta,s)$. We set $V_0=D_{N,d,p}^{-1}U_0$ with $d=\frac{N-p}{N-p-\mu}$. It means that $U_0=D_{N,d,p}V_0$. We will show that V_0 is a maximizer of $CKN(N,\mu,\theta,s,p,q,r)$. Indeed, for any v, we need to show

$$\frac{\left(\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \leq \frac{\left(\int\limits_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla V_0|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}.$$

By Lemma 2.2, we get by noting that when $d = \frac{N-p}{N-p-\mu}$: $d\left(p+\mu-N\right)+N-p=0$:

$$\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s} = d^{1+\frac{p-1}{p}r} \int\limits_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^r}{|x|^{N+sd-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta} = d^{1+\frac{p-1}{p}q} \int\limits_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^q}{|x|^{N+\theta d-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} \frac{|\nabla v|^p}{|x|^\mu} dx \geq \int\limits_{\mathbb{R}^N} |\nabla D_{N,d,p}v|^p \, dx.$$

and

$$\int\limits_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s} = d^{1+\frac{p-1}{p}r} \int\limits_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta} = d^{1+\frac{p-1}{p}q} \int\limits_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu} dx = \int\limits_{\mathbb{R}^N} |\nabla U_0|^p \, dx.$$

Hence

$$\begin{aligned}
&\frac{\left(\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\
&\leq \frac{\left(d^{1+\frac{p-1}{p}r} \int\limits_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla D_{N,d,p}v|^p \, dx\right)^{\frac{a}{p}} \left(d^{1+\frac{p-1}{p}q} \int\limits_{\mathbb{R}^N} \frac{|D_{N,d,p}v|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}} \\
&\leq d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla U_0|^p \, dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}} \\
&= \frac{\left(\int |V_0|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int |\nabla V_0|^p \frac{dx}{|x|^\mu}\right)^{\frac{a}{p}} \left(\int |V_0|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}.
\end{aligned}$$

We note that we have the equality in the last row because U_0 and V_0 are radial. Hence $CKN(N,\mu,\theta,s,p,q,r)$ is attained. Moreover, it is easy to see that

$$CKN(N,\mu,\theta,s,p,q,r)=d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}GN(N,p,q,r,\mu,\theta,s)$$

□

Lemma 3.4. *Assume (C1). If V_0 is a maximizer of $CKN(N,\mu,\theta,s,p,q,r)$, then V_0 is radially symmetric.*

Proof. let V_0 be a maximizer of $CKN(N,\mu,\theta,s,p,q,r)$. We set $U_0=D_{N,d,p}V_0$ where $d=\frac{N-p}{N-p-\mu}$. We will show that U_0 is a maximizer of $GN(N,p,q,r,\mu,\theta,s)$. Indeed for any radial function u (we can just choose radial functions because of the symmetrization arguments), we define

$$v=D_{N,d,p}^{-1}u,\text{ i.e. }u=D_{N,d,p}v.$$

By Lemma 2.2, we get

$$\int\limits_{\mathbb{R}^N}|v|^r\frac{dx}{|x|^s}=d^{1+\frac{p-1}{p}r}\int\limits_{\mathbb{R}^N}\frac{|u|^r}{|x|^{N+sd-Nd}}dx$$

$$\int\limits_{\mathbb{R}^N}|v|^q\frac{dx}{|x|^\theta}=d^{1+\frac{p-1}{p}q}\int\limits_{\mathbb{R}^N}\frac{|u|^q}{|x|^{N+\theta d-Nd}}dx$$

$$\int\limits_{\mathbb{R}^N}\frac{|\nabla v|^p}{|x|^\mu}dx=\int\limits_{\mathbb{R}^N}|\nabla u|^p\,dx$$

and

$$\int\limits_{\mathbb{R}^N}|V_0|^r\frac{dx}{|x|^s}=d^{1+\frac{p-1}{p}r}\int\limits_{\mathbb{R}^N}\frac{|U_0|^r}{|x|^{N+sd-Nd}}dx$$

$$\int\limits_{\mathbb{R}^N}|V_0|^q\frac{dx}{|x|^\theta}=d^{1+\frac{p-1}{p}q}\int\limits_{\mathbb{R}^N}\frac{|U_0|^q}{|x|^{N+\theta d-Nd}}dx$$

$$\int\limits_{\mathbb{R}^N}\frac{|\nabla V_0|^p}{|x|^\mu}dx\geq\int\limits_{\mathbb{R}^N}|\nabla U_0|^p\,dx.$$

Hence

$$
\begin{aligned}
& d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla U_0|^p \, dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}} \\
& \geq \frac{\left(\int\limits_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu} dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\
& \geq \frac{\left(\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \frac{|\nabla v|^p}{|x|^\mu} dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}} \\
& = d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{|u|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}}.
\end{aligned}
$$

Moreover, it is easy to see that

$$
\begin{aligned}
& d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)} \frac{\left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^r}{|x|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} |\nabla U_0|^p \, dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} \frac{|U_0|^q}{|x|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}} \\
& = \frac{\left(\int\limits_{\mathbb{R}^N} |V_0|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu} dx\right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^\theta}\right)^{\frac{1-a}{q}}}.
\end{aligned}
$$

Hence

$$
\int\limits_{\mathbb{R}^N} \frac{|\nabla V_0|^p}{|x|^\mu} dx = \int\limits_{\mathbb{R}^N} |\nabla U_0|^p \, dx.
$$

So V_0 is radial. □

Lemma 3.5. *Assume (C1) with $s=\theta=\frac{N\mu}{N-p}$. If $r=p\frac{q-1}{p-1}$, then with $\delta=Np-q(N-p)$:*

$$
\begin{aligned}
&CKN(N,s,\mu,p,q,r)\\
&=\left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\\
&\times\left(\frac{q-p}{p\sqrt{\pi}}\right)^{a}\left(\frac{pq}{N(q-p)}\right)^{\frac{a}{p}}\left(\frac{\delta}{pq}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}
\end{aligned}
$$

and all the maximizers have the form

$$V_0(x)=A\left(1+B|x|^{\frac{1}{d}\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}\ \textit{for some } A\in\mathbb{R},\ B>0$$

where $d=\frac{N-p}{N-p-\mu}$.

