J Geom Anal (2010) 20: 670–689
DOI 10.1007/s12220-010-9123-6

Discrete Calderón's Identity, Atomic Decomposition and Boundedness Criterion of Operators on Multiparameter Hardy Spaces

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Received: 7 April 2009 / Published online: 23 March 2010

Abstract In this paper we establish a discrete Calderón's identity which converges in both $L^q(\mathbb{R}^{n+m})$ $(1 < q < \infty)$ and Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ $(0 < p \le 1)$. Based on this identity, we derive a new atomic decomposition into (p,q)-atoms $(1 < q < \infty)$ on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $0 < p \le 1$. As an application, we prove that an operator T, which is bounded on $L^q(\mathbb{R}^{n+m})$ for some $1 < q < \infty$, is bounded from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^{n+m})$ if and only if T is bounded uniformly on all (p,q)-product atoms in $L^p(\mathbb{R}^{n+m})$. The similar result from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is also obtained.

Keywords Boundedness · Calderón-Zygmund operator · Calderón's identity · Multiparameter Hardy spaces · Atomic decomposition · Boundedness criterion of operators

Mathematics Subject Classification (2000) 42B30 · 42B20

Communicated by Richard Rochberg.

G. Lu is partly supported by US NSF grants DMS0500853 and DMS0901761 and NSFC of China grant No. 10710207. K. Zhao is partly supported by NNSF-China No. 10671115.

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1 Introduction

The "atoms" and "molecules" were introduced (see Coifman [6], Latter [19], Coifman-Weiss [7], etc.) and were used in harmonic and wavelet analysis (see Meyer [21], Meyer-Coifman [22] and Stein [24], etc.) to form the basic building blocks of various function spaces. Since then, the study of the operators acting on a space of functions or distributions can become very simple when the elements of the space admit "atomic decompositions". Indeed, many problems in analysis have natural formulations as questions of continuity of linear operators defined on spaces of functions or distributions. Such questions can often be answered by rather straightforward techniques if they can first be reduced to the study of the operator on an appropriate class of simple functions which generate the entire space in some appropriate sense. This fundamental principle was applied by many people to problems where atomic decompositions exist. However, recently Bownik in [1] gave an example of a linear functional defined on a dense subspace of Hardy space $H^1(\mathbb{R}^n)$, which maps all $(1,\infty)$-atoms into bounded scalars, but it cannot be extended to a bounded functional on the whole space $H^1(\mathbb{R}^n)$. The construction in [1] is based on the fact due to Meyer [21], which states that quasi-norms corresponding to finite and infinite atomic decompositions in H^p, $0 < p \leq 1$, are not equivalent. We note that $H^1(\mathbb{R}^n)$ can be generated by $(1,\infty)$-atoms. This example shows that one has to be careful when the above fundamental principle is used. It is then natural to ask under what extra conditions the above fundamental principle can be used. Nevertheless, Meda et al. in [20] have recently proved that if T is a linear operator defined on the subspace $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^n)$ of finite linear combinations of $(1,q)$ atoms in $\mathbb{R}^n$ with the property that

$$\sup\{\|Ta\|_Y : a \text{ is a } (1,q)\text{-atom}\} < \infty, \quad 1 < q < \infty,$$

then T admits a (unique) continuous extension to a bounded linear operator from $H^1(\mathbb{R}^n)$ to Y (Banach space) (see also Han and Zhao [13] for related results). Furthermore, Ricci and Verdera have shown that if $0 < p < 1$ and T is a linear operator defined on the subspace $H^{p,\infty}_{\mathrm{fin}}(\mathbb{R}^n)$ of finite linear combinations of (p,∞) atoms in $\mathbb{R}^n$ with the property that

$$\sup\{\|Ta\|_Y : a \text{ is a } (p,\infty)\text{-atom}\} < \infty,$$

then T admits a (unique) continuous extension to a bounded linear operator from $H^p(\mathbb{R}^n)$ to Y (Banach space). Moreover, the structure of the space F^p of finite linear combinations of (p,∞) atoms, the completion of F^p and the dual spaces were thoroughly studied in [23].

The main purpose of this paper is to establish a discrete Calderón's identity in multiparameter setting and apply this identity to provide a new atomic decomposition of (p,q)-atoms of multiparameter Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. As an application of this new atomic decomposition, we will prove the uniform boundedness criterion of linear operators in multiparameter setting by considering its action on all (p,q)-atoms for $0 < p \leq 1$ and $1 < q < \infty$, which is similar to the one-parameter setting.

Atomic decomposition in multiparameter Hardy spaces is much more complicated. It is well-known that there is a basic obstacle to the pure product Hardy and

BMO space theory associated with multiparameter product dilations. Indeed it was conjectured that the product atomic Hardy space on $\mathbb{R} \times \mathbb{R}$ could be defined by rectangle atoms. Here a rectangle atom is a function $a(x, y)$ supported on a rectangle $R = I \times J$ having the property that

$$\|a\|_2 \le |R|^{1/2}, \qquad \int_I a(x, y)\, dx = \int_J a(x, y)\, dy = 0$$

for every $(x, y) \in \mathbb{R}$. Then $H^1_{\text{rect}}(\mathbb{R} \times \mathbb{R})$ is the space of functions $\sum_k \lambda_k a_k$ with each a_k a rectangle atom and $\sum_k |\lambda_k| < \infty$. However, this conjecture was disproved by Carleson by constructing a counter-example of a measure satisfying the product form of the Carleson measure, that is, the measure μ satisfies

$$\int_{S(I)\times S(J)} d\mu \le C|I \times J|$$

for all intervals I, J in $\mathbb{R}$ and $S(I)$ is the Carleson region associated with I. Carleson [2] showed that the measure he constructed is not bounded on the product Hardy space $H^1(\mathbb{R} \times \mathbb{R})$.

This led to the role of cubes in the classical atomic decomposition of $H^p(\mathbb{R}^n)$ being replaced by arbitrary open sets of finite measures in the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and the Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ theory was finally developed by Chang and Fefferman [3–5]. Chang and Fefferman [4] proved the following

Theorem *$f \in H^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ if and only if $f(x, y) = \sum_k \lambda_k a_k(x, y)$ where $\sum_k |\lambda_k|^p < \infty$ and $a_k(x, y)$ are $(p, 2)$-atoms, that is, each $a_k(x, y)$ is supported in an open set Ω with finite measure satisfying the following properties*:

$$\|a_k\|_2 \le |\Omega|^{1/2-1/p};$$

each $a_k(x, y)$ can be further decomposed by

$$a_k(x, y) = \sum_{R\in\Omega} a_R(x, y)$$

where $R = I \times J \subset \Omega$ are dyadic rectangles in $\mathbb{R}^2$, and $a_R(x, y)$ satisfy

$$\int_I a_R(x, y)x^\alpha\, dx = \int_J a_R(x, y)y^\beta\, dy = 0$$

for $0 \le |\alpha|, |\beta| \le N_p = [2/p - 4/3]$, and a_R is a C^α, $\alpha \le N_p + 1$, function satisfying

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} a_R(x, y)\right| \le d_R|I|^{-\alpha}, \qquad \left|\frac{\partial^\alpha}{\partial y^\alpha} a_R(x, y)\right| \le d_R|J|^{-\alpha}$$

with

$$\sum_{R\in m(\Omega)} |R|d_R^2 \le |\Omega|^{1-2/p},$$

where $m(\Omega)$ is the set of all rectangles in Ω which are maximal in both directions of x and y.

