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# FLAG HARDY SPACES AND MARCINKIEWICZ MULTIPLIERS ON THE HEISENBERG GROUP

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Marcinkiewicz multipliers are $L^p$ bounded for $1 < p < \infty$ on the Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$, as shown by D. Müller, F. Ricci, and E. M. Stein. This is surprising in that these multipliers are invariant under a two-parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, while there is *no* two-parameter group of *automorphic* dilations on $\mathbb{H}^n$. This lack of automorphic dilations underlies the failure of such multipliers to be in general bounded on the classical Hardy space $H^1$ on the Heisenberg group, and also precludes a pure product Hardy space theory.

We address this deficiency by developing a theory of *flag* Hardy spaces $H^p_{\text{flag}}$ on the Heisenberg group, $0 < p \leq 1$, that is in a sense "intermediate" between the classical Hardy spaces $H^p$ and the product Hardy spaces $H^p_{\text{product}}$ on $\mathbb{C}^n \times \mathbb{R}$ developed by A. Chang and R. Fefferman. We show that flag singular integral operators, which include the aforementioned Marcinkiewicz multipliers, are bounded on $H^p_{\text{flag}}$, as well as from $H^p_{\text{flag}}$ to $L^p$, for $0 < p \leq 1$. We also characterize the dual spaces of $H^1_{\text{flag}}$ and $H^p_{\text{flag}}$, and establish a Calderón–Zygmund decomposition that yields standard interpolation theorems for the flag Hardy spaces $H^p_{\text{flag}}$. In particular, this recovers some $L^p$ results of Müller, Ricci, and Stein (but not their sharp versions) by interpolating between those for $H^p_{\text{flag}}$ and $L^2$.



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## 1. Introduction

Classical Calderón–Zygmund theory centers around singular integrals associated with the Hardy–Littlewood maximal operator $M$ that commutes with the usual dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta x_1, \dots, \delta x_n)$ for $\delta > 0$. On the other hand, *product* Calderón–Zygmund theory centers around singular integrals associated with the *strong* maximal function $M_S$ that commutes with the multiparameter dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta_1 x_1, \dots, \delta_n x_n)$ for $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n_+$. The strong maximal function [Jessen et al. 1935] is given by

$$M_S(f)(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy, \tag{1-1}$$

where the supremum is taken over the family of all rectangles $R$ with sides parallel to the axes.

For Calderón–Zygmund theory in the product setting, one considers operators of the form $Tf = K * f$, where $K$ is homogeneous, that is, $\delta_1 \cdots \delta_n K(\delta \cdot x) = K(x)$, or, more generally, $K(x)$ satisfies certain differential inequalities and cancellation conditions such that the kernels $\delta_1 \cdots \delta_n K(\delta \cdot x)$ also satisfy the same bounds. Such operators have been studied, for example, in [Gundy and Stein 1979; Fefferman and Stein 1982; Fefferman 1986; 1987; 1999; Chang 1979; Chang and Fefferman 1985; 1982; 1980; Journé 1985; 1986; Pipher 1986; Ferguson and Lacey 2002], where both the $L^p$ theory for $1 < p < \infty$ and $H^p$ theory for $0 < p \le 1$ were developed. More precisely, Fefferman and Stein [1982] studied the $L^p$ boundedness ($1 < p < \infty$) for the product convolution singular integral operators. Journé [1985; 1988] introduced non-convolution-product singular integral operators, established the product $T1$ theorem, and proved the $L^\infty \to \mathrm{BMO}$ boundedness of such operators. The product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ was first introduced by Gundy and Stein [1979]. Chang and Fefferman [1985; 1982; 1980] developed the theory of atomic decomposition and established the dual space of the Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$, namely the product $\mathrm{BMO}(\mathbb{R}^n \times \mathbb{R}^m)$ space. Another characterization of such product BMO space was given in conjunction with Hankel theorems and commutators in the product setting by Ferguson and Lacey [2002] and Lacey and Terwilleger [2005]. Carleson [1974] disproved by a counterexample the conjecture that the product atomic Hardy space on $\mathbb{R}^n \times \mathbb{R}^m$ could be defined by rectangle atoms. This motivated Chang and Fefferman to replace the role of cubes in the classical atomic decomposition of $H^p(\mathbb{R}^n)$ by arbitrary open sets of finite measures in the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Subsequently, Fefferman [1987] established the criterion for the $H^p \to L^p$ boundedness of singular integral operators in Journé's class by considering its action only on rectangle atoms by using Journé's lemma. However, Fefferman's criterion cannot be extended to three or more parameters without further assumptions on the nature of $T$, as shown in [Journé 1985; Journé 1988]. In fact, Journé provided a counterexample in the three-parameter setting of singular integral operators such that Fefferman's criterion breaks down. Subsequently, the $H^p$ to $L^p$ boundedness for Journé's class of singular integral operators with arbitrary number of parameters was established by J. Pipher [1986] by considering directly the action of the operator on (nonrectangle) atoms and an extension of Journé's geometric lemma to higher dimensions.

