

HARDY SPACE ESTIMATES FOR BILINEAR SQUARE FUNCTIONS AND CALDERÓN-ZYGMUND OPERATORS

JAROD HART AND GUOZHEN LU

ABSTRACT. In this work we prove Hardy space estimates for bilinear Littlewood-Paley-Stein square function and Calderón-Zygmund operators. Sufficient Carleson measure type conditions are given for square functions to be bounded from $H^{p_1} \times H^{p_2}$ into L^p for indices smaller than 1, and sufficient *BMO* type conditions are given for a bilinear Calderón-Zygmund operator to be bounded from $H^{p_1} \times H^{p_2}$ into H^p for indices smaller than 1. Subtle difficulties arise in the bilinear nature of these problems that are related to frequency properties of products of functions. Moreover, three types of bilinear paraproducts are defined and shown to be bounded from $H^{p_1} \times H^{p_2}$ into H^p for indices smaller than 1. The first is a bilinear Bony type paraproduct that was defined in [33]. The second is a paraproduct that resembles the product of two Hardy space functions. The third class of paraproducts are operators given by sums of molecules, which were introduced in [2].

1. INTRODUCTION

There is a rich theory of Hardy spaces in harmonic analysis. Some of the early groundbreaking work in the area are from Stein, Weiss, Coifman, and C. Fefferman, among many others, see for example [49, 48, 18, 10]. In more recent years, Hardy space theory has been studied in product (aka multiparameter) settings and to a lesser extent in multilinear settings. The challenges in these areas are formidable. There are many instances where results from the classical theory, which one may initially expect to hold in the product and multilinear setting, fail. This phenomenon has been observed many times in the multiparameter setting, for example in the absence of a weak $(1, 1)$ estimate for the strong maximal operator (see for example [37]) and the difference between various definitions of Hardy and *BMO* spaces in the product setting (see for example [5, 6, 39, 19, 40, 31]). This type of difficulty presents in multilinear analysis as well; of interest in this work are the failure of some boundedness properties for multilinear operators on products of Hardy spaces.

The study of multilinear Calderón-Zygmund operators was initiated by Coifman and Meyer [11, 12, 13]. A fruitful theory has grown around these operators, see for example [7, 26, 27, 23, 2, 43, 29, 36, 34, 41]. A multilinear and multi-parameter version of the Coifman-Meyer Fourier multiplier theorem was established in [44, 45] using time-frequency analysis (see also [9] using the Littlewood-Paley analysis) and a pseudo-differential analogue was carried out in [15]. There has also been some work done for the operators in the context of distributions spaces (Triebel-Lizorkin and Besov spaces), see for example [28, 1, 43, 2]. For appropriate indices, some of these distribution spaces coincide with Hardy spaces, which are the focus of this work. One of the main results we prove in this work is a bilinear *T1* type theorem that extends the bilinear *T1* theorem

Date: August 30, 2015.

1991 Mathematics Subject Classification. 42B20, 42B25, 42B30.

Key words and phrases. Square Function, Littlewood-Paley-Stein, Bilinear, Calderón-Zygmund Operators.

Hart was partially supported by an AMS-Simons Travel Grant. Lu was supported by NSF grant #DMS1301595.

from [7, 26, 27, 33] to a Hardy space setting. This work also provides bilinear versions of the linear results in [18, 50, 22, 20, 35]; in particular there is a close parallel with the Hardy space results in [35]. There has also been a considerable development of bilinear Littlewood-Paley-Stein theory in recent years, see for example [42, 43, 32, 29, 25, 8, 24]. All of these works deal with Littlewood-Paley-Stein operator mapping properties from $L^{p_1} \times L^{p_2}$ into L^p for $p_1, p_2 > 1$. Here we extend this to $H^{p_1} \times H^{p_2}$ to L^p boundedness properties for indices $0 < p_1, p_2 \leq 1$.

To highlight a challenge that arises in bilinear Hardy space theory that is not present in the linear theory, consider the following problem, to which we give a solution in this work. Given a bilinear operator T acting on a product of Hardy spaces $H^{p_1} \times H^{p_2}$, what conditions are sufficient for T to map $H^{p_1} \times H^{p_2}$ into H^p ? At the root of this problem is a simple, but menacing, fact. Given a nonzero real-valued function $f \in H^1$, it follows that $f \cdot f = f^2 \notin H^{1/2}$. Note that it is that fact that f^2 can have non-zero integral that bars it from membership in $H^{1/2}$ (any integrable function in $H^{1/2}$ must have mean zero). This is a failure of a bilinear analogue of classical theory in the following sense. The operator $P(f_1, f_2)(x) = f_1(x)f_2(x)$ is analogous to the identity operator $If(x) = f(x)$ in the linear theory. Clearly the identity operator I is bounded on any reasonable function space, including H^p for all $0 < p \leq 1$, and the bilinear operator P enjoys many similar properties to I . For example P is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $0 < p_1, p_2, p < \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ (this is just Hölder's inequality), and by a result in [23], P is also bounded from $H^{p_1} \times H^{p_2}$ into L^p for any $0 < p_1, p_2, p \leq 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Although, the product operator P fails to satisfy many Hardy space bounds that one might initially expect, and this presents a much more difficult problem. Namely, P is not bounded from $H^{p_1} \times H^{p_2}$ into H^p , whenever $0 < p_1, p_2, p \leq 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. The product structure of bilinear operators severely complicates oscillatory properties of functions. For this reason, addressing Hardy space $H^{p_1} \times H^{p_2}$ to H^p bounds for bilinear operators is connected to the deep mathematical problem of understanding the oscillatory behavior of products of functions. One result in this work (Theorem 2.5) is a paraproduct operator $\Pi(f_1, f_2)$ that resembles the product $P(f_1, f_2) = f_1 \cdot f_2$ in some senses, but satisfies $\Pi(f_1, f_2) \in H^{1/2}$ for all $f_1, f_2 \in H^1$, along with other properties.

The example in the last paragraph gives some initial insight into what we can expect as far as results for the bilinear singular integral operators we work with in this article. Consider again the pointwise product operator $P(f_1, f_2)(x) = f_1(x)f_2(x)$. This is arguably the simplest bilinear operator that one can consider, but it is not bounded from $H^{p_1} \times H^{p_2}$ into H^p for $0 < p \leq 1$. So the conditions we impose on our bilinear operators below must not be verified by P . The appropriate bilinear operators to have the Hardy space boundedness properties must "improve" the functions f_1 and f_2 in the following sense. Formally, $f_1 \in H^{p_1}$ and $f_2 \in H^{p_2}$ must have some regularity and vanishing moment properties up to order (comparable to) $1/p_1$ and $1/p_2$ respectively. Since we wish to obtain $T(f_1, f_2) \in H^p$, $T(f_1, f_2)$ must satisfy better regularity and vanishing moment properties than either f_1 or f_2 (roughly speaking, up to order $1/p = 1/p_1 + 1/p_2$). This precludes the typical paradigm for interpreting the regularity properties for inputs and outputs of a bilinear Calderón-Zygmund operators as set by the product operator P . That is, one typically considers $T(f_1, f_2)$ to satisfy the regularity properties that $f_1(x) \cdot f_2(x)$ would satisfy; which means that a general bilinear Calderón-Zygmund operator cannot be smoothing. We look at a smaller class of bilinear Calderón-Zygmund operators that are smoothing in some sense; hence from this point of view we can again rule out the product operator P as a representative example. An example of a smoothing bilinear Calderón-Zygmund operator was given in [2] by a certain paraproduct, in which

the authors prove boundedness properties on homogeneous Sobolev spaces, from $\dot{L}_s^{p_1} \times \dot{L}_t^{p_2}$ to \dot{L}_{s+t}^p . We consider these smoothing operators since the output has regularity $s+t$, whereas the inputs, $f_1 \in \dot{L}_s^{p_1}$ and $f_2 \in \dot{L}_t^{p_2}$, have lesser regularity, s and t respectively. A general Calderón-Zygmund operator that is bounded $L^2 \times L^2$ to L^1 does not satisfy this type of smoothing boundedness property. Later we apply our main Calderón-Zygmund operator result (Theorem 2.2) to the paraproducts from [2] to prove that they are also bounded from $H^{p_1} \times H^{p_2}$ into H^p under appropriate conditions, see Theorem 2.6.

There has been work done on bilinear operators that are related to the Hardy space mapping problem we are considering that also hint at what conditions may be needed for bilinear operators to be bounded in H^p for $0 < p \leq 1$. In [23], Grafakos and Kalton give conditions for a bilinear Calderón-Zygmund operator to be bounded from $H^{p_1} \times H^{p_2}$ into L^p (we restrict to the bilinear setting for this discussion). In fact, they only require that T is bounded from $L^2 \times L^2$ into L^1 and additional kernel regularity for this conclusion. There is no further cancellation required for T beyond what is necessary for Calderón-Zygmund operators to be bounded on Lebesgue spaces with indices larger than 1, see for example [7, 26, 27, 33]. Note that the product operator still falls into the class of operators to which the results in [23] apply; as previously mentioned P is bounded from $H^{p_1} \times H^{p_2}$ into L^p for indices smaller than 1. This highlights the drastic difference between the work in [23] and this article. Another work that is closely related to this article is [36]. In that work, the authors prove some $H^{p_1} \times H^{p_2}$ to H^p estimates when $p \leq 1$. Their result, which is for p close to 1, relies heavily on atomic decompositions of Hardy spaces and the boundedness results from [23]. We give sufficient conditions for a bilinear operator T to be decomposed in terms of Littlewood-Paley-Stein theory, and prove estimates for Hardy spaces with indices p ranging all the way down to zero without using atomic decompositions. This is inspired by the works in the multi-parameter Hardy space theory in [30, 31] where a discrete Littlewood-Paley theory is carried out to prove boundedness of singular integral operators on multi-parameter Hardy spaces.

2. MAIN RESULTS

We take a moment to describe our general approach to the results in this article. Our primary goal is to prove $H^{p_1} \times H^{p_2}$ to H^p estimates for bilinear Calderón-Zygmund operators. The way we approach this is to decompose T into smooth truncation operators $\Theta_k = Q_k T$ for $k \in \mathbb{Z}$, and reduce the H^p estimates for T to estimates in for Θ_k using the Littlewood-Paley characterization of H^p from [18]. That is, we choose Q_k to be Littlewood-Paley-Stein type operators, sufficient for the following semi-norm equivalence,

$$\|T(f_1, f_2)\|_{H^p} \approx \left\| \left(\sum_{k \in \mathbb{Z}} |Q_k T(f_1, f_2)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \left\| \left(\sum_{k \in \mathbb{Z}} |\Theta_k(f_1, f_2)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} = \|S_{\Theta}(f_1, f_2)\|_{L^p},$$

where S_{Θ} is the square function associated to the collection $\Theta_k = Q_k T$. These operators Θ_k for $k \in \mathbb{Z}$ define what we call a collection of bilinear Littlewood-Paley-Stein operators. In this way we reduce our $H^{p_1} \times H^{p_2}$ to H^p estimates for T to $H^{p_1} \times H^{p_2}$ to L^p estimates for S_{Θ} . The latter estimates for S_{Θ} are the square function type estimates that we will prove, thereby yielding the Calderón-Zygmund operator estimates as well. We also obtain paraproduct boundedness properties by applying our Calderón-Zygmund operator estimates.

2.1. Bilinear Littlewood-Paley-Stein Theory. Given kernel functions $\theta_k : \mathbb{R}^{3n} \rightarrow \mathbb{C}$ for $k \in \mathbb{Z}$, define

$$\Theta_k(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2$$

for appropriate functions $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{C}$. Define the square function associated to $\{\Theta_k\}$

$$S_{\Theta}(f_1, f_2)(x) = \left(\sum_{k \in \mathbb{Z}} |\Theta_k(f_1, f_2)(x)|^2 \right)^{\frac{1}{2}}.$$

We say that a collection of operators Θ_k for $k \in \mathbb{Z}$ is a collection of bilinear Littlewood-Paley-Stein operators with decay and smoothness (N, L) , written $\{\Theta_k\} \in BLPSO(N, L)$, for an integer $L \geq 0$ and $N > 0$ if there exists a constant C such that

$$(2.1) \quad |D_1^{\alpha} D_2^{\beta} \theta_k(x, y_1, y_2)| \leq C 2^{k(|\alpha|+|\beta|)} \Phi_k^N(x-y_1, x-y_2) \text{ for all } |\alpha|, |\beta| \leq L.$$

Here we use the notation $\Phi_k^N(x, y) = 2^{2kn} (1 + 2^k|x| + 2^k|y|)^{-N}$ for $N > 0$, $x, y \in \mathbb{R}^n$, and $k \in \mathbb{Z}$. We also use the notation $D_0^{\alpha} F(x, y_1, y_2) = \partial_x^{\alpha} F(x, y_1, y_2)$, $D_1^{\alpha} F(x, y_1, y_2) = \partial_{y_1}^{\alpha} F(x, y_1, y_2)$, and $D_2^{\alpha} F(x, y_1, y_2) = \partial_{y_2}^{\alpha} F(x, y_1, y_2)$ for $F : \mathbb{R}^{3n} \rightarrow \mathbb{C}$ and $\alpha \in \mathbb{N}_0^n$.

Given $\{\Theta_k\} \in BLPSO(N, L)$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| < N - 2n$, define the (α, β) order moment function associated to $\{\Theta_k\}$ by

$$[[\Theta_k]]_{\alpha, \beta}(x) = 2^{k(|\alpha|+|\beta|)} \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) (x-y_1)^{\alpha} (x-y_1)^{\beta} dy_1 dy_2$$

for $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. It is worth noting that $[[\Theta_k]]_0(x) = \Theta_k(1, 1)(x)$, which is an object that is closely related to boundedness properties of S_{Θ} , see for example [42, 43, 32, 25, 29, 24]. Our main square function boundedness result is the following theorem.

Theorem 2.1. *Let $L \geq 1$ be an integer and $N = 2n + L(2n + L + 5)/2$. If $\{\Theta_k\} \in BLPSO(N, L)$ and*

$$(2.2) \quad d\mu(x, t) = \sum_{k \in \mathbb{Z}} \sum_{|\alpha|, |\beta| \leq L-1} |[[\Theta_k]]_{\alpha, \beta}(x)|^2 \delta_{t=2^{-k}} dx$$

is a Carleson measure, then S_{Θ} can be extended to a bounded operator from $H^{p_1} \times H^{p_2}$ into L^p for all $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Here $\delta_{t=2^{-k}}$ is the delta point mass measure on $(0, \infty)$ concentrated at 2^{-k} . That is, for a set $E \subset (0, \infty)$, the measure $\delta_{t=2^{-k}}$ is defined by $\delta_{t=2^{-k}}(E) = 1$ if $2^{-k} \in E$ and $\delta_{t=2^{-k}}(E) = 0$ if $2^{-k} \notin E$. Also, a non-negative measure $d\mu(x, t)$ on $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ is a Carleson measure if there is a $C > 0$ such that $d\mu(Q \times (0, \ell(Q))) \leq C|Q|$ for all cubes $Q \subset \mathbb{R}^n$, where $\ell(Q)$ is the side length of Q and $|Q|$ is the Lebesgue measure of Q .