Proof. When $r=p\frac{q-1}{p-1}$ and $s=\theta=\frac{N\mu}{N-p}$, we have from [1, 2, 12, 13] that

$$
\begin{aligned}
&GN(N,p,q,r)\\
&=\left(\frac{q-p}{p\sqrt{\pi}}\right)^{a}\left(\frac{pq}{N(q-p)}\right)^{\frac{a}{p}}\left(\frac{\delta}{pq}\right)^{\frac{1}{r}}\left(\frac{\Gamma\left(q\frac{p-1}{q-p}\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{p-1}{p}\frac{\delta}{q-p}\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}
\end{aligned}
$$

and all the maximizers have the form

$$U_0(x)=A\left(1+B|x-\overline{x}|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}\ \text{for some } A\in\mathbb{R},\ B>0,\ \overline{x}\in\mathbb{R}^N.$$

Now, let V_0 be a maximizer of $CKN(N,s,\mu,p,q,r)$. By Lemma 3.4, $D_{N,d,p}V_0$ is a maximizer of $GN(N,p,q,r)$. Hence

$$D_{N,d,p}V_0(x)=A\left(1+B|x-\overline{x}|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

It means that

$$V_0(x)=A'\left(1+B\left||x|^{\frac{1}{d}-1}x-\overline{x}\right|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

Noting that V_0 is radial, we conclude that $\overline{x}=0$. ☐

Lemma 3.6. *Assume (C1) with $s=\theta=\frac{N\mu}{N-p}$. If $q=p\frac{r-1}{p-1}$, then with $\delta=Np-r(N-p)$:*

$$
\begin{aligned}
&CKN(N,s,\mu,p,q,r)\\
&=\left(\frac{N-p}{N-p-\mu}\right)^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\\
&\times\left(\frac{p-r}{p\sqrt{\pi}}\right)^{a}\left(\frac{pr}{N(p-r)}\right)^{\frac{a}{p}}\left(\frac{pr}{\delta}\right)^{\frac{1-a}{q}}\left(\frac{\Gamma\left(\frac{p-1}{p}\frac{\delta}{p-r}+1\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(r\frac{p-1}{p-r}+1\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}.
\end{aligned}
$$

If $r>2-\frac{1}{p}$, then all the maximizers of $GN(N,p,q,r)$ have the form

$$V_0(x)=A\left(1-B|x|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}}\ \textit{for some } A\in\mathbb{R},\ B>0.$$

Proof. When $q = p\frac{r-1}{p-1}$ and $s = \theta = \frac{N\mu}{N-p}$, we have from [1, 2, 12, 13] that

$$\begin{aligned} &GN\left(N,p,q,r\right) \\ &= \left(\frac{p-r}{p\sqrt{\pi}}\right)^{a}\left(\frac{pr}{N\left(p-r\right)}\right)^{\frac{a}{p}}\left(\frac{pr}{\delta}\right)^{\frac{1-a}{q}}\left(\frac{\Gamma\left(\frac{p-1}{p}\frac{\delta}{p-r}+1\right)\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(r\frac{p-1}{p-r}+1\right)\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{\frac{a}{N}}. \end{aligned}$$

Also, when $r > 2 - \frac{1}{p}$, all the maximizers of $GN\left(N,p,q,r\right)$ have the form

$$U_0\left(x\right) = A\left(1 - B\left|x-\overline{x}\right|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}} \text{ for some } A \in \mathbb{R},\ B > 0,\ \overline{x} \in \mathbb{R}^N.$$

Now, let V_0 be a maximizer of $CKN\left(N,s,\mu,p,q,r\right)$. By Lemma 3.4, $D_{N,d,p}V_0$ is a maximizer of $GN\left(N,p,q,r\right)$. Hence

$$D_{N,d,p}V_0\left(x\right) = A\left(1 - B\left|x-\overline{x}\right|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}}.$$

It means that

$$V_0\left(x\right) = A'\left(1 - B\left|\left|x\right|^{\frac{1}{d}-1}x-\overline{x}\right|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}}.$$

Noting that V_0 is radially symmetric, we conclude that $\overline{x} = 0$. □

4. CKN inequalities in the regime (C2)

In this section, we will concern the CKN inequalities in the class (C2):

$$\begin{aligned} &p = 2 < 2+\mu < N,\ 2 < r = 2(q-1) < \frac{2N}{N-2} \\ &\mu + 2 > s = \theta > \frac{N\mu}{N-2};\ a = \frac{(N-s)\left[q-2\right]}{\left[(N-s)\,2-(N-\mu-2)\,q\right](q-1)} \end{aligned} \tag{C2}$$

Recall that

$$CKN\left(N,\mu,s,q\right) = \sup_{u\in D^{2,q}_{\mu,s}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N}\left|u\right|^{2(q-1)}\frac{dx}{\left|x\right|^{s}}\right)^{\frac{1}{2(q-1)}}}{\left(\int\limits_{\mathbb{R}^N}\left|\nabla u\right|^2\frac{dx}{\left|x\right|^{\mu}}\right)^{\frac{a}{2}}\left(\int\limits_{\mathbb{R}^N}\left|u\right|^{q}\frac{dx}{\left|x\right|^{s}}\right)^{\frac{1-a}{q}}}.$$

We also denote

$$CKN_1\left(N,\mu,s,q\right) = \sup_{u\in D^{2,q}_{0,s}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N}\left|u\right|^{2(q-1)}\frac{dx}{\left|x\right|^{N+sd-Nd}}\right)^{\frac{1}{2(q-1)}}}{\left(\int\limits_{\mathbb{R}^N}\left|\nabla u\right|^2 dx\right)^{\frac{a}{2}}\left(\int\limits_{\mathbb{R}^N}\left|u\right|^{q}\frac{dx}{\left|x\right|^{N+sd-Nd}}\right)^{\frac{1-a}{q}}}$$

where $d = \frac{N-2}{N-2-\mu}$.

Proof of Theorem 1.4. For any $v \in D^{2,q}_{\mu,s}\left(\mathbb{R}^N\right)$, we have with $u = D_{N,d,2}v$, that

$$\int\limits_{\mathbb{R}^N}\left|\nabla v\right|^2\frac{dx}{\left|x\right|^{\mu}} \geq \int\limits_{\mathbb{R}^N}\left|\nabla u\right|^2 dx$$

$$\int\limits_{\mathbb{R}^N}\left|v\right|^{2(q-1)}\frac{dx}{\left|x\right|^{s}} = d^q\int\limits_{\mathbb{R}^N}\frac{\left|u\right|^{2(q-1)}}{\left|x\right|^{N+sd-Nd}}dx$$

$$\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^s} = d^{1+\frac{1}{2}q} \int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{N+sd-Nd}} dx.$$