It is worthwhile to point out that the key tool, using the maximal characterization of the one-parameter $H^p(\mathbb{R}^n)$, to get the atomic decomposition is a refined Calderón-Zygmund decomposition; see [6, 7, 19]. This maximal characterization of the one-parameter $H^p(\mathbb{R}^n)$ is the main tool used in the works of [20, 23]. However, it seems to be difficult to carry this method out to the multiparameter product spaces. Using the classical version of continuous Calderón's identity on the product space, A. Chang and R. Fefferman established the above atomic decomposition for $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, by use of $(p,2)$-atoms. We would like to point out that this classical version of Calderón's identity does not seem to lead to an atomic decomposition by (p,q)-atoms for $1 < q < \infty, q \neq 2$. Thus, deriving an atomic decomposition in the product spaces using (p,q)-atoms for $q \neq 2$ becomes interesting.

One of the main purposes of this paper is to establish all $(p,q), 1 < q < \infty$, atomic decomposition for multiparameter Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $0 < p \leq 1$ with the convergence in both $L^q(\mathbb{R}^{n+m})$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ norms (see Theorem 1.2 below). It should be pointed out that the series in $(p,2)$-atomic decomposition in [3] converges only in the sense of distributions. As we pointed out earlier, this classical continuous version of Calderón's identity is not applicable to provide an atomic decomposition by (p,q)-atoms for $1 < q < \infty, q \neq 2$. The crucial idea to prove $(p,q), 0 < p \leq 1 < q < \infty$, atomic decomposition is to establish a new discrete Calderón's identity on the product space, which converges both in $L^q(\mathbb{R}^{n+m})$ norm for $1 < q < \infty$ and the $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ norm (see Theorem 1.1 below).

The idea of using discrete Calderón's identity in product spaces was developed earlier in [10, 11], but in a quite different way. To be more precise, to develop multiparameter Hardy space theory associated with flag singular integrals and the Zygmund dilation on $\mathbb{R}^3$, the first two authors introduced the multiparameter test function spaces associated with flag singular integrals and the Zygmund dilation, respectively, and provided the discrete Calderón's identities on these test function spaces. Using these discrete identities, they proved the Min-Max comparison inequalities and finally the Hardy spaces $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ and $H^p_Z(\mathbb{R}^3)$ were well defined; see [10, 11] for more details. Moreover, boundedness of singular integral operators from H^p to H^p and from H^p to L^p were proved in [10, 11] without using Journé's covering lemma. In particular, we developed a general method to derive $H^p \to L^p$ boundedness from $H^p \to H^p$ boundedness of linear operators for the product spaces using discrete Calderón's identity and discrete Littlewood-Paley theory. The crucial idea there is to prove the following inequality

$$\|f\|_{L^p} \leq C\|f\|_{H^p} \quad \text{for } f \in L^q \cap H^p,\ 1 < q < \infty,\ 0 < p \leq 1. \tag{1.1}$$

Using this inequality, the H^p to L^p boundedness follows immediately from the boundedness on H^p spaces. This general philosophy in the product spaces of homogeneous type has also been established in [17]. The duality theory for multiparameter Hardy spaces has been established in [16] using the idea of discretization.

In this present paper, using a different approach from those in [10, 11], namely Coifman's decomposition of the identity operator and the Calderón-Zygmund operator theory on the multiparameter product space, we first derive the following discrete Calderón-type identity:

Theorem 1.1 *Suppose* $0 < p \le 1$. *Let* $\psi^{(1)}(x_1)$, $\psi^{(2)}(x_2)$ *be Schwartz functions with support on the unit ball satisfying the conditions*: *for a fixed large integer* M *depending on* p,

$$\sum_j |\widehat{\psi}^{(1)}(2^{-j}\xi_1)|^2 = 1$$

for all $\xi_1 \in \mathbb{R}^n \setminus \{0\}$, *and* $\int_{\mathbb{R}^n} \psi^{(1)}(x_1)x_1^{\alpha}\, dx_1 = 0$ *for all* $0 \le |\alpha| \le 2M$; *and*

$$\sum_k |\widehat{\psi}^{(2)}(2^{-k}\xi_2)|^2 = 1$$

for all $\xi_2 \in \mathbb{R}^m \setminus \{0\}$, *and* $\int_{\mathbb{R}^m} \psi^{(2)}(x_2)x_2^{\beta}\, dx_2 = 0$ *for all* $0 \le |\beta| \le 2M$.

Let $\psi_{j,k}(x_1, x_2) = 2^{jn+km}\psi^{(1)}(2^j x_1)\psi^{(2)}(2^k x_2)$. Then for each function $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^q(\mathbb{R}^{n+m})$, $1 < q < \infty$, there exists a function $g \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^q(\mathbb{R}^{n+m})$ with $c_1\|f\|_q \le \|g\|_q \le c_2\|f\|_q$ and $c_1\|f\|_{H^p} \le \|g\|_{H^p} \le c_2\|f\|_{H^p}$, where the constants c_1 and c_2 are independent of f, such that

$$f(x_1, x_2) = \sum_j \sum_k \sum_I \sum_J |I \times J|\psi_{j,k}(x_1 - x_I, x_2 - x_J)\psi_{j,k} * g(x_I, y_J), \quad (1.1)$$

where $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes, $l(I) = 2^{-j-N}, l(J) = 2^{-k-N}$ for some large integer N depending on ψ and p, q, M; x_I, y_J are any fixed points in I and J, respectively; the series in (1.1) converges in both the norms of $L^q(\mathbb{R}^{n+m})$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.

As a consequence, this discrete Calderón-type identity gives the following atomic decomposition in terms of (p, q)-atoms for the multiparameter product Hardy space.

Theorem 1.2 *Suppose* $0 < p \le 1$. *Let* $f \in L^q(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 < q < \infty$. *Then there exists a sequence of* (p, q)-*product atoms* $\{a_j\}$ *of* $H^p(\mathbb{R}^n \times \mathbb{R}^n)$ *and a sequence of scalars* $\{\lambda_j\}$ *with* $(\sum_j |\lambda_j|^p)^{\frac{1}{p}} \le C\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}$, *where the constant* C *is independent of* f, *such that* $f = \sum_j \lambda_j a_j$, *where the series converges to* f *in both the* $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ *and* $L^q(\mathbb{R}^{n+m})$ *norms.*

Here, a function a is said to be a (p, q)-product atom of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, $0 < p \le 1 < q < \infty$, if a satisfies

(1) $\operatorname{Supp} a \subset \Omega$, where Ω is an open set of $\mathbb{R}^{n+m}$ with finite measure.
(2) $\|a\|_{L^q(\mathbb{R}^{n+m})} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}}$.

Moreover, a can be further decomposed into a rectangle (p, q)-atom a_R associated to the rectangle $R = I \times J$ which is supported in $3R$ and such that

(3a) For $2 \le q < \infty$, $a = \sum_{R\in m(\Omega)} a_R$, and $(\sum_{R\in m(\Omega)} \|a_R\|^q_{L^q(\mathbb{R}^{n+m})})^{1/q} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}}$.