On the other hand, multiparameter analysis has only recently been developed for $L^p$ theory with $1 < p < \infty$ when the underlying multiparameter structure is not explicit, but *implicit*, as in the flag multiparameter structure studied in [Nagel et al. 2001] and its counterpart on the Heisenberg group $\mathbb{H}^n$

studied in [Müller et al. 1995; 1996]. In these latter two papers the authors obtained the surprising result that certain Marcinkiewicz multipliers, invariant under a two-parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, are bounded on $L^p(\mathbb{H}^n)$, *despite* the absence of a two-parameter automorphic group of dilations on $\mathbb{H}^n$. This striking result exploited an implicit product, or *semiproduct,* structure underlying the group multiplication in $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$. In contrast to this, it is not hard to see that the class of flag singular integrals considered there is *not* in general bounded on the standard one-parameter Hardy space $H^1(\mathbb{H}^n)$ as in [Fefferman and Stein 1972] (see, for example, Theorem 67 in Section 11 below). The lesson learned here is that Hardy space theories for $0 < p \le 1$ must be tailored to the invariance properties of the class of singular integral operators under consideration.

The goal of this paper is to develop for the Heisenberg group a theory of *flag* Hardy spaces $H^p_{\text{flag}}$ with $0 < p \le 1$. The first two authors have treated the Euclidean flag structure in [Han and Lu 2008]; see also the multiparameter setting associated with the Zygmund dilation [Han and Lu 2010]. The ideas developed in this paper and [Han and Lu 2008; Han and Lu 2010] have been adapted to some other multiparameter cases, such as the product spaces of Carnot–Carathéodory spaces [Han et al. 2013a], where the $L^p$ theory was established in [Nagel and Stein 2004], and the composition of two singular integrals with different homogeneity [Han et al. 2013b].

This flag theory for the Heisenberg group is most conveniently explained when $p = 1$ in the more general context of spaces $(X, \rho, d\mu)$ of homogeneous type [Coifman and Weiss 1976], which already include Euclidean spaces $\mathbb{R}^N$ and stratified graded nilpotent Lie groups such as the Heisenberg groups $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. We may assume here that $\rho$ and $d\mu$ are connected by the equivalence

$$\mu(B_\rho(x, r)) \approx r, \quad \text{where } B_\rho(x, r) = \{y \in X : \rho(x, y) < r\}. \tag{1-2}$$

In particular, the usual structure on Euclidean space $\mathbb{R}^n$ is given by $\rho(x, y) = |x - y|^n$ and $d\mu(x) = dx$.