By a result in [22], we have that $U(x) = D_{N,d,2} V_0(x) = C\left(1 + D|x|^{2-N-sd+Nd}\right)^{-\frac{1}{q-2}}$ for some $C \in \mathbb{R}$, $D > 0$ is the maximizer for $CKN_1(N, \mu, s, q)$ for $0 < N + sd - Nd < 2$ small enough. Hence we have by Lemma 2.2 that

$$\begin{aligned}
\frac{\left(\int_{\mathbb{R}^N} |v|^{2(q-1)} \frac{dx}{|x|^s}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N} |\nabla v|^2 \frac{dx}{|x|^\mu}\right)^{\frac{a}{2}} \left(\int_{\mathbb{R}^N} |v|^q \frac{dx}{|x|^s}\right)^{\frac{1-a}{q}}} &\leq \frac{d^{\frac{q}{2(q-1)}}}{d^{\frac{1+\frac{1}{2}q}{\frac{1-a}{q}}}} \frac{\left(\int_{\mathbb{R}^N} |u|^{2(q-1)} \frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{a}{2}} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1-a}{q}}} \\
&\leq \frac{d^{\frac{q}{2(q-1)}}}{d^{\frac{1+\frac{1}{2}q}{\frac{1-a}{q}}}} \frac{\left(\int_{\mathbb{R}^N} |U|^{2(q-1)} \frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N} |\nabla U|^2 dx\right)^{\frac{a}{2}} \left(\int_{\mathbb{R}^N} |U|^q \frac{dx}{|x|^{N+sd-Nd}}\right)^{\frac{1-a}{q}}} \\
&= \frac{\left(\int_{\mathbb{R}^N} |V_0|^{2(q-1)} \frac{dx}{|x|^s}\right)^{\frac{1}{2(q-1)}}}{\left(\int_{\mathbb{R}^N} |\nabla V_0|^2 \frac{dx}{|x|^\mu}\right)^{\frac{a}{2}} \left(\int_{\mathbb{R}^N} |V_0|^q \frac{dx}{|x|^s}\right)^{\frac{1-a}{q}}}.
\end{aligned}$$

In other words, V_0 is the optimizer for $CKN(N, \mu, s, q)$. $\square$

5. Caffarelli-Kohn-Nirenberg inequality without the interpolation term: the case $a = 1$

In this section, we will also consider the CKN inequalities without the interpolation term for all $1 < p < N$ and will concern the following range:

$$\begin{aligned}
&1 < p < \ p + \mu < N, \ \frac{\mu}{p} \leq \frac{s}{r} < \frac{\mu}{p} + 1, \\
&r = \frac{(N-s)p}{N-\mu-p}; \ a = 1.
\end{aligned} \tag{C3}$$

Noting that the condition $\frac{\mu}{p} \leq \frac{s}{r} \leq \frac{\mu}{p} + 1$ comes from the constraints of the CKN inequality. In this case, we denote

$$D_\mu^{1,p}\left(\mathbb{R}^N; \frac{dx}{|x|^s}\right) = \left\{u \in L^r\left(\frac{dx}{|x|^s}\right) : \int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu} < \infty\right\},$$

and

$$CKN(N, p, \mu, s) = \sup_{u \in D_\mu^{1,p}\left(\mathbb{R}^N; \frac{dx}{|x|^s}\right)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{|x|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{dx}{|x|^\mu}\right)^{1/p}}.$$

Then we will prove in this section the following result:

Theorem 5.1. *Assume (C3). Then $CKN(N,p,\mu,s)$ is achieved with the extremals being of the following form:*

$$V_{c,\lambda}(x)=c\left(\lambda+|x|^{\frac{p+\mu-s}{p-1}}\right)^{-\frac{N-p-\mu}{p+\mu-s}}$$

for some $c\neq 0$, $\lambda>0$.

Theorem 5.1 was studied in [36] by solving the corresponding ODE. In this section, we will provide another proof using the transform $D_{N,d,p}$.

We note that $\frac{\mu}{p}\leq\frac{s}{r}<\frac{\mu}{p}+1$ means

$$\frac{N\mu}{N-p}\leq s<p+\mu.$$

We also denote

$$HS(N,p,\mu,s)=\sup_{D_0^{1,p}\left(\mathbb{R}^N;\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)}\frac{\left(\int\limits_{\mathbb{R}^N}|u|^{p^*\left(\frac{s(N-p)-N\mu}{N-p-\mu}\right)}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{1/p^*\left(\frac{s(N-p)-N\mu}{N-p-\mu}\right)}}{\left(\int\limits_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{1/p}}.$$

Noting that

$$p^*\left(\frac{s(N-p)-N\mu}{N-p-\mu}\right)=\frac{N-\frac{s(N-p)-N\mu}{N-p-\mu}}{N-p}p=\frac{(N-s)p}{N-\mu-p}$$

and

$$0\leq\frac{s(N-p)-N\mu}{N-p-\mu}<p.$$

Lemma 5.1. *$CKN(N,p,\mu,s)$ can be attained*

Proof. We will use the fact that $HS(N,p,\mu,s)$ is attained by some radial functions U_0. Set $V_0=D_{N,d,p}^{-1}U_0$ with $d=\frac{N-p}{N-p-\mu}$. It means that $U_0=D_{N,d,p}V_0$. We will show that V_0 is a maximizer of $CKN(N,p,\mu,s)$. Indeed, for any $v\in D_\mu^{1,p}\left(\mathbb{R}^N\right)$, we set $u=D_{N,d,p}v$. Then by Lemma 2.2, we get

$$\int\limits_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}\geq\int\limits_{\mathbb{R}^N}\frac{|\nabla D_{N,d,p}v|^p}{|x|^{d(p+\mu-N)+N-p}}dx=\int\limits_{\mathbb{R}^N}|\nabla u|^pdx$$

$$\begin{aligned}\int\limits_{\mathbb{R}^N}|v|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}&=dd^{\frac{p-1}{p}\frac{(N-s)p}{N-\mu-p}}\int\limits_{\mathbb{R}^N}|u|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{N+sd-Nd}}\\&=d^{1+\frac{(N-s)(p-1)}{N-\mu-p}}\int\limits_{\mathbb{R}^N}|u|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.\end{aligned}$$

$$\int\limits_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}=\int\limits_{\mathbb{R}^N}|\nabla U_0|^pdx$$

and

$$\int\limits_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}=d^{1+\frac{(N-s)(p-1)}{N-\mu-p}}\int\limits_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.$$

Hence

$$\frac{\left(\int\limits_{\mathbb{R}^N}|v|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}\right)^{1/p}} \leq d^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}\frac{\left(\int\limits_{\mathbb{R}^N}|u|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla u|^p dx\right)^{1/p}}$$

$$\leq d^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}\frac{\left(\int\limits_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla U_0|^p dx\right)^{1/p}}$$

$$=\frac{\left(\int\limits_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\right)^{1/p}}.$$

In other words, V_0 is the maximizer for $CKN\left(N,p,\mu,s\right)$. Moreover, we also deduce that

$$CKN\left(N,p,\mu,s\right)=d^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}HS\left(N,p,\mu,s\right).$$

□

Lemma 5.2. *All the optimizers for $CKN\left(N,p,\mu,s\right)$ are radially symmetric.*

Proof. Assume that V_0 is a maximizer for $CKN\left(N,p,\mu,s\right)$ and $U_0=D_{N,d,p}V_0$ where $d=\frac{N-p}{N-p-\mu}$. Again by Lemma 2.2, we get

$$\int\limits_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\geq\int\limits_{\mathbb{R}^N}|\nabla U_0|^p dx$$

and

$$\int\limits_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}=d^{1+\frac{(N-s)(p-1)}{N-\mu-p}}\int\limits_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.$$

