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A GEOMETRIC COVERING LEMMA AND NODAL SETS OF EIGENFUNCTIONS

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Abstract. The main purpose of this paper is two-fold. On one hand, we prove a sharper covering lemma in Euclidean space $\mathbb{R}^n$ for all $n \geq 2$ (see Theorem 1.5). On the other hand, we apply this covering lemma to improve existing results for BMO and volume estimates of nodal sets for eigenfunctions u satisfying $\triangle u + \lambda u = 0$ on n-dimensional Riemannian manifolds when λ is large (see Theorems 1.7, 1.8). We also improve the BMO estimates for the function $q = |\nabla u|^2 + \frac{\lambda}{n} u^2$ (see Theorem 1.10). Our covering lemma sharpens substantially earlier results and is fairly close to the optimal one we can expect (Conjecture 1.6).

1. Introduction

Let M be a smooth, compact and connected Riemannian manifold without boundary. Let $\triangle$ denote the Laplacian on M. Assume throughout this paper that u is the solution to $\triangle u + \lambda u = 0$, $\lambda > 1$, i.e., u is an eigenfunction with eigenvalue λ. The nodal set $\mathcal{N}$ of u is defined to be the set of points $x \in M$ where $u(x) = 0$. Then outside the singular set $\mathcal{S} = \{x | u(x) = 0, \nabla u(x) = 0\}$, $\mathcal{N}$ is a regular $(n-1)$-dimensional submanifold of M. The main focus of the current paper concerns a geometric covering lemma in the Euclidean space $\mathbb{R}^n$ and applying it to the BMO norm and volume estimates of nonzero sets of eigenfunctions on Riemannian manifolds.

Let D be the diameter of the manifold M, and H the upper bound of the absolute value of the sectional curvature of M. It was conjectured by S.T. Yau (see Problem 73 of [Yau]) that

$$c_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}) \leq c_2\sqrt{\lambda},$$

where the constants c_1 and c_2 depend only on the geometry of the manifold (D and H) and $\mathcal{H}^{n-1}(\mathcal{N})$ is the $(n-1)$-dimensional Hausdorff measure of $\mathcal{N}$. When M is real analytic, Donnelly and Fefferman have proved that this conjecture holds for all dimensions [DF1]. In two dimensional smooth manifold, Brűning [B] derived the optimal lower bound and Donnelly and Fefferman proved [DF2] that $\mathcal{H}^{n-1}(\mathcal{N}) \leq c_3\lambda^{3/4}$, and subsequently Dong gave an alternative proof of this result in [D1].

For smooth n-dimensional manifolds, it was proved by Hardt and Simon [HS] that $\mathcal{H}^{n-1}(\mathcal{N}) \leq c_3 \exp(\sqrt{\lambda}\log\lambda)$ where c_3 depends on D and H. In fact, this estimate holds for solutions to a general class of elliptic equations of second order with smooth coefficients (see [HS]). F. Lin [Lin] further obtained an optimal upper bound estimate

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of the Hausdorff measures of nodal sets for solutions to second order elliptic equations with analytic coefficients and parabolic equations with time independent analytic coefficients. Furthermore, Han and Lin [HL1] derived such estimates for a general class (time dependent) of second order parabolic equations without the assumption of real analyticity. We also refer to the works by Han, Hardt and Lin [HHL], Han [H1] on estimates of Hausdorff measure of singular sets of solutions to elliptic differential equations, Han [H2] and the forthcoming book by Han and Lin [HL2] for more detailed exposition on this subject.

In this paper, we are mainly interested in the BMO norm estimates and volume estimates of nonzero sets of eigenfunctions on Riemannian manifolds. In [DF3], Donnelly and Fefferman established the growth property, estimates of the BMO norm for $\log|u|$ and lower bounds for the volume of the nonzero set of eigenfunctions for all $n \geq 2$. The results of Donnelly and Fefferman were improved subsequently by Chanillo and Muckenhoupt [CM] by employing a covering lemma intrinsic to the growth property of the eigenfunctions and further improvement was obtained by the second author of the current paper in [L1] and [L2] by deriving a sharper covering lemma.

Concerning the BMO norm and volume estimates of nodal sets of eigenfunctions, Donnelly and Fefferman proved the following in [DF3]:

Theorem 1.1. *(BMO estimate for* $\log|u|$*) Assume that* u *satisfies* $\triangle u + \lambda u = 0$ *on* M*, then for* $\dim M = n \geq 2$*,*

$$\|\log|u|\|_{BMO} \leq c\lambda^{\frac{n(n+2)}{4}},$$

where $c = c(M)$ *independent of* λ*.*

Theorem 1.2. *(Estimates for nodal sets) Assume that* u *satisfies* $\triangle u + \lambda u = 0$ *on* M*, let* $B \subset M$ *be any ball and* $\Omega \subset B$ *by any of the connected components of* $\{x \in B : u(x) \neq 0\}$*. If* Ω *intersects the middle half of* B*, then for* $n \geq 2$

$$|\Omega| \geq C(M,n)\lambda^{-\frac{n+n^2(n+2)}{2}}(\log\lambda)^{-4n}|B|,$$

where $C(M,n,\epsilon)$ *is independent of* λ *and* u*.*

The BMO estimate for eigenfunctions in the work [DF3] employs the growth property of the eigenfunctions together with a Vitali's covering lemma, and then the volume estimates of nodal sets follows from the BMO estimates and the growth property. The growth property proved in [DF3] reads as follows:

Theorem 1.3. *Assume that* u *satisfies* $\triangle u + \lambda u = 0$ *on* M*. Let* $B(x,R)$ *denote the ball centered at* x *of radius* R*. Then*

$$\int_{B\left(x,(1+\lambda^{-\frac{1}{2}})R\right)} |u|^2 \leq c\int_{B(x,R)} |u|^2, \tag{1.1}$$

$$\left[\int_{B(x,R)} |\bigtriangledown u|^2\right]^{1/2} \leq c\frac{\sqrt{\lambda}}{R}\left[\int_{B(x,R)} |u|^2\right]^{1/2}. \tag{1.2}$$

Soon after Donnelly and Fefferman's work [DF3], Chanillo and Muckenhoupt [CM] discovered the idea of using a more delicate covering lemma which is more suitably adapted to the growth property of eigenfunctions. They introduced a covering lemma akin to, but quite different than, the classical Besicovitch covering lemma and used such a covering lemma to sharpen the BMO and volume estimates of nodal sets of eigenfunctions by Donnelly and Fefferman [DF3].