Recall that one of several equivalent definitions of the Hardy space $H^1(X)$ is as the set of $f \in (C^\eta(X))^*$ with

$$\|f\|_{H^1(X)} \equiv \|g(f)\|_{L^1(d\mu)} < \infty,$$

where the Littlewood–Paley $g$-function $g(f)$ is given by

$$g(f) = \left\{ \sum_{j=-\infty}^{\infty} |E_j f|^2 \right\}^{\frac{1}{2}},$$

where $\{E_j\}_{j=-\infty}^{\infty}$ is an appropriate Littlewood–Paley decomposition of the identity on $L^2(d\mu)$.

The *product* Hardy space $H^1_{\text{product}}(X \times X')$ corresponding to a product of homogeneous spaces $(X, \rho, d\mu)$ and $(X', \rho', d\mu')$ is given as the set of $f \in (C^\eta(X \times X'))^*$ with

$$\|f\|_{H^1_{\text{product}}(X \times X')} \equiv \|g_{\text{product}}(f)\|_{L^1(d\mu \times d\mu')} < \infty,$$

where the *product* Littlewood–Paley $g$-function $g_{\text{product}}(f)$ is given by

$$g_{\text{product}}(f) = \left\{ \sum_{j,j'=-\infty}^{\infty} |D_j D'_{j'} f|^2 \right\}^{\frac{1}{2}},$$

and where $\{D_j\}_{j=-\infty}^{\infty}$ and $\{D'_{j'}\}_{j'=-\infty}^{\infty}$ are Littlewood–Paley decompositions of the identities on $L^2(d\mu)$ and $L^2(d\mu')$, respectively (and act separately on the respective distinct variables). Note that if $j=j'$, then $D_jD'_{j'}=D_jD'_j$ satisfies estimates similar to those for $E_j$ in the standard *one-parameter* Hardy space $H^1(X\times X')$. Thus, we see that

$$g_{\text{product}}(f)=\left\{\sum_{j,j'=-\infty}^{\infty}|D_jD'_{j'}f|^2\right\}^{\frac{1}{2}}\geq\left\{\sum_{j}|D_jD'_jf|^2\right\}^{\frac{1}{2}}\approx\left\{\sum_{j}|E_jf|^2\right\}^{\frac{1}{2}}=g(f),$$

and so we have the inclusion

$$H^1_{\text{product}}(X\times X')\subset H^1(X\times X').$$

Now we specialize the space of homogeneous type $X$ to be the Heisenberg group $\mathbb{H}^n=\mathbb{C}^n\times\mathbb{R}$. The *flag* structure on the Heisenberg group $\mathbb{H}^n$ arises in an intermediate manner, namely, as a homogeneous space structure derived from the Heisenberg multiplication law that is adapted to the product of the homogeneous spaces $\mathbb{C}^m$ and $\mathbb{R}$. The appropriate definition of the flag Hardy space $H^1_{\text{flag}}(\mathbb{H}^n)$ is already suggested in [Müller et al. 1996], where a Littlewood–Paley $g$-function $g_{\text{flag}}$ is introduced that is adapted to the flag structure on the Heisenberg group $\mathbb{H}^n=\mathbb{C}^n\times\mathbb{R}$:

$$g_{\text{flag}}(f)=\left\{\sum_{j,k=-\infty}^{\infty}|E_kD_jf|^2\right\}^{\frac{1}{2}},$$

where $\{D_j\}_{j=-\infty}^{\infty}$ is the standard Littlewood–Paley decomposition of the identity on $L^2(\mathbb{H}^n)$, and $\{E_k\}_{k=-\infty}^{\infty}$ is the standard Littlewood–Paley decomposition of the identity on $L^2(\mathbb{R})$. One can then define $H^1_{\text{flag}}(\mathbb{H}^n)$ to consist of appropriate "distributions" $f$ on $\mathbb{H}^n$ with

$$\|f\|_{H^1_{\text{flag}}(\mathbb{H}^n)}\equiv\|g_{\text{flag}}(f)\|_{L^1(\mathbb{H}^n)}<\infty.$$