We will now prove that U_0 is a maximizer for $HS\left(N,p,\mu,s\right)$. Indeed, for any radial function u (we can assume u is radial by the Schwarz rearrangement argument), and set $v=D_{N,d,p}^{-1}u$, that is $u=D_{N,d,p}v$, we obtain by Lemma 2.2:

$$\int\limits_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}=\int\limits_{\mathbb{R}^N}|\nabla u|^p dx$$

and

$$\int\limits_{\mathbb{R}^N}|v|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}=d^{1+\frac{(N-s)(p-1)}{N-\mu-p}}\int\limits_{\mathbb{R}^N}|u|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}.$$

Hence,

$$\begin{aligned}\frac{\left(\int\limits_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla U_0|^p dx\right)^{1/p}} &\geq \left(\frac{1}{d}\right)^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}\frac{\left(\int\limits_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\right)^{1/p}}\\
&\geq \left(\frac{1}{d}\right)^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}\frac{\left(\int\limits_{\mathbb{R}^N}|v|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla v|^p\frac{dx}{|x|^\mu}\right)^{1/p}}\\
&= \frac{\left(\int\limits_{\mathbb{R}^N}|u|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla u|^p dx\right)^{1/p}}.\end{aligned}$$

Hence U_0 is a maximizer for $HS\left(N,p,\mu,s\right)$. Moreover, it is easy to see that the equality must happen in the first line, that is

$$\frac{\left(\int\limits_{\mathbb{R}^N}|U_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^{\frac{s(N-p)-N\mu}{N-p-\mu}}}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla U_0|^p dx\right)^{1/p}} = \left(\frac{1}{d}\right)^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}\frac{\left(\int\limits_{\mathbb{R}^N}|V_0|^{\frac{(N-s)p}{N-\mu-p}}\frac{dx}{|x|^s}\right)^{\frac{N-\mu-p}{(N-s)p}}}{\left(\int\limits_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}\right)^{1/p}}.$$

It means that

$$\int\limits_{\mathbb{R}^N}|\nabla V_0|^p\frac{dx}{|x|^\mu}=\int\limits_{\mathbb{R}^N}|\nabla U_0|^p dx$$

and thus, V_0 is radial. □

Proof of Theorem 5.1. From Lemma 5.1 and Lemma 5.2, we see that $CKN\left(N,p,\mu,s\right)$ is attained,

$$CKN\left(N,p,\mu,s\right)=d^{\left(1+\frac{(N-s)(p-1)}{N-\mu-p}\right)\frac{N-\mu-p}{(N-s)p}}HS\left(N,p,\mu,s\right),$$

and all maximizers for $CKN\left(N,p,\mu,s\right)$ are radially symmetric. Futhermore, we can conclude that V_0 is a maximizer for $CKN\left(N,p,\mu,s\right)$ only if $U_0=D_{N,d,p}V_0$ is a maximizer for $HS\left(N,p,\mu,s\right)$ where $d=\frac{N-p}{N-p-\mu}$. It is known (see [26] for instance) that $HS\left(N,p,\mu,s\right)$ is attained with the maximizers being the functions

$$\begin{aligned}U_{c,\lambda}\left(x\right)&=c\left(\lambda+|x|^{\frac{p-\frac{s(N-p)-N\mu}{N-p-\mu}}{p-1}}\right)^{-\frac{N-p}{p-\frac{s(N-p)-N\mu}{N-p-\mu}}}\\
&=c\left(\lambda+|x|^{\frac{(N-p)(p+\mu-s)}{(p-1)(N-p-\mu)}}\right)^{-\frac{N-p-\mu}{p+\mu-s}}\end{aligned}$$

for some $c \neq 0$, $\lambda > 0$. Hence $CKN(N,p,\mu,s)$ could be achieved with the optimizers being the functions

$$\begin{aligned} V_{c,\lambda}(x) &= D_{N,d,p}^{-1} U_{c,\lambda}(x) \\ &= c\left(\lambda + |x|^{\frac{p+\mu-s}{p-1}}\right)^{-\frac{N-p-\mu}{p+\mu-s}} \end{aligned}$$

for some $c \neq 0$, $\lambda > 0$. □

Remark 1. *If we have $\frac{s}{r} = \frac{\mu}{p} + 1$ in the condition (C2), then $s = p + \mu$ and $\frac{s(N-p)-N\mu}{N-p-\mu} = p$. So in this case, after applying the transform $D_{N,d,p}$ where $d = \frac{N-p}{N-p-\mu}$, the CKN inequality corresponds to the Hardy inequality. Hence, the best constant in this case is*

$$CKN(N,p,\mu,s) = \frac{p}{N-p-\mu},$$

and it is never achieved.

6. Caffarelli-Kohn Nirenberg inequalities with arbitrary norm

In this section, we will investigate the CKN under arbitrary norms in $\mathbb{R}^N$ in the spirit of Cordero-Erausquin, Nazaret and Villani [10]. More precisely, let $E = \left(\mathbb{R}^N, \|\cdot\|\right)$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^N$. Then its dual space $E^* = \left(\mathbb{R}^N, \|\cdot\|_*\right)$ where for $X \in E^*$:

$$\|X\|_* = \sup_{Y \in E : \|Y\| \leq 1} X \cdot Y.$$

For simplicity, we will assume that $|\{\|x\|_* \leq 1\}| = \omega_N$ and denote $\kappa_N = |\{\|x\| \leq 1\}|$. We will asume that for any $X \in \mathbb{R}^N$, there exists a unique $X^* \in \mathbb{R}^N$ such that $\|X^*\|_* = 1$ and

$$X \cdot X^* = \|X\| = \sup_{Y \in \mathbb{R}^N : \|Y\|_* \leq 1} X \cdot Y.$$

It is clear that $\|\cdot\|$ is Lipschitz with the Lipschitz constant 1, and thus, differentiable a.e. From the property that

$$\|\lambda x\| = \lambda \|x\| \text{ for all } \lambda > 0,$$

we can see that the gradient of $\|\cdot\|$ at $x \in \mathbb{R}^N$ is the unique vector $\nabla(\|\cdot\|)(x) = x^*$. Recall that

$$\|x^*\|_* = 1,\ x \cdot x^* = \|x\| = \sup_{\|y\|_* \leq 1} x \cdot y.$$

Actually, at first, we will consider a more general situation. More precisely, we suppose that C is $q-$homogeneous, that is there exists $q > 1$ such that

$$C(\lambda x) = \lambda^q C(x) \ \ \forall \lambda \geq 0,\ \forall x \in \mathbb{R}^N. \tag{6.1}$$

Then C^*, the Legendre transform of C, defined by

$$C^*(x) = \sup_y \{\langle x, y\rangle - C(y)\},$$

is even, strictly convex function and is $p-$homogeneous with $p = \frac{q}{q-1}$.