For the classical Besicovitch lemma in $\mathbb{R}^n$, we refer to the books by Wheeden and Zygmund [WZ] for such a lemma in terms of cubes and by Ziemer [Z] in terms of balls.

We first recall the covering lemma introduced in [CM] which is of its independent interest.

Theorem 1.4. *Let $n \geq 2$ and $\delta > 0$ be small enough, then given any finite collection of balls $\{B_\alpha\}_{\alpha \in I}$ in $\mathbb{R}^n$, one can select a subcollection $B_1, \cdots, B_N$ such that*

$$(i) \quad \bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (1+\delta) B_i$$

and

$$(ii) \quad \sum_{i=1}^{N} \chi_{B_i}(x) \leq c(n)\delta^{-n},$$

where $c(n)$ is a constant only dependent on the dimension n but independent of δ and the given collection of balls, $(1+\delta)B$ denotes a ball concentric with the ball B but with $1+\delta$ times of the radius of B.

Incorporating the above covering lemma when $\delta = \lambda^{-\frac{1}{2}}$ with the growth property, Chanillo and Muckenhoupt [CM] were able to sharpen the BMO estimates for $\log |u|$ in Theorem (1.1) to

$$\| \log |u| \|_{BMO} \leq c\lambda^n \log \lambda$$

and the volume estimates in Theorem (1.2) to

$$|\Omega| \geq C(M,n)\lambda^{-2n^2-\frac{n}{2}}(\log \lambda)^{-2n}|B|$$

which improve those of Donnelly and Fefferman [DF3].

The above covering lemma Theorem 1.4 is not sharp in the sense that the bounds in (ii) are not best possible when $\delta \to 0$.

It was subsequently improved by the second author in [L1], [L2] and the bound in (ii) was sharpened to $c\delta^{-n+\frac{1}{2}} \log(\frac{1}{\delta})$ for all $n \geq 2$. Furthermore, improved BMO and volume estimates for nodal sets of eigenfunctions were derived in [L1], [L2]. More precisely, it was proved in [L1] that in the two-dimensional case

$$\| \log |u| \|_{BMO} \leq C(M,\epsilon)\lambda^{\frac{15}{8}+\epsilon}$$

and in [L2] that for all $n \geq 3$,

$$\| \log |u| \|_{BMO} \leq C(M)\lambda^{n-\frac{1}{8}}(\log \lambda)^2,$$

$$|\Omega| \geq C(M,n)\lambda^{-2n^2-\frac{n}{4}}(\log \lambda)^{-4n}|B|.$$

Though the improvement in [L1] on the upper bounds in (ii) of Theorem 1.4 in two dimension to $\delta^{-\frac{7}{4}} \log(\frac{1}{\delta})$ from the estimate δ^{-2} appears to be minor, its proof given in

[L1] was fairly nontrivial and rather complicated, and was consequently very lengthy. It involved many geometric considerations and quite some steps to accomplish.

One of the primary purposes of the current paper is to sharpen further the upper bounds in (ii) of the covering lemma in [L1], [L2], and in the meantime we carry out a completely different but considerably simpler argument of its proof.

We now state our new covering lemma as

Theorem 1.5. *(Covering Lemma) Let $n \geq 2$ and $\delta > 0$ be small enough, then given any finite collection of balls $\{B_\alpha\}_{\alpha\in I}$ in $\mathbb{R}^n$, one can select a subcollection $B_1, \cdots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1+\delta)B_i \tag{1.3}$$

and

$$\sum_{i=1}^N \chi_{B_i}(x) \leq c\delta^{-\frac{n}{2}} \log(\frac{1}{\delta}), \tag{1.4}$$

where c is a constant independent of δ.

Though we do not know yet if the above lemma is the best possible result, we believe that the above lemma is fairly sharp and close to the optimal one. Indeed, we conjecture the following covering lemma.

Conjecture 1.6. *The upper bound in (1.4) can be improved to $c\delta^{-\frac{n-1}{2}}$ for all $n \geq 2$ and it is sharp.*

The sharpness of the above conjecture, if it is true, is demonstrated by an example (see Example 5.1 in Section 5). However, we still do not have a proof that it is indeed true.

The second main purpose of this paper is to get a better BMO estimate of eigenfunctions and volume estimates of nodal sets. The main results are the following

Theorem 1.7. *(BMO estimate for $\log|u|$) For u, λ as above, then for $n \geq 3$,*

$$\|\log|u|\|_{BMO} \leq c\lambda^{\frac{3n}{4}}(\log\lambda)^2,$$

where $c = c(M)$ independent of λ; and for $n = 2$,

$$\|\log|u|\|_{BMO} \leq c\lambda^{3/2+\epsilon},$$

where $c = c(\epsilon, M)$ independent of λ.