(3b) For $1 < q < 2, a = \sum_{R\in m_1(\Omega)} a_R + \sum_{R\in m_2(\Omega)} a_R$, and for any $\delta > 0$, there exists a constant $C_{q,\delta>0}$, where $C_{q,\delta}$ depends only on q and δ, such that

$$\left(\sum_{R\in m_1(\Omega)} \gamma_2^{-\delta} \|a_R\|^q_{L^q(\mathbb{R}^{n+m})} + \sum_{R\in m_2(\Omega)} \gamma_1^{-\delta} \|a_R\|^q_{L^q(\mathbb{R}^{n+m})}\right)^{1/q} \leq C_{q,\delta}|\Omega|^{\frac{1}{q}-\frac{1}{p}}.$$

Here and in the sequel, $m(\Omega)$ is the set of all maximal rectangles contained in Ω in both directions x_1 and x_2 and $\widetilde{\Omega} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m : M_s(\chi_\Omega)(x_1, x_2) > \frac{1}{100}\}$, where M_s is the strong maximal function. $m_1(\Omega)$ is the set of all rectangles contained in Ω and maximal in the x_1 direction and $m_2(\Omega)$ is defined similarly. γ_1 is defined by $\gamma_1(R) = \frac{|l|}{|I|}$, where $R = I \times J \in m_2(\Omega)$ and l is the maximal dyadic cube such that $l \times J \in \widetilde{\Omega}$. $\gamma_2(R)$ is similarly defined.

(4) (Cancellation conditions): for $0 \leq |\alpha|, |\beta| \leq N_p = n[\frac{1}{p} - 1] + m[\frac{1}{p} - 1]$,

$$\int_{3I} a_R(x, y)x^\alpha\, dx = 0 \quad \text{for a.e. } y \in 3J,$$

$$\int_{3J} a_R(x, y)y^\beta\, dy = 0 \quad \text{for a.e. } x \in 3I.$$

We would like to point out that the condition given in (3b) for (p, q)-atoms when $1 < q < 2$ is different from the original one as given in (3a) for $q = 2$ by [9]. However, Fefferman's boundedness criterion of an operator by considering its action on rectangle atoms [9] still works for (p, q)-atoms for $1 < q < 2$ under the condition of (3b). We refer the reader to the Appendix for the outline of the proof. We remark here that it is easy to see that $L^q(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $0 < p \leq 1 < q < \infty$.

Finally, we are able to prove a uniform boundedness criterion for an operator on the product Hardy spaces by considering its action on (p, q)-atoms. This is the following

Theorem 1.3 *Suppose* $0 < p \leq 1$. *Let* T *be a linear operator which is bounded on* $L^q(\mathbb{R}^{n+m})$ *for some* $1 < q < \infty$. *Then*

(1) *T is bounded from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^{n+m})$ if and only if $\|Ta\|_{L^p(\mathbb{R}^{n+m})} \leq C$ for all (p, q)-product atoms*;
(2) *T is bounded on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if $\|Ta\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C$ for all (p, q)-product atoms*, *where the constant C is independent of a.*

The organization of the paper is as follows. In Sect. 2 we prove a new discrete Calderón-type identity on the product space which converges both in $L^q(\mathbb{R}^{n+m})$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ norms (Theorem 1.1). This is the key ingredient of this paper. Next, we provide an atomic decomposition in terms of (p, q)-atoms for the multiparameter product Hardy space (Theorem 1.2). Section 3 gives the proof of Theorem 1.3.

2 Discrete Calderón's Identity on Product Spaces

As we pointed out in the introduction, the discrete Calderón's identity in product spaces plays a crucial role in establishing the (p,q)-atoms in product spaces for all $1 < q < \infty$. The main purpose of this section is to prove this identity which is of independent interest.

We first recall some basic definitions.

Let $\psi^{(1)}(x_1), \psi^{(2)}(x_2)$ be Schwartz functions satisfying $\sum_j |\widehat{\psi}^{(1)}(2^{-j}\xi_1)|^2 = 1$ for all $\xi_1 \in \mathbb{R}^n \setminus \{0\}$, and $\int_{\mathbb{R}^n} \psi^{(1)}(x_1)x_1^\alpha \, dx_1 = 0$ for all $|\alpha| \geq 0$; and $\sum_k |\widehat{\psi}^{(2)}(2^{-k}\xi_2)|^2 = 1$ for all $\xi_2 \in \mathbb{R}^m \setminus \{0\}$, and $\int_{\mathbb{R}^m} \psi^{(2)}(x_2)x_2^\beta \, dx_2 = 0$ for all $|\beta| \geq 0$.

Definition 2.1 Suppose that $f(x_1,x_2) \in \mathcal{S}'(\mathbb{R}^{n+m})$, the space of tempered distributions. Let $\psi^{(1)}(x_1), \psi^{(2)}(x_2)$ be functions as above. The Littlewood-Paley function of f, $g(f)$, is defined by

$$g(f)(x_1,x_2) = \left\{ \sum_{j,k} |\psi_{jk} * f(x_1,x_2)|^2 \right\}^{\frac{1}{2}},$$

where $\psi_{jk}(x_1,x_2) = 2^{jn+km}\psi^{(1)}(2^j x_1)\psi^{(2)}(2^k x_2)$.

Similar to [4], the above Littlewood-Paley function can be used to characterize the product Hardy spaces as follows.

Definition 2.2 The product Hardy space $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is defined by

$$H^p(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in \mathcal{S}'(\mathbb{R}^{n+m}) : g(f) \in L^p(\mathbb{R}^{n+m})\}$$

with $\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} = \|g(f)\|_{L^p(\mathbb{R}^{n+m})}$.

We now recall a variant of the boundedness result of operators on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ proved in [14, 15], which is sufficient for our purpose here.

Theorem 2.1 *For given $0 < p \leq 1$, suppose that T is a bounded operator on $L^2(\mathbb{R}^{n+m})$ and it is associated with the kernel $K(x_1,x_2,y_1,y_2)$ given by*

$$Tf(x_1,x_2) = \int K(x_1,x_2,y_1,y_2)f(y_1,y_2)\,dy_1\,dy_2, \tag{2.1}$$

where the kernel $K(x_1,x_2,y_1,y_2)$ is defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m$ and there exist constants $C > 0$ and a large integer $M > \frac{n+m}{p} + 100$ such that for all $0 \leq |\alpha|, |\beta|, |\theta|, |\gamma| \leq M$,

$$|\partial_{x_1}^\alpha \partial_{x_2}^\beta \partial_{y_1}^\theta \partial_{y_2}^\gamma K(x_1,x_2,y_1,y_2)| \leq \frac{C}{|x_1-y_1|^{n+|\alpha|+|\theta|}|x_2-y_2|^{m+|\beta|+|\gamma|}}, \tag{2.2}$$

and

$$\int K(x_1, x_2, y_1, y_2)x_1^{\alpha}\, dx_1 = 0, \quad \textit{for all } x_2 \in \mathbb{R}^m,\ y_1 \in \mathbb{R}^n,\ y_2 \in \mathbb{R}^m,$$

$$\int K(x_1, x_2, y_1, y_2)x_2^{\beta}\, dx_2 = 0, \quad \textit{for all } x_1 \in \mathbb{R}^n,\ y_1 \in \mathbb{R}^n,\ y_2 \in \mathbb{R}^m,$$

$$\int K(x_1, x_2, y_1, y_2)y_1^{\theta}\, dy_1 = 0, \quad \textit{for all } x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m,\ y_2 \in \mathbb{R}^m,$$

and

$$\int K(x_1, x_2, y_1, y_2)y_2^{\gamma}\, dy_2 = 0, \quad \textit{for all } x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m,\ y_1 \in \mathbb{R}^n.$$

Then T is bounded on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.