We have that $\langle X, Y\rangle \leq C^*(X) + C(Y)$ for all X, Y. Hence $\langle X, Y\rangle \leq \lambda^p C^*(X) + \lambda^{-q} C(Y)$ for all $\lambda > 0$, X, Y. Minimizing the right hand side with respect to λ gives the Cauchy-Schwarz inequality

$$X \cdot Y \leq [qC(Y)]^{\frac{1}{q}} [pC^*(X)]^{\frac{1}{p}}.$$

By Young's inequality, we have

$$X \cdot Y \leq [qC(Y)]^{\frac{1}{q}} [pC^*(X)]^{\frac{1}{p}} \leq C^*(x) + C(y).$$

Hence, we also have that

$$[pC^*(X)]^{\frac{1}{p}} = \sup_Y \frac{X \cdot Y}{[qC(Y)]^{\frac{1}{q}}}.$$

In other words,

$$C^*(X) = \sup_Y \frac{|X \cdot Y|^p}{p\,[qC(Y)]^{\frac{p}{q}}}$$

We will assume that for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that

$$x \cdot x^* = qC(x) \text{ and } C^*(x^*) = (q-1)\,C(x) = \frac{q}{p}C(x).$$

In other words, for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that the equality in the Cauchy-Schwarz inequality happens.

Noting that from (6.1), we get that $C(\cdot)$ is differentiable a.e. We will assume that the gradient of $C(\cdot)$ at $x \in \mathbb{R}^N$ is the unique vector x^*. The example that we have in mind are $C(x) = \frac{1}{q}|x|^q$ and $C^*(x) = \frac{1}{p}|x|^p$ with $|\cdot|$ is the regular Euclidean norm on $\mathbb{R}^N$. Another example is the pair $C(x) = \frac{1}{q}\|x\|^q$ and $C^*(x) = \frac{1}{p}\|x\|_*^p$.

6.1. **A change of variables.** Similarly as Lemma 2.1, we have

Lemma 6.1. *We have*

$$|x \cdot \nabla u(x)| = [qC(x)]^{\frac{1}{q}}\,[pC^*(\nabla u)]^{\frac{1}{p}} \text{ for a.e.} x \in \mathbb{R}^N,$$

if and only if u *is* $C-$*radial, i.e.* $u(x) = u(y)$ *when* $C(x) = C(y)$.

Proof. If u is $C-$radial, then recalling that $\nabla(C(\cdot))(x) = x^*$, we have

$$\frac{\partial u}{\partial x_j}(x) = u'(C(x))\,x_j^*.$$

Hence,

$$C^*(\nabla u) = C^*(u'(C(x))\,x^*) = |u'(C(x))|^p\,C^*(x^*) = |u'(C(x))|^p\,\frac{q}{p}C(x)$$

and

$$[qC(x)]^{\frac{1}{q}}\,[pC^*(\nabla u)]^{\frac{1}{p}} = [qC(x)]^{\frac{1}{q}}\left[|u'(C(x))|^p\,qC(x)\right]^{\frac{1}{p}} = |u'(C(x))|\,qC(x).$$

Also,

$$|x \cdot \nabla u(x)| = \left|\sum_{j=1}^N x_j \frac{\partial u}{\partial x_j}(x)\right| = |u'(\|x\|)|\left|\sum_{j=1}^N x_j x_j^*\right| = |u'(\|x\|)|\,qC(x).$$

Now, if for all $x \in \mathbb{R}^N$:

$$|x \cdot \nabla u(x)| = [qC(x)]^{\frac{1}{q}}\,[pC^*(\nabla u)]^{\frac{1}{p}},$$

then $\nabla u(x)$ has the same direction with x^*. That is we can find a function $f(x)$ such that $\nabla u(x) = f(x)\,x^*$. Now let a and b be two points on the $C-$sphere with radius $r > 0$. That is $C(a) = C(b) = r$. We connect a and b by a piecewise smooth curve $r(t)$ on the sphere, i.e. $C(r(t)) = r$ and $C(r(0)) = a$, $C(r(1)) = b$. Then we have

$$\nabla u(r(t)) = f(r(t))\,(r(t))^*.$$

Using that fact that $C(r(t)) = r$ for all t, we get

$$(r(t))^* \cdot \nabla r(t) = 0.$$

Hence

$$\int_0^1 \nabla u(r(t)) \cdot \nabla r(t)\,dt = \int_0^1 f(r(t))\,(r(t))^* \cdot \nabla r(t)\,dt = 0.$$

In other words,

$$u(b) - u(a) = u(C(r(1))) - u(C(r(0))) = 0.$$

□

Let $d>0$. We define the vector-valued function $L_{N,d}:\mathbb{R}^N\rightarrow\mathbb{R}^N$ by

$$L_{N,d}(x)=C\left(x\right)^{d}x.$$

The Jacobian matrix of this function $L_{N,d}$ is

$$\mathbf{J}_{\mathbf{L_{N,d}}}=C\left(x\right)^{d}\mathbb{I}_N+A$$

where

$$\mathbf{A}=\left(\begin{array}{cccc} dC\left(x\right)^{d-1}x_1x_1^{\ast} & dC\left(x\right)^{d-1}x_1x_2^{\ast} & \dots & dC\left(x\right)^{d-1}x_1x_N^{\ast} \\ dC\left(x\right)^{d-1}x_2x_1^{\ast} & dC\left(x\right)^{d-1}x_2x_2^{\ast} & \dots & dC\left(x\right)^{d-1}x_2x_N^{\ast} \\ \vdots & \vdots & \ddots & \vdots \\ dC\left(x\right)^{d-1}x_Nx_1^{\ast} & dC\left(x\right)^{d-1}x_Nx_2 & \dots & dC\left(x\right)^{d-1}x_Nx_N^{\ast} \end{array}\right).$$

Then we get

$$\det(J_{L_{N,d}})=(-1)^N\det\left(-C\left(x\right)^{d}\mathbb{I}_N-A\right)=\left(1+dq\right)C\left(x\right)^{Nd}.$$

We now also define mappings $D_{N,d,p}$ with $p>1$ by

$$D_{N,d,p}u(x):=\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}}u(L_{N,d}(x))=\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}}u(C\left(x\right)^{d}x).$$

We also define $D_{N,d,p}^{-1}$

$$D_{N,d,p}^{-1}u=v\text{ if }u=D_{N,d,p}v.$$

Under the transform $D_{N,d,p}$, we also have the following result:

Lemma 6.2. *(1) For continuous function f, we have*

$$\int\limits_{\mathbb{R}^N}\frac{f\left(\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}}u\left(x\right)\right)}{C\left(x\right)^{t}}dx=\left(1+dq\right)\int\limits_{\mathbb{R}^N}\frac{f\left(D_{N,d,p}u\left(x\right)\right)}{C\left(x\right)^{t(dq+1)-Nd}}dx.$$

In particular, we obtain that $u\in L^s\left(\frac{dx}{C(x)^t}\right)$ if and only if $D_{N,d,p}u\in L^s\left(\frac{dx}{C(x)^{t(dq+1)-Nd}}\right)$.