2.2. Bilinear Calderón-Zygmund Theory. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of smooth, rapidly decreasing functions endowed with the standard Schwartz semi-norm topology, and let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be its continuous dual, the class of tempered distributions. We say that a continuous bilinear operator T from $\mathcal{S} \times \mathcal{S}$ into \mathcal{S}' is a bilinear Calderón-Zygmund operator with smoothness M , written $T \in BCZO(M)$, if T has function kernel $K : \mathbb{R}^{3n} \setminus \{(x, x, x) : x \in \mathbb{R}^n\} \rightarrow \mathbb{C}$ such

that

$$\langle T(f_1, f_2), g \rangle = \int_{\mathbb{R}^{3n}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) g(x) dy_1 dy_2 dx$$

whenever $f_1, f_2, g \in C_0^\infty$ have disjoint support, $\text{supp}(f_1) \cap \text{supp}(f_2) \cap \text{supp}(g) = \emptyset$, and there is a constant $C > 0$ such that the kernel function K satisfies

$$|D_0^\alpha D_1^\beta D_2^\mu K(x, y_1, y_2)| \leq \frac{C}{(|x - y_1| + |x - y_2|)^{2n + |\alpha| + |\beta| + |\mu|}} \text{ for all } |\alpha|, |\beta|, |\mu| \leq M$$

for $x, y_1, y_2 \in \mathbb{R}^n$ such that $|x - y_1| + |x - y_2| > 0$. Define the transposes of T by the dual relations

$$\langle T(f_1, f_2), g \rangle = \langle T^{*1}(g, f_2), f_1 \rangle = \langle T^{*2}(f_1, g), f_2 \rangle$$

for $f_1, f_2, g \in \mathcal{S}$, which also satisfy $T^{*j} \in BCZO(M)$ for $j = 1, 2$ whenever $T \in BCZO(M)$. We also define moment distributions for an operator $T \in BCZO(M)$, but we require some notation first. For an integer $M \geq 0$, define the collections of functions

$$O_M = O_M(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |f(x)| \cdot (1 + |x|)^{-M} < \infty \right\} \text{ and}$$

$$\mathcal{D}_M = \mathcal{D}_M(\mathbb{R}^n) = \left\{ f \in C_0^\infty(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x) x^\alpha dx = 0 \text{ for all } |\alpha| \leq M \right\}.$$

Define the topology of \mathcal{D}_M by the sequential characterization, for $f_k, f \in \mathcal{D}_M$ for $k \in \mathbb{N}$, $f_k \rightarrow f$ in \mathcal{D}_M if there exists a compact set K such that $\text{supp}(f_k), \text{supp}(f) \subset K$ for all $k \in \mathbb{N}$ and

$$\lim_{k \rightarrow \infty} \|D^\alpha f_k - D^\alpha f\|_{L^\infty} = 0$$

for all $\alpha \in \mathbb{N}_0^n$. Then \mathcal{D}'_M is defined to be all linear functionals $W : \mathcal{D}_M \rightarrow \mathbb{C}$ such that

$$f_k \rightarrow f \text{ in } \mathcal{D}_M \text{ implies } \langle W, f_k \rangle \rightarrow \langle W, f \rangle.$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be supported in $B(0, 2)$ and $\eta(x) = 1$ for $x \in B(0, 1)$. Define for $R > 0$, $\eta_R(x) = \eta(x/R)$. We reserve the notation η_R for functions constructed in this way. In [1], Bényi defined $T(f_1, f_2)$ for $f_1 \in O_{M_1}$ and $f_2 \in O_{M_2}$ where T is a bilinear singular integral operator. We give an equivalent definition. Let T be a $BCZO(M+1)$ and $f_1 \in O_{M_1}$ and $f_2 \in O_{M_2}$ for some integers $M_1, M_2 \geq 0$ such that $M_1 + M_2 \leq M$. For $\psi \in \mathcal{D}_M$, define

$$\langle T(f_1, f_2), \psi \rangle = \lim_{R \rightarrow \infty} \langle T(\eta_R f_1, \eta_R f_2), \psi \rangle.$$

Also, for $f_1 \in O_M$, $f_2 \in C_0^\infty$, and $\psi \in \mathcal{D}_M$, define

$$\langle T(f_1, f_2), \psi \rangle = \lim_{R \rightarrow \infty} \langle T(\eta_R f_1, f_2), \psi \rangle.$$

These limits exist based on the kernel representation and kernel properties for $T \in BCZO(M+1)$, see [1] for proof of this fact. Now we define the moment distribution $[[T]]_{\alpha, \beta} \in \mathcal{D}'_{|\alpha| + |\beta|}$ for $T \in BCZO(M+1)$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq M$ by

$$\langle [[T]]_{\alpha, \beta}, \psi \rangle = \lim_{R \rightarrow \infty} \int_{\mathbb{R}^{3n}} \mathcal{K}(x, y_1, y_2) (x - y_1)^\alpha \eta_R(y_1) (x - y_2)^\beta \eta_R(y_2) \psi(x) dy_1 dy_2 dx$$

for $\psi \in \mathcal{D}_M$. Here $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^{3n})$ is the distribution kernel of T , and the integral above is interpreted as a dual pairing between $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^{3n})$ and elements of $\mathcal{S}(\mathbb{R}^{3n})$. This distributional moment associated to T generalizes the notion of $T(1, 1)$ as used in [7, 26, 27, 33] in the sense that

$\langle [[T]]_{0,0}, \psi \rangle = \langle T(1, 1), \psi \rangle$ for all $\psi \in \mathcal{D}_0$. We will also use a generalized notion of BMO to extend the cancellation conditions $T(1, 1), T^{*1}(1, 1), T^{*2}(1, 1) \in BMO$, which were used in the bilinear $T1$ theorems from [7, 26, 27, 33]. Let $M \geq 0$ be an integer and $F \in \mathcal{D}'_M/\mathcal{P}$, distributions \mathcal{D}'_M modulo polynomials. We say that $F \in BMO_M$ if the non-negative measure on \mathbb{R}_+^{n+1} ,

$$\sum_{k \in \mathbb{Z}} 2^{2Mk} |Q_k F(x)|^2 dx \delta_{t=2^{-k}}$$

is a Carleson measure. This definition agrees with the classical definition of BMO by the characterization of BMO in terms of Carleson measures in [4, 38]. It should be noted that we defined this polynomial growth BMO_M space in [35], and a similar polynomial growth BMO_M was defined by Youssfi [51]. We use this polynomial growth BMO_M to quantify cancellation conditions for operators $T \in BCZO(M)$ in the following result.

Theorem 2.2. *Assume that $T \in BCZO(M)$, where $M = L(2n + L + 5)/2$ for some integer $L \geq 1$, and that T is bounded from $L^2 \times L^2$ into L^1 . If*

$$(2.3) \quad T^{*1}(x^\alpha, \psi) = T^{*2}(\psi, x^\alpha) = 0 \text{ in } \mathcal{D}'_{|\alpha|} \text{ for all } |\alpha| \leq 2L + n \text{ and } \psi \in \mathcal{D}_{2L+n},$$

$$(2.4) \quad [[T]]_{\alpha,\beta} \in BMO_{|\alpha|+|\beta|} \text{ for all } |\alpha|, |\beta| \leq L-1,$$

then T can be extended to bounded operator from $H^{p_1} \times H^{p_2}$ into H^p for all $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

We prove Theorem 2.2 by decomposing an operator $T \in BCZO(M)$ into a collection of operators $\{\Theta_k\} \in BLPSO(2n + M, L)$ in Theorem 2.1, where L depends on the regularity parameter M . This decomposition of T into a collection of bilinear Littlewood-Paley-Stein operators is stated precisely in the next theorem.

Theorem 2.3. *Let $L \geq 1$ be an integer, $N > 2n$, and $T \in BCZO(M)$ for some integer $M \geq \max(N - 2n, 2L + 1)$. Also assume that T is bounded from $L^2 \times L^2$ into L^1 . If T satisfies (2.3), then $\{\Theta_k\} = \{Q_k T\} \in BLPSO(N, L)$ for any $\psi \in \mathcal{D}_M$, where $Q_k f = \psi_k * f$ and $\psi_k(x) = 2^{kn} \psi(2^k x)$.*

2.3. Applications to Paraproducts. In this article, we use Theorem 2.2 to prove that three types of paraproducts are bounded on Hardy spaces. The first is a bilinear Bony type paraproduct, which is a bilinear version of Bony's paraproduct in [3] and was originally introduced in [33]. It is constructed as follows.

Let $\psi \in \mathcal{D}_M$ and $\varphi \in C_0^\infty$, and define $\psi_k(x) = 2^{kn} \psi(2^k x)$, $\varphi_k(x) = 2^{kn} \varphi(2^k x)$, $Q_k f = \psi_k * f$, and $P_k f = \varphi_k * f$. For $b \in BMO$, define the bilinear Bony paraproduct

$$(2.5) \quad \Pi_b(f_1, f_2)(x) = \sum_{j \in \mathbb{Z}} Q_j (Q_j b \cdot P_j f_1 \cdot P_j f_2)(x).$$

If ψ and φ are chosen appropriately, then this paraproduct satisfies $\Pi_b(1, 1) = b$, see [33]. We are not concerned with this particular property here, so we allow for a more general selection of ψ and φ . We will apply Theorem 2.2 to Π_b to prove the following theorem.

Theorem 2.4. *Let L be a non-negative integer, $b \in BMO$, $\psi \in \mathcal{D}_{3L+n}$, and $\varphi \in C_0^\infty$, then Π_b , as defined in (2.5), is bounded from $H^{p_1} \times H^{p_2}$ into H^p for all $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

The motivation for our second paraproduct operator is to find a replacement for the product $f_1(x)f_2(x)$ that is in H^p for $0 < p \leq 1$ for appropriate f_1 and f_2 . Once again, we construct this operator a bit more generally. We will use the following definition for the Fourier transform; for $f \in L^1$, define

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\xi} dx.$$

Let $\psi \in \mathcal{S}$ such that its Fourier transform $\widehat{\psi}$ is supported away from the origin, and let $\varphi \in \mathcal{S}$. Let Ψ_k, Φ_k, Q_k , and P_k be as above. Define

$$(2.6) \quad \Pi(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k f_2)(x).$$

We prove the following Hardy space bounds for Π .

Theorem 2.5. *Let $\Pi(f_1, f_2)$ be as in (2.6). Then Π is bounded from $H^{p_1} \times H^{p_2}$ into H^p for all $0 < p_1, p_2, p \leq 1$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

In order to construct the paraproduct $\Pi(f_1, f_2)$ to resemble the product $f_1 \cdot f_2$, we choose ψ and φ in the following way. Let $\varphi \in \mathcal{S}$ such that $\widehat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$ and $\text{supp}(\widehat{\varphi}) \subset B(0, 2)$. Define $\psi(x) = 2^{-2n}\varphi(2^{-2}x) - 2^{-3n}\varphi(2^{-3}x)$. With this choice of ψ and φ , it follows that $Q_k = P_k Q_k$. Under these conditions, it also follows that

$$\begin{aligned} \Pi(1, f_2)(x) &= \sum_{k \in \mathbb{Z}} Q_k(P_k(1) \cdot P_k f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k P_k f_2(x) = \sum_{k \in \mathbb{Z}} Q_k f_2(x) = f_2(x) \text{ and} \\ \Pi(f_1, 1)(x) &= \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k(1))(x) = \sum_{k \in \mathbb{Z}} Q_k P_k f_1(x) = \sum_{k \in \mathbb{Z}} Q_k f_1(x) = f_1(x) \end{aligned}$$

in H^p for all $f_1, f_2 \in H^p \cap L^2$. This gives precise meaning to how $\Pi(f_1, f_2)(x)$ ‘‘resembles’’ the product $f_1(x) \cdot f_2(x)$.

We should remark here that the constructions in Theorems 2.4 and 2.5 use slightly different techniques, but are interchangeable in some senses. In Theorem 2.4, we choose the convolution kernels $\psi_k \in \mathcal{D}_{3L+n}$, and conclude bounds for Hardy spaces with indices bounded below by $\frac{n}{2n+L}$ and $\frac{n}{n+L}$, where L can be taken arbitrarily large. One can construct the bilinear Bony paraproduct Π_b with $\psi \in \mathcal{S}$ such that $\widehat{\psi}$ is supported away from the origin, and the conclusion is strengthened to all $0 < p_1, p_2, p \leq 1$ similar to Theorem 2.5. Similarly, the paraproduct Π can be constructed with functions $\psi_k \in \mathcal{D}_M$, and obtain the same conclusion as in Theorem 2.4 with the appropriate lower bounds for $0 < p_1, p_2, p \leq 1$.

Finally, we consider a class of paraproducts defined in [2]. For a dyadic cube Q , a function $\phi_Q : \mathbb{R}^n \rightarrow \mathbb{C}$ is an (M, N) -smooth molecule associated to Q if there exists a constant $C = C_{M,N}$ independent of Q such that

$$|D^\alpha \phi_Q(x)| \leq C \frac{\ell(Q)^{-n/2} \ell(Q)^{-|\alpha|}}{(1 + \ell(Q)^{-1} |x - x_Q|)^N}$$

for all $|\alpha| \leq M$, where x_Q denotes the lower-left corner of Q . A family of (M, N) -smooth molecules ψ_Q indexed by dyadic cubes Q is an (M, N, L) -smooth family of molecules with cancellation if ψ_Q is an (M, N) -smooth molecule associated to Q and

$$(2.7) \quad \int_{\mathbb{R}^n} \psi_Q(x) x^\alpha dx = 0$$

for all $|\alpha| \leq L$. Let $\phi_Q^1, \phi_Q^2, \phi_Q^3$ be three families of molecules indexed by dyadic cubes Q , and define the paraproduct T , as in [2], by

$$T(f_1, f_2)(x) = \sum_Q \langle f_1, \phi_Q^1 \rangle \langle f_2, \phi_Q^2 \rangle \phi_Q^3(x).$$

The sum here indexed by Q is over all dyadic cubes. In [2], the authors prove the boundedness of these operators on many different function spaces, for example from $L^{p_1} \times L^{p_2} \rightarrow L^p, H^{p_1} \times H^{p_2} \rightarrow L^p, L^\infty \times L^\infty \rightarrow BMO$, as well as a number of estimates for weak L^p spaces and weighted spaces. Here we extend their results to boundedness properties from products of Hardy spaces into Hardy spaces.

Theorem 2.6. *Let $L \geq 0$ be an integer, $M = L(2n + L + 5)/2$, and $N > 10n + 15M + 5$. Assume that $\phi_Q^1, \phi_Q^2, \phi_Q^3$ are three families of (M, N) -smooth molecules. Furthermore assume that ϕ_Q^3 is a collection of $(M, N, 2L + n)$ -smooth molecules with cancellation, and either ϕ_Q^1 or ϕ_Q^2 is a family of $(M, N, L - 1)$ -smooth molecules with cancellation. Then T can be extended to a bounded operator from $H^{p_1} \times H^{p_2}$ into H^p for all $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L}$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.*

The selection of M and N in this result are not optimal. Here we use the selection of parameters that are sufficient to show that $T \in BCZO(M)$ based on the work in [2], in which the authors note the parameters are not chosen to be optimal.

This article is organized in the following way. In Section 3, we establish some notation, and state some preliminary results. Section 4 is dedicated to proving the bilinear square function estimates in Theorem 2.1. Section 5 is used to prove our bilinear Calderón-Zygmund operator estimates in Theorems 2.2 and 2.3. Finally in Section 6, we apply Theorem 2.2 to the paraproduct operators in Theorems 2.4, 2.5, and 2.6.

3. PRELIMINARIES

We use the notation $A \lesssim B$ to mean that $A \leq CB$ for some constant C . The constant C is allowed to depend on the ambient dimension, smoothness and decay parameters of our operators, indices of function spaces etc.; in context, the dependence of the constants is clear.

We will use the following Frazier and Jawerth type discrete Calderón reproducing formula [21] (see also [30] for a multiparameter formulation of this reproducing formula): there exist $\phi_j, \tilde{\phi}_j \in \mathcal{S}$ for $j \in \mathbb{Z}$ with infinite vanishing moment such that

$$(3.1) \quad f(x) = \sum_{j \in \mathbb{Z}} \sum_{\ell(Q)=2^{-(j+N_0)}} |Q| \phi_j(x - c_Q) \tilde{\phi}_j * f(c_Q) \text{ in } L^2$$

for $f \in L^2$. The summation in Q here is over all dyadic cubes with side length $\ell(Q) = 2^{-(j+N_0)}$ for some $N_0 > n$, and c_Q denotes the center of cube Q .

We will also use a more traditional formulation of Calderón's reproducing formula: fix $\varphi \in C_0^\infty(B(0, 1))$ such that

$$(3.2) \quad \sum_{k \in \mathbb{Z}} Q_k f = f \text{ in } L^2$$

for $f \in L^2$, where $\psi(x) = 2^n \varphi(2x) - \varphi(x)$, $\psi_k(x) = 2^{kn} \psi(2^k x)$, and $Q_k f = \psi_k * f$. Furthermore, we can assume that ψ has an arbitrarily large, but fixed, number of vanishing moments.

There are many equivalent definitions of the real Hardy spaces $H^p = H^p(\mathbb{R}^n)$ for $0 < p < \infty$. We use the following one. Define the non-tangential maximal function

$$\mathcal{N}^\varphi f(x) = \sup_{t>0} \sup_{|x-y|\leq t} \left| \int_{\mathbb{R}^n} t^{-n} \varphi(t^{-1}(y-u)) * f(u) du \right|,$$

where $\varphi \in \mathcal{S}$ with non-zero integral. It was proved by Fefferman and Stein in [18] that one can define $\|f\|_{H^p} = \|\mathcal{N}^\varphi f\|_{L^p}$ to obtain the classical real Hardy spaces H^p for $0 < p < \infty$. It was also proved in [18] that for any $\varphi \in \mathcal{S}$ and $f \in H^p$ for $0 < p < \infty$,

$$\left\| \sup_{k \in \mathbb{Z}} |\varphi_k * f| \right\|_{L^p} \lesssim \|f\|_{H^p}.$$

We will use a number of equivalent quasi-norms for H^p . Let $\psi \in \mathcal{D}_M$ for some integer $M > n(1/p - 1)$, and let ψ_k and Q_k be as above. For $f \in \mathcal{S}'/\mathcal{P}$ (tempered distributions modulo polynomials), $f \in H^p$ if and only if

$$\left\| \left(\sum_{k \in \mathbb{Z}} |Q_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} < \infty,$$

and this quantity is comparable to $\|f\|_{H^p}$. The space H^p can also be characterized by the operators ϕ_j and $\tilde{\phi}_j$ from the discrete Littlewood-Paley-Stein decomposition in (3.1). This characterization is given by the following, which can be found in [30]. Given $0 < p < \infty$

$$\left\| \left(\sum_{j \in \mathbb{Z}} \sum_{\ell(Q)=2^{-(j+N_0)}} |\tilde{\phi}_j * f(c_Q)|^2 \chi_Q \right)^{\frac{1}{2}} \right\|_{L^p} \approx \|f\|_{H^p},$$

where $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \notin E$ for a subset $E \subset \mathbb{R}^n$. For a continuous function $f \in L^1_{loc}(\mathbb{R}^n)$ and $0 < r < \infty$, define

$$(3.3) \quad \mathcal{M}_j^r f(x) = \left\{ \mathcal{M} \left[\left(\sum_{\ell(Q)=2^{-(j+N_0)}} f(c_Q) \chi_Q \right)^r \right] (x) \right\}^{\frac{1}{r}},$$

where \mathcal{M} is the Hardy-Littlewood maximal operator. The following estimate was also proved by Han and Lu in [30].

Proposition 3.1. *For $0 < r < p \leq 1$ and $f \in H^p$*

$$\left\| \left(\sum_{j \in \mathbb{Z}} (\mathcal{M}_j^r(\tilde{\phi}_j * f))^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{H^p},$$

where \mathcal{M}_j^r is defined as in (3.3).