We remark that the necessary and sufficient conditions of the product H^p boundedness of operators of Journé's class nonconvolution type whose kernels satisfy weaker smoothness conditions are given in [15] for the two-parameter setting and in [14] for an arbitrary number of parameters. We also mention that the proof of Theorem 1.1 follows from the same idea in [11] of the discrete Calderón's identity in the case of flag singular integrals (but simpler in our pure product case here). But we will provide a somewhat different proof here by invoking Theorem 2.1.

Proof of Theorem 1.1 Based on the conditions on $\psi_{j,k}$, where M is chosen by $M > \frac{n+m}{p} + 100$, the classical Calderón's identity on $L^2(\mathbb{R}^{n+m})$ is given by

$$f(x_1, x_2) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x_1, x_2). \tag{2.3}$$

Using Coifman's decomposition of the identity, we write

$$\begin{aligned} f(x_1, x_2) &= \sum_j \sum_k \int_{\mathbb{R}^n \times \mathbb{R}^m} \psi_{j,k}(x_1 - u, x_2 - v)\psi_{j,k} * f(u, v)\, du\, dv \\ &= \sum_j \sum_k \sum_I \sum_J \int_{I \times J} \psi_{j,k}(x_1 - u, x_2 - v)\psi_{j,k} * f(u, v)\, du\, dv \\ &= \sum_j \sum_k \sum_I \sum_J |I \times J|\psi_{j,k}(x_1 - x_I, x_2 - x_J)\psi_{j,k} * f(x_I, x_J) \\ &\quad + R_N(f)(x_1, x_2), \end{aligned} \tag{2.4}$$

where $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are dyadic cubes , $l(I) = 2^{-j-N}, l(J) = 2^{-k-N}$ for some large integer N which will be chosen later; x_I, y_J are any fixed points in I and J, respectively; and

$$R_N(f)(x_1,x_2)=\sum_j\sum_k\sum_I\sum_J\int_{I\times J}[\psi_{j,k}(x_1-u,x_2-v)\psi_{j,k}*f(u,v)$$
$$-\psi_{j,k}(x_1-x_I,x_2-x_J)\psi_{j,k}*f(x_I,x_J)]\,du\,dv.$$

Now, we prove that R_N is bounded on $L^2(\mathbb{R}^{n+m})$ and $H^p(\mathbb{R}^n\times\mathbb{R}^m)$. More precisely, we show

$$\|R_N(f)\|_{L^2(\mathbb{R}^{n+m})}\le C2^{-N}\|f\|_{L^2(\mathbb{R}^{n+m})} \tag{2.5}$$

and

$$\|R_N(f)\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}\le C2^{-N}\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{2.6}$$

To do this, we rewrite $R_N(f)(x_1,x_2)$ for

$$\begin{aligned}
&R_N(f)(x_1,x_2)\\
&=\sum_{j,k}\sum_{I,J}\int_{I\times J}\{[\psi_{j,k}(x_1-u,x_2-v)-\psi_{j,k}(x_1-x_I,x_2-x_J)]\psi_{j,k}*f(u,v)\\
&\quad+\psi_{j,k}(x_1-x_I,x_2-x_J)[\psi_{j,k}*f(u,v)-\psi_{j,k}*f(x_I,x_J)]\}\,du\,dv\\
&=\int_{\mathbb{R}^n\times\mathbb{R}^m}R_N(x_1,x_2,y_1,y_2)f(y_1,y_2)\,dy_1\,dy_2,
\end{aligned}$$

where $R_N(x_1,x_2,y_1,y_2)$, the kernel of $R_N(f)$, is given by

$$\begin{aligned}
&R_N(x_1,x_2,y_1,y_2)\\
&=\sum_{j,k}\sum_{I,J}\int_{I\times J}[\psi_{j,k}(x_1-u,x_2-v)-\psi_{j,k}(x_1-x_I,x_2-x_J)]\\
&\quad\times\psi_{j,k}(u-y_1,v-y_2)\,du\,dv\\
&\quad+\sum_{j,k}\sum_{I,J}\int_{I\times J}[\psi_{j,k}(u-y_1,v-y_2)-\psi_{j,k}(x_I-y_1,x_J-y_2)]\\
&\quad\times\psi_{j,k}(x_1-x_I,x_2-x_J)\,du\,dv\\
&=R_{N,1}(x_1,x_2,y_1,y_2)+R_{N,2}(x_1,x_2,y_1,y_2).
\end{aligned}$$

We now verify that the kernel $R_N(x_1,x_2,y_1,y_2)$ satisfies the condition (2.2) with constant $C2^{-N}$ and then prove (2.5).

For $R_{N,1}(x_1,x_2,y_1,y_2)$, since ψ is a Schwartz function, thus

$$\begin{aligned}
&|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\psi_{j,k}(x_1-u,x_2-v)-\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\psi_{j,k}(x_1-x_I,x_2-x_J)|\\
&\le|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\psi_{j,k}(x_1-u,x_2-v)-\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\psi_{j,k}(x_1-x_I,x_2-v)|\\
&\quad+|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\psi_{j,k}(x_1-x_I,x_2-v)-\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\psi_{j,k}(x_1-x_I,x_2-x_J)|
\end{aligned}$$

$$\begin{aligned}
&\leq C\left[\left(\frac{|u-x_I|}{2^{-j}+|x_1-u|}\right)\frac{2^{j|\alpha|}2^{-j(2M+1)}}{(2^{-j}+|x_1-u|)^{n+2M+1}}\frac{2^{-k(2M+1)}}{(2^{-k}+|x_2-v|)^{m+2M+1}}\right]\\
&\quad+C\left[\frac{2^{-j}}{(2^{-j}+|x_1-x_I|)^{n+2M+1}}\left(\frac{|v-x_J|}{2^{-k}+|x_2-v|}\right)\frac{2^{k|\beta|}2^{-k(2M+1)}}{(2^{-k}+|x_2-v|)^{m+2M+1}}\right]\\
&\leq C2^{-N}\frac{2^{j|\alpha|}2^{-j}}{(2^{-j}+|x_1-u|)^{n+2M+1}}\frac{2^{k|\beta|}2^{-k}}{(2^{-k}+|x_2-v|)^{m+2M+1}},
\end{aligned}$$

where we use the facts that $u, x_I \in I$, $v, x_J \in J$ and $l(I)=2^{-j-N}$, $l(J)=2^{-k-N}$.

Noting that $0\leq|\alpha|, |\beta|\leq M$, then

$$\begin{aligned}
&|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\partial_{y_1}^{\theta}\partial_{y_2}^{\gamma}R_{N,1}(x_1,x_2,y_1,y_2)|\\
&\quad\leq C2^{-N}\sum_{j,k}\int_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{j|\alpha|}2^{-j(2M+1)}}{(2^{-j}+|x_1-y_1|)^{n+2M+1}}\\
&\qquad\times\frac{2^{k|\beta|}2^{-k(2M+1)}}{(2^{-k}+|x_2-y_2|)^{m+2M+1}}|\partial_{y_1}^{\theta}\partial_{y_2}^{\gamma}\psi_{j,k}(u-y_1,v-y_2)|\,du\,dv\\
&\quad\leq C2^{-N}\frac{1}{|x_1-y_1|^{n+|\alpha|+|\theta|}}\frac{1}{|x_2-y_2|^{m+|\beta|+|\gamma|}}.
\end{aligned}$$

This implies that $R_{N,1}(x_1,x_2,y_1,y_2)$ satisfies (2.2) with constant $C2^{-N}$. The same results hold for $R_{N,2}(x_1,x_2,y_1,y_2)$, and hence (2.2) holds for $R_N(x_1,x_2,y_1,y_2)$ with the constant $C2^{-N}$ where C depends only on ψ and M.