(2) For smooth functions u :

$$\int\limits_{\mathbb{R}^N}\frac{C^{\ast}\left(\nabla D_{N,d,p}u(x)\right)}{C\left(x\right)^{(qd+1)\mu+pd-Nd}}dx\leq\int\limits_{\mathbb{R}^N}\frac{C^{\ast}\left(\nabla u(y)\right)}{C\left(y\right)^{\mu}}dy.$$

The equality occurs if and only if u is $C-$radially symmetric.

Proof. (1)

$$\int\limits_{\mathbb{R}^N}\frac{f\left(D_{N,d,p}u\left(x\right)\right)}{C\left(x\right)^{t(dq+1)-Nd}}dx=\frac{1}{1+dq}\int\limits_{\mathbb{R}^N}\frac{f\left(\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}}u\left(y\right)\right)}{C\left(y\right)^{t}}dy.$$

Using change of variables $y_i=C\left(x\right)^{d}x_i$, $i=1,2,...,N$, we have

$$dy=\det(J_{L_{N,d}})dx=\left(1+dq\right)C\left(x\right)^{Nd}dx,$$

and

$$dx=\frac{1}{\left(1+dq\right)}\frac{dy}{C\left(y\right)^{\frac{Nd}{dq+1}}}.$$

Hence

$$\int_{\mathbb{R}^N} \frac{f\left(D_{N,d,p}u\left(x\right)\right)}{C\left(x\right)^{t(dq+1)-Nd}}dx = \int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}} u(C\left(x\right)^d x)\right)}{C\left(x\right)^{t(dq+1)-Nd}}dx$$

$$= \frac{1}{1+dq}\int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}} u\left(y\right)\right)}{C\left(y\right)^{\frac{t(dq+1)-Nd}{dq+1}}}\frac{dy}{C\left(y\right)^{\frac{Nd}{dq+1}}} = \frac{1}{1+dq}\int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}} u\left(y\right)\right)}{C\left(y\right)^{t}}dy.$$

(2) Now we begin to consider the gradient of $D_{N,d,p}u$. After calculations, we have

$$\begin{pmatrix} \frac{\partial D_{N,d,p}u}{\partial x_1}(x) \\ \frac{\partial D_{N,d,p}u}{\partial x_2}(x) \\ \vdots \\ \frac{\partial D_{N,d,p}u}{\partial x_N}(x) \end{pmatrix} = \nabla D_{N,d,p}u(x) = \left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}} \nabla(u(C\left(x\right)^d x))$$

$$= \left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}} J^T_{L_{N,d}} \begin{pmatrix} \frac{\partial u}{\partial x_1}(C\left(x\right)^d x) \\ \frac{\partial u}{\partial x_2}(C\left(x\right)^d x) \\ \vdots \\ \frac{\partial u}{\partial x_N}(C\left(x\right)^d x) \end{pmatrix}.$$

Hence we have

$$\frac{\partial u(C\left(x\right)^d x)}{\partial x_i} = \left(C\left(x\right)^d \frac{\partial u}{\partial x_i}(C\left(x\right)^d x) + A_i\right),$$

for $i = 1, 2, ...N$, where

$$A_i := \sum_{j=1}^{N} dC\left(x\right)^{d-1} x_i^* x_j \frac{\partial u}{\partial x_j}(C\left(x\right)^d x).$$

$$C^*\left(X\right) = \sup \frac{\left|X\cdot Y\right|^p}{p\left[qC\left(Y\right)\right]^{\frac{p}{q}}}$$

Hence, we obtain

$$C^*\left(\nabla D_{N,d,p}u(x)\right) = C^*\left(\left(\frac{1}{1+dq}\right)^{\frac{p-1}{p}} \nabla(u(C\left(x\right)^d x))\right) = \left(\frac{1}{1+dq}\right)^{p-1} C^*\left(\nabla(u(C\left(x\right)^d x))\right)$$

$$= \left(\frac{1}{1+dq}\right)^{p-1} \sup_y \left\{\frac{\left(\nabla(u(C\left(x\right)^d x))\cdot y\right)^p}{p\left[qC\left(y\right)\right]^{\frac{p}{q}}}\right\}$$

$$= \left(\frac{1}{1+dq}\right)^{p-1} \sup_y \left\{\frac{\left[\sum_{i=1}^N \left[C\left(x\right)^d \frac{\partial u}{\partial x_i}(C\left(x\right)^d x)y_i + A_i y_i\right]\right]^p}{p\left[qC\left(y\right)\right]^{\frac{p}{q}}}\right\}.$$

The first term is easy to compute:

$$\begin{aligned} I_1 &= \sum_{i=1}^{N} C\left(x\right)^d \frac{\partial u}{\partial x_i}(C\left(x\right)^d x)y_i \\ &= C\left(x\right)^d \nabla u\left(C\left(x\right)^d x\right)\cdot y \\ &\leq C\left(x\right)^d \left[qC\left(y\right)\right]^{\frac{1}{q}} \left[pC^*\left(\nabla u\left(C\left(x\right)^d x\right)\right)\right]^{\frac{1}{p}} \end{aligned}$$

Applying the Cauchy-Schwarz inequality

$$X \cdot Y \leq [qC(Y)]^{\frac{1}{q}} [pC^*(X)]^{\frac{1}{p}},$$

we can estimate the second term:

$$\begin{aligned}
I_2 &= \sum_{i=1}^{N} A_i y_i \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} dC(x)^{d-1} x_i^* x_j \frac{\partial u}{\partial x_j}(C(x)^d x) y_i \\
&= dC(x)^{d-1} \sum_{i=1}^{N} x_i^* y_i \sum_{j=1}^{N} x_j \frac{\partial u}{\partial x_j}(C(x)^d x) \\
&\leq dC(x)^{d-1} |x^* \cdot y| \left| x \cdot \nabla u(C(x)^d x) \right| \\
&\leq dC(x)^{d-1} [qC(y)]^{\frac{1}{q}} [pC^*(x^*)]^{\frac{1}{p}} [qC(x)]^{\frac{1}{q}} \left[pC^*\left(\nabla u(C(x)^d x) \right) \right]^{\frac{1}{p}} \\
&\leq dC(x)^{d-1} [qC(y)]^{\frac{1}{q}} [qC(x)]^{\frac{1}{p}} [qC(x)]^{\frac{1}{q}} \left[pC^*\left(\nabla u(C(x)^d x) \right) \right]^{\frac{1}{p}} \\
&\leq qdC(x)^d [qC(y)]^{\frac{1}{q}} \left[pC^*\left(\nabla u(C(x)^d x) \right) \right]^{\frac{1}{p}}
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_y \left\{ \frac{\left[\sum_{i=1}^{N} \left[C(x)^d \frac{\partial u}{\partial x_i}(C(x)^d x) y_i + A_i y_i \right] \right]^p}{p [qC(y)]^{\frac{p}{q}}} \right\} \\
&\leq \sup_y \left\{ \frac{[(1+qd)]^p C(x)^{pd} [qC(y)]^{\frac{p}{q}} pC^*\left(\nabla u(C(x)^d x) \right)}{p [qC(y)]^{\frac{p}{q}}} \right\} \\
&= [(1+qd)]^p C(x)^{pd} C^*\left(\nabla u(C(x)^d x) \right).
\end{aligned}$$