Theorem 1.8. *(Estimates for nodal sets) For u, λ as above, let $B \subset M$ be any ball, and let $\Omega \subset B$ by any of the connected components of $\{x \in B : u(x) \neq 0\}$. If Ω intersects the middle half of B, then for $n \geq 3$,*

$$|\Omega| \geq C(M,n)\lambda^{-\frac{3n^2}{4}-\frac{n}{2}}(\log\lambda)^{-2n}|B|,$$

and for $n = 2$ and any given $\epsilon > 0$,

$$|\Omega| \geq C(M,n,\epsilon)\lambda^{-4-\epsilon}|B|,$$

where $C(M,n)$ and $C(M,n,\epsilon)$ are independent of λ and u.

Some lower bound estimates for the volume in real analytic manifolds or two-dimensional C^∞ surfaces have also been recently obtained by Mangoubi in [M].

It was proved by Dong [D2] the following

Theorem 1.9. *Assume that* $\triangle u+\lambda u=0$ *on* M *and* $\dim M=n$. *Let* $q=|\nabla u|^2+\frac{\lambda}{n}u^2$. *Then*

$$\mathcal{H}^{n-1}(\mathcal{N})=\frac{1}{2}\int_M \frac{|\nabla u|^2+\lambda|u|}{\sqrt{q}}.$$

He further proved in [D2] that for $n\geq 3$,

$$\|\log q\|_{BMO}\leq c\lambda^n\log\lambda.$$

As another application of our covering Theorem (1.5), we also improve the BMO estimate for the functions q as follows.

Theorem 1.10. *(BMO estimate for* $\log q$*) For* q, λ *as above, then for* $n\geq 3$,

$$\|\log q\|_{BMO}\leq c\lambda^{\frac{3n}{4}}(\log\lambda)^2,$$

where $c=c(M)$ *independent of* λ.

We end this introduction with the following remark. The major contribution of this paper is really the covering Theorem 1.5. The covering lemma itself is of great interest in harmonic analysis and it seems to be a challenging task to derive the optimal bounds (see Conjecture 1.6). Applying this covering lemma to estimate the BMO norms of the eigenfunctions and the volume of the nodal sets is the motivation to establish this covering lemma. Nevertheless, to make this paper self-contained we have also chosen to include sufficient details to deduce theses estimates of applications. If the reader is only interested in the applications to BMO and volume estimates of eigenfunctions, one only needs to read Sections 2 and 3. A reader who is only interested in the covering lemma and its proof only needs to read Sections 4 and 5.

This paper is organized as follows: Section 2 gives the proofs of Theorems 1.7 and 1.8 in two dimensional case; Section 3 deals with the proofs of Theorems 1.7 and 1.8 when the dimension $n\geq 3$ and Theorem 1.10. Section 4 is devoted to the proof of the new covering Theorem 1.5; Section 5 justifies the sharpness of the bound $c\delta^{-\frac{n-1}{2}}$ in Conjecture 1.6.

One word about notations: throughout this note, C and c will always denote generic positive constants independent of the given balls $\{B_\alpha\}_{\alpha\in I}$ and $\delta>0$; $\rho(B)$ will denote the radius of the ball B; $B(x,r)$ will denote the ball centered at x and of radius r and B^δ denotes the concentric ball with B and with radius $(1+\delta)\rho(B)$.

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2. Proof of Theorems 1.7 and 1.8 in dimension two

This section is devoted to the proof of Theorems 1.7 and 1.8 in dimension two. Now we start the proof of Theorem 1.7 by showing the following lemmas.

Lemma 2.1. *Let u, λ as before, $1 \leq q < \infty$, then u satisfies the Reverse-Holder inequality:*

$$\left[\frac{1}{|B|}\int_B |u|^q\right]^{1/q} \leq c\sqrt{\lambda}\left[\frac{1}{|B|}\int_B |u|^2\right]^{1/2}, \tag{2.1}$$

where c depends on q.

Proof. By the Poincaré inequality, for any ball B, we have

$$\left[\frac{1}{|B|}\int_B |u-u_B|^q\right]^{1/q} \leq c|B|^{1/2}\left[\frac{1}{|B|}\int_B |\bigtriangledown u|^p\right]^{1/p}, \tag{2.2}$$

where $u_B = \frac{1}{|B|}\int_B u$ and $1 < p < 2, 1/q = 1/p - 1/2$, and $c = c(p,q)$. Applying Hőlder's inequality and (1.2) to the right side of (2.2), we obtain

$$\left[\frac{1}{|B|}\int_B |u-u_B|^q\right]^{1/q} \leq c\sqrt{\lambda}\left[\frac{1}{|B|}\int_B |u|^2\right]^{1/2}.$$

By Minkowski's inequality, Lemma 2.1 follows for $2 < q < \infty$, for the case $1 < q \leq 2$, we can apply Hőlder inequality again. □

Our Theorem 1.7 in dimension two will follow frow the following:

Lemma 2.4. *Suppose $w > 0$, $q > 2$, $\epsilon > 0$ and $\frac{1+\epsilon}{q'} \geq 1$, where $1/q' = 1 - 1/q$. Assume also that for any ball $B = B(x,R) \subset \mathbb{R}^2$,*

$$\int_{B(x,(1+\lambda^{-1/2})R)} w \leq c_0 \int_{B(x,R)} w, \tag{2.3}$$

$$\left(\frac{1}{|B|}\int_B w^q\right)^{1/q} \leq c_1\lambda\frac{1}{|B|}\int_B w, \tag{2.4}$$

then

$$||\log w||_{BMO} \leq c\lambda^{3/2+\epsilon},$$

where $c = c(c_0, c_1, \epsilon)$.

Theorem 1.7 in dimension two will follow from Lemma 2.4 if we choose $w = |u|^2$.