To show (2.5), we use Cotlar-Stein's lemma on $L^2(\mathbb{R}^{n+m})$. To be more precise, let $R_{N,1}(f)=\sum_{j,k}R_{j,k}(f)$ where

$$\begin{aligned}
&R_{j,k}(f)(x_1,x_2)\\
&\quad=\sum_{I,J}\int_{I\times J}[\psi_{j,k}(x_1-u,x_2-v)-\psi_{j,k}(x_1-x_I,x_2-x_J)]\psi_{j,k}*f(u,v)\,du\,dv.
\end{aligned}$$

By the proof of the estimates for the kernel of R_N given above, it is easy to see that $\|R_{j,k}(f)\|_{L^2(\mathbb{R}^{n+m})}\leq C2^{-N}\|f\|_{L^2(\mathbb{R}^{n+m})}$ for all j,k. Using an almost orthogonality argument, see [12, 22], yields

$$\begin{aligned}
&|R_{j,k}R^*_{j',k'}(x_1,x_2,y_1,y_2)|\\
&\quad\leq C2^{-N}2^{|j-j'|}2^{|k-k'|}\frac{2^{-(j\wedge j')}}{(2^{-(j\wedge j')}+|x_1-y_1|)^{n+1}}\frac{2^{-(k\wedge k')}}{(2^{-(k\wedge k')}+|x_2-y_2|)^{m+1}},
\end{aligned}$$

where $a\wedge b$ is the minimum of a and b, and $R_{j,k}R^*_{j',k'}(x_1,x_2,y_1,y_2)$ is the kernel of operator $R_{j,k}R^*_{j',k'}$.

This implies

$$\|R_{j,k}R^*_{j',k'}(f)\|_{L^2(\mathbb{R}^{n+m})}\leq C2^{-N}2^{|j-j'|}2^{|k-k'|}\|f\|_{L^2(\mathbb{R}^{n+m})}.$$

Similarly, we obtain

$$\|R_{j,k}^*R_{j',k'}(f)\|_{L^2(\mathbb{R}^{n+m})} \le C2^{-N}2^{|j-j'|}2^{|k-k'|}\|f\|_{L^2(\mathbb{R}^{n+m})}.$$

Thus the proof of the estimate in (2.5) follows from Cotlar-Stein's lemma. We remark that the estimate in (2.5) can also be obtained by use of Journé's $T1$ theorem on product space $L^2(\mathbb{R}^{n+m})$. Indeed, by the regularity and cancellation conditions of R_N, one only needs to check the weak boundedness conditions with the bound by $C2^{-N}$; see [8] for a similar proof for one-parameter case. We leave the details to the interested reader. Applying Theorem 2.1 yields (2.6).

If we choose N big enough so that $C2^{-N} < 1$, since the identity operator $I = T_N + R_N$, then T_N^{-1} is also bounded on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and $L^2(\mathbb{R}^{n+m})$, where

$$T_N f(x_1,x_2) = \sum_j\sum_k\sum_I\sum_J |I\times J|\psi_{j,k}(x_1-x_I,x_2-x_J)\psi_{j,k}*f(x_I,x_J).$$

By interpolation, R_N is bounded on $L^q(\mathbb{R}^{n+m})$ with the bound $C2^{-N}$ for all $1 < q < \infty$. Therefore, T_N^{-1} is bounded on $L^q(\mathbb{R}^{n+m})$. Set $g(x_1,x_2) = T_N^{-1}(f)(x_1,x_2)$, then $\|g\|_q \le C\|f\|_q$, $1 < q < \infty$, and moreover,

$$f(x_1,x_2) = \sum_j\sum_k\sum_I\sum_J |I\times J|\psi_{j,k}(x_1-x_I,x_2-x_J)\psi_{j,k}*g(x_I,x_J), \quad (2.7)$$

where the above series converges in both $L^2(\mathbb{R}^{n+m})$ and $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ norms.

To complete the proof of Theorem 1.1, we only need to show that the series (2.7) converges also in $L^q(\mathbb{R}^{n+m})$ for any $1 < q < \infty$. Note first that by the Littlewood-Paley estimates on $L^q(\mathbb{R}^{n+m})$, $1 < q < \infty$, it is easy to see that

$$\begin{aligned}&\Big\|\sum_j\sum_k\sum_I\sum_J |I\times J|\psi_{j,k}(x_1-x_I,x_2-x_J)\psi_{j,k}*f(x_I,x_J)\Big\|_{L^q(\mathbb{R}^{n+m})}\\ &\quad\le C\|f\|_{L^q(\mathbb{R}^{n+m})}.\end{aligned}$$

Thus it suffices to show that the series (2.7) converges in $L^q(\mathbb{R}^{n+m})$ for each function $f \in L^q(\mathbb{R}^{n+m})\cap L^2(\mathbb{R}^{n+m})$ since the subspace $L^q(\mathbb{R}^{n+m})\cap L^2(\mathbb{R}^{n+m})$ is dense in $L^q(\mathbb{R}^{n+m})$. Indeed, the convergence of the series (2.7) in $L^2(\mathbb{R}^{n+m})$ implies the convergence for almost every $(x_1,x_2)\in\mathbb{R}^{n+m}$. For $f\in L^q(\mathbb{R}^{n+m})\cap L^2(\mathbb{R}^{n+m})$ set

$$\begin{aligned}B_l = \{R = I\times J : l(I) = 2^{-j-N},\ l(J) = 2^{-k-N},\ I\subset B_1(0,l),\\ J\subset B_2(0,l),\ |j|,|k|\le l\},\end{aligned}$$

where $B_1(0,l)$ and $B_2(0,l)$ are balls centered at 0 with the radius l in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Write $\psi_R = \psi_{j,k}$, then it suffices to show that for each function $f \in L^q(\mathbb{R}^{n+m})\cap L^2(\mathbb{R}^{n+m})$ and any positive integer L,

$$\Big\|\sum_{l>L}\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*(g)(x_I,x_J)\Big\|_{L^q(\mathbb{R}^{n+m})}$$

tends to zero as L goes to infinity.