The next result is a reformulation of an estimate proved by Han and Lu in [30]; this version of the result was proved in [35].

Proposition 3.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ a non-negative continuous function, $\nu > 0$, and $\frac{n}{n+\nu} < r \leq 1$. Then*

$$\sum_{\ell(Q)=2^{-(j+N_0)}} |Q| \Phi_{\min(j,k)}^{n+\nu}(x-c_Q) f(c_Q) \lesssim 2^{\max(0,j-k)(N-n)} \mathcal{M}_j^r f(x)$$

for all $x \in \mathbb{R}^n$, where \mathcal{M}_j^r is defined in (3.3) and the summation indexed by $\ell(Q) = 2^{-(j+N_0)}$ is the sum over all dyadic cubes with side length $2^{-(j+N_0)}$ and c_Q denotes the center of cube Q .

The next result is proved using some well-known techniques for Carleson measure. The proof can be found in [35].

Proposition 3.3. *Suppose*

$$(3.4) \quad d\mu(x,t) = \sum_{k \in \mathbb{Z}} \mu_k(x) \delta_{t=2^{-k}} dx$$

is a Carleson measure, where μ_k is a non negative, locally integrable function for all $k \in \mathbb{Z}$. Also let $\varphi \in \mathcal{S}$, and define $P_k f = \varphi_k * f$, where $\varphi_k(x) = 2^{kn} \varphi(2^k x)$ for $k \in \mathbb{Z}$. Then

$$\left\| \left(\sum_{k \in \mathbb{Z}} |P_k f|^p \mu_k \right)^{\frac{1}{p}} \right\|_{L^p} \lesssim \|f\|_{H^p} \quad \text{for all } 0 < p < \infty$$

and

$$\left\| \left(\sum_{k \in \mathbb{Z}} |P_k f|^2 \mu_k \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{H^p} \quad \text{for all } 0 < p \leq 2.$$

We will also need Hardy space estimates for linear Littlewood-Paley-Stein square function operators that was proved in [35]. First we set some notation for linear Littlewood-Paley-Stein operators. Given kernel functions $\lambda_k : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ for $k \in \mathbb{Z}$, define

$$\Lambda_k f(x) = \int_{\mathbb{R}^n} \lambda_k(x,y) f(y) dy$$

for appropriate functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$. Define the square function associated to $\{\Lambda_k\}$

$$S_\Lambda f(x) = \left(\sum_{k \in \mathbb{Z}} |\Lambda_k f(x)|^2 \right)^{\frac{1}{2}}.$$

We say that a collection of operators Λ_k for $k \in \mathbb{Z}$ is a collection of Littlewood-Paley-Stein operators with decay and smoothness (N, L) , written $\{\Lambda_k\} \in LPSO(N, L)$, for an integer $L \geq 0$ and $N > 0$ if there exists a constant C such that

$$(3.5) \quad |D_1^\alpha \lambda_k(x,y)| \leq C 2^{|\alpha|k} \Phi_k^N(x-y) \text{ for all } |\alpha| \leq L.$$

Here we use the notation $\Phi_k^N(x) = 2^{kn} (1 + 2^k |x|)^{-N}$ for $N > 0$, $x \in \mathbb{R}^n$, and $k \in \mathbb{Z}$. We also write $D_0^\alpha F(x,y) = \partial_x^\alpha F(x,y)$ and $D_1^\alpha F(x,y) = \partial_y^\alpha F(x,y)$ for $F : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ and $\alpha \in \mathbb{N}_0^n$. This is a slightly different definition for $LPSO(N, L)$ than what was used in [35]. In that article, we defined $LPSO(N, L + \delta)$ for integers $L \geq 0$ and $0 < \delta \leq 1$ in terms of δ -Hölder conditions on $\lambda_k(x,y)$ in the y variable. Here we require $\{\Lambda_k\} \in LPSO(N, L)$ to have kernel that is L times differentiable in y instead of $L - 1$ times differentiable with Lipschitz $L - 1$ order derivatives as in [35]. It is not

hard to see that the definition we use for $LPSO(N, L)$ here is contained in the class defined in [35]; hence all results from [35] for $LPSO(N, L)$ are still applicable in the current work.

Given $\{\Lambda_k\} \in LPSO(N, L)$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < N - n$, define

$$[[\Lambda_k]]_\alpha(x) = 2^{k|\alpha|} \int_{\mathbb{R}^n} \lambda_k(x, y)(x - y)^\alpha dy$$

for $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. The next theorem was also proved in [35].

Theorem 3.4. *Let $\{\Lambda_k\} \in LPSO(n + 2L, L)$ for some integer $L \geq 1$. If*

$$(3.6) \quad d\mu_\alpha(x, t) = \sum_{k \in \mathbb{Z}} |[[\Lambda_k]]_\alpha(x)|^2 \delta_{t=2^{-k}} dx$$

is a Carleson measure for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq L - 1$, then S_Λ is bounded from H^p into L^p for all $\frac{n}{n+L} < p \leq 1$.

Note that we use Λ_k to denote linear operators and Θ_k to denote bilinear operators. We will keep this convention throughout the paper.

4. HARDY SPACE ESTIMATES FOR BILINEAR SQUARE FUNCTIONS

To start this section, we prove a reduced version of Theorem 2.1, where we strengthen the cancellation assumptions on Θ_k from the ones in (2.2) to the conditions in the next lemma. Once we establish Lemma 4.1, we extend to the general situation in Theorem 2.1 by using a paraproduct type decomposition.

Lemma 4.1. *Let $\{\Theta_k\} \in BLPSO(N, L)$, where $N = 2n + 3L$ for some integer $L \geq 1$. If*

$$\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) y_1^\alpha dy_1 = \int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) y_2^\alpha dy_2 = 0$$

for all $k \in \mathbb{Z}$ and $|\alpha| \leq L - 1$, then S_{Θ} can be extended to a bounded operator from $H^{p_1} \times H^{p_2}$ into L^p for all $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Proof. Let p, p_1, p_2 be as above. Choose v such that $\frac{n}{p} - 2n < v < L$, which is possible since $\frac{n}{2n+L} < p$. Since $v > \frac{n}{p} - 2n$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, there exists $s \in (0, 1)$ such that $sv > \frac{n}{p_1} - n$ and $(1-s)v > \frac{n}{p_2} - n$. Finally fix $r_1, r_2 > 0$ such that $\frac{n}{n+sv} < r_1 < p_1$ and $\frac{n}{n+(1-s)v} < r_2 < p_2$. Note that this also implies that $n + sv \leq n/p_1 < n + L$ and $n + (1-s)v \leq n/p_2 < n + L$, which will be used later to conclude that $\Phi_k^{n+L}(x) \leq \Phi_k^{n+sv}(x)$ and $\Phi_k^{n+L}(x) \leq \Phi_k^{n+(1-s)v}(x)$.

By density, it is sufficient to prove the appropriate estimate for $f_i \in H^{p_i} \cap L^2$ for $i = 1, 2$. We decompose $\Theta_k(f_1, f_2)$ using (3.1) for each f_1 and f_2 ,

$$\begin{aligned} \Theta_k(f_1, f_2)(x) &= \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{Q_1, Q_2} |Q_1| |Q_2| \tilde{\Phi}_{j_1} * f_1(c_{Q_1}) \tilde{\Phi}_{j_2} * f_2(c_{Q_2}) \Theta_k(\Phi_{j_1}^{c_{Q_1}}, \Phi_{j_2}^{c_{Q_2}})(x) \\ &= \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{Q_1, Q_2} |Q_1| |Q_2| \tilde{\Phi}_{j_1} * f_1(c_{Q_1}) \tilde{\Phi}_{j_2} * f_2(c_{Q_2}) \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) \Phi_{j_1}^{c_{Q_1}}(y_1) \Phi_{j_2}^{c_{Q_2}}(y_2) dy_1 dy_2. \end{aligned}$$

The summation in Q_1 and Q_2 are over all dyadic cubes with side lengths $\ell(Q_1) = 2^{-(j_1+N_0)}$ and $\ell(Q_2) = 2^{-(j_2+N_0)}$ respectively. Now we establish an almost orthogonally estimate for the integral term in the previous equation. Using the vanishing moment properties of θ_k and the regularity of

ϕ_{j_1} , we obtain the following.

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) \phi_{j_1}^{c_{Q_1}}(y_1) \phi_{j_2}^{c_{Q_2}}(y_2) dy_1 dy_2 \right| \\
&= \left| \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) \left(\phi_{j_1}^{c_{Q_1}}(y_1) - \sum_{|\alpha| \leq L-1} \frac{D^\alpha \phi_{j_1}^{c_{Q_1}}(x)}{\alpha!} (y_1 - x)^\alpha \right) \phi_{j_2}^{c_{Q_2}}(y_2) dy_1 dy_2 \right| \\
&\lesssim \int_{\mathbb{R}^{2n}} \Phi_k^N(x - y_1, x - y_2) (2^{j_1} |x - y_1|)^L \\
&\quad \times \left(\Phi_{j_1}^{n+L}(y_1 - c_{Q_1}) + \Phi_{j_1}^{n+L}(x - c_{Q_1}) \right) \Phi_{j_2}^{n+L}(y_2 - c_{Q_2}) dy_1 dy_2 \\
&\lesssim 2^{L(j_1-k)} \int_{\mathbb{R}^{2n}} \Phi_k^{2n+2L}(x - y_1, x - y_2) \\
&\quad \times \left(\Phi_{j_1}^{n+L}(y_1 - c_{Q_1}) + \Phi_{j_1}^{n+L}(x - c_{Q_1}) \right) \Phi_{j_2}^{n+L}(y_2 - c_{Q_2}) dy_1 dy_2 \\
&\lesssim 2^{L(j_1-k)} \Phi_{\min(j_1, k)}^{n+sv}(x - c_{Q_1}) \Phi_{\min(j_2, k)}^{n+(1-s)v}(x - y_2).
\end{aligned}$$

A similar estimate holds for $2^{L(j_2-k)}$ in place of $2^{L(j_1-k)}$. Using the vanishing moment properties of ϕ_{j_1} , we have the following estimate,

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) \phi_{j_1}^{c_{Q_1}}(y_1) \phi_{j_2}^{c_{Q_2}}(y_2) dy_1 dy_2 \right| \\
&= \left| \int_{\mathbb{R}^{2n}} \left(\theta_k(x, y_1, y_2) - \sum_{|\alpha| \leq L-1} \frac{D_1^\alpha \theta_k(x, c_{Q_1}, y_2)}{\alpha!} (y_1 - c_{Q_1})^\alpha \right) \phi_{j_1}^{c_{Q_1}}(y_1) \phi_{j_2}^{c_{Q_2}}(y_2) dy_1 dy_2 \right| \\
&\lesssim \int_{\mathbb{R}^{2n}} \Phi_k^N(x - y_1, x - y_2) (2^k |y_1 - c_{Q_1}|)^L \Phi_{j_1}^{N+L}(y_1 - c_{Q_1}) \Phi_{j_2}^N(y_2 - c_{Q_2}) dy_1 dy_2 \\
&\quad \times \int_{\mathbb{R}^{2n}} \Phi_k^N(x - c_{Q_1}, x - y_2) (2^k |y_1 - c_{Q_1}|)^L \Phi_{j_1}^{N+L}(y_1 - c_{Q_1}) \Phi_{j_2}^N(y_2 - c_{Q_2}) dy_1 dy_2 \\
&\lesssim 2^{L(k-j_1)} \int_{\mathbb{R}^{2n}} \Phi_k^N(x - y_1, x - y_2) \Phi_{j_1}^N(y_1 - c_{Q_1}) \Phi_{j_2}^N(y_2 - c_{Q_2}) dy_1 dy_2 \\
&\lesssim 2^{L(k-j_1)} \Phi_{\min(j_1, k)}^{n+sv}(x - c_{Q_1}) \Phi_{\min(j_2, k)}^{n+(1-s)v}(x - c_{Q_2}).
\end{aligned}$$

Once again, this estimate holds with $2^{L(k-j_2)}$ in place of $2^{L(k-j_1)}$. Therefore

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) \phi_{j_1}^{c_{Q_1}}(y_1) \phi_{j_2}^{c_{Q_2}}(y_2) dy_1 dy_2 \right| \\
&\lesssim 2^{-L \max(|j_1-k|, |j_2-k|)} \Phi_{\min(j_1, k)}^{n+sv}(x - c_{Q_1}) \Phi_{\min(j_2, k)}^{n+(1-s)v}(x - c_{Q_2}).
\end{aligned}$$

Applying Proposition 3.2 yields

$$\begin{aligned}
 |\Theta_k(f_1, f_2)(x)| &\lesssim \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{Q_1, Q_2} |Q_1| |Q_2| \tilde{\Phi}_{j_1} * f_1(c_{Q_1}) \tilde{\Phi}_{j_2} * f_2(c_{Q_2}) \\
 &\quad \times 2^{-L \max(|j_1-k|, |j_2-k|)} \Phi_{\min(j_1, k)}^{n+sv}(x-c_{Q_1}) \Phi_{\min(j_2, k)}^{n+(1-s)v}(x-c_{Q_2}) \\
 &\lesssim \sum_{j_1, j_2 \in \mathbb{Z}} 2^{-L \max(|j_1-k|, |j_2-k|)} 2^{sv \max(0, k-j_1)} 2^{(1-s)v \max(0, k-j_2)} \mathcal{M}_{j_1}^{r_1}(\tilde{\Phi}_{j_1} * f_1)(x) \mathcal{M}_{j_2}^{r_2}(\tilde{\Phi}_{j_2} * f_2)(x) \\
 &\lesssim \sum_{j_1, j_2 \in \mathbb{Z}} 2^{-\varepsilon \max(|j_1-k|, |j_2-k|)} \mathcal{M}_{j_1}^{r_1}(\tilde{\Phi}_{j_1} * f_1)(x) \mathcal{M}_{j_2}^{r_2}(\tilde{\Phi}_{j_2} * f_2)(x),
 \end{aligned}$$

where $\varepsilon = L - v > 0$. Applying Proposition 3.1 for both $\mathcal{M}_{j_1}^{r_1}(\tilde{\Phi}_{j_1} * f_1)$ and $\mathcal{M}_{j_2}^{r_2}(\tilde{\Phi}_{j_2} * f_2)$ yields the appropriate estimate below,

$$\begin{aligned}
 \|S_\Theta(f_1, f_2)\|_{L^p} &\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} \left[\sum_{j_1, j_2 \in \mathbb{Z}} 2^{-\frac{\varepsilon}{2}|j_1-k|} 2^{-\frac{\varepsilon}{2}|j_2-k|} \mathcal{M}_{j_1}^{r_1}(\tilde{\Phi}_{j_1} * f_1) \mathcal{M}_{j_2}^{r_2}(\tilde{\Phi}_{j_2} * f_2) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\
 &\lesssim \left\| \left(\sum_{j_1, j_2, k \in \mathbb{Z}} 2^{-\frac{\varepsilon}{2}|j_1-k|} 2^{-\frac{\varepsilon}{2}|j_2-k|} \left[\mathcal{M}_{j_1}^{r_1}(\tilde{\Phi}_{j_1} * f_1) \mathcal{M}_{j_2}^{r_2}(\tilde{\Phi}_{j_2} * f_2) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\
 &\lesssim \left\| \left(\sum_{j_1 \in \mathbb{Z}} \left[\mathcal{M}_{j_1}^{r_1}(\tilde{\Phi}_{j_1} * f_1) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^{p_1}} \left\| \left(\sum_{j_2 \in \mathbb{Z}} \left[\mathcal{M}_{j_2}^{r_2}(\tilde{\Phi}_{j_2} * f_2) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^{p_2}} \\
 &\lesssim \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}.
 \end{aligned}$$

This completes the proof of Lemma 4.1. \square

Next we construct paraproducts to decompose Θ_k . These are the same operators that were constructed in [35], although we modify the decomposition to fit the bilinear framework. Fix an approximation to identity operator $P_k f = \varphi_k * f$, where $\varphi_k(x) = 2^{kn} \varphi(2^k x)$ and $\varphi \in \mathcal{S}$ with integral 1. Define for $\alpha, \beta \in \mathbb{N}_0^n$

$$M_{\alpha, \beta} = \begin{cases} (-1)^{|\alpha|} \frac{\beta!}{(\beta - \alpha)!} \int_{\mathbb{R}^n} \varphi(y) y^{\beta - \alpha} dy & \alpha \leq \beta \\ 0 & \alpha \not\leq \beta \end{cases}.$$

Here we say $\alpha \leq \beta$ for $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ if $\alpha_i \leq \beta_i$ for all $i = 1, \dots, n$. It is clear that $|M_{\alpha, \beta}| < \infty$ for all $\alpha, \beta \in \mathbb{N}_0^n$ since $\varphi \in \mathcal{S}$. It is not hard to verify that when $|\alpha| = |\beta|$,

$$(4.1) \quad M_{\alpha, \beta} = \begin{cases} (-1)^{|\beta|} \beta! & \alpha = \beta \\ 0 & \alpha \neq \beta \text{ and } |\alpha| = |\beta| \end{cases}.$$

Consider the operators $P_k D^\alpha f$ defined for $f \in \mathcal{S}'$, where D^α is the distributional derivative on \mathcal{S}' . Hence $P_k D^\alpha f(x)$ is well defined for $f \in \mathcal{S}'$ since $P_k D^\alpha f(x) = \langle \varphi_k^x, D^\alpha f \rangle = (-1)^{|\alpha|} \langle D^\alpha(\varphi_k^x), f \rangle$ and $D^\alpha(\varphi_k^x) \in \mathcal{S}$. In fact, this gives a kernel representation for $P_k D^\alpha$; estimates for this kernel are

addressed in the proof of Proposition 4.3. Also,

$$[[P_k D^\alpha]]_\beta(x) = 2^{|\beta|k} \int_{\mathbb{R}^n} \varphi_k(x-y) \partial_y^\alpha ((x-y)^\beta) dy = 2^{k|\alpha|} M_{\alpha,\beta}.$$

For $\{\Theta_k\} \in BLPSO(N, L)$ and $k \in \mathbb{Z}$, define

$$(4.2) \quad \begin{aligned} \Theta_k^{(0)}(f_1, f_2)(x) &= \Theta_k(f_1, f_2)(x) - [[\Theta_k(\cdot, f_2)]]_0(x) \cdot P_k f_1(x) \\ &\quad - [[\Theta_k(f_1, \cdot)]]_0(x) \cdot P_k f_2(x) + [[\Theta_k]]_{0,0}(x) \cdot P_k f_1(x) P_k f_2(x), \text{ and} \end{aligned}$$

$$(4.3) \quad \begin{aligned} \Theta_k^{(m)}(f_1, f_2)(x) &= \Theta_k^{(m-1)}(f_1, f_2)(x) - \sum_{|\alpha|=m} (-1)^{|\alpha|} \frac{[[\Theta_k^{(m-1)}(\cdot, f_2)]]_\alpha(x)}{\alpha!} \cdot 2^{-k|\alpha|} P_k D^\alpha f_1(x) \\ &\quad - \sum_{|\beta|=m} (-1)^{|\beta|} \frac{[[\Theta_k^{(m-1)}(f_1, \cdot)]]_\beta(x)}{\beta!} \cdot 2^{-k|\beta|} P_k D^\beta f_2(x) \\ &\quad + \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|+|\beta|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\beta}(x)}{\alpha!\beta!} \cdot 2^{-k(|\alpha|+|\beta|)} P_k D^\alpha f_1(x) P_k D^\beta f_2(x) \end{aligned}$$

for $1 \leq m \leq L$.