In order to prove Lemma 2.4, we need the following

Lemma 2.5. *Let w, q, $0 < \epsilon < 1$ satisfy the hypothesis of Lemma 2.4, k is an integer, let B be a fixed ball, $E \subset B$,then there exist c_2, c_3 such that if*

$$|E| \geq \left(1 - c_2\lambda^{-3/2-\epsilon}(\log\lambda)^{-1}\right)^k |B|, \tag{2.5}$$

then

$$\int_E w \geq \left(c_3\lambda^{-1/2}(\log\lambda)^{-1}\right)^k \int_B w, \tag{2.6}$$

where $c_2 = c_2(c_1), c_3 = c_3(c_0)$.

Proof. We will use induction on k as done in [CM] in higher dimension $n \geq 3$ (see [L1] for the two dimension case). We first verify the lemma for k=1. To do so, we claim that if $\epsilon > 0$, $|E| \geq (1-\bar{c}\lambda^{-1-\epsilon})|B|$ for some appropriate $\bar{c} = \bar{c}(c_1)$, then $\int_E w \geq 1/2 \int_B w$. To show this, we first note that $|B \backslash E| \leq \bar{c}\lambda^{-1-\epsilon}|B|$. If we choose $q > 2$ such that $\frac{1+\epsilon}{q'} \geq 1$, thus by (2.4),

$$\int_{B\backslash E} w \leq \left(\int_B w^q\right)^{1/q} |B\backslash E|^{1/q'}$$
$$\leq c_1 \bar{c}^{1/q'} \lambda^{-\frac{1+\epsilon}{q'}+1} \int_B w \leq c_1 \bar{c}^{1/q'} \int_B w.$$

If we choose $\bar{c}$ such that $c_1\bar{c}^{1/q'} < 1/2$, then $\int_{B\backslash E} w \leq 1/2 \int_B w$, this implies $\int_E w > 1/2 \int_B w$. Note that the choice of $\bar{c}$ is dependent on ϵ since $c_1 = c_1(q)$ and q is dependent on ϵ. Thus if $c_2 \leq \bar{c}$, and $|E| \geq (1 - c_2\lambda^{-3/2-\epsilon}(\log\lambda)^{-1})|B|$, then

$$\int_E w \geq 1/2 \int_B w \geq c_3 \left(\lambda^{-1/2}(\log\lambda)^{-1}\right) \int_B w,$$

and we are done for the case $k = 1$. Now we assume the statement is true for $k-1$. We may assume $|E| \leq (1-\bar{c}\lambda^{-1-\epsilon})|B|$, otherwise, there is nothing to prove. Thus for each density point x of E, we can select a ball $B_x \subset B$ such that $x \in B_x$, and

$$\frac{|B_x \bigcap E|}{|B_x|} = 1 - \bar{c}\lambda^{-1-\epsilon}.$$

Applying the cover lemma Theorem 1.5 when $n = 2$ to the balls B_x with the choice $\delta = \lambda^{-1/2}$, and without loss of generality, assume B_x are finite, and define

$$E_1 = \left[\bigcup_{i=1}^N (1+\lambda^{-1/2})B_i\right] \bigcap B.$$

Then $E_1 \subset B$, and as the proof given in [L1], we can show

$$|E| \leq \left(1 - c_2\lambda^{-3/2-\epsilon}(\log\lambda)^{-1}\right)|E_1|,$$
$$\int_E w \geq c_3\lambda^{-1/2}(\log\lambda)^{-1} \int_{E_1} w$$

for some $c_2 = c_2(c_1), c_3 = c_3(c_0)$.

This suffices to complete the proof of Lemma 2.4. □

Now we prove Theorem 1.7 in dimension two. The proof will be the same as that given in [CM] in higher dimension. In order to get the precise estimate, we need to carry out the details.

Proof. It will be enough to assume $\frac{1}{|B|}\int_B w = 1$. It is also sufficient to show

$$|\{x \in B : w^{-1}(x) > t\}| \leq \frac{|B|}{t^{c\lambda^{-3/2-\epsilon}(\log\lambda)^{-2}}}.$$

It is equivalent to show

$$|\{x \in B : w(x) < t\}| \leq t^{c\lambda^{-3/2-\epsilon}(\log\lambda)^{-2}}|B|.$$

Let us denote by $E = \{x \in B : w(x) < t\}$. Select k_0 such that

$$|E| \approx [1 - c_2\lambda^{-3/2-\epsilon}(\log\lambda)^{-1}]^{k_0}|B|.$$

Thus

$$k_0 \approx c\left(\lambda^{3/2+\epsilon}\log\lambda\right)\log\left(\frac{|B|}{|E|}\right).$$

Then by Lemma (2.4), and the normalization $\frac{1}{|B|}\int_B w = 1$, we have

$$|B| = \int_B w \le \left(c_3^{-1}\lambda^{1/2}\log\lambda\right)^{k_0}\int_E w$$

$$\le \left(c_3^{-1}\lambda^{1/2}\log\lambda\right)^{k_0} t|E|.$$

Thus

$$\frac{|B|}{|E|} \le t\ e^{k_0\log\left(c_3^{-1}\lambda^{1/2}\log\lambda\right)}$$

$$\le t\ \left(\frac{|B|}{|E|}\right)^{\left(c\lambda^{3/2+\epsilon}\log\lambda\right)\log\left(c_3^{-1}\lambda^{1/2}\log\lambda\right)}$$

$$\le t\ \left(\frac{|B|}{|E|}\right)^{c'\lambda^{3/2+\epsilon}(\log\lambda)^2}.$$

Thus

$$|E| \le t^{c'\lambda^{-3/2-\epsilon}(\log\lambda)^{-2}}|B|,$$

where c' is dependent on the constant c. Since ϵ is arbitrary, we can then have

$$|E| \le t^{c''\lambda^{-3/2-\epsilon}}|B|.$$

$\square$

Applying Theorem 1.7 in dimension two, we can derive Theorem 1.8 in dimension two as done in [DF3]. We shall not present the proof here.