Now, for $k\leq 2j$, it turns out that $E_kD_j$ is essentially the *one-parameter* Littlewood–Paley function $D_j$; while, for $k>2j$, it turns out that $E_kD_j$ is essentially the *product* Littlewood–Paley function $E_kF_j$, where $\{F_j\}_{j=-\infty}^{\infty}$ is the standard Littlewood–Paley decomposition of the identity on $L^2(\mathbb{C}^n)$. Thus we see that $g_{\text{flag}}(f)$ is a *semiproduct* Littlewood–Paley function satisfying

$$g_{\text{product}}(f)\gtrsim g_{\text{flag}}(f)\gtrsim g(f),\quad H^1_{\text{product}}(X\times X')\subset H^1_{\text{flag}}(X\times X')\subset H^1(X\times X').$$

We describe this structure as "semiproduct", since only *vertical* Heisenberg rectangles (which are essentially unions of contiguous Heisenberg balls of fixed radius stacked one on top of the other) arise essentially as the supports of the components $E_kD_j$, when $k>2j$. When $k\leq 2j$, the support of $E_kD_j$ is essentially a Heisenberg cube. Thus no *horizontal* rectangles arise, and the structure is "semiproduct".

Of course, we must also address the nature of the "distributions" referred to above, and for this we will use a lifting technique introduced in [Müller et al. 1995] to define *projected* flag molecular spaces $\mathcal{M}_{\text{flag}}(\mathbb{H}^n)$, and then the aforementioned distributions will be elements of the dual space $\mathcal{M}_{\text{flag}}(\mathbb{H}^n)'$. We also show that these distributions are essentially the same as those obtained from the dual of a more familiar *moment* flag molecular space $\mathcal{M}_F(\mathbb{H}^n)$. Finally, we mention that a theory of flag Hardy spaces can also be developed with the techniques used here, but without recourse to any notion of "distributions", by simply defining $H^p_{\text{abstract}}(\mathbb{H}^n)$ to be the abstract completion of the metric space

$$X^p(\mathbb{H}^n) \equiv \{f \in L^2(\mathbb{H}^n) : g_{\text{flag}}(f) \in L^p(\mathbb{H}^n)\}$$

with metric

$$d(f_1, f_2) \equiv \|g_{\text{flag}}(f_1 - f_2)\|^p_{L^p(\mathbb{H}^n)}, \quad f_j \in X^p(\mathbb{H}^n).$$

We show that the abstract space $H^p_{\text{abstract}}(\mathbb{H}^n)$, whose elements are realized only as equivalence classes of Cauchy sequences, is in fact isomorphic to the space $H^p_{\text{flag}}(\mathbb{H}^n)$, whose elements have the advantage of being realized as a subspace of distributions, namely those $f$ in $\mathcal{M}_{\text{flag}}(\mathbb{H}^n)'$ whose flag Littlewood–Paley function $g_{\text{flag}}(f)$ belongs to $L^p(\mathbb{H}^n)$. Here $\mathcal{M}_{\text{flag}}(\mathbb{H}^n)$ is a molecule space with implicit product structure.

In Part I of the paper we define flag Hardy spaces and state our results. In Part II we give the proofs, and in Part III we construct a dyadic grid adapted to the flag structure.

**Remark 1.** Some of the proofs we need in this paper are straightforward modifications of arguments already in the literature, and in order not to interrupt the flow of the paper, we have left these proofs out. However, all the details are included in the expanded version of this paper [Han et al. 2012].