Let $h \in L^{q'}(\mathbb{R}^{n+m}) \cap L^2(\mathbb{R}^{n+m})$ $(\frac{1}{q} + \frac{1}{q'} = 1)$. By duality argument, Cauchy's inequality and Hölder's inequality, then we have

$$\begin{aligned}
&\Big\| \sum_{l>L} \sum_{R\in B_l} |R|\psi_R(x_1 - x_I, x_2 - x_J)\psi_R * g(x_I, x_J) \Big\|_q \\
&= \sup_{\|h\|_{q'}\le 1} \Big| \Big\langle \sum_{l>L} \sum_{R\in B_l} |R|\psi_R(x_1 - x_I, x_2 - x_J)\psi_R * g(x_I, x_J), h \Big\rangle \Big| \\
&= \sup_{\|h\|_{q'}\le 1} \Big| \sum_{l>L} \sum_{R\in B_l} |R|\psi_R * h(x_I, x_J)\psi_R * g(x_I, x_J) \Big| \\
&\le \sup_{\|h\|_{q'}\le 1} \int \sum_{l>L} \sum_{R\in B_l} |\psi_R * h(x_I, x_J)| \cdot |\psi_R * g(x_I, x_J)| \chi_R(x_1, x_2)\, dx_1\, dx_2 \\
&\le \sup_{\|h\|_{q'}\le 1} \int \Big\{ \sum_{l>L} \sum_{R\in B_l} |\psi_R * h(x_I, x_J)|^2 \chi_R(x_1, x_2) \Big\}^{\frac{1}{2}} \\
&\quad \times \Big\{ \sum_{l>L} \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \Big\}^{\frac{1}{2}} dx_1\, dx_2 \\
&\le \sup_{\|h\|_{q'}\le 1} \Big\| \Big\{ \sum_{l>L} \sum_{R\in B_l} |\psi_R * h(x_I, x_J)|^2 \chi_R(x_1, x_2) \Big\}^{\frac{1}{2}} \Big\|_{q'} \\
&\quad \times \Big\| \Big\{ \sum_{l>L} \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \Big\}^{\frac{1}{2}} \Big\|_q \\
&\le C \sup_{\|h\|_{q'}\le 1} \|h\|_{q'} \Big\| \Big\{ \sum_{l>L} \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \Big\}^{\frac{1}{2}} \Big\|_q .
\end{aligned}$$

By the $L^q(\mathbb{R}^{n+m})$ norm inequality of the discrete Littlewood-Paley function, that is, for $1 < q < \infty$,

$$\Big\| \Big\{ \sum_j \sum_k \sum_I \sum_J |I \times J| \psi_{j,k} * f(x_I, x_J)|^2 \chi_R(x_1, x_2) \Big\}^{\frac{1}{2}} \Big\|_q \le C\|f\|_q,$$

the last term tends to zero as L goes to infinity and this completes the proof of Theorem 1.1.

Before ending this section, we remark that by use of discrete Calderón-type identity given in Theorem 1.1, one can provide the following Min-Max type inequality:

for $0 < p \le 1 < q < \infty$ and each $f \in L^q(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left\| \left[\sum_{j,k} \sum_{I,J} \sup_{(u,v)\in I\times J} |\psi_{j,k} * f(u,v)|^2 \chi_I(x_1)\chi_J(x_2) \right]^{\frac{1}{2}} \right\|_p \approx \left\| \left[\sum_{j,k} \sum_{I,J} \inf_{(u,v)\in I\times J} |\psi_{j,k} * f(u,v)|^2 \chi_I(x_1)\chi_J(x_2) \right]^{\frac{1}{2}} \right\|_p, \tag{2.8}$$

where ψ, I, J and p are the same as in Theorem 1.1; see [10, 11] for the similar but more difficult proofs in the flag and Zygmund dilation settings.

The Min-Max type inequality in (2.8) easily implies that for $0 < p \le 1 < q < \infty$ and each $f \in L^q(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left\| \left\{ \sum_{j,k} |\psi_{j,k} * f(x_1,x_2)|^2 \right\}^{\frac{1}{2}} \right\|_p \approx \left\| \left\{ \sum_{j,k} \sum_{I,J} |\psi_{j,k} * f(x_I,x_J)|^2 \chi_I(x_1)\chi_J(x_2) \right\}^{\frac{1}{2}} \right\|_p. \tag{2.9}$$

This provides the discrete Littlewood-Paley characterization of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, that is, for each $f \in L^q(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$, $0 < p \le 1 < q < \infty$,

$$\left\| \left\{ \sum_{j,k} \sum_{I,J} |\psi_{j,k} * f(x_I,x_J)|^2 \chi_I(x_1)\chi_J(x_2) \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{n+m})} \approx \|f\|_{H^p(\mathbb{R}^{n+m})}. \tag{2.10}$$

□

3 Atomic Decomposition and Boundedness Criterion of Operators in Product Hardy Spaces

By the discrete Calderón's identity in (1.1), we can first establish the atomic decomposition into (p,q)-atoms for $0 < p \le 1 < q < \infty$, namely Theorem 1.2.

Proof of Theorem 1.2 Let $f \in L^q(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$, $0 < p \le 1 < q < \infty$, and $\psi_R = \psi_{j,k}$ be functions mentioned in the proof of Theorem 1.1. Set, by Theorem 1.1,

$$\Omega_l = \left\{ (x_1,x_2) : \left[\sum_{j,k} \sum_{I,J} |\psi_{j,k} * g(x_I,x_J)|^2 \chi_I(x_1)\chi_J(x_2) \right]^{\frac{1}{2}} > 2^l \right\}$$

and

$$B_l = \left\{ R = I \times J : |R \cap \Omega_l| > \frac{1}{2}|R| \text{ and } |R \cap \Omega_{l+1}| \le \frac{1}{2}|R| \right\},$$

where χ_I and χ_J are the characteristic functions of I and J, respectively, and I and J are dyadic rectangles in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then, by reproducing the discrete

Calderón formula in Theorem 1.1, we have

$$f(x_1,x_2)=\sum_j\sum_k\sum_I\sum_J|I\times J|\psi_{j,k}(x_1-x_I,x_2-x_J)\psi_{j,k}*g(x_I,x_J)$$
$$=\sum_l\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J),$$

where $\psi_R=\psi_{j,k}$ and $x_I\in I$ and $x_J\in J$.

To see the decomposition above gives an atomic decomposition in terms of (p,q)-atoms, rewrite

$$f(x_1,x_2)=\sum_l\lambda_l a_l(x_1,x_2), \tag{3.1}$$

where

$$a_l=\frac{1}{\lambda_l}\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J),$$

and when $2\le q<\infty$,

$$\lambda_l=C\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac12}\right\|_q|\widetilde{\Omega}_l|^{\frac1p-\frac1q}$$

and when $1<q<2$,

$$\lambda_l=C\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac12}\right\|_2|\widetilde{\Omega}_l|^{\frac1p-\frac12}.$$

In the above expressions we have set $\widetilde{\Omega}_l=\{(x_1,x_2):M_s(\chi_{\Omega_l})(x_1,x_2)>\frac{1}{100}\}$. Since $R\in B_l\Longrightarrow 4R\subset\widetilde{\Omega}_l$, thus $\bigcup_{R\in B_l}4R\subseteq\widetilde{\Omega}_l$, this implies that a_l is supported in an open set $\widetilde{\Omega}_l$, and hence a_l satisfies (1) of Theorem 1.2.