Lemma 4.2. *Let $\{\Theta_k\} \in BLPSO(N, L)$ for some integer $L \geq 1$ and $N > 2n$. Also let $0 \leq M < N - n$ be an integer. Then*

$$(4.4) \quad \int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) y_1^\alpha dy_1 = 0$$

for all $x, y_2 \in \mathbb{R}^n$ and $|\alpha| \leq M$ if and only if $[[\Theta_k(\cdot, f)]]_\alpha(x) = 0$ for all $x \in \mathbb{R}^n$, $|\alpha| \leq M$, and $f \in C_0^\infty$. Likewise for integrals in y_2 and $[[\Theta_k(f, \cdot)]]_\alpha$.

Here we define $[[\Theta_k(\cdot, f)]]_\alpha$ by applying the definition of $[[\cdot]]_\alpha$ for linear operator to the linear operator $g \mapsto \Theta_k(g, f)$ with f fixed. A similar notation is used for $[[\Theta_k(f, \cdot)]]_\alpha$.

Proof. Note that the condition $0 \leq M < N - n$ implies that $[[\Theta_k(\cdot, f)]]_\alpha$ is well defined for $|\alpha| \leq M$. Assume that (4.4) holds. Then for any $|\alpha| \leq M$

$$\begin{aligned} [[\Theta_k(\cdot, f)]]_\alpha &= 2^{|\alpha|k} \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) (x - y_1)^\alpha f(y_2) dy_1 dy_2 \\ &= 2^{|\alpha|k} \sum_{\mu+\nu=\alpha} (-1)^{|\nu|} c_{\mu,\nu} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) y_1^\nu dy_1 \right) x^\mu f(y_2) dy_2 = 0. \end{aligned}$$

Here $c_{\mu,\nu}$ are binomial coefficients. Now assume that $[[\Theta_k(\cdot, f)]]_\alpha(x) = 0$ for all $x \in \mathbb{R}^n$, $|\alpha| \leq M$, and $f \in C_0^\infty$. Let $\varphi \in C_0^\infty$ be radial with integral 1 and define $\varphi_k(x) = 2^{kn} \varphi(2^k x)$. The radial condition

here is not strictly necessary, but it simplifies notation. Then for $|\alpha| \leq M$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) y_1^\alpha dy_1 &= \sum_{\mu+v=\alpha} \int_{\mathbb{R}^n} \theta_k(x, y_1, y_2) (-1)^{|\alpha|+|\mu|} x^\mu (x-y_1)^v dy_1 \\ &= \sum_{\mu+v=\alpha} (-1)^{|\alpha|+|\mu|} x^\mu \lim_{N \rightarrow \infty} \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, u) (x-y_1)^v \varphi_N(u-y_2) dy_1 du \\ &= \sum_{\mu+v=\alpha} (-1)^{|\alpha|+|\mu|} x^\mu \lim_{N \rightarrow \infty} [[\Theta_k(\cdot, \varphi_N^{y_1})]]_v(x) = 0. \end{aligned}$$

We use that $\theta_k(x, y_1, y_2)$ is a bounded $L^1(\mathbb{R}^n)$ continuous function in x for $y_1 \neq y_2$ to use the approximation to identity property $\varphi_N * \theta_k(x, \cdot, y_2)(y_2) \rightarrow \theta_k(x, y_1, y_2)$ as $N \rightarrow \infty$ pointwise for $y_1 \neq y_2$ and in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This completes the proof. \square

Proposition 4.3. *Let $\{\Theta_k\} \in BLPSO(N, L)$ for some integer $L \geq 1$ and $N \geq 2n + L(2n + L - 1)/2$, and assume that*

$$(4.5) \quad d\mu(x, t) = \sum_{k \in \mathbb{Z}} \sum_{|\alpha|, |\beta| \leq L-1} |[[\Theta_k]]_{\alpha, \beta}(x)|^2 \delta_{t=2^{-k}} dx$$

is a Carleson measure. Also let $\Theta_k^{(m)}$ be as in (4.2) and (4.3) for $0 \leq m \leq L-1$. Then $\Theta_k^{(m)} \in BLPSO(\tilde{N}_m, L)$ where $\tilde{N}_m = N - L(2n + L - 1)/2$, and they satisfy the following for $0 \leq m \leq L-1$:

(1) For all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m \leq L-1$, we have

$$\int_{\mathbb{R}^n} \theta_k^{(m)}(x, y_1, y_2) y_1^\alpha dy_1 = \int_{\mathbb{R}^n} \theta_k^{(m)}(x, y_1, y_2) y_2^\alpha dy_2 = 0.$$

(2) $d\mu(x, t)$ is a Carleson measure, where $d\mu$ is defined by

$$d\mu(x, t) = \sum_{k \in \mathbb{Z}} \sum_{m=0}^{L-1} \sum_{|\alpha|, |\beta| \leq L-1} |[[\Theta_k^{(m)}]]_{\alpha, \beta}(x)|^2 \delta_{t=2^{-k}} dx.$$

Proof. We will show that $\{\Theta_k^{(m)}\} \in BLPSO(\tilde{N}_m, L)$ by induction. First we check that $\{\Theta_k^{(0)}\} \in BLPSO(\tilde{N}_0, L) = BLPSO(N-n, L)$. Using the definition in (4.2), it is sufficient to show that $\Theta_k(f_1, f_2)$, $[[\Theta_k(\cdot, f_2)]]_0 \cdot P_k f_1$, $[[\Theta_k(f_1, \cdot)]]_0 \cdot P_k f_2$, and $[[\Theta_k]]_{0,0} \cdot P_k f_1 \cdot P_k f_2$ each define operators of type $BLPSO(\tilde{N}_0, L)$. The first and last terms trivially satisfy these properties; note that $\{\Theta_k\} \in BLPSO(N, L) \subset BLPSO(\tilde{N}_0, L)$ by assumption and $[[\Theta_k]]_{0,0} \cdot P_k f_1 \cdot P_k f_2$ is in any $BPLSO(N, L)$ class since $[[\Theta_k]]_{0,0} \lesssim 1$ and the convolution kernel of P_k is in Schwartz class \mathcal{S} . We consider the second term $[[\Theta_k(\cdot, f_2)]]_0 \cdot P_k f_1$, whose kernel is

$$\varphi_k(x-y_1) \int_{\mathbb{R}^n} \theta_k(x, u_1, y_2) du.$$

It is not hard to verify that this kernel satisfies the appropriate estimates to be a member of $BLPSO(\tilde{N}_0, L)$. The kernel condition (2.1) is verified by the following. For $|\alpha|, |\beta| \leq L$

$$\begin{aligned} \left| D_1^\alpha D_2^\beta \left(\varphi_k(x-y_1) \int_{\mathbb{R}^n} \theta_k(x, u_1, y_2) du_1 \right) \right| &= \left| \partial_{y_1}^\alpha \varphi_k(x-y_1) \int_{\mathbb{R}^n} \partial_{y_2}^\beta \theta_k(x, u_1, y_2) du_1 \right| \\ &\lesssim 2^{|\alpha|k} \Phi_k^{N-n}(x-y_1) \int_{\mathbb{R}^n} 2^{|\beta|k} \Phi_k^N(x-u_1, x-y_2) du_1 \\ &\lesssim 2^{(|\alpha|+|\beta|)k} \Phi_k^{N-n}(x-y_1) \Phi_k^{N-n}(x-y_2) \\ &\leq 2^{(|\alpha|+|\beta|)k} \Phi_k^{\tilde{N}_0}(x-y_1, x-y_2). \end{aligned}$$

Here we use that

$$\Phi_k^R(x) \Phi_k^R(y) = \frac{2^{kn}}{(1+2^k|x|)^R} \frac{2^{kn}}{(1+2^k|y|)^R} \leq \frac{2^{2kn}}{(1+2^k|x|+2^k|y|)^R} = \Phi_k^R(x, y).$$

By symmetry $[[\Theta_k(f_1, \cdot)]]_0 \cdot P_k f_2$ defines an operator of type $BLPSO(\tilde{N}_0, L)$ as well. Now we proceed by induction. Assume that $\{\Theta_k^{(m-1)}\} \in BLPSO(\tilde{N}_{m-1}, L)$. Similar to the reduction for the $m=0$ case, this can be easily reduced to showing that $[[\Theta_k^{(m-1)}(\cdot, f_2)]]_\alpha \cdot 2^{-k|\alpha|} P_k D^\alpha f_1$ for $|\alpha| = m$ and the symmetric term define a collections of type $BLPSO(\tilde{N}_m, L)$. The kernel of $[[\Theta_k^{(m-1)}(\cdot, f_2)]]_\alpha \cdot 2^{-k|\alpha|} P_k D^\alpha f_1$ is

$$(-1)^{|\alpha|} (D^\alpha \varphi)_k(x-y_1) \int_{\mathbb{R}^n} \theta_k^{(m-1)}(x, u, y_2) 2^{k|\alpha|} (x-u)^\alpha du.$$

The kernel condition (2.1) for $[[\Theta_k^{(m-1)}(\cdot, f_2)]]_\alpha \cdot 2^{-k|\alpha|} P_k D^\alpha f_1$ is verified as follows using the inductive hypothesis. For $|\mu|, |\nu| \leq L$ and $|\alpha| = m$

$$\begin{aligned} &\left| D_1^\mu D_2^\nu \left((-1)^{|\alpha|} (D^\alpha \varphi)_k(x-y_1) \int_{\mathbb{R}^n} \theta_k^{(m-1)}(x, u, y_2) 2^{k|\alpha|} (u-y_1)^\alpha du \right) \right| \\ &\lesssim 2^{|\mu|k} |(D^{\alpha+\mu} \varphi)_k(x-y_1)| \int_{\mathbb{R}^n} |D_2^\nu \theta_k^{(m-1)}(x, u, y_2) 2^{k|\alpha|} (u-y_1)^\alpha| du \\ &\lesssim 2^{|\mu|k} \Phi_k^{\tilde{N}_m}(x-y_1) \int_{\mathbb{R}^n} 2^{|\nu|k} \Phi_k^{\tilde{N}_{m-1}}(x-u, x-y_2) 2^{k|\alpha|} (u-y_1)^\alpha | du \\ &\leq 2^{(|\mu|+|\nu|)k} \Phi_k^{\tilde{N}_m}(x-y_1) \int_{\mathbb{R}^n} \Phi_k^{\tilde{N}_{m-1}-|\alpha|}(x-u, x-y_2) du \\ &\leq 2^{(|\mu|+|\nu|)k} \Phi_k^{\tilde{N}_m}(x-y_1) \Phi_k^{\tilde{N}_{m-1}-n-m}(x-y_2) \\ &\leq 2^{(|\mu|+|\nu|)k} \Phi_k^{\tilde{N}_m}(x-y_1, x-y_2). \end{aligned}$$

This gives the appropriate formula \tilde{N}_m , defined recursively by $\tilde{N}_0 = N - n$ and $\tilde{N}_m = \tilde{N}_{m-1} - n - m$ for $m \geq 1$. This yields the formula $\tilde{N}_m = N - (m+1)(2n+m)/2$. Since $\tilde{N}_m \leq \tilde{N}_{m-1}$ for all $1 \leq m \leq L-1$, this gives the requirement $\tilde{N}_{L-1} = N - L(2n+L-1)/2$ as in the statement of Proposition 4.3.

Since $\{\Theta_k\} \in BLPSO(N, L)$ for some $N \geq 2n + 2L$, we know that $[[\Theta_k]]_{\alpha, \beta}(x)$ exists for all $|\alpha|, |\beta| \leq L-1$. We prove (1) by induction and using the reduction in Lemma 4.2 for each $\Theta_k^{(m)}$:

the $m = 0$ case for (1) is not hard to verify; for all $f_1 \in C_0^\infty$

$$\begin{aligned} [[\Theta_k^{(0)}(f_1, \cdot)]_0(x) &= [[\Theta_k(f_1, \cdot)]_0(x) - [[\Theta_k(f_1, \cdot)]_0(x) \\ &\quad - [[\Theta_k]_{0,0}(x)P_k f_1(x) + [[\Theta_k]_{0,0}(x)P_k f_1(x) = 0, \end{aligned}$$

and likewise $[[\Theta_k^{(0)}(\cdot, f_2)]_0 = 0$ for all $f_2 \in C_0^\infty$. Now assume that (1) holds for $m - 1$. Then for $|\mu| \leq m - 1$ and $f_1 \in C_0^\infty$

$$\begin{aligned} [[\Theta_k^{(m)}(f_1, \cdot)]_\mu &= [[\Theta_k^{(m-1)}(f_1, \cdot)]_\mu - \sum_{|\alpha|=m} (-1)^{|\alpha|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\mu}}{\alpha!} \cdot 2^{-k|\alpha|} P_k D^\alpha f_1 \\ &\quad - \sum_{|\beta|=m} (-1)^{|\beta|} \frac{[[\Theta_k^{(m-1)}(f_1, \cdot)]_\beta}{\beta!} \cdot 2^{-k|\beta|} [[P_k D^\beta]]_\mu \\ &\quad + \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|+|\beta|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\beta}}{\alpha!\beta!} \cdot 2^{-k(|\alpha|+|\beta|)} P_k D^\alpha f_1 [[P_k D^\beta]]_\mu = 0 \end{aligned}$$

by the inductive hypothesis and the fact that $[[P_k D^\alpha]]_\mu = [[P_k D^\beta]]_\mu = 0$ for $|\mu| \leq m - 1 < m = |\alpha| = |\beta|$. For $|\mu| = m$,

$$\begin{aligned} [[\Theta_k^{(m)}(f_1, \cdot)]_\mu &= [[\Theta_k^{(m-1)}(f_1, \cdot)]_\mu - \sum_{|\alpha|=m} (-1)^{|\alpha|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\mu}}{\alpha!} \cdot 2^{-k|\alpha|} P_k D^\alpha f_1 \\ &\quad - \sum_{|\beta|=m} (-1)^{|\beta|} \frac{[[\Theta_k^{(m-1)}(f_1, \cdot)]_\beta}{\beta!} \cdot 2^{-k|\beta|} [[P_k D^\beta f_2]]_\mu \\ &\quad + \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|+|\beta|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\beta}}{\alpha!\beta!} \cdot 2^{-k(|\alpha|+|\beta|)} P_k D^\alpha f_1 [[P_k D^\beta f_2]]_\mu \\ &= [[\Theta_k^{(m-1)}(f_1, \cdot)]_\mu - \sum_{|\alpha|=m} (-1)^{|\alpha|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\mu}}{\alpha!} \cdot 2^{-k|\alpha|} P_k D^\alpha f_1 \\ &\quad - [[\Theta_k^{(m-1)}(f_1, \cdot)]_\mu + \sum_{|\alpha|=m} (-1)^{|\alpha|} \frac{[[\Theta_k^{(m-1)}]]_{\alpha,\mu}}{\alpha!} \cdot 2^{-k|\alpha|} P_k D^\alpha f_1 = 0, \end{aligned}$$

where the summations in β collapse using (4.1) and that $M_{\beta,\mu} = (-1)^{|\beta|}$ when $\mu = \beta$ and $M_{\beta,\mu} = 0$ when $|\beta| = |\mu|$ but $\beta \neq \mu$. By symmetry the same holds for $[[\Theta_k(\cdot, f_2)]_\mu$ and hence by induction and Lemma 4.2 this verifies (1) for all $m \leq L - 1$. Given the Carleson measure assumption for $d\mu(x, t)$ in (4.5), one can easily prove (2) if the following statement holds: for all $0 \leq m \leq L - 1$

$$(4.6) \quad \sum_{|\alpha|, |\beta| \leq L-1} |[[\Theta_k^{(m)}]]_{\alpha,\beta}(x)| \leq C_0^{2(m+1)} \sum_{|\alpha|, |\beta| \leq L-1} |[[\Theta_k]]_{\alpha,\beta}(x)|,$$

where $C_0 = \sum_{|\alpha|, |\beta| \leq L-1} (1 + |M_{\alpha,\beta}|)$.