## Part I. Flag Hardy spaces: definitions and results

Our point of departure is to develop a *wavelet* Calderón reproducing formula associated with the given two-parameter structure as in [Müller et al. 1996], and then to prove a Plancherel–Pólya-type inequality in this setting. This will provide the flexibility needed to define flag Hardy spaces and prove boundedness of flag singular integrals, duality, and interpolation theorems for these spaces. To explain the novelty in this approach more carefully, we point out the following three types of reproducing formulas derived from the original idea of Calderón:

$$f(x) = \int_0^\infty \psi_t * \psi_t * f(x) \frac{dt}{t},$$

$$f(x) = \sum_{j\in\mathbb{Z}} \widetilde{D}_j D_j f(x),$$

$$f(x) = \sum_j \sum_I \{|I|(\psi_j * f)(x_I)\} \tilde{\psi}_j(x, x_I).$$

We refer to the first formula as a *continuous* Calderón reproducing formula, its advantage being the use of *compactly* supported components $\psi_t$ that are repeated. We refer to the second formula as a *discrete* Calderón reproducing formula, in which $D_j$ is generally a compactly supported *nonconvolution* operator in a space of homogeneous type, and $\widetilde{D}_j$ is no longer compactly supported but satisfies *molecular* estimates. In certain cases, such as in Euclidean space, it is possible to use the Fourier transform to obtain a discrete decomposition with repeated convolution operators $D_j = \psi_j$.

Finally, we refer to the third formula as a *wavelet* Calderón reproducing formula, which can also be developed in a space of homogeneous type. For example, such formulas were first developed in certain situations in [Frazier and Jawerth 1990]. The advantage of the third formula is that it expresses $f$ as a sum of molecular, or wavelet-like, functions $\tilde{\psi}_j(x, x_I)$ with coefficients $|I|(\psi_j * f)(x_I)$ that are obtained by evaluating $\psi_j * f$ at *any* convenient point in the set $I$ from a dyadic decomposition at scale $2^j$ of the space. As a consequence, we can replace the coefficient $|I|(\psi_j * f)(x_I)$ with either the supremum or infimum of such choices and retain appropriate estimates(see Theorem 19 below). We note in passing that the collection of functions $\{\tilde{\psi}_j(x, x_I)\}_{j,I}$ forms a *Riesz* basis for $L^2$. In certain cases when such functions form an *orthogonal* basis, the decomposition is referred to as a *wavelet* decomposition, and it is from this that we borrow our terminology.

This "wavelet" scheme is particularly useful in dealing with the Hardy spaces $H^p$ for $0 < p \leq 1$, and using this, we will show that flag singular integral operators are bounded on $H^p_{\text{flag}}$ for all $0 < p \leq 1$, and furthermore that these operators are bounded from $H^p_{\text{flag}}$ to $L^p$ for all $0 < p \leq 1$. These ideas can also be applied in the pure product setting to provide a different approach to proving $H^p_{\text{product}}$ to $L^p$ boundedness than that used by Fefferman, and thus to bypass both the action of singular integral operators on rectangle atoms, and the use of Journé's covering lemma.

We now recall the example of implicit multiparameter structure that provides the main motivation for this paper. In [Müller et al. 1995], Müller, Ricci, and Stein uncovered a new class of flag singular integrals on Heisenberg(-type) groups, which arose in the investigation of Marcinkiewicz multipliers. To be more precise, let $m(\mathcal{L}, iT)$ be the Marcinkiewicz multiplier operator, where $\mathcal{L}$ is the sub-Laplacian, $T$ is the central element of the Lie algebra on the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, and $m$ satisfies Marcinkiewicz conditions as in [Müller et al. 1995]. It was proved in [Müller et al. 1995] that the kernel of $m(\mathcal{L}, iT)$ satisfies the standard one-parameter Calderón–Zygmund-type estimates associated with automorphic dilations in the region where $|t| < |z|^2$, and the multiparameter Calderón–Zygmund-type estimates in the region where $|t| \geq |z|^2$.