In conclusion, we get

$$C^*(\nabla D_{N,d,p} u(x)) \leq (1+qd) C(x)^{pd} C^*\left(\nabla u(C(x)^d x) \right).$$

Using the change of variables again, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{C^*(\nabla u(y))}{C(y)^\mu} dy &= \int_{\mathbb{R}^N} \frac{C^*\left(\nabla u(C(x)^d x) \right)}{C\left(C(x)^d x \right)^\mu} (1+dq) C(x)^{Nd} dx \\
&\geq \int_{\mathbb{R}^N} \frac{C^*(\nabla D_{N,d,p} u(x))}{C(x)^{(qd+1)\mu} C(x)^{pd}} C(x)^{Nd} dx \\
&= \int_{\mathbb{R}^N} \frac{C^*(\nabla D_{N,d,p} u(x))}{C(x)^{(qd+1)\mu + pd - Nd}} dx
\end{aligned}$$

Finally, it is easy to check that the equalities hold if and only if the equality in the Cauchy-Schwarz inequality occur. It means that u is $C-$radially symmetric. □

We note here that we will mainly apply the above change of variables with $C(x) = \frac{1}{q}\|x\|^q$ and $C^*(x) = \frac{1}{p}\|x\|_*^p$. In this case, for the easy references, we will use the transform

$$T_{N,d,p}u(x) := \left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(\|x\|^{d-1} x).$$

We also define $T_{N,d,p}^{-1}$

$$T_{N,d,p}^{-1}u = v \text{ if } u = T_{N,d,p}v.$$

The following lemma is a restatement of Lemma 6.2.

Lemma 6.3. *(1) For continuous function f, we have*

$$\int_{\mathbb{R}^N} \frac{f\left(\left(\frac{1}{d}\right)^{\frac{p-1}{p}} u(x)\right)}{\|x\|^t} dx = d \int_{\mathbb{R}^N} \frac{f(T_{N,d,p}u(x))}{\|x\|^{N+td-Nd}} dx.$$

In particular, we obtain that $u \in L^s\left(\frac{dx}{\|x\|^t}\right)$ if and only if $T_{N,d,p}u \in L^s\left(\frac{dx}{\|x\|^{N+td-Nd}}\right)$.

(2) If $\nabla u \in L^p\left(\frac{dx}{\|x\|^\mu}\right)$, then $\nabla T_{N,d,p} \in L^p\left(\frac{dx}{\|x\|^{d(p+\mu-N)+N-p}}\right)$. Moreover,

$$\int_{\mathbb{R}^N} \frac{\|\nabla T_{N,d,p}u(x)\|_*^p}{\|x\|^{d(p+\mu-N)+N-p}} dx \le \int_{\mathbb{R}^N} \frac{\|\nabla u(x)\|_*^p}{\|x\|^\mu} dx.$$

The equality occurs if and only if u is $\|\cdot\|$ −radial.

6.2. **Maximizers for the CKN inequalities with arbitrary norms.** Consider the following class:

$$\begin{aligned} &1 < p < \ p+\mu < N,\ \theta \le \frac{N\mu}{N-p} \le s < N, \\ &1 \le q < r < \frac{Np}{N-p};\ a = \frac{[(N-\theta)r - (N-s)q]p}{[(N-\theta)p - (N-\mu-p)q]r}. \end{aligned} \tag{C4}$$

Denote $D_{\mu,\theta}^{p,q}(\mathbb{R}^N)$ the completion of the space of smooth compactly supported functions with the norm $\left(\int_{\mathbb{R}^N} \|\nabla u\|_*^p \frac{dx}{\|x\|^\mu}\right)^{1/p} + \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^\theta}\right)^{1/q}$, and set

$$CKN(N,\mu,\theta,s,p,q,r) = \sup_{u\in D_{\mu,\theta}^{p,q}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \frac{dx}{\|x\|^s}\right)^{1/r}}{\left(\int_{\mathbb{R}^N} \|\nabla u\|_*^p \frac{dx}{\|x\|^\mu}\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^\theta}\right)^{\frac{1-a}{q}}};$$

$$GN(N,p,q,r,\mu,\theta,s) = \sup_{u\in D_{0,N+\theta d-Nd}^{p,q}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} \frac{|u|^r}{\|x\|^{N+sd-Nd}} dx\right)^{1/r}}{\left(\int_{\mathbb{R}^N} \|\nabla u\|_*^p dx\right)^{\frac{a}{p}} \left(\int_{\mathbb{R}^N} \frac{|u|^q}{\|x\|^{N+\theta d-Nd}} dx\right)^{\frac{1-a}{q}}}$$

Then, similarly as in Section 3, we can prove that

Theorem 6.1. *Assume (C3). Then $CKN(N,\mu,\theta,s,p,q,r)$ can be achieved. Moreover, all the extremal functions of $CKN(N,\mu,\theta,s,p,q,r)$ are $\|\cdot\|$ −radially symmetric.*

Furthermore, we can provide the maximizers for $CKN(N,\mu,\theta,s,p,q,r)$ in the following two classes:

Theorem 6.2. *Assume (C2) with $\theta=s=\frac{N\mu}{N-p}$. If $r=p\frac{q-1}{p-1}$, then $CKN(N,\mu,\theta,s,p,q,r)$ is achieved by maximizers of the form*

$$V_0(x)=A\left(1+B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}\ \textit{for some } A\in\mathbb{R},\ B>0.$$

Theorem 6.3. *Assume (C2) with $\theta=s=\frac{N\mu}{N-p}$. If $q=p\frac{r-1}{p-1}$, then if $r>2-\frac{1}{p}$, $CKN(N,\mu,\theta,s,p,q,r)$ is achieved by maximizers of the form*

$$V_0(x)=A\left(1-B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}}\ \textit{for some } A\in\mathbb{R},\ B>0.$$

Proof. The proof of Theorem 6.1 is similar to that of Theorem 1.1 and will be omitted.