We verify (4.6) by induction. For $m = 0$, let $|\alpha|, |\beta| \leq L - 1$, and it follows that

$$[[\Theta_k^{(0)}]]_{\alpha,\beta} = [[\Theta_k]]_{\alpha,\beta} - [[\Theta_k]]_{0,\beta}[[P_k]]_{\alpha} - [[\Theta_k]]_{\alpha,0}[[P_k]]_{\beta} + [[\Theta_k]]_{0,0}[[P_k]]_{\alpha}[[P_k]]_{\beta}$$

Then

$$\begin{aligned} \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k^{(0)}]]_{\alpha,\beta}| &\leq \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k]]_{\alpha,\beta}| + |[[\Theta_k]]_{0,\beta}M_{0,\alpha}| \\ &\quad + \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k]]_{\alpha,0}M_{0,\beta}| + |[[\Theta_k]]_{0,0}M_{0,\alpha}M_{0,\beta}| \\ &\leq \left(\sum_{|\alpha|,|\beta| \leq L-1} (1 + |M_{0,\alpha}| + |M_{0,\beta}| + |M_{0,\alpha}M_{0,\beta}|) \right) \left(\sum_{|\mu|,|\nu| \leq L-1} |[[\Theta_k]]_{\alpha,\beta}| \right) \\ &\leq \left(\sum_{|\alpha|,|\beta| \leq L-1} (1 + |M_{0,\alpha}|)(1 + |M_{0,\beta}|) \right) \left(\sum_{|\mu|,|\nu| \leq L-1} |[[\Theta_k]]_{\alpha,\beta}| \right) \\ &\leq C_0^2 \sum_{|\mu|,|\nu| \leq L-1} |[[\Theta_k]]_{\alpha,\beta}|. \end{aligned}$$

Now assume that (4.6) holds for $m - 1$, and consider

$$\begin{aligned} \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k^{(m)}]]_{\alpha,\beta}| &\leq \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k^{(m-1)}]]_{\alpha,\beta}| + \sum_{|\alpha|,|\beta| \leq L-1} \sum_{|\mu|=m} |[[\Theta_k^{(m-1)}]]_{0,\beta}M_{\mu,\alpha}| \\ &\quad + \sum_{|\alpha|,|\beta| \leq L-1} \sum_{|\nu|=m} |[[\Theta_k^{(m-1)}]]_{\alpha,0}M_{\nu,\beta}| + \sum_{|\alpha|,|\beta| \leq L-1} \sum_{|\mu|=|\nu|=m} |[[\Theta_k^{(m-1)}]]_{0,0}M_{\mu,\alpha}M_{\nu,\beta}| \\ &\leq \left(\sum_{|\alpha|,|\beta| \leq L-1} \left[1 + \sum_{|\mu|=m} |M_{\mu,\alpha}| + \sum_{|\nu|=m} |M_{\nu,\beta}| + \sum_{|\mu|=|\nu|=m} |M_{\mu,\alpha}M_{\nu,\beta}| \right] \right) \\ &\quad \times \left(\sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k^{(m-1)}]]_{\alpha,\beta}| \right) \\ &\leq \left(\sum_{|\alpha|,|\beta| \leq L-1} \sum_{|\mu|,|\nu| \leq m} (1 + |M_{\mu,\alpha}|)(1 + |M_{\nu,\beta}|) \right) \left(\sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k^{(m-1)}]]_{\alpha,\beta}| \right) \\ &\leq C_0^2 \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k^{(m-1)}]]_{\alpha,\beta}| \leq C_0^{2(m+1)} \sum_{|\alpha|,|\beta| \leq L-1} |[[\Theta_k]]_{\alpha,\beta}| \end{aligned}$$

We use the inductive hypothesis in the last inequality here to bound the $[[\Theta_k^{(m-1)}]]_{\alpha,\beta}$. Then by induction, the estimate in (4.6) holds for all $0 \leq m \leq L - 1$, and this completes the proof. \square

Define the linear operators

$$(4.7) \quad \Lambda_k^{1,0,\alpha} f(x) = [[\Theta_k(f, \cdot)]]_{\alpha}(x), \quad \Lambda_k^{2,0,\alpha} f(x) = [[\Theta_k(\cdot, f)]]_{\alpha}(x),$$

$$(4.8) \quad \Lambda_k^{1,m,\alpha} f(x) = [[\Theta_k^{(m-1)}(f, \cdot)]]_{\alpha}(x), \quad \text{and} \quad \Lambda_k^{2,m,\alpha} f(x) = [[\Theta_k^{(m-1)}(\cdot, f)]]_{\alpha}(x)$$

for $1 \leq m \leq L - 1$.

Lemma 4.4. *Let $L \geq 1$ be an integer, and assume that $\{\Theta_k\} \in BLPSO(N, L)$, where $N = 2n + L(2n + L + 3)/2$. Define $\Theta_k^{(m)}$ and $\Lambda_k^{i,m,\alpha}$ for $i = 1, 2$, $0 \leq m \leq L - 1$, and $|\alpha| \leq L - 1$ as in (4.2), (4.3), (4.7), and (4.8). Then $\{\Lambda_k^{i,m,\alpha}\} \in LPSO(n + 2L, L)$ for each $i = 1, 2$, $0 \leq m \leq L - 1$, and $|\alpha| \leq L - 1$. Furthermore, if Θ_k satisfies the Carleson condition in (4.5), then*

$$(4.9) \quad d\nu(x, t) = \sum_{k \in \mathbb{Z}} \sum_{i=1}^2 \sum_{m=0}^{L-1} \sum_{|\alpha|, |\beta| \leq L-1} |[[\Lambda_k^{i,m,\alpha}]]_{\beta}(x)|^2 \delta_{t=2^{-k}} dx$$

is a Carleson measure and $S_{\Lambda^{i,m,\alpha}}$ is bounded from H^p into L^p for all $\frac{n}{n+L} < p \leq 1$, $i = 1, 2$, $m = 0, 1, \dots, L - 1$, and $|\alpha| \leq L$, where $S_{\Lambda^{i,m,\alpha}}$ is the square function operator associated to $\{\Lambda^{i,m,\alpha}\}$

$$S_{\Lambda^{i,m,\alpha}} f(x) = \left(\sum_{k \in \mathbb{Z}} |\Lambda_k^{i,m,\alpha} f(x)|^2 \right)^{\frac{1}{2}}.$$

Proof. We first look at $\Lambda_k^{1,0,\alpha} f = [[\Theta_k(f, \cdot)]]_{\alpha}$ for $i = 1, 2$ and $|\alpha| \leq L - 1$. We wish to show that $\{\Lambda_k^{1,0,\alpha}\} \in LPSO(n + 2L, L)$. The kernel of $\Lambda_k^{1,0,\alpha}$ is bounded in the following way. For $|\beta| \leq L$

$$\begin{aligned} |D_1^{\beta} \lambda_k^{1,0,\alpha}(x, y)| &\leq \int_{\mathbb{R}^n} |D_1^{\beta} \theta_k(x, y, y_2) 2^{|\alpha|k} (x - y_2)^{\alpha}| dy_2 \\ &\lesssim 2^{|\beta|k} \int_{\mathbb{R}^n} \Phi_k^N(x - y, x - y_2) (2^k |x - y_2|)^{|\alpha|} dy_2 \\ &\lesssim 2^{|\beta|k} \int_{\mathbb{R}^n} \Phi_k^{N-|\alpha|}(x - y, x - y_2) dy_2 \\ &\lesssim 2^{|\beta|k} \Phi_k^{N-n-(L-1)}(x - y) \leq 2^{|\beta|k} \Phi_k^{n+2L}(x - y). \end{aligned}$$

Here we use that our hypotheses on N imply that $N \geq 2n + 3L - 1$. Then $\{\Lambda_k^{1,0,\alpha}\} \in LPSO(n + 2L, L)$ for all $|\alpha| \leq L - 1$, and by symmetry it follows that $\{\Lambda_k^{2,0,\alpha}\} \in LPSO(n + 2L, L)$ for $|\alpha| \leq L - 1$ as well. If $L = 1$, this completes the estimates.

Now we continue to estimate these operators for $1 \leq m \leq L - 1$, where $L \geq 2$. Fix $0 \leq m \leq L - 1$ and $|\alpha| \leq L - 1$. The kernel $\lambda_k^{1,m,\alpha} f(x)$ is given by

$$\lambda_k^{1,m,\alpha}(x, y) = \int_{\mathbb{R}^n} \theta_k^{(m-1)}(x, y, y_2) (x - y_2)^{\alpha} dy_2.$$

By Proposition 4.3, we know that $\{\Theta_k^{(m-1)}\} \in BLPSO(\tilde{N}_{m-1}, L)$ where $\tilde{N}_{m-1} = N - m(2n + m - 1)/2$, and we now show that $\{\Lambda_k^{1,m,\alpha}\} \in LPSO(n + 2L, L)$. We check the kernel conditions for $\lambda_k^{1,m,\alpha}$ hold. For $1 \leq m \leq L - 1$, $|\alpha| \leq L - 1$, and $|\beta| \leq L$

$$\begin{aligned} |D_1^{\beta} \lambda_k^{1,m,\alpha}(x, y)| &\leq \int_{\mathbb{R}^n} |D_1^{\beta} \theta_k^{(m-1)}(x, y, y_2) 2^{|\alpha|k} (x - y_2)^{\alpha}| dy_2 \\ &\leq 2^{|\beta|k} \int_{\mathbb{R}^n} \Phi_k^{\tilde{N}_{m-1}}(x - y, x - y_2) 2^{|\alpha|k} |x - y_2|^{\alpha} dy_2 \\ &\leq 2^{|\beta|k} \int_{\mathbb{R}^n} \Phi_k^{\tilde{N}_{m-1}-|\alpha|}(x - y, x - y_2) dy_2 \\ &\leq 2^{|\beta|k} \Phi_k^{\tilde{N}_{m-1}-n-(L-1)}(x - y) \leq 2^{|\beta|k} \Phi_k^{n+2L}(x - y). \end{aligned}$$

The last inequality is given by our assumption that $N \geq 2n + L(2n + L + 3)/2$ since for $0 \leq m \leq L - 1$, we have

$$\begin{aligned} \tilde{N}_{m-1} - n - (L - 1) &\geq \tilde{N}_{L-2} - n - (L - 1) = \tilde{N}_{L-1} = N - L(2n + L - 1)/2 \\ &\geq 2n + L(2n + L + 3)/2 - L(2n + L - 1)/2 = 2n + 2L \geq n + 2L. \end{aligned}$$

Also, by (2) in Proposition 4.3 it follows that $dv(x, t)$ defined in (4.9) is a Carleson measure; this is because

$$\begin{aligned} [[\Lambda_k^{1,0,\alpha}]]_\beta &= [[\Theta_k]]_{\beta,\alpha}, & [[\Lambda_k^{2,0,\alpha}]]_\beta &= [[\Theta_k]]_{\alpha,\beta}, \\ [[\Lambda_k^{1,m,\alpha}]]_\beta &= [[\Theta_k^{(m-1)}]]_{\beta,\alpha}, & \text{and} & \quad [[\Lambda_k^{2,m,\alpha}]]_\beta = [[\Theta_k^{(m-1)}]]_{\alpha,\beta} \quad \text{for } 0 < m \leq L - 1. \end{aligned}$$

By Theorem 2.1 from [35], it follows that $S_{\Lambda^i, m, \alpha}$ is bounded from H^p into L^p for all $\frac{n}{n+L} < p \leq 1$. \square

Now we use Lemma 4.1 and the paraproduct operators $\Theta_k^{(m)}$ along with Propositions 3.3 and 4.3 to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Using the definitions in (4.2), (4.3), (4.7), and (4.8), we can rewrite $\Theta_k^{(m)}$ in terms of $\Lambda_k^{i,m,\alpha}$ in the following way;

$$\begin{aligned} \Theta_k^{(0)}(f_1, f_2)(x) &= \Theta_k(f_1, f_2)(x) - \Lambda_k^{2,0,0} f_2(x) \cdot P_k f_1(x) - \Lambda_k^{1,0,0} f_1(x) \cdot P_k f_2(x) \\ &\quad + [[\Theta_k]]_{0,0}(x) \cdot P_k f_1(x) P_k f_2(x) \\ &= \Theta_k^{(m-1)}(f_1, f_2)(x) - \sum_{|\alpha|=m} \frac{(-1)^{|\alpha|}}{\alpha!} \Lambda_k^{2,m,\alpha} f_2(x) \cdot 2^{-k|\alpha|} P_k D^\alpha f_1(x) \\ &\quad - \sum_{|\beta|=m} \frac{(-1)^{|\beta|}}{\beta!} \Lambda_k^{1,m,\beta} f_1(x) \cdot 2^{-k|\beta|} P_k D^\beta f_2(x) \\ &\quad + \sum_{|\alpha|=|\beta|=m} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha! \beta!} [[\Theta_k^{(m-1)}]]_{\alpha,\beta}(x) \cdot 2^{-k(|\alpha|+|\beta|)} P_k D^\alpha f_1(x) P_k D^\beta f_2(x). \end{aligned}$$

Then Θ_k is bounded in the following way;

$$\begin{aligned} |\Theta_k(f_1, f_2)| &\leq |\Theta_k^{(0)}(f_1, f_2)| + |\Lambda_k^{1,0,0} f_1 P_k f_2| + |\Lambda_k^{2,0,0} f_2 P_k f_1| + |[[\Theta_k]]_{0,0} P_k f_1 P_k f_2| \\ &\leq |\Theta_k^{(1)}(f_1, f_2)| + |\Lambda_k^{1,0,0} f_1 P_k f_2| + |P_k f_1 \Lambda_k^{2,0,0} f_2| + |[[\Theta_k]]_{0,0} P_k f_1 P_k f_2| \\ &\quad + \sum_{|\alpha|=1} |2^{-k|\alpha|} P_k D^\alpha f_1 \Lambda_k^{2,1,\alpha} f_2| + \sum_{|\beta|=1} |\Lambda_k^{1,1,\beta} f_1 2^{-k|\beta|} P_k D^\beta f_2| \\ &\quad + \sum_{|\alpha|=|\beta|=1} |[[\Theta_k^{(0)}]]_{\alpha,\beta} 2^{-k|\alpha|} P_k D^\alpha f_1 2^{-k|\beta|} P_k D^\beta f_2| \end{aligned}$$

$$\begin{aligned}
 (4.10) \quad &\leq |\Theta_k^{(L-1)}(f_1, f_2)| + \sum_{m=0}^{L-1} \sum_{|\alpha|=m} |2^{-k|\alpha|} P_k D^\alpha f_1 \Lambda_k^{2,m,\alpha} f_2| \\
 &+ \sum_{m=0}^{L-1} \sum_{|\beta|=m} |\Lambda_k^{1,m,\beta} f_1 2^{-k|\beta|} P_k D^\beta f_2| \\
 &+ \sum_{m=0}^{L-1} \sum_{|\alpha|=|\beta|=m} |[[\Theta_k^{(m)}]]_{\alpha,\beta} 2^{-k|\alpha|} P_k D^\alpha f_1 2^{-k|\beta|} P_k D^\beta f_2| + |[[\Theta_k]]_{0,0} P_k f_1 P_k f_2|.
 \end{aligned}$$

We verify that the square functions associated to each of the terms on the right hand side of (4.10) is bounded by a constant times $\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$.

The $\Theta_k^{(L-1)}(f_1, f_2)$ term: First we note that since $\{\Theta_k\} \in BLPSO(N, L)$ for $N = 2n + L(2n + L + 5)/2$, by Proposition 4.3 it follows that $\{\Theta_k^{(L-1)}\} \in BLPSO(\tilde{N}_{L-1}, L)$ where

$$\tilde{N}_{L-1} = N - L(2n + L - 1)/2 = 2n + L(2n + L + 5)/2 - L(2n + L - 1)/2 = 2n + 3L.$$

This inequality for \tilde{N}_{L-1} is the binding restriction for the requirement $N = 2n + L(2n + L + 5)/2$. The other terms require N to be large, but none as large as the requirement for $\{\Theta_k^{(L-1)}\}$ to be a member of $BLPSO(2n + 3L, L)$. Also by property (1) in Proposition 4.3, we have that

$$\int_{\mathbb{R}^n} \Theta_k^{(L-1)}(x, y_1, y_2) y_1^\alpha dy_1 = \int_{\mathbb{R}^n} \Theta_k^{(L-1)}(x, y_1, y_2) y_2^\alpha dy_2 = 0$$

for all $|\alpha| \leq L - 1$. By Lemma 4.1, it follows that

$$\|S_{\Theta^{(L-1)}}(f_1, f_2)\|_{L^p} = \left\| \left(\sum_{k \in \mathbb{Z}} |\Theta_k^{(L-1)}(f_1, f_2)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$$

since $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$.