The proof of $L^p$ boundedness of $m(\mathcal{L}, iT)$ given in [Müller et al. 1995] requires lifting the operator to a larger group, $\mathbb{H}^n \times \mathbb{R}$. This lifts $K$, the kernel of $m(\mathcal{L}, iT)$ on $\mathbb{H}^n$, to a *product* kernel $\widetilde{K}$ on $\mathbb{H}^n \times \mathbb{R}$. The lifted kernel $\widetilde{K}$ is constructed so that it projects to $K$ by

$$K(z, t) = \int_{-\infty}^{\infty} \widetilde{K}(z, t - u, u)\, du,$$

taken in the sense of distributions. The operator $\widetilde{T}$ corresponding to the product kernel $\widetilde{K}$ can be dealt with in terms of tensor products of operators, and one can obtain their $L^p$ boundedness from the known pure product theory. Finally, the $L^p$ boundedness of the operator with kernel $K$ follows from the transference method of [Coifman and Weiss 1976], using the projection $\pi : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n$ given by $\pi((z, t), u) = (z, t + u)$. One of our main results, Corollary 27 below, is an extension of the boundedness of $m(\mathcal{L}, iT)$ to flag Hardy spaces $H^p_{\text{flag}}$ for all $0 < p \leq 1$, and follows from the boundedness of flag singular integrals on $H^p_{\text{flag}}$.

In [Müller et al. 1996], the authors obtained the same boundedness results, but with optimal regularity on the multipliers. This required working directly on the group without lifting to a product, and led to the introduction of a *continuous* flag Littlewood–Paley $g$-function and a corresponding *continuous* Calderón reproducing formula. We remark that one of the main features of our extension of these results to $H^p$ for $0 < p \leq 1$ is the construction of a *wavelet* Calderón reproducing formula.

We note that the regularity satisfied by flag singular kernels is better than that of the product singular kernels. More precisely, the singularity of the standard pure product kernel on $\mathbb{C}^n \times \mathbb{R}$ is contained in the union $\{(z, 0)\} \cup \{(0, u)\}$ of two subspaces, while the singularity of $K(z, u)$, the flag singular kernel on $\mathbb{H}^n \times \mathbb{R}$ defined by Definition 7 below, is contained in a single subspace $\{(0, u)\}$, but is more singular on yet a smaller subspace $\{(0, 0)\}$, a situation described neatly in terms of the flag (or filtration) of subspaces, $\{(0, 0)\} \subsetneqq \{(0, u)\} \subsetneqq \mathbb{H}^n$. In the following, we describe some natural questions that arise.

**Question 1.** What is the correct definition of a flag Hardy space $H^p_{\text{flag}}$ associated with flag singular integral operators for $0 < p \le 1$ so that both (1) flag singular integral operators are bounded, and (2) a satisfactory theory of interpolation emerges?

**Question 2.** What is the correct definition of spaces $\text{BMO}_{\text{flag}}$ of bounded mean oscillation for flag singular integral operators, and are the singular integrals bounded on them?

**Question 3.** What is the duality theory for $H^p_{\text{flag}}$? Is there an analogue of BMO and Carleson measure-type function spaces which are dual spaces of the flag Hardy spaces $H^p_{\text{flag}}$ as in the pure product setting?

**Question 4.** Is there a Calderón–Zygmund decomposition adapted to functions in flag Hardy spaces $H^p_{\text{flag}}$ that leads, for example, to an appropriate theory of interpolation?

**Question 5.** What is the relationship between classical Hardy spaces $H^p$ and the flag Hardy spaces $H^p_{\text{flag}}$?

We address these five questions as follows. As in the $L^p$ theory for $p > 1$ considered in [Müller et al. 1995], one is naturally tempted to establish Hardy space theory under the implicit two-parameter structure associated with the flag singular kernel by invoking the method of lifting to the pure product setting together with the transference method in [Coifman and Weiss 1976]. However, this direct lifting method is not readily adaptable to the case of $p \le 1$ because the transference method is not known to be valid. A different approach centering on the use of a continuous flag Littlewood–Paley $g$-function was carried out in [Müller et al. 1996]. This suggests that the flag Hardy space $H^p_{\text{flag}}$ associated with this implicit two-parameter structure for $0 < p \le 1$ should be defined in terms