When $r=p\frac{q-1}{p-1}$ and $s=\theta=\frac{N\mu}{N-p}$, we have from [10] that $GN(N,p,q,r)$ is achieved by maximizers of the form

$$U_0(x)=A\left(1+B\|x-\overline{x}\|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}\ \text{for some } A\in\mathbb{R},\ B>0,\ \overline{x}\in\mathbb{R}^N.$$

Now, let V_0 be a maximizer of $CKN(N,s,\mu,p,q,r)$. Then $T_{N,d,p}V_0$ is a maximizer of $GN(N,p,q,r)$ with $d=\frac{N-p}{N-p-\mu}$. Hence $CKN(N,s,\mu,p,q,r)$ can be attained by

$$V_0(x)=T_{N,d,p}^{-1}A\left(1+B\|x-\overline{x}\|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

It means that

$$V_0(x)=A'\left(1+B\left\||x|^{\frac{1}{d}-1}x-\overline{x}\right\|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

Noting that V_0 is $\|\cdot\|$ −radial, we conclude that $\overline{x}=0$. That is

$$V_0(x)=A'\left(1+B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)^{-\frac{p-1}{q-p}}.$$

Simiarly, when $\theta=s=\frac{N\mu}{N-p}$. If $q=p\frac{r-1}{p-1}$ and $r>2-\frac{1}{p}$, $CKN(N,\mu,\theta,s,p,q,r)$ is achieved by maximizers of the form

$$V_0(x)=A\left(1-B\|x\|^{\frac{N-p-\mu}{N-p}\frac{p}{p-1}}\right)_+^{-\frac{p-1}{r-p}}\ \text{for some } A\in\mathbb{R},\ B>0.$$

□

7. Further comments

Let $d>1$. Then under the transform $T_{N,d,p}$, the CKN inequality with the triple (s,μ,θ) would be converted to the one with the triple $(N+sd-Nd, d(p+\mu-N)+N-p, N+\theta d-Nd)$. We should note that

$$\begin{aligned} a&=\frac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r}\\ &=\frac{[(N-(N+\theta d-Nd))r-(N-(N+sd-Nd))q]p}{[(N-(N+\theta d-Nd))p-(N-(d(p+\mu-N)+N-p)-p)q]r}. \end{aligned}$$

This fact may be used to simplify the study of symmetry/symmetry breaking phenomena. For instance, we could prove that

Theorem 7.1. *Assume* $d = \frac{N-p}{N-p-\mu} > 1$ *and* $0 < a = \frac{[(N-\theta)r-(N-s)q]p}{[(N-\theta)p-(N-\mu-p)q]r} \leq 1$. *If*

$$CKN_1 = \sup_{u \in D^{p,q}_{0,N+\theta d-Nd}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N} |u|^r \frac{dx}{\|x\|^{N+sd-Nd}} \right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \|\nabla u\|_*^p \, dx \right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^{N+\theta d-Nd}} \right)^{\frac{1-a}{q}}}$$

has a $\|\cdot\|$ *−radially symmetric maximizer, then*

$$CKN_2 = \sup_{u \in D^{p,q}_{\mu,\theta}(\mathbb{R}^N)} \frac{\left(\int\limits_{\mathbb{R}^N} |u|^r \frac{dx}{\|x\|^s} \right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N} \|\nabla u\|_*^p \frac{dx}{\|x\|^\mu} \right)^{\frac{a}{p}} \left(\int\limits_{\mathbb{R}^N} |u|^q \frac{dx}{\|x\|^\theta} \right)^{\frac{1-a}{q}}}$$

is attained by some $\|\cdot\|$ *−radial optimizers.*

Proof. Assume that U_0 is a $\|\cdot\|$ −radial maximizer of CKN_1. We set $V_0 = T^{-1}_{N,d,p} U_0$. It means that $U_0 = T_{N,d,p} V_0$. We note that V_0 is $\|\cdot\|$ −radial. Then for any v, we get

$$\int\limits_{\mathbb{R}^N} |v|^r \frac{dx}{\|x\|^s} = d^{1+\frac{p-1}{p}r} \int\limits_{\mathbb{R}^N} \frac{|T_{N,d,p}v|^r}{\|x\|^{N+sd-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} |v|^q \frac{dx}{\|x\|^\theta} = d^{1+\frac{p-1}{p}q} \int\limits_{\mathbb{R}^N} \frac{|T_{N,d,p}v|^q}{\|x\|^{N+\theta d-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} \frac{\|\nabla v\|_*^p}{\|x\|^\mu} dx \geq \int\limits_{\mathbb{R}^N} \|\nabla T_{N,d,p} v\|_*^p \, dx.$$

and

$$\int\limits_{\mathbb{R}^N} |V_0|^r \frac{dx}{\|x\|^s} = d^{1+\frac{p-1}{p}r} \int\limits_{\mathbb{R}^N} \frac{|U_0|^r}{\|x\|^{N+sd-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} |V_0|^q \frac{dx}{\|x\|^\theta} = d^{1+\frac{p-1}{p}q} \int\limits_{\mathbb{R}^N} \frac{|U_0|^q}{\|x\|^{N+\theta d-Nd}} dx$$

$$\int\limits_{\mathbb{R}^N} \frac{\|\nabla V_0\|_*^p}{\|x\|^\mu} dx = \int\limits_{\mathbb{R}^N} \|\nabla U_0\|_*^p \, dx.$$

Hence

$$\frac{\left(\int\limits_{\mathbb{R}^N}|v|^r\frac{dx}{\|x\|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}\frac{\|\nabla v\|_*^p}{\|x\|^\mu}dx\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}|v|^q\frac{dx}{\|x\|^\theta}\right)^{\frac{1-a}{q}}}\leq\frac{\left(d^{1+\frac{p-1}{p}r}\int\limits_{\mathbb{R}^N}\frac{\left|T_{N,d,p}v\right|^r}{\|x\|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}\|\nabla T_{N,d,p}v\|_*^p\,dx\right)^{\frac{a}{p}}\left(d^{1+\frac{p-1}{p}q}\int\limits_{\mathbb{R}^N}\frac{\left|T_{N,d,p}v\right|^q}{\|x\|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}}$$

$$\leq d^{\frac{1}{r}+\frac{p-1}{p}-\frac{1-a}{q}-\frac{p-1}{p}(1-a)}\frac{\left(\int\limits_{\mathbb{R}^N}\frac{|U_0|^r}{\|x\|^{N+sd-Nd}}dx\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}\|\nabla U_0\|_*^p\,dx\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}\frac{|U_0|^q}{\|x\|^{N+\theta d-Nd}}dx\right)^{\frac{1-a}{q}}}$$

$$=\frac{\left(\int\limits_{\mathbb{R}^N}|V_0|^r\frac{dx}{\|x\|^s}\right)^{1/r}}{\left(\int\limits_{\mathbb{R}^N}\frac{\|\nabla V_0\|_*^p}{\|x\|^\mu}dx\right)^{\frac{a}{p}}\left(\int\limits_{\mathbb{R}^N}|V_0|^q\frac{dx}{\|x\|^\theta}\right)^{\frac{1-a}{q}}}.$$

We note that we have the equality in the last row because U_0 and V_0 are $\|\cdot\|$ −radial. Hence V_0 is a $\|\cdot\|$ −radial maximizer of CKN_2. $\square$

As an application of Theorem 7.1, to study the symmetry problem of maximizers for CKN inequality (with the assumption that $\frac{N-p}{N-p-\mu}>1$), we can assume that $\mu=0$.

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