To see that a_l satisfies (2) of Theorem 1.2, as in the proof of Theorem 1.1, let $h\in L^{q'}(\mathbb{R}^{n+m})\cap L^2(\mathbb{R}^{n+m})$ $(\frac1q+\frac1{q'}=1)$. By Cauchy's and Hölder's inequalities and the discrete Littlewood-Paley square function estimates on L^q, $1<q<\infty$, then we have

$$\left\|\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J)\right\|_q$$
$$=\sup_{\|h\|_{q'}\le1}\left|\left\langle\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J),h\right\rangle\right|$$
$$\le\sup_{\|h\|_{q'}\le1}\int\sum_{R\in B_l}|\psi_R*h(x_I,x_J)|\cdot|\psi_R*g(x_I,x_J)|\chi_R(x_1,x_2)\,dx_1\,dx_2$$

$$\leq \sup_{\|h\|_{q'}\leq 1}\left\|\left\{\sum_{R\in B_l}|\psi_R*h(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_{q'}$$

$$\times\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_{q}$$

$$\leq C\sup_{\|h\|_{q'}\leq 1}\|h\|_{q'}\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_{q},$$

which yields that when $2\leq q<\infty$,

$$\|a_l\|_q=\left(C\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_{q}|\widetilde{\Omega}_l|^{\frac{1}{p}-\frac{1}{q}}\right)^{-1}$$

$$\times\left\|\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J)\right\|_q$$

$$\leq|\widetilde{\Omega}_l|^{\frac{1}{q}-\frac{1}{p}}.$$

Note that a_l is supported in $\widetilde{\Omega}_l$. Thus if $1<q<2$, the similar estimate and the definition of λ_l yield

$$\|a_l\|_q=\left(C\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_{2}|\widetilde{\Omega}_l|^{\frac{1}{p}-\frac{1}{2}}\right)^{-1}$$

$$\times\left\|\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J)\right\|_q$$

$$\leq\left(C\left\|\left\{\sum_{R\in B_l}|\psi_R*g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_{q}|\widetilde{\Omega}_l|^{\frac{1}{p}-\frac{1}{2}}\right)^{-1}$$

$$\times|\widetilde{\Omega}_l|^{\frac{1}{q}-\frac{1}{2}}\left\|\sum_{R\in B_l}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J)\right\|_2$$

$$\leq|\widetilde{\Omega}_l|^{\frac{1}{q}-\frac{1}{p}},$$

which implies that a_l satisfies the size condition (2) of (p,q)-atoms in Theorem 1.2.

To verify that a_l satisfies the conditions (3) and (4) of Theorem 1.2, note that $a_l=\sum_{Q\in m(\widetilde{\Omega}_l)}a_{l,Q}$, where

$$a_{l,Q}=\frac{1}{\lambda_l}\sum_{R\in B_l,R\subset Q}|R|\psi_R(x_1-x_I,x_2-x_J)\psi_R*g(x_I,x_J).$$

It suffices to verify that each $a_{l,Q}$ is a rectangle (p,q)-atom satisfying (3) and (4) in Theorem 1.2. The support and cancellation conditions follow directly from the conditions on ψ and thus (4) follows easily. Therefore, it remains to show (3).

To see that when $2 \le q < \infty$, a_l satisfies the estimate of (3a), by the same proof for the estimate of $\|a_l\|_q$, we have

$$\|a_{l,Q}\|_q \le C\left\|\left\{\sum_{R\in B_l, R\subset Q} |\psi_R * g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_q$$

and hence, the fact that $2 \le q < \infty$ and the definition of λ_l yield

$$\sum_{Q\in m(\widetilde{\Omega}_l)} \|a_{l,Q}\|_q^q \le \|a_l\|_q^q$$

which, by the estimate (2) for a_l, implies that a_l satisfies (3a) of Theorem 1.2. When $1 < q < 2$, we have

$$\begin{aligned}
&\sum_{Q\in m_1(\widetilde{\Omega}_l)} \gamma_2^{-\delta}(Q)\|a_{l,Q}\|_q^q\\
&\quad\le \frac{C}{\lambda_l^q}\sum_{Q\in m_1(\widetilde{\Omega}_l)} \gamma_2^{-\delta}(Q)\left\|\left\{\sum_{R\in B_l, R\subset Q} |\psi_R * g(x_I,x_J)|^2\chi_R(x_1,x_2)\right\}^{\frac{1}{2}}\right\|_q^q\\
&\quad\le \frac{C}{\lambda_l^q}\sum_{Q\in m_1(\widetilde{\Omega}_l)} \gamma_2^{-\delta}(Q)|Q|^{1-\frac{q}{2}}\\
&\qquad\times\left\{\int\sum_{R\in B_l, R\subset Q} |\psi_R * g(x_I,x_J)|^2\chi_R(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{q}{2}}\\
&\quad\le \frac{C}{\lambda_l^q}\left\{\sum_{Q\in m_1(\widetilde{\Omega}_l)} \gamma_2^{-\delta'}(Q)|Q|\right\}^{1-\frac{q}{2}}\\
&\qquad\times\left\{\int\sum_{R\in B_l} |\psi_R * g(x_I,x_J)|^2\chi_R(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{q}{2}}\\
&\quad\le C_{q,\delta'}|\widetilde{\Omega}_l|^{(1-\frac{q}{2})}|\widetilde{\Omega}_l|^{\frac{q}{2}-\frac{q}{p}} = C_{\delta'}|\widetilde{\Omega}_l|^{1-\frac{q}{p}},
\end{aligned}$$

where the last inequality follows from Journé's Lemma [18]. The other summation in (3b) can be proved by the same manner. This shows that a_l satisfies (3b) of Theorem 1.2.

Note that by the maximal theorem $|\widetilde{\Omega}_l| \le C|\Omega_l|$. Since if $(x_1,x_2)\in R\in B_l$ then $M_s(\chi_{R\cap\widetilde{\Omega}_l\backslash\Omega_{l+1}})(x_1,x_2) > \frac{1}{2}$, we have

$$\begin{aligned}
&\chi_R(x_1,x_2) \le 2M_s(\chi_{R\cap\widetilde{\Omega}_l\backslash\Omega_{l+1}})(x_1,x_2)\\
&\quad\Longrightarrow\quad \chi_R(x_1,x_2) \le 4M_s^2(\chi_{R\cap\widetilde{\Omega}_l\backslash\Omega_{l+1}})(x_1,x_2).
\end{aligned}$$

Thus, by the Fefferman-Stein vector valued inequality, for all $1 < q < \infty$,

$$
\begin{aligned}
&\left\| \left\{ \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{1}{2}} \right\|_q^q \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^m} \left\{ \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{q}{2}} dx_1\, dx_2 \\
&\le C \int_{\mathbb{R}^n\times\mathbb{R}^m} \left\{ \sum_{R\in B_l} |\psi_R * g(x_I, x_J) M_s(\chi_{R\cap\widetilde{\Omega}_l\setminus\Omega_{l+1}})(x_1, x_2)|^2 \right\}^{\frac{q}{2}} dx_1\, dx_2 \\
&\le C \int_{\widetilde{\Omega}_l\setminus\Omega_{l+1}} \left\{ \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{q}{2}} dx_1\, dx_2 \\
&\le C 2^{lq} |\widetilde{\Omega}_l|.
\end{aligned}
$$

Therefore, when $2 \le q < \infty$, we have

$$
\begin{aligned}
\sum_l |\lambda_l|^p &= C \sum_l \left\| \left\{ \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{1}{2}} \right\|_q^p |\widetilde{\Omega}_l|^{1-\frac{p}{q}} \\
&\le C \sum_l 2^{lp} |\widetilde{\Omega}_l|^{\frac{p}{q}} |\widetilde{\Omega}_l|^{1-\frac{p}{q}} \\
&= C \sum_l 2^{lp} |\widetilde{\Omega}_l| \le C \sum_l 2^{lp} |\Omega_l| \\
&\le C \left\| \left\{ \sum_l \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{1}{2}} \right\|_p^p \\
&\le C \|g\|_{H^p}^p \le C \|f\|_{H^p}^p,
\end{aligned}
$$

and when $1 < q < 2$,

$$
\begin{aligned}
\sum_l |\lambda_l|^p &= C \sum_l \left\| \left\{ \sum_{R\in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{1}{2}} \right\|_2^p |\widetilde{\Omega}_l|^{1-\frac{p}{2}} \\
&\le C \sum_l 2^{lp} |\widetilde{\Omega}_l|^{\frac{p}{2}} |\widetilde{\Omega}_l|^{1-\frac{p}{2}} \\
&= C \sum_l 2^{lp} |\widetilde{\Omega}_l| \le C \sum_l 2^{lp} |\Omega_l|
\end{aligned}
$$

$$\leq C \left\| \left\{ \sum_{l} \sum_{R \in B_l} |\psi_R * g(x_I, x_J)|^2 \chi_R(x_1, x_2) \right\}^{\frac{1}{2}} \right\|_p^p$$

$$\leq C \|g\|_{H^p}^p \leq C \|f\|_{H^p}^p.$$

Finally, the fact that the atomic decomposition in (3.1) converges in $L^q(\mathbb{R}^{n+m})$ follows from the same proof as given in Theorem 1.1. This ends the proof of Theorem 1.2. □