3. Proof of Theorems 1.7, 1.10 and 1.8 in higher dimensions

We begin with the following lemma which is really the key to derive the BMO estimate Theorem 1.7 and 1.10 using the covering Theorem 1.5.

Lemma 3.1. *Suppose $n \ge 3$ and $w > 0$ and assume that*

$$\int_{B(x,(1+\lambda^{-1/2})R)} w \le c_0\int_{B(x,R)} w, \tag{3.1}$$

$$\left(\frac{1}{|B|}\int_B w^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \le c_1\lambda\frac{1}{|B|}\int_B w, \tag{3.2}$$

then

$$||\log w||_{BMO} \le c\lambda^{\frac{3n}{4}}(\log\lambda)^2,$$

where $c = c(c_0, c_1)$.

Using the sharp Poincaré inequality of type $L^2 \to L^{\frac{2n}{n-2}}$ and the growth property Theorem 1.3, for any eigenfunction u it satisfies

$$\left(\frac{1}{|B|}\int_B |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \le c\sqrt{\lambda}\left(\frac{1}{|B|}\int_B |u|^2\right)^{\frac{1}{2}}.$$

Theorem 1.7 will follow if we choose $w = |u|^2$ in Lemma 3.1, and Theorem 1.10 follows by choosing $w = q = |\nabla u|^2 + \frac{\lambda}{n}u^2$ in Lemma 3.1 (we point out here it was verified in [D2] that w satisfies the assumptions (3.1) and (3.2) of Lemma 3.1).

To show Lemma 3.1, we first need to show the following

Lemma 3.3. *Let w satisfy the hypothesis of Lemma 3.1, k is an integer, let B be a fixed ball, $E \subset B$, then there exist c_2, c_3 such that if*

$$|E| \ge \left(1 - c_2\lambda^{-\frac{3n}{4}}(\log\lambda)^{-1}\right)^k |B|,$$

then

$$\int_E w \ge \left(c_3\lambda^{-\frac{n}{4}}(\log\lambda)^{-1}\right)^k \int_B w,$$

where $c_2 = c(c_1), c_3 = c(c_0)$.

The proof of Lemma 3.3 is similar to the case in dimension two and that in [CM] by adapting the covering Theorem 1.5 proved in this paper. We shall only sketch the proof of Lemma 3.3.

Proof. We first want to show if $|E| \ge (1 - \bar{c}\lambda^{-\frac{n}{2}})|B|$ for some appropriate $\bar{c} = \bar{c}(c_1)$, then $\int_E w \ge 1/2\int_B w$. To show this, we first note that $|B\backslash E| \le \bar{c}\lambda^{-\frac{n}{2}}|B|$. Thus,

$$\int_{B\backslash E} w \le \left(\int_B w^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} |B\backslash E|^{\frac{2}{n}}$$
$$\le c_1\bar{c}^{1/q'}\lambda^{-\frac{1+\epsilon}{q'}+1}\int_B w \le c_1\bar{c}^{\frac{2}{n}}\int_B w.$$

If we choose $\bar{c}$ such that $c_1\bar{c}^{\frac{2}{n}} < 1/2$, then $\int_{B\backslash E} w \le 1/2\int_B w$, this implies $\int_E w > 1/2\int_B w$. Thus if $c_2 \le \bar{c}$, and $|E| \ge (1 - c_2\lambda^{-\frac{3n}{4}}(\log\lambda)^{-1})|B|$, then

$$\int_E w \ge 1/2\int_B w \ge c_3\left(\lambda^{-\frac{n}{4}}(\log\lambda)^{-1}\right)\int_B w,$$

and we are done for the case $k = 1$. Now we assume the statement is true for $k - 1$. We may assume $|E| \le (1 - \bar{c}\lambda^{-\frac{n}{2}})|B|$, otherwise, there is nothing to prove. Thus for each density point x of E, we can select a ball $B_x \subset B$ such that $x \in B_x$, and

$$\frac{|B_x \bigcap E|}{|B_x|} = 1 - \bar{c}\lambda^{-\frac{n}{2}}.$$

Applying the cover lemma Theorem 1.5 when $n \ge 3$ to the balls B_x with the choice $\delta = \lambda^{-1/2}$, and without loss of generality, assume B_x are finite, thus there exist a finite number of balls $\{B_i\}_{i=1}^N$ such that

$$(i)\quad \bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^N (1 + \lambda^{-\frac{1}{2}})B_i$$

and

$$(ii)\quad \sum_{i=1}^{N} \chi_{B_i}(x) \leq c\lambda^{\frac{n}{4}} \log \lambda,$$

where c is a constant independent of λ.

We then define

$$E_1 = \left[\bigcup_{i=1}^{N} (1+\lambda^{-1/2}) B_i \right] \bigcap B.$$

Then $E_1 \subset B$, and as the proof given in dimension two in Section 2 (see also [CM]), we can show

$$|E| \leq \left(1 - c_2 \lambda^{-\frac{3n}{4}} (\log \lambda)^{-1}\right) |E_1|.$$

Using the growth property (3.1) of w as assumed in Lemma 3.1 and combining it with the covering Theorem 1.5, we get

$$\int_{E_1} w \leq \sum_{i=1}^{N} \int_{(1+\lambda^{-\frac{1}{2}})B_i} w \leq c_0 \sum_{i=1}^{N} \int_{B_i} w.$$