The $2^{-k|\alpha|} P_k D^\alpha f_1 \Lambda_k^{2,m,\alpha} f_2$ terms: By Lemma 4.4, we know that $S_{\Lambda^{i,m,\alpha}}$ is bounded from H^q into L^q for any $\frac{n}{n+L} < q \leq 1$ and each $0 \leq m \leq L - 1$, $i = 1, 2$, and $|\alpha| \leq L - 1$. It is also not hard to see that $2^{-k|\alpha|} P_k D^\alpha f(x) = (-1)^{|\alpha|} (D^\alpha \varphi)_k * f(x)$, where $D^\alpha \varphi \in \mathcal{S}$ since $\varphi \in \mathcal{S}$. Therefore

$$\begin{aligned}
 \left\| \left(\sum_{k \in \mathbb{Z}} |2^{-k|\alpha|} P_k D^\alpha f_1 \Lambda_k^{2,m,\alpha} f_2|^2 \right) \right\|_{L^p} &\leq \left\| \sup_{k \in \mathbb{Z}} |(D^\alpha \varphi)_k * f_1| \right\|_{L^{p_1}} \left\| \left(\sum_{k \in \mathbb{Z}} |\Lambda_k^{2,m,\alpha} f_2|^2 \right) \right\|_{L^{p_1}} \\
 &\lesssim \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}.
 \end{aligned}$$

Note that here we use that $\frac{n}{n+L} < p_1 \leq 1$. Then the $\Lambda_k^{1,m,\beta} f_1 2^{-k|\beta|} P_k D^\beta f_2$ terms are bounded appropriately by a similar argument and the assumption $\frac{n}{n+L} < p_2 \leq 1$.

The $[[\Theta_k^{(m)}]]_{\alpha,\beta} 2^{-k|\alpha|} P_k D^\alpha f_1 2^{-k|\beta|} P_k D^\beta f_2$ terms: For these terms we use the Carleson measure property (2) from Proposition 4.3. Also using Proposition 3.3, it follows that

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} |[[\Theta_k^{(m)}]]_{\alpha,\beta} 2^{-k|\alpha|} P_k D^\alpha f_1 2^{-k|\beta|} P_k D^\beta f_2|^2 \right) \right\|_{L^p} \\ & \leq \left\| \left(\sum_{k \in \mathbb{Z}} |[[\Theta_k^{(m)}]]_{\alpha,\beta} (D^\alpha \varphi)_k * f_1|^2 \right) \right\|_{L^{p_1}} \left\| \sup_{k \in \mathbb{Z}} |(D^\beta \varphi)_k * f_2| \right\|_{L^{p_2}} \lesssim \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \end{aligned}$$

for $0 \leq m \leq L-1$. In fact, this estimate holds for p, p_1, p_2 that range all the way down to zero, but the bounds for the other terms require the assumed lower bounds for p, p_1, p_2 .

The $[[\Theta_k]]_{0,0} P_k f_1 P_k f_2$ term: This term is bounded in the same way as the ones from the previous case. By Proposition 3.3 and the assumed Carleson measure properties for $[[\Theta_k]]_{0,0}$ it follows that

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} |[[\Theta_k]]_{0,0} P_k f_1 P_k f_2|^2 \right) \right\|_{L^p} \leq \left\| \left(\sum_{k \in \mathbb{Z}} |[[\Theta_k]]_{0,0} P_k f_1|^2 \right) \right\|_{L^{p_1}} \left\| \sup_{k \in \mathbb{Z}} |P_k f_2| \right\|_{L^{p_2}} \\ & \lesssim \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}. \end{aligned}$$

Therefore the H^p norm of the square function associated to the right hand side of (4.10) is bounded by a constant times $\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$ as desired, and hence S_Θ is bounded from $H^{p_1} \times H^{p_2}$ into L^p . \square

5. HARDY SPACE BOUNDS FOR SINGULAR INTEGRAL OPERATORS

As described above, we will apply Theorem 2.1 to prove Theorem 2.2. In order to do so, we prove the decomposition result in Theorem 2.3. First we prove a quick lemma that show that if (2.3) holds for $\psi \in \mathcal{D}_{2L+n}$ as stated, then we can replace $\psi \in \mathcal{D}_{2L+n}$ by $\varphi \in C_0^\infty$ in (2.3).

Lemma 5.1. *Assume that $T \in BCZO(M)$ for an integer $M = L(2n + L + 5)/2$ for some integer $L \geq 1$, and that T is bounded from $L^2 \times L^2$ into L^1 . If $T^{*1}(x^\alpha, \psi) = 0$ in $\mathcal{D}'_{|\alpha|}$ for all $\psi \in \mathcal{D}_{|\alpha|}$, then $T^{*1}(x^\alpha, \varphi) = 0$ for all $\varphi \in C_0^\infty$. Likewise for $T^{*2}(\psi, x^\alpha)$ and $T^{*2}(\varphi, x^\alpha)$.*

Proof. Let $\varphi \in C_0^\infty$ and $\psi \in \mathcal{D}_{2L+n}$ such that

$$f = \sum_{k \in \mathbb{Z}} \psi_k * \varphi$$

in L^2 for $f \in L^2$. Then for $|\alpha| \leq 2L + n$ and $\phi \in \mathcal{D}_{|\alpha|}$, we have

$$\begin{aligned} \langle T^{*1}(x^\alpha, \varphi), \phi \rangle &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^n} T(\phi, \varphi)(x) x^\alpha \eta_R(x) dx \\ &= \lim_{R \rightarrow \infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} T(\phi, \psi_k * \varphi)(x) x^\alpha \eta_R(x) dx \\ &= \sum_{k \in \mathbb{Z}} \lim_{R \rightarrow \infty} \int_{\mathbb{R}^n} T(\phi, \psi_k * \varphi)(x) x^\alpha \eta_R(x) dx = 0. \end{aligned}$$

This holds if the series above is absolutely convergent. Using the $L^2 \times L^2$ to L^1 bound and kernel representation for T , it is not hard to show that there exists a constant $C_{\varphi, T} < \infty$ (independent of

$k \in \mathbb{Z}$ and R for R sufficiently large) such that

$$\left| \int_{\mathbb{R}^n} T(\phi, \psi_k * \phi)(x) x^\alpha \eta_R(x) dx \right| \leq C_{\phi, T} \|\psi_k * \phi\|_{L^2}.$$

Since $\psi \in \mathcal{D}_{2L+n} \subset \mathcal{D}_n$, it follows that $\|\psi_k * \phi\|_{L^2} \lesssim \min(2^{kn/2}, 2^{-kn/2})$. This estimate is easy when $k \leq 0$ since $\|\psi_k * \phi\|_{L^2} \leq \|\psi_k\|_{L^2} \|\phi\|_{L^1} \lesssim 2^{kn/2}$. When $k > 0$, we use $\psi \in \mathcal{D}_n$ to conclude that $|\psi_k * \phi(x)| \lesssim 2^{-nk} (\Phi_k^{n+1}(x) + \Phi_0^{n+1}(x))$. Hence for $k > 0$, we have

$$\|\psi_k * \phi\|_{L^2} \lesssim 2^{-nk} (\|\Phi_k^{n+1}\|_{L^2} + \|\Phi_0^{n+1}\|_{L^2}) \lesssim 2^{-nk} (2^{nk/2} + 1) \lesssim 2^{-nk/2}.$$

Then the above sum is absolutely convergent, and the proof of the lemma is complete. \square

Proof of Theorem 2.3. Assume that T and ψ are as in the statement of Theorem 2.3. Without loss of generality we assume that $\text{supp}(\psi) \subset B(0, 1)$. To compute the kernel for $\Theta_k f = Q_k T$, we consider the following decomposition: Let $\phi \in C_0^\infty(B(0, 1))$ with integral 1 such that

$$\int_{\mathbb{R}^n} \phi(x) x^\alpha dx = 0$$

for $0 < |\alpha| \leq M$. Define $\tilde{\psi}_k(x) = \phi_{k+1}(x) - \phi_k(x)$, and it follows that $\tilde{\psi} \in \mathcal{D}_M$. Then

$$\begin{aligned} \Theta_k(f_1, f_2)(x) &= \lim_{R \rightarrow \infty} \sum_{\ell_1=k}^{R-1} \left\langle T(\tilde{Q}_{\ell_1} f_1, P_R f_2), \Psi_k^x \right\rangle + \left\langle T(P_k f_1, P_R f_2), \Psi_k^x \right\rangle \\ &= \lim_{R \rightarrow \infty} \sum_{\ell_1=k}^{R-1} \sum_{\ell_2=k}^{R-1} \left\langle T(\tilde{Q}_{\ell_1} f_1, \tilde{Q}_{\ell_2} f_2), \Psi_k^x \right\rangle + \sum_{\ell_1=k}^{R-1} \left\langle T(\tilde{Q}_{\ell_1} f_1, P_k f_2), \Psi_k^x \right\rangle \\ &\quad + \sum_{\ell_2=k}^{R-1} \left\langle T(P_k f_1, \tilde{Q}_{\ell_2} f_2), \Psi_k^x \right\rangle + \left\langle T(P_k f_1, P_k f_2), \Psi_k^x \right\rangle \\ (5.1) \quad &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^{2n}} \left(\sum_{\ell_1=k}^{R-1} \sum_{\ell_2=k}^{R-1} I_{\ell_1, \ell_2, k}(x, y_1, y_2) + \sum_{\ell_1=k}^{R-1} II_{\ell_1, k}(x, y_1, y_2) \right. \\ &\quad \left. + \sum_{\ell_2=k}^{R-1} III_{\ell_2, k}(x, y_1, y_2) + IV_k(x, y_1, y_2) \right) f_1(y_1) f_2(y_2) dy_1 dy_2 \end{aligned}$$

where

$$\begin{aligned} I_{\ell_1, \ell_2, k}(x, y_1, y_2) &= \left\langle T(\tilde{\psi}_{\ell_1}^{y_1}, \tilde{\psi}_{\ell_2}^{y_2}), \Psi_k^x \right\rangle, & II_{\ell_2, k}(x, y_1, y_2) &= \left\langle T(\phi_k^{y_1}, \tilde{\psi}_{\ell_2}^{y_2}), \Psi_k^x \right\rangle, \\ III_{\ell_1, k}(x, y_1, y_2) &= \left\langle T(\tilde{\psi}_{\ell_1}^{y_1}, \phi_k^{y_2}), \Psi_k^x \right\rangle, & \text{and} \quad IV_k(x, y_1, y_2) &= \left\langle T(\phi_k^{y_1}, \phi_k^{y_2}), \Psi_k^x \right\rangle. \end{aligned}$$

To show that Θ_k satisfies (2.1), we first assume that $|x - y_1| + |x - y_2| > 2^{2-k}$. Without loss of generality, we assume that $\ell_1 \geq \ell_2$. For $|\alpha|, |\beta| \leq L$ we have

$$\begin{aligned}
\left| D_1^\alpha D_2^\beta I_{\ell_1, \ell_2, k}(x, y_1, y_2) \right| &= 2^{|\alpha|\ell_1 + |\beta|\ell_2} \left| \left\langle T((D_1^\alpha \tilde{\psi})_{\ell_1}^{y_1}, (D_2^\beta \tilde{\psi})_{\ell_2}^{y_2}), \Psi_k^x \right\rangle \right| \\
&= 2^{|\alpha|\ell_1 + |\beta|\ell_2} \left| \int_{\mathbb{R}^{3n}} \left(K(u, v_1, v_2) - \sum_{|\mu| \leq M-1} \frac{D_1^\mu K(u, y_1, v_2)}{\mu!} (v_1 - y_1)^\mu \right) \right. \\
&\quad \left. \times (D^\alpha \tilde{\psi})_{\ell_1}^{y_1}(v_1) (D^\beta \tilde{\psi})_{\ell_2}^{y_2}(v_2) \Psi_k^x(u) du dv_1 dv_2 \right| \\
&\lesssim 2^{|\alpha|\ell_1 + |\beta|\ell_2} \int_{\mathbb{R}^{3n}} \frac{|v_1 - y_1|^M}{(|u - v_1| + |u - v_2|)^{2n+M}} |(D^\alpha \tilde{\psi})_{\ell_1}^{y_1}(v_1) (D^\beta \tilde{\psi})_{\ell_2}^{y_2}(v_2) \Psi_k^x(u)| du dv_1 dv_2 \\
&\lesssim 2^{-\ell_1(M-|\alpha|-|\beta|)} \frac{2^{(2n+M)k}}{(1 + 2^k|x - y_1| + 2^k|x - y_2|)^{2n+M}} \\
&\lesssim 2^{(k - \max(\ell_1, \ell_2))(M-|\alpha|-|\beta|)} 2^{(|\alpha|+|\beta|)k} \Phi_k^{2n+M}(x - y_1, x - y_2).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{\ell_1=k}^{\infty} \sum_{\ell_2=k}^{\infty} \left| D_1^\alpha D_2^\beta I_{\ell_1, \ell_2, k}(x, y_1, y_2) \right| \\
\lesssim 2^{(|\alpha|+|\beta|)k} \Phi_k^{2n+M}(x - y_1, x - y_2) \sum_{\ell_1=k}^{\infty} \sum_{\ell_2=k}^{\infty} 2^{(k-\ell_1)\frac{M-|\alpha|-|\beta|}{2}} 2^{(k-\ell_1)\frac{M-|\alpha|-|\beta|}{2}} \\
\lesssim 2^{(|\alpha|+|\beta|)k} \Phi_k^N(x - y_1, x - y_2).
\end{aligned}$$

The second term satisfies a similar estimate

$$\begin{aligned}
\left| D_1^\alpha D_2^\beta II_{\ell_1, k}(x, y_1, y_2) \right| &= 2^{|\alpha|\ell_1} 2^{|\beta|k} \left| \int_{\mathbb{R}^{3n}} K(u, v_1, v_2) (D^\alpha \tilde{\psi})_{\ell_1}^{y_1}(v_1) (D^\beta \varphi)_k^{y_2}(v_2) \Psi_k^x(u) du dv_1 dv_2 \right| \\
&= 2^{|\alpha|\ell_1} 2^{|\beta|k} \left| \int_{\mathbb{R}^{3n}} \left(K(u, v_1, v_2) - \sum_{|\mu| \leq M-1} \frac{D_1^\mu K(u, y_1, v_2)}{\mu!} (v_1 - y_1)^\mu \right) \right. \\
&\quad \left. \times (D^\alpha \tilde{\psi})_{\ell_1}^{y_1}(v_1) (D^\beta \varphi)_k^{y_2}(v_2) \Psi_k^x(u) du dv_1 dv_2 \right| \\
&\lesssim 2^{|\alpha|\ell_1} 2^{|\beta|k} \int_{\mathbb{R}^{3n}} \frac{|v_1 - y_1|^M}{(|u - v_1| + |u - v_2|)^{2n+M}} |(D^\alpha \tilde{\psi})_{\ell_1}^{y_1}(v_1) (D^\beta \varphi)_k^{y_2}(v_2) \Psi_k^x(u)| du dv_1 dv_2 \\
&\lesssim 2^{(|\alpha|+|\beta|)k} 2^{(k-\ell_1)(M-|\alpha|)} \Phi_k^{2n+M}(x - y_1, x - y_2).
\end{aligned}$$

Again summing over $\ell_1 \geq k$, we obtain

$$\begin{aligned}
\sum_{\ell_1=k}^{\infty} \left| D_1^\alpha D_2^\beta II_{\ell_1, k}(x, y_1, y_2) \right| &\lesssim 2^{(|\alpha|+|\beta|)k} \Phi_k^{2n+M}(x - y_1, x - y_2) \sum_{\ell_1=k}^{\infty} 2^{(k-\ell_1)(M-|\alpha|)} \\
&\lesssim 2^{(|\alpha|+|\beta|)k} \Phi_k^N(x - y_1, x - y_2).
\end{aligned}$$

The third term $III_{\ell_2, k}(x, y_1, y_2)$ is also bounded by a constant times $2^{(|\alpha|+|\beta|)k} \Phi_k^N(x - y_1, x - y_2)$ by symmetry. We estimate the last term.

$$\begin{aligned}
 |D_1^\alpha D_2^\beta IV_k(x, y_1, y_2)| &= 2^{(|\alpha|+|\beta|)k} \left| \left\langle T((D^\alpha \varphi)_k^{y_1}, (D^\beta \varphi)_k^{y_2}), \Psi_k^x \right\rangle \right| \\
 &= 2^{(|\alpha|+|\beta|)k} \left| \int_{\mathbb{R}^{3n}} \left(K(u, v_1, v_2) - \sum_{|\mu| \leq M-1} \frac{D_0^\mu K(x, v_1, v_2)}{\mu!} (u-x)^\mu \right) \right. \\
 &\quad \left. \times (D^\alpha \varphi)_k^{y_1}(v_1) (D^\beta \varphi)_k^{y_2}(v_2) \Psi_k^x(u) du dv_1 dv_2 \right| \\
 &\lesssim 2^{(|\alpha|+|\beta|)k} \int_{\mathbb{R}^{3n}} \frac{|u-x|^M}{(|x-v_1|+|x-v_2|)^{2n+M}} |(D^\alpha \varphi)_k^{y_1}(v_1) (D^\beta \varphi)_k^{y_2}(v_2) \Psi_k^x(u)| du dv_1 dv_2 \\
 &\lesssim 2^{(|\alpha|+|\beta|)k} \frac{2^{-Mk}}{(2^{-k} + |x-y_1| + |x-y_2|)^{2n+M}} = 2^{(|\alpha|+|\beta|)k} \Phi_k^N(x-y_1, x-y_2).
 \end{aligned}$$