We are now ready to provide the

Proof of Theorem 1.3 We only need to prove the "if" parts of Theorem 1.3 because by the same proof as in [9], it is easy to check that if a is a (p, q)-product atom then $a \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and $\|a\|_{H^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C$ where C is a constant uniformly for all (p, q)-product atoms. See the Appendix for the outline of this proof. If $\|T(a)\|_{L^p(\mathbb{R}^{n+m})} \leq C$ uniformly on all (p, q)-product atoms in $L^p(\mathbb{R}^{n+m})$, then by Theorem 1.2, for $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^q(\mathbb{R}^{n+m})$,

$$Tf = \sum_{l} \lambda_l T a_l$$

since T is bounded and $f = \sum_l \lambda_l a_l$ on $L^q(\mathbb{R}^{n+m})$.

Thus

$$\|Tf\|_p^p \leq \sum_{l} |\lambda_l|^p \|Ta_l\|_p^p \leq C^p \sum_{l} |\lambda_l|^p \leq C \|f\|_{H^p}^p.$$

(2) If $\|T(a)\|_{H^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C$ uniformly on all (p, q)-product atoms in $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, then by Theorem 1.2 and notes that the H^p norm of f is the L^p norm of the Littlewood-Paley function $g(f)$, for $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^q(\mathbb{R}^{n+m})$,

$$\|Tf\|_{H^p}^p \leq \sum_{l} |\lambda_l|^p \|Ta_l\|_{H^p}^p \leq C^p \sum_{l} |\lambda_l|^p \leq C \|f\|_{H^p}^p.$$

Since $H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^q(\mathbb{R}^{n+m})$ is dense in $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, the proof of Theorem 1.3 is complete. □

Acknowledgements The authors wish to sincerely thank Professor Fulvio Ricci for his encouragement to complete this work and his suggestions. The authors are also grateful to the referee's very careful reading and many useful suggestions and comments which improve the exposition of the paper.

Appendix

We give the outline of the proof that if a is a (p, q)-product atom for $1 < q < 2$, then $\|a\|_{H^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C$, where C is a constant independent of a. The proof for this fact when $q \geq 2$ is basically done in [9].

For simplicity, we only consider the case where $n = m = 1$. But from the proof given below, the proof for the general n and m can be done in the same manner with minimal modifications. To show $\|a\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C$, it will be enough to prove that $\|g(a)\|_p \le C$, where $g(a)$ is the Littlewood-Paley g function of a given in Definition 2.1 and C is a constant independent of a.

Recall that a is an atom supported in Ω satisfying conditions (1), (2) in Theorem 1.2 and the following (i.e., (3b) in Theorem 1.2):

$$a = \sum_{R\in m_1(\Omega)} a_R + \sum_{R\in m_2(\Omega)} a_R,$$

and for any $\delta > 0$, there exists a constant $C_{q,\delta>0}$, where $C_{q,\delta}$ depends only on q and δ and a_R satisfies condition (4) in Theorem 1.2 such that

$$\left(\sum_{R\in m_1(\Omega)} \gamma_2^{-\delta}\|a_R\|^q_{L^q(\mathbb{R}^{n+m})} + \sum_{R\in m_2(\Omega)} \gamma_1^{-\delta}\|a_R\|^q_{L^q(\mathbb{R}^{n+m})}\right)^{1/q} \le C_{q,\delta}|\Omega|^{\frac{1}{q}-\frac{1}{p}}.$$

We will use the same notation and follow the similar outline as given on page 120 of [9]. Let $\widetilde{\Omega} = \{(x_1,x_2) : M_s(\chi_\Omega)(x_1,x_2) > \frac{1}{2}\}$ and $\widetilde{\widetilde{\Omega}} = (\widetilde{\widetilde{\Omega}})$.

To estimate $\int_{\widetilde{\widetilde{\Omega}}} |g(a)|^p\,dx_1\,dx_2$, applying Hölder's inequality and the boundedness of g on L^q we get

$$\int_{\widetilde{\widetilde{\Omega}}} |g(a)|^p\,dx_1\,dx_2 \le \left\{\int_{\widetilde{\widetilde{\Omega}}} |g(a)|^q\,dx_1\,dx_2\right\}^{\frac{p}{q}} |\widetilde{\widetilde{\Omega}}|^{1-\frac{p}{q}} \le C\|a\|_q^p|\Omega|^{1-\frac{p}{q}} \le C.$$

To handle $\int_{\widetilde{\widetilde{\Omega}}^c} |g(a)|^p\,dx_1\,dx_2$, we note the properties of ψ_{jk} in the definition of the g function and follow the outline given on p. 121 of [9]. Let $R = I\times J \in m_2(\Omega)$, let $\hat{I}$ be the maximal dyadic cube in $\mathbb{R}^n$ such that $\hat{I}\times J \subset \widetilde{\Omega}$, let $\widetilde{\hat{I}}$ be the double of $\hat{I}$ and $(\widetilde{\hat{I}})^c$ be the complement of $\widetilde{\hat{I}}$. Thus, we have

$$\int_{\widetilde{\hat{I}}^c\times\mathbb{R}} |g(a_R)|^p\,dx_1\,dx_2 \le C\gamma_1^{-\delta}(R)\|a_R\|_q^p|R|^{1-\frac{p}{q}}.$$

Summing over R gives

$$\begin{aligned}&\sum_{R\in m_2(\Omega)} \gamma_1^{-\delta}(R)\|a_R\|_q^p|R|^{1-\frac{p}{q}}\\ &\quad\le \left\{\sum_{R\in m_2(\Omega)} \gamma_1^{-\delta'}(R)\|a_R\|_q^q\right\}^{\frac{p}{q}}\left\{\sum_{R\in m_2(\Omega)} \gamma_1^{-\delta''}(R)|R|\right\}^{1-\frac{p}{q}} \le C,\end{aligned}$$

where $\delta' = -\frac{\delta}{2}\frac{q}{p}$ and $\delta'' = -\frac{\delta}{2}(\frac{q}{p})'$. Here the last inequality above follows from Journé's Lemma and the condition of (3b) of Theorem 1.2.

The similar proof works when the Littlewood-Paley square function g is replaced by the operator T as given in [9]. We leave the details to the interested reader. This means if we assume Fefferman's condition of the operator on the rectangle atom, then we can show $\|T(a)\|_p \le C$ for any (p,q)-atom when $1 < q < 2$. Therefore, Fefferman's criterion holds in this case as well. This demonstrates that our definition of (p,q)-atoms is consistent with the existing multiparameter H^p theory.

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