By the choice of each B_i and the reverse Holder inequality assumption (3.2) in Lemma 3.1, one can show that

$$\int_{B_i \setminus E} w < \frac{1}{2} \int_{B_i \cap E} w.$$

Thus,

$$\begin{aligned}
\int_E w &\leq 2c_0 \sum_{i=1}^{N} \int_{B_i \cap E} w \\
&= 2c_0 \sum_{i=1}^{N} \sum_{i=1}^{N} \int_E \chi_{B_i} w \\
&= 2c_0 \int_E (\sum_{i=1}^{N} \chi_{B_i}) w \\
&\leq c_0 c \lambda^{\frac{n}{4}} \log(\lambda) \int_E w.
\end{aligned}$$

Therefore,

$$\int_E w \geq c_3 \lambda^{-\frac{n}{4}} (\log \lambda)^{-1} \int_{E_1} w$$

for some $c_2 = C(c_1)$, $c_3 = C(c_0)$. This completes the proof of Lemma 3.3. $\square$

Remark: The proof of Theorem 1.8 follows from Theorem 1.7 using the same techniques as done in [DF3]. We shall not repeat it here.

4. Proof of the covering lemma: Theorem 1.5

Before we give the proof of the new covering Theorem 1.5, let us give some definitions we will use in the proof. All the collections of balls here are finite. We also use the notation $B^\delta = (1+\delta)B$.

Definition 4.1 (δ-proper cover). *Given $\delta \geq 0$, a subcollection of balls $\{B_1, ..., B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ in $\mathbb{R}^n$ is called a δ-proper cover of $\{B_\alpha\}_{\alpha\in I}$ if*

$$\bigcup_{\alpha\in I} B_\alpha \subset \bigcup_{i=1}^{N} B_i^\delta \tag{4.1}$$

and

$$B_j \not\subset \bigcup_{i=1, i\neq j}^{N} B_i^\delta \tag{4.2}$$

for every $j = 1, ..., N$.

If $C = \{B_1, ..., B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ satisfies (4.1), then it is called a $\delta-$cover of $\{B_\alpha\}_{\alpha\in I}$. If $S = \{B_1, ..., B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ satisfies (4.2), then it is called a $\delta-$proper subcollection of $\{B_\alpha\}_{\alpha\in I}$.

Lemma 4.2. *Given a collection of balls $\{B_\alpha\}_{\alpha\in I}$ in $\mathbb{R}^n$, there exists $\delta_0 > 0$ such that for $\forall\delta \in [0, \delta_0]$, there exists a δ-proper cover of $\{B_\alpha\}_{\alpha\in I}$.*

Proof. We will prove Lemma 4.2 by induction on the cardinality of the collection of balls, namely $|I|$.

It is obvious that a collection of a single ball has a δ-proper cover for any $\delta \geq 0$. If this lemma is true for every collection $\{B_\alpha\}_{\alpha\in I}$ with $|I| \leq k$, we will prove that it is also true for $\{B_\alpha\}_{\alpha\in I}$ with $|I| = k+1$.

Case 1: If $\exists\ C_0 = \{B_1, ..., B_N\} \subsetneqq \{B_\alpha\}_{\alpha\in I}$ is a 0-cover of $\{B_\alpha\}_{\alpha\in I}$ (then $1 \leq N \leq k$), i.e.,

$$\bigcup_{\alpha\in I} B_\alpha \subset \bigcup_{i=1}^{N} B_i$$

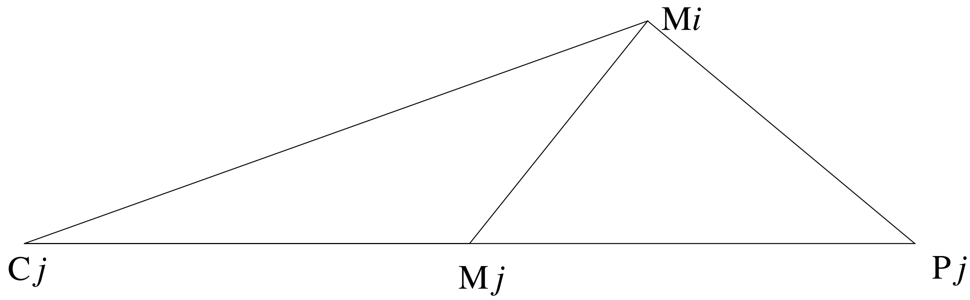
then by induction, $\exists\delta_0 > 0$, such that for $\forall\delta \in [0, \delta_0]$, there exists a δ-proper cover of C_0, which is then also a δ-proper cover of $\{B_\alpha\}_{\alpha\in I}$.

Case 2: Assume that there is no 0-cover of $\{B_\alpha\}_{\alpha\in I}$ with $|I| \leq k$. Let $\{B_\alpha\}_{\alpha\in I} = \{B_{\alpha_1}, ..., B_{\alpha_{k+1}}\}$. Then

$$B_{\alpha_j} \not\subset \bigcup_{i=1, i\neq j}^{N} B_{\alpha_i}$$

for every $j = 1, .., k+1$. Therefore, there exists a sufficiently small $\delta_j > 0$ such that

$$B_{\alpha_j} \not\subset \bigcup_{i=1, i\neq j}^{N} B_{\alpha_i}^{\delta_j}.$$



Let $\delta_0 = \min(\delta_1, ..., \delta_{k+1}) > 0$, we can see

$$B_{\alpha_j} \not\subset \bigcup_{i=1, i\neq j}^{N} B_{\alpha_i}^{\delta_0} \subset \bigcup_{i=1, i\neq j}^{N} B_i^{\delta_j}$$

for every $j = 1, .., k+1$, then $\{B_{\alpha_1}, ..., B_{\alpha_{k+1}}\}$ is the δ-proper subcollection of itself for $\delta \in [0, \delta_0]$, and thus the δ-proper cover of itself for $\delta \in [0, \delta_0]$. □

Lemma 4.3. *Let $\delta > 0$ be small enough, then given any collection of balls $\{B_\alpha\}_{\alpha\in I}$ in $\mathbb{R}^n$, and $S = \{B_1, ..., B_N\} \subset \{B_\alpha\}_{\alpha\in I}$ is a $\delta - proper$ subcollection of $\{B_\alpha\}_{\alpha\in I}$ with $r \le r_i \le 2r$ for $i = 1, ..., N$, where $r > 0$, then*

$$\sum_{i=1}^{N} \chi_{B_i}(x) \le c\delta^{-\frac{n}{2}} \tag{4.3}$$

for all $x \in \mathbb{R}^n$, where c is only dependent on n.