So we that that $\theta_k(x, y_1, y_2)$ as given in (5.1) satisfies (2.1) for $|x-y_1| + |x-y_2| \geq 2^{2-k}$. Now we assume that $|x-y_1| + |x-y_2| \leq 2^{2-k}$. In this situation $2^{2kn} \lesssim \Phi_k^N(x-y_1, x-y_2)$, so it is sufficient to prove (2.1) with 2^{2kn} in place of $\Phi_k^N(x-y_1, x-y_2)$. We again use the decomposition in (5.1). To bound $I_{\ell_1, \ell_2, k}$, we assume without loss of generality that $\ell_1 \geq \ell_2$. Since we have assumed that $\langle T(\tilde{\Psi}_{\ell_1}^{y_1}, \tilde{\Psi}_{\ell_2}^{y_2}), x^\mu \rangle = 0$ for $|\mu| \leq 2L+n$, we can write

$$\begin{aligned}
 |D_1^\alpha D_2^\beta \langle T(\tilde{\Psi}_{\ell_1}^{y_1}, \tilde{\Psi}_{\ell_2}^{y_2}), \Psi_k^x \rangle| &\leq |A_{\ell_1, \ell_2, k}(x, y_1, y_2)| + |B_{\ell_1, \ell_2, k}(x, y_1, y_2)|, \text{ where} \\
 A_{\ell_1, \ell_2, k}(x, y_1, y_2) &= 2^{\ell_1|\alpha|+\ell_2|\beta|} \int_{|u-y_1| \leq 2^{1-\ell_1}} T((D^\alpha \tilde{\Psi})_{\ell_1}^{y_1}, (D^\beta \tilde{\Psi})_{\ell_2}^{y_2})(u) \\
 &\quad \times \left(\Psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (u-y_1)^\alpha \right) du, \\
 B_{\ell_1, \ell_2, k}(x, y_1, y_2) &= 2^{\ell_1|\alpha|+\ell_2|\beta|} \int_{|u-y_1| > 2^{1-\ell_1}} T((D^\alpha \tilde{\Psi})_{\ell_1}^{y_1}, (D^\beta \tilde{\Psi})_{\ell_2}^{y_2})(u) \\
 &\quad \times \left(\Psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (u-y_1)^\alpha \right) du.
 \end{aligned}$$

The $A_{\ell_1, \ell_2, k}$ term is bounded as follows,

$$\begin{aligned}
 |A_{\ell_1, \ell_2, k}(x, y_1, y_2)| &\leq 2^{\ell_1|\alpha|+\ell_2|\beta|} \|T((D^\alpha \tilde{\Psi})_{\ell_1}^{y_1}, (D^\beta \tilde{\Psi})_{\ell_2}^{y_2}) \cdot \chi_{B(y_1, 2^{1-\ell_1})}\|_{L^1} \\
 &\quad \times \left\| \left(\Psi_k^x - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (\cdot - y_1)^\alpha \right) \cdot \chi_{B(y_1, 2^{1-\ell_1})} \right\|_{L^\infty} \\
 &\lesssim 2^{\ell_1|\alpha|+\ell_2|\beta|} 2^{-\ell_1 n/2} \|T((D^\alpha \tilde{\Psi})_{\ell_1}^{y_1}, (D^\beta \tilde{\Psi})_{\ell_2}^{y_2})\|_{L^2} 2^{(n+2L+1)(k-\ell_1)} 2^{kn} \\
 &\lesssim 2^{\ell_1|\alpha|+\ell_2|\beta|} 2^{-\ell_1 n/2} \|(D^\alpha \tilde{\Psi})_{\ell_1}^{y_1}\|_{L^2} \|(D^\beta \tilde{\Psi})_{\ell_2}^{y_2}\|_{L^\infty} 2^{(n+2L+1)(k-\ell_1)} 2^{kn} \\
 &\lesssim 2^{\ell_1|\alpha|+\ell_2|\beta|} 2^{(n+2L+1)(k-\ell_1)} 2^{kn} 2^{\ell_2 n} \lesssim 2^{k(|\alpha|+|\beta|)} 2^{k-\max(\ell_1, \ell_2)}.
 \end{aligned}$$

Let $0 < \delta < 1$. The $B_{\ell_1, \ell_2, k}$ term is bounded using the kernel representation of T , which is viable since $|u - y_1| > 2^{1-\ell_1}$ implies $u \notin \text{supp}(\tilde{\Psi}_{\ell_1}^{y_1})$. Since $\tilde{\Psi}_{\ell_1} \in \mathcal{D}_{2L}$, we have

$$\begin{aligned}
|B_{\ell_1, \ell_2, k}(x, y_1, y_2)| &= 2^{\ell_1|\alpha| + \ell_2|\beta|} \left| \int_{|u-y_1| > 2^{1-\ell_1}} \int_{\mathbb{R}^{2n}} K(u, v_1, v_2) (D^\alpha \tilde{\Psi}_{\ell_1}^{y_1})^{y_1}(v_1) (D^\beta \tilde{\Psi}_{\ell_2}^{y_2})^{y_2}(v_2) dv_1 v_2 \right. \\
&\quad \times \left. \left(\Psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (u - y_1)^\alpha \right) du \right. \\
&= 2^{\ell_1|\alpha| + \ell_2|\beta|} \left| \int_{|u-y_1| > 2^{1-\ell_1}} \int_{\mathbb{R}^{2n}} \left(K(u, v_1, v_2) - \sum_{|\mu| \leq 2L} \frac{K(u, y_1, v_2)}{\mu!} (v_1 - y_1)^\mu \right) \right. \\
&\quad \times (D^\alpha \tilde{\Psi}_{\ell_1}^{y_1})^{y_1}(v_1) (D^\beta \tilde{\Psi}_{\ell_2}^{y_2})^{y_2}(v_2) dv_1 v_2 \\
&\quad \times \left. \left(\Psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (u - y_1)^\alpha \right) du \right. \\
&\lesssim 2^{\ell_1|\alpha| + \ell_2|\beta|} \sum_{m=1}^{\infty} \int_{2^{m-\ell_1} < |u-y_1| \leq 2^{m+1-\ell_1}} \int_{\mathbb{R}^{2n}} \frac{|v_1 - y_1|^{2L+1}}{(|u - y_1| + |u - v_2|)^{2n+2L+1}} \\
&\quad \times |(D^\alpha \tilde{\Psi}_{\ell_1}^{y_1})^{y_1}(v_1) (D^\beta \tilde{\Psi}_{\ell_2}^{y_2})^{y_2}(v_2)| \left| \Psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (u - y_1)^\alpha \right| dudv_1 dv_2 \\
&\lesssim 2^{\ell_1|\alpha| + \ell_2|\beta|} \sum_{m=1}^{\infty} 2^{n(m-\ell_1)} \frac{2^{-(2L+1)\ell_1}}{2^{(2n+2L+1)(m-\ell_1)}} 2^{kn} 2^{(n+2L+\delta)(k+m-\ell_1)} \\
&\lesssim 2^{\ell_1|\alpha| + \ell_2|\beta|} 2^{2L(k-\ell_1)} 2^{\delta(k-\ell_1)} 2^{2kn} \sum_{m=1}^{\infty} 2^{-(1-\delta)m} \\
&\lesssim 2^{k(|\alpha| + |\beta|)} 2^{\delta(k-\ell_1)} 2^{2kn}
\end{aligned}$$

Recall that by symmetry, we assume that $\ell_1 \geq \ell_2$, so $2^{\delta(k-\ell_1)} = 2^{\delta(k-\max(\ell_1, \ell_2))}$. It follows that

$$\sum_{\ell_1=k+1}^{R-1} \sum_{\ell_2=k+1}^{R-1} \left| D_1^\alpha D_2^\beta I_{\ell_1, \ell_2, k}(x, y_1, y_2) \right| \lesssim 2^{k(|\alpha| + |\beta|)} 2^{2kn} \sum_{\ell_1, \ell_2 > k} 2^{\delta(k-\max(\ell_1, \ell_2))} \lesssim 2^{k(|\alpha| + |\beta|)} 2^{2kn}$$

for all $|\alpha|, |\beta| \leq L$. The estimate for $II_{\ell_1, \ell_2, k}$ is obtained by essentially the same argument. That is, we estimate $II_{\ell_2, k}$ in the following way using the hypothesis $\langle T(\tilde{\Psi}_{\ell_1}^{y_1}, \tilde{\Psi}_{\ell_2}^{y_2}), x^\mu \rangle = 0$ for $|\mu| \leq 2L+n$,

$$\left| D_1^\alpha D_2^\beta \langle T(\Phi_k^{y_1}, \tilde{\Psi}_{\ell_2}^{y_2}), \Psi_k^x \rangle \right| \leq |A_{\ell_2, k}(x, y_1, y_2)| + |B_{\ell_2, k}(x, y_1, y_2)|,$$

where

$$\begin{aligned}
A_{\ell_2, k}(x, y_1, y_2) &= 2^{k|\alpha| + \ell_2|\beta|} \int_{|u-y_1| \leq 2^{1-\ell_2}} T((D^\alpha \Phi_k^{y_1}), (D^\beta \tilde{\Psi}_{\ell_2}^{y_2})(u)) \\
&\quad \times \left(\Psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \Psi_k^x(y_1)}{\alpha!} (u - y_1)^\alpha \right) du
\end{aligned}$$

and

$$B_{\ell_2, k}(x, y_1, y_2) = 2^{k|\alpha| + \ell_2|\beta|} \int_{|u-y_1| > 2^{1-\ell_2}} T((D^\alpha \varphi)_k^{y_1}, (D^\beta \tilde{\psi})_{\ell_2}^{y_2})(u) \\ \times \left(\psi_k^x(u) - \sum_{|\alpha| \leq 2L+n} \frac{D^\alpha \psi_k^x(y_1)}{\alpha!} (u-y_1)^\alpha \right) du.$$

By the argument above, we obtain the estimate

$$\sum_{\ell_2=k+1}^{R-1} \left| D_1^\alpha D_2^\beta III_{\ell_2, k}(x, y_1, y_2) \right| \lesssim 2^{k(|\alpha|+|\beta|)} 2^{2kn} \sum_{\ell_2 > k} 2^{\delta(k-\ell_2)} \lesssim 2^{k(|\alpha|+|\beta|)} 2^{2kn}.$$

A similar argument can be applied to $III_{\ell_1, k}$ as well. The estimate for IV_k trivially follows from the $L^2 \times L^2$ to L^1 boundedness of T ,

$$|D_1^\alpha D_2^\beta \langle T(\varphi_k^{y_1}, \varphi_k^{y_2}), \psi_k^x \rangle| = 2^{k(|\alpha|+|\beta|)} \left| \langle T((D_1^\alpha \varphi)_k^{y_1}, (D_2^\beta \varphi)_k^{y_2}), \psi_k^x \rangle \right| \lesssim 2^{k(|\alpha|+|\beta|)} 2^{2nk}.$$

Therefore all of the sums on the right hand side of (5.1) converge absolutely, and hence

$$\Theta_k(x, y_1, y_2) = \sum_{\ell_1=k}^{\infty} \sum_{\ell_2=k}^{\infty} I_{\ell_1, \ell_2, k}(x, y_1, y_2) + \sum_{\ell_2=k}^{\infty} II_{\ell_2, k}(x, y_1, y_2) + \sum_{\ell_1=k}^{\infty} III_{\ell_1, k}(x, y_1, y_2) + IV_k(x, y_1, y_2)$$

satisfies (2.1). Hence $\{\Theta_k\} = \{Q_k T\} \in BLPSO(N, L)$. \square

The proof of Theorem 2.2 follows immediately from Theorems 2.1 and 2.3.

6. APPLICATIONS TO PARAPRODUCT

6.1. Bilinear Bony Paraproducts. Let $b \in BMO$, and recall the bilinear Bony paraproduct

$$\Pi_b(f_1, f_2)(x) = \sum_{j \in \mathbb{Z}} Q_j (Q_j b \cdot P_j f_1 \cdot P_j f_2)(x)$$

Now we prove Theorem 2.4

Proof. It is easily shown that $\Pi_b \in BCZO(M)$ for all $M \in \mathbb{N}$. So it is sufficient to show conditions (2.3) and (2.4) from Theorem 2.2. We first show (2.3). For $f_1 \in \mathcal{D}_{2L+n}$, $f_2 \in \mathcal{D}_{|\alpha|}$, and $|\alpha| \leq 2L+n$

$$\langle \Pi_b(f_1, f_2), x^\alpha \rangle = \lim_{R \rightarrow \infty} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} Q_j b(u) P_j f_1(u) P_j f_2(u) Q_j(\eta_{R x^\alpha})(u) du \\ = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} Q_j b(u) P_j f_1(u) P_j f_2(u) Q_j(x^\alpha)(u) du = 0,$$

and likewise $\langle \Pi_b(f_2, f_1), x^\alpha \rangle = 0$. The $BMO_{|\alpha|+|\beta|}$ conditions in (2.4) are more difficult to verify, but hold nonetheless. For $|\alpha|, |\beta| \leq L-1$, we first compute

$$\begin{aligned}
2^{k(|\alpha|+|\beta|)} Q_k[[\Pi_b]]_{\alpha,\beta}(x) &= 2^{k(|\alpha|+|\beta|)} \langle [[\Pi_b]]_{\alpha,\beta}, \Psi_k^x \rangle \\
&= \lim_{R \rightarrow \infty} 2^{k(|\alpha|+|\beta|)} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{4n}} \Psi_j(u-v) Q_j b(v) \varphi_j(v-y_1) \varphi_j(v-y_2) \eta_R(y_1) \eta_R(y_2) \\
&\quad \times (u-y_1)^\alpha (u-y_2)^\beta \Psi_k^x(u) du dv dy_1 dy_2 \\
&= \lim_{R \rightarrow \infty} \sum_{\mu \leq \alpha} \sum_{\nu \leq \beta} c_{\alpha,\mu} c_{\beta,\nu} 2^{k(|\alpha|+|\beta|)} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{4n}} \Psi_j(u-v) Q_j b(v) \varphi_j(v-y_1) \varphi_j(v-y_2) \eta_R(y_1) \eta_R(y_2) \\
&\quad \times (u-v)^{\mu+\nu} (v-y_1)^{\alpha-\mu} (v-y_2)^{\beta-\nu} \Psi_k^x(u) du dv dy_1 dy_2 \\
&= \sum_{\mu \leq \alpha} \sum_{\nu \leq \beta} c_{\alpha,\mu} c_{\beta,\nu} C_{\alpha-\mu} C_{\beta-\nu} 2^{(k-j)(|\alpha|+|\beta|)} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \Psi_j^{(\mu+\nu)}(u-v) Q_j b(v) \Psi_k^x(u) du dv \\
&= \sum_{\mu \leq \alpha} \sum_{\nu \leq \beta} c_{\alpha,\mu} c_{\beta,\nu} C_{\alpha-\mu} C_{\beta-\nu} 2^{(k-j)(|\alpha|+|\beta|)} \sum_{j \in \mathbb{Z}} Q_k Q_j^{(\mu+\nu)} Q_j b(x) \\
&= \sum_{\mu \leq \alpha} \sum_{\nu \leq \beta} c_{\alpha,\mu} c_{\beta,\nu} C_{\alpha-\mu} C_{\beta-\nu} W_k * (T_{V^{(\mu+\nu)}} b)(x),
\end{aligned}$$

where $c_{\alpha,\mu}$ are binomial coefficients, $C_\mu = \int_{\mathbb{R}^n} \varphi(x) x^\mu dx$, and V, W , and T_V are defined via $\widehat{V^{(\mu+\nu)}}(\xi) = |\xi|^{|\alpha|+|\beta|} \widehat{\Psi^{(\mu+\nu)}}(\xi) \widehat{\Psi}(\xi)$, $\widehat{W}(\xi) = |\xi|^{-(|\alpha|+|\beta|)} \widehat{\Psi}(\xi)$, $V_j^{(\mu+\nu)}(x) = 2^{jn} V^{(\mu+\nu)}(2^j x)$, $W_j(x) = 2^{jn} W(2^j x)$, and

$$T_{V^{(\mu+\nu)}} f(x) = \sum_{j \in \mathbb{Z}} V_j^{(\mu+\nu)} * f(x).$$

This is verified by the following computation.

$$\begin{aligned}
2^{(k-j)(|\alpha|+|\beta|)} \mathcal{F} \left[Q_k Q_j^{(\mu+\nu)} Q_j f \right] (\xi) &= 2^{(k-j)(|\alpha|+|\beta|)} \widehat{\Psi}(2^{-k}\xi) \widehat{\Psi^{(\mu+\nu)}}(2^{-j}\xi) \widehat{\Psi}(2^{-j}\xi) \widehat{f}(\xi) \\
&= (2^{-k}|\xi|)^{-(|\alpha|+|\beta|)} (2^{-j}|\xi|)^{|\alpha|+|\beta|} \widehat{\Psi}(2^{-k}\xi) \widehat{\Psi^{(\mu+\nu)}}(2^{-j}\xi) \widehat{\Psi}(2^{-j}\xi) \widehat{f}(\xi) \\
&= \widehat{W}(2^{-k}\xi) \widehat{V^{(\mu+\nu)}}(2^{-j}\xi) \widehat{f}(\xi) \\
&= \mathcal{F} \left[W_k * V_j^{(\mu+\nu)} * f \right] (\xi).
\end{aligned}$$

It was shown in [35] that $T_{V^{(\mu+\nu)}}$ is bounded on BMO . Then $T_{V^{(\mu+\nu)}} b \in BMO$ with $\|T_{V^{(\mu+\nu)}} b\|_{BMO} \lesssim \|b\|_{BMO}$, and

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}} 2^{2k(|\alpha|+|\beta|)} |Q_k[[\Pi_b]]_{\alpha,\beta}(x)|^2 \delta_{t=2^{-k}} dx \\
&= \sum_{k \in \mathbb{Z}} \left| \sum_{\mu \leq \alpha} \sum_{\nu \leq \beta} c_{\alpha,\mu} c_{\beta,\nu} C_{\alpha-\mu} C_{\beta-\nu} W_k * (T_{V^{(\mu+\nu)}} b)(x) \right|^2 \delta_{t=2^{-k}} dx
\end{aligned}$$

is a Carleson measure since $T_{V^{(\mu+\nu)}} b \in BMO$ and $W_k(x) = 2^{kn} W(2^k x)$ has integral zero. \square

6.2. Product Function Paraproducts. Now we prove Theorem 2.5. First recall the definition of $\Pi(f_1, f_2)$.

$$\Pi(f_1, f_2) = \sum_{k \in \mathbb{Z}} \Pi_k(f_1, f_2), \quad \text{where} \quad \Pi_k(f_1, f_2) = Q_k(P_k f_1 \cdot P_k f_2).$$

Proof of Theorem 2.5. It is easy to verify that $\Pi \in BCZO(M)$ for all $M \geq 1$. Then to show that Π is bounded as described in Theorem 2.5, it is sufficient to show (2.3) and (2.4) from Theorem 2.2. The kernel of Π_k is given by

$$\pi_k(x, y_1, y_2) = \int_{\mathbb{R}^n} \Psi_k(x-u) \Phi_k(u-y_1) \Phi_k(u-y_2) du.$$

We first prove (2.3); for $\alpha \in \mathbb{N}_0^n$, $f_1 \in \mathcal{D}_{|\alpha|}$, and $f_2 \in \mathcal{D}_{2L+n}$, we have

$$\begin{aligned} \langle \Pi(f_1, f_2), x^\alpha \rangle &= \lim_{R \rightarrow \infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^{4n}} \Psi_k(x-u) \Phi_k(u-y_1) \Phi_k(u-y_2) f_1(y_1) f_2(y_2) \eta_R(x) x^\alpha du dx dy_1 dy_2 \\ &= \lim_{R \rightarrow \infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \Phi_k(u-y_1) \Phi_k(u-y_2) f_1(y_1) f_2(y_2) \left(\int_{\mathbb{R}^n} \Psi_k(x-u) \eta_R(x) x^\alpha dx \right) du dy_1 dy_2 = 0 \end{aligned}$$

To prove (2.4), let $\alpha, \beta \in \mathbb{N}_0^n$ and $f \in \mathcal{D}_{|\alpha|+|\beta|}$. It follows that

$$\begin{aligned} \langle [[\Pi]]_{\alpha, \beta}, f \rangle &= \lim_{R \rightarrow \infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^{4n}} \Psi_k(x-u) \Phi_k(u-y_1) \Phi_k(u-y_2) \eta_R(y_1) \eta_R(y_2) \\ &\quad \times (x-y_1)^\alpha (x-y_2)^\beta f(x) du dx dy_1 dy_2 \\ &= \lim_{R \rightarrow \infty} \sum_{k \in \mathbb{Z}} \sum_{\mu_1 + \nu_1 = \alpha} \sum_{\mu_2 + \nu_2 = \beta} c_{\mu_1, \nu_1} c_{\mu_2, \nu_2} \int_{\mathbb{R}^{4n}} \Psi_k(x-u) \Phi_k(u-y_1) \Phi_k(u-y_2) \eta_R(y_1) \eta_R(y_2) \\ &\quad \times (x-u)^{\mu_1} (u-y_1)^{\nu_1} (x-u)^{\mu_2} (u-y_2)^{\nu_2} f(x) du dx dy_1 dy_2 \\ &= \lim_{R \rightarrow \infty} \sum_{k \in \mathbb{Z}} \sum_{\mu_1 + \nu_1 = \alpha} \sum_{\mu_2 + \nu_2 = \beta} c_{\mu_1, \nu_1} c_{\mu_2, \nu_2} \int_{\mathbb{R}^{2n}} \Psi_k(x-u) (x-u)^{\mu_1 + \mu_2} f(x) \\ &\quad \times \left(\int_{\mathbb{R}^n} \Phi_k(u-y_1) (u-y_1)^{\nu_1} \eta_R(y_1) dy_1 \right) \\ &\quad \times \left(\int_{\mathbb{R}^n} \Phi_k(u-y_2) (u-y_2)^{\nu_2} \eta_R(y_2) dy_2 \right) du dx \\ &= \sum_{k \in \mathbb{Z}} \sum_{\mu_1 + \nu_1 = \alpha} \sum_{\mu_2 + \nu_2 = \beta} c_{\mu_1, \nu_1} c_{\mu_2, \nu_2} C_{\nu_1} C_{\nu_2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \Psi_k(u) u^{\mu_1 + \mu_2} du \right) f(x) dx = 0. \end{aligned}$$

Here we define C_ν for $\nu \in \mathbb{N}_0^n$

$$C_\nu = \int_{\mathbb{R}^n} \Phi(x) x^\nu dx,$$

and $c_{\nu, \mu}$ are binomial coefficients. Therefore Π is bounded from $H^{p_1} \times H^{p_2}$ into H^p for all $\frac{n}{2n+L} < p \leq 1$ and $\frac{n}{n+L} < p_1, p_2 \leq 1$, as in the statement of Theorem 2.5. \square

6.3. Molecular Paraproducts. Finally, we address the paraproducts in Theorem 2.6. Recall the paraproduct operators

$$T(f_1, f_2)(x) = \sum_Q |Q|^{-1/2} \langle f_1, \phi_Q^1 \rangle \langle f_2, \phi_Q^2 \rangle \phi_Q^3(x),$$

where $\phi_Q^1, \phi_Q^2, \phi_Q^3$ are (M, N) -smooth molecules indexed by dyadic cubes.

Proof of Theorem 2.6. It is easy to see that $T \in BCZO(M)$ and is bounded from $L^2 \times L^2$ into L^1 , which was proved in [2]. So we must verify conditions (2.3) and (2.4) of Theorem 2.2. As was shown in [2], the kernel of T is

$$K(x, y_1, y_2) = \sum_Q |Q|^{-1/2} \phi_Q^1(y_1) \phi_Q^2(y_2) \phi_Q^3(x).$$

For $f_1 \in \mathcal{D}_{|\alpha|}$ and $f_2 \in \mathcal{D}_{2L+n}$ with $|\alpha| \leq L$, we have

$$\begin{aligned} \langle T(f_1, f_2), x^\alpha \rangle &= \lim_{R \rightarrow \infty} \sum_Q |Q|^{-1/2} \langle f_1, \phi_Q^1 \rangle \langle f_2, \phi_Q^2 \rangle \langle \phi_Q^3, x^\alpha \eta_R \rangle \\ &= \sum_Q |Q|^{-1/2} \langle f_1, \phi_Q^1 \rangle \langle f_2, \phi_Q^2 \rangle \int_{\mathbb{R}^n} \phi_Q^3(x) x^\alpha dx = 0. \end{aligned}$$

It is not hard to show that this series is absolutely summable independent of R when $f_1 \in \mathcal{D}_{|\alpha|}$ and $f_2 \in C_0^\infty$; hence we can bring the limit in R inside the sum. A similar computation shows that $\langle T(f_1, f_2), x^\alpha \rangle = 0$ for $f_1 \in \mathcal{D}_{2L+n}$ and $f_2 \in \mathcal{D}_{|\alpha|}$. To show (2.4), we first verify that we can use an alternate formula to compute $[[T]]_{\alpha, \beta}$. For $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq L - 1$ and $\psi \in \mathcal{D}_{|\alpha| + |\beta|}$, fix R_0 large enough so that $\text{supp}(\psi) \subset B(0, R_0/4)$. Then for $S, R > R_0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{3n}} \mathcal{K}(x, y_1, y_2) (x - y_1)^\alpha (x - y_2)^\beta \eta_R(y_1) \eta_S(y_2) \psi(x) dy_1 dy_2 dx \\ &= \int_{\mathbb{R}^{3n}} K(x, y_1, y_2) (x - y_1)^\alpha (x - y_2)^\beta (\eta_R(y_1) \eta_S(y_2) - \eta_{R_0}(y_1) \eta_{R_0}(y_2)) \psi(x) dy_1 dy_2 dx \\ &\quad + \int_{\mathbb{R}^{3n}} \mathcal{K}(x, y_1, y_2) (x - y_1)^\alpha (x - y_2)^\beta \eta_{R_0}(y_1) \eta_{R_0}(y_2) \psi(x) dy_1 dy_2 dx, \end{aligned}$$

where the integrand of the first integral is bounded by an $L^1(\mathbb{R}^{3n})$ function independent of both R and S . Then the limit as $S, R \rightarrow \infty$ exists (as a two dimensional limit), and

$$\lim_{R \rightarrow \infty} \lim_{S \rightarrow \infty} \int_{\mathbb{R}^{3n}} \mathcal{K}(x, y_1, y_2) (x - y_1)^\alpha (x - y_2)^\beta \eta_R(y_1) \eta_S(y_2) \psi(x) dy_1 dy_2 dx = \langle [[T]]_{\alpha, \beta}, \psi \rangle.$$

Therefore

$$\begin{aligned} &\langle [[T]]_{\alpha, \beta}, \psi \rangle \\ &= \lim_{R \rightarrow \infty} \lim_{S \rightarrow \infty} \sum_Q |Q|^{-1/2} \int_{\mathbb{R}^{3n}} \phi_Q^1(y_1) (x - y_1)^\alpha \eta_S(y_1) \phi_Q^2(y_2) (x - y_2)^\beta \eta_R(y_2) \phi_Q^3(x) \psi(x) dy_1 dy_2 dx \\ &= \lim_{R \rightarrow \infty} \sum_Q |Q|^{-1/2} \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} \phi_Q^1(y_1) (x - y_1)^\alpha dy_1 \right) \phi_Q^2(y_2) (x - y_2)^\beta \eta_R(y_2) \phi_Q^3(x) \psi(x) dy_2 dx = 0. \end{aligned}$$

Then we apply Theorem 2.2 to complete the proof. \square

REFERENCES

1. Á. Bényi, *Bilinear singular integral operators, smooth atoms and molecules*, J. Fourier Anal. Appl. **9**, (2003), no. 3, 301-319.
2. Á. Bényi, D. Maldonado, A. Nahmod, and R. H. Torres, *Bilinear paraproducts revisited*, Math. Nachr. **283**, (2010), no. 9, 1257-1276.
3. J. M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup., **14**, (1981), no. 2, 109-246.
4. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80**, (1958), 921-930.
5. S.-Y. Chang and R. Fefferman, *The Calderón-Zygmund decomposition on product domains*, Amer. J. Math., **104**, (1982), no. 3, 455-468.
6. S.-Y. Chang and R. Fefferman, *Some recent developments in Fourier analysis and H^p -theory on product domains*, Bull. Amer. Math. Soc. (N.S.) **12**, (1985), no. 1, 1-43.
7. M. Christ and J. L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. **159**, (1987), no. 1-2, 51-80.
8. L. Chaffee, J. Hart, and L. Oliviera, *Weighted multilinear square function bounds*, Michigan Math. J. **63**, (2014), no. 2, 371-400.
9. J. Chen and G. Lu, *Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness*, Nonlinear Anal. **101** (2014), 98-112.
10. R. R. Coifman, *A real variable characterization of H^p* , Studia math., **51**, (1974), 269-274.
11. R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Am. Math. Soc., **212**, (1975), 315-331.
12. R. R. Coifman and Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier, Grenoble, **28**, (1978), 177-202.
13. R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, 1978.
14. R. R. Coifman and L. Grafakos, *Hardy space estimates for multilinear operators I*, Rev. Mat. Iberoamericana **8** (1992), no. 1, 45-67.
15. W. Dai and G. Lu, *L^p estimates for multi-linear and multi-parameter pseudo-differential operators*, Bull. Soc. Math. Fr., arXiv:1308.4062.
16. G. David and J. L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2), **120**, (1984), no. 2, 371-397.
17. G. David, J. L. Journé, and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana **1**, (1985), no. 4, 1-56.
18. C. Fefferman, E. Stein, *H^p spaces of several variables*, Acta Math., **129**, (1972), no. 3-4, 137-193.
19. S. Ferguson, M. Lacey, *A characterization of product BMO by commutators*, Acta Math. **189** (2002), no. 2, 143-160.
20. M. Frazier, Y.S. Han, B. Jawerth, and G. Weiss, *The T1 theorem for Triebel-Lizorkin spaces*, Lecture Notes in Math., 1384, Springer, Berlin, 1989.
21. M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34**, (1985), no. 4, 777-799.
22. M. Frazier, R. H. Torres, and G. Weiss, *The boundedness of Calderón-Zygmund operators on the spaces $\dot{F}_p^{\alpha,q}$* , Rev. Mat. Iberoamericana **4**, (1988), no. 1, 41-72.
23. L. Grafakos and N. Kalton, *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. **52**, (2001), no. 2, 169-179.
24. L. Grafakos, L. Liu, D. Maldonado, and D. Yang, *Multilinear analysis on metric spaces*, Dissertationes Math. (Rozprawy Mat.) **497**, (2014), 121.
25. L. Grafakos and L. Oliveira, *Carleson measures associated with families of multilinear operators*, Studia Math. **211**, (2012), no. 1, 71-94.
26. L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math., **165**, (2002), 1, 124-164.
27. L. Grafakos and R. H. Torres, *On multilinear singular integrals of Calderón-Zygmund type*, Publ. Mat., Vol. Extra, (2002), 57-91.
28. L. Grafakos and R. H. Torres, *Discrete decompositions for bilinear operators and almost diagonal conditions*, Trans. Amer. Math. Soc. **354**, (2002), no. 3, 1153-1176.

29. A. Grau de la Herrán, J. Hart, and L. Oliveira, *Multilinear local Tb theorem for square functions*, Ann. Acad. Sci. Fenn. Math. **38**, (2013), no. 2, 697-720.
30. Y. Han and G. Lu, *Some recent works on multiparameter Hardy space theory and discrete Littlewood-Paley-Stein analysis*, Trends in partial differential equations, 99-191, Adv. Lect. Math. (ALM), **10**, Int. Press, Somerville, MA, 2010.
31. Y. Han, G. Lu, and E. Sawyer, *Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group*, Anal. PDE **7** (2014), no. 7, 1465-1534.
32. J. Hart, *Bilinear square functions and vector-valued Calderón-Zygmund operators*, J. Fourier Analysis Appl. **18**, (2012), no.6, 1291-1313.
33. J. Hart, *A new proof of the bilinear $T(1)$ theorem*, Proc. Amer. Math. Soc. **142**, (2014), no. 9, 3169-3181.
34. J. Hart, *A bilinear Tb theorem for singular integral operators*, J. Funct. Anal. **268**, (2015), no. 12, 3680-3733.
35. J. Hart and G. Lu, *Hardy space estimates for Littlewood-Paley square functions and singular integral operators*, to appear in J. Fourier Anal. Appl.
36. G. Hu and Y. Meng, *Multilinear Calderón-Zygmund operator on products of Hardy spaces*, Acta Math. Sin. (Engl. Ser.) **28**, (2012), no. 2, 281-294.
37. B. Jessen, J. Marcinkiewicz, and A. Zygmund, *Note on the differentiability of multiple integrals*, Funda. Math., **25**, (1935), 217-234.
38. P. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure*, Lecture Notes in Math., 1384, Springer, Berlin, 1989.
39. J.-L. Journé, *Calderón-Zygmund operators on product spaces*, Rev. Mat. Iberoamericana, **1**, (1985), no. 3, 55-91.
40. G. Lu and Z. Ruan, *Duality theory of weighted Hardy spaces with arbitrary number of parameters*, Forum Math. **26** (2014), no. 5, 1429-1457.
41. G. Lu and P. Zhang, *Multilinear Calderón-Zygmund operators with kernels of Dini's type and applications*, Non-linear Anal. **107** (2014), 92-117.
42. D. Maldonado, *Multilinear Singular Integrals and Quadratic Estimates*, Doctoral Dissertation, University of Kansas, 2005.
43. D. Maldonado and V. Naibo, *On the boundedness of bilinear operators on products of Besov and Lebesgue spaces*, J. Math. Anal. Appl. **352**, (2009), no. 2, 591-603.
44. C. Muscalu, J. Pipher, T. Tao and C. Thiele, *Bi-parameter paraproducts*, Acta Math. **193** (2004), no. 2, 269-296.
45. C. Muscalu, J. Pipher, T. Tao and C. Thiele, *Multi-parameter paraproducts*, Rev. Mat. Iberoam. **22** (2006), no. 3, 963-976.
46. S. Semmes, *Square function estimates and the $T(b)$ theorem*, Proc. Amer. Math. Soc., **110**, no. 3, (1990), 721-726.
47. E. Stein, *On the functions of Littlewood-Paley-Stein; Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc., **88**, (1958), 430-466.
48. E. Stein, *On the theory of harmonic functions of several variables. II. Behavior near the boundary*, Acta Math. **106**, (1961), 137-174.
49. E. Stein and G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p -spaces*, Acta Math. **103**, (1960), 25-62.
50. R. H. Torres, *Boundedness results for operators with singular kernels on distribution spaces*, Mem. Amer. Math. Soc. **90**, (1991), 442.
51. A. Youssfi, *Bilinear operators and the Jacobian-determinant on Besov spaces*, Indiana Univ. Math. J. **45**, (1996), 381-396.

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202

E-mail address: jarod.hart@wayne.edu

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202

E-mail address: gzlu@wayne.edu