Proof. Let $x_0 \in \bigcap_{i=1}^{M} B_i$, $M = M(x_0)$. By a translation we may suppose $x_0 = 0$. For $x \in \mathbb{R}^n$, define $T_r(x) = x/r$, then $\{T_r(B_1), ..., T_r(B_M)\}$ is a $\delta - proper$ subcollection with the radius between 1 and 2. Without loss of generality, we may assume $1 \le r_i \le 2$ for $i = 1, ..., M$.

Now $0 \in \bigcap_{i=1}^{M} B_i$ and

$$B_j \not\subset \bigcup_{i=1, i\neq j}^{M} B_i^{\delta}, \tag{4.4}$$

which means $\forall\ j = 1, ..., M$, $\exists\ P_j \in B_j$ and $dist(P_j, C_i) \ge (1+\delta)r_i$, for all $i = 1, ..., M, i \neq j$, in which C_i is the center of B_i. Thus we have

$$|C_i| < 2,\ |P_i| < 4,\ for\ i = 1, ..., M.$$

Denote M_j as the midpoint of C_j and P_j:

$$|M_j| = |\frac{C_j + P_j}{2}| \le 4.$$

$$\overline{C_jM_i} + \overline{M_iP_i} \geq \overline{C_jP_i} \geq (1+\delta)r_j,$$
$$\overline{P_jM_i} + \overline{M_iC_i} \geq \overline{P_jC_i} \geq (1+\delta)r_i,$$
$$\overline{M_iP_i} = \overline{M_iC_i} \leq \frac{1}{2}r_i.$$

Thus,

$$\overline{P_jM_i} \geq (1+\delta)r_i - \frac{1}{2}r_i > 0$$

and

$$\overline{C_jM_i} \geq (1+\delta)r_j - \frac{1}{2}r_i > 0.$$

$$\overline{C_jM_j}^2 + \overline{M_jM_i}^2 - 2\cos(\pi-\theta)\cdot\overline{C_jM_j}\cdot\overline{M_jM_i} = \overline{C_jM_i}^2, \tag{4.5}$$

$$\overline{P_jM_j}^2 + \overline{M_jM_i}^2 - 2\cos\theta\cdot\overline{P_jM_j}\cdot\overline{M_jM_i} = \overline{P_jM_i}^2. \tag{4.6}$$

Combining (4.5) and (4.6), and noticing that $\overline{M_jP_j} = \overline{M_jC_j} \leq \frac{1}{2}r_j$,

$$\begin{aligned}
2\overline{M_jM_i}^2 &= \overline{C_jM_i}^2 + \overline{P_jM_i}^2 - \overline{C_jM_j}^2 - \overline{P_jM_j}^2 \\
&\geq ((1+\delta)r_i - \frac{1}{2}r_i)^2 + ((1+\delta)r_j - \frac{1}{2}r_i)^2 - 2(\frac{1}{2}r_j)^2 \\
&\geq (\frac{1}{2}+\delta)^2r_i^2 + (1+\delta)r_j^2 + \frac{1}{4}r_i^2 - (1+\delta)r_ir_j \\
&\geq (\frac{\delta}{2}+\delta^2)r_i^2 + \frac{\delta}{2}r_j^2 \\
&\geq \delta + \delta^2,
\end{aligned}$$

where in the above we have used the inequality that $r_ir_j \leq \frac{r_i^2+r_j^2}{2}$ and $r_i \geq 1$, $r_j \geq 1$. Then we get

$$\overline{M_jM_i} \geq \frac{\sqrt{\delta}}{\sqrt{2}}.$$

Hence, $M(\sqrt{\delta})^n \leq (4\sqrt{2})^n$, i.e., $M \leq 4^{n+\frac{1}{4}}\delta^{-\frac{n}{2}}$, and (4.3) of Lemma 4.3 follows. □

Now we will prove the main covering lemma: Theorem 1.5.

Proof. By Lemma 4.2, there $\exists\delta_0 > 0$ such that for $\forall\delta \in [0,\delta_0]$, there exists

$$\{B_1, ..., B_N\} \subset \{B_\alpha\}_{\alpha\in I}$$

as its δ-proper cover. Then it clearly satisfies (i) of Theorem (1.5), we now prove (ii) of Theorem (1.5).

Let $x_0 \in \bigcap_{i=1}^M B_i$, $M = M(x_0)$. By a translation we may suppose $x_0 = 0$. Now $0 \in \bigcap_{i=1}^M B_i$ and

$$B_j \not\subset \bigcup_{i=1, i\neq j}^{M} B_i^\delta. \tag{4.7}$$

Without loss of generality, we may assume $r_1 \leq ... \leq r_M$, since $0 \in B_1 \cap B_M$ and $B_1 \subsetneq B_M^\delta$, we have $2r_1 \geq \delta r_M$, then let $K = K(0) = 2 + \lfloor \log_2 \frac{1}{\delta} \rfloor$, where $\lfloor \cdot \rfloor$ denote the largest integer part, and let

$$S_j = \{B_i | 2^{j-1} r_1 \leq r_i < 2^j r_1, i = 1, ...M\}$$

for j=1,...,K. We can see S_j is a $\delta - proper$ subcollection of $\{B_\alpha\}_{\alpha \in I}$ for j=1,...,K, and since $2^K r_1 \geq r_M$, we have

$$\{B_1, ..., B_M\} \subset \bigcup_{j=1}^{K} S_j.$$

Denote $K_j = |S_j|$, thus $K_j \leq 4^n \delta^{-\frac{n}{2}}$ by Lemma 4.2. Then

$$M = \sum_{j=1}^{K} K_j \leq K 4^n \delta^{-\frac{n}{2}} \leq c\delta^{-\frac{n}{2}} \log \frac{1}{\delta} \tag{4.8}$$

and (ii) of Theorem 1.5 follows. □

5. The sharpness of Conjecture 1.6

In this section, we will show that if it is replaced by $c\delta^{-\frac{n-1}{2}}$ in the upper bound in Conjecture 1.6, then it is the sharp estimate. Let $S(x_0, r)$ denote the boundary of the ball $B(x_0, r)$ in $\mathbb{R}^n$. First, let us give two unit balls centered at C_1 and C_2 such that $C_1, C_2 \in S(0, \frac{1}{2}) \subset \mathbb{R}^n$, the sphere centered at the origin O with radius $\frac{1}{2}$.

Example 5.1. *In the figure below, we take C_1 and C_2 in $S(O, \frac{1}{2})$, and P_1 such that O, C_1, P_1 are on the same line and*

$$\overline{OC_1} = \overline{OC_2} = \frac{1}{2},$$

$$\overline{C_1P_1} = 1,\ \overline{C_2P_1} = 1 + \delta.$$

Consider the plane formed by O, C_1, C_2, P_1 and let $\theta = \angle C_1OC_2$, we have

$$\cos\theta = \frac{\overline{OC_2}^2 + \overline{OP_1}^2 - \overline{C_2P_1}^2}{2\overline{OC_2} \cdot \overline{OP_1}} \sim 1 - \frac{\theta^2}{2},$$

$$\theta \sim \sqrt{\frac{8\delta + 4\delta^2}{3}} \sim 2\sqrt{\frac{2\delta}{3}},$$

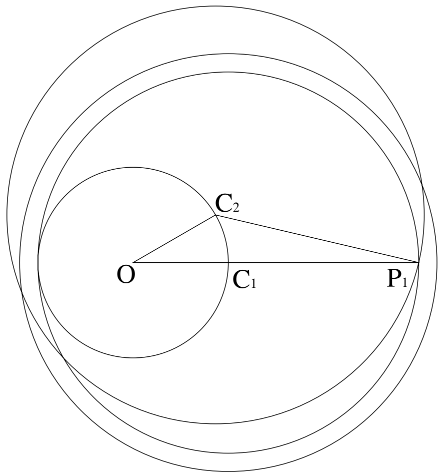
$$\overline{C_1C_2} \sim \frac{1}{2}\theta \sim \sqrt{\frac{2\delta}{3}}.$$

We now consider the given family of unit balls $\{B_1, ..., B_N\}$ with centers of lattices $C_i \in S(0, \frac{1}{2})$ for i=1,...,N of size $\sqrt{\delta}$ (roughly speaking), thus it satisfies

$$\overline{C_iC_j} \geq \sqrt{\delta} \tag{5.1}$$

for any two centers C_i and C_j.

Since $\overline{C_iC_j} \geq \sqrt{\delta} > \sqrt{2\delta/3}$, $\exists P_j \in B_j$ such that $P_j \notin \bigcup_{i=1, i\neq j}^{M} B_i^\delta$ for j=1,...,N, and each ball must be selected such that the requirements in the covering lemma are satisfied.



Obviously,

$$N \sim \frac{c(\frac{1}{2})^{n-1}}{(\sqrt{\delta})^{n-1}} \sim c\delta^{-\frac{n-1}{2}}$$

and

$$\sum_{i=1}^{N} \chi_{B_i}(O) = N \sim c\delta^{-\frac{n-1}{2}}.$$

We can provide a more precise example in $\mathbb{R}^2$*:*

Consider a family of unit discs centered at $(\frac{1}{2}\cos\theta_i, \frac{1}{2}\sin\theta_i) \in S(0, \frac{1}{2}) \subset \mathbb{R}^2$, $\theta_i = 2i\sqrt{\delta}$, *i=1,...,N=*$\lfloor \frac{\pi}{\sqrt{\delta}} \rfloor - 1$. *Then for any two centers* C_i *and* C_j, $\overline{C_iC_j} \gtrsim \sqrt{\delta} > \sqrt{2\delta/3}$ *when* δ *is small enough, and each disc must be selected to satisfy the criterion in the covering Theorem 1.5. Then*

$$\sum_{i=1}^{N} \chi_{B_i}(O) = \lfloor \frac{\pi}{\sqrt{\delta}} \rfloor - 1 \sim c\delta^{-\frac{1}{2}}.$$

Added in Proof: After our paper was submitted for publication, there have been some recent progress on lower bounds estimates of Hausdorff measures on nodal sets of eigenfunctions by C. Sogge and S. Zelditch (Lower bounds on the Hausdorff measure of nodal sets, Math. Research Letters, 18 (2011), 25-37), T. Colding and W. Minicozzi (Lower bounds for nodal sets of eigenfunctions, To appear in Communications in Mathematical Physics.) and D. Mangoubi (A remark on recent lower bounds for nodal sets, `http://arxiv.org/PS_cache/arxiv/pdf/1010/1010.4579v2.pdf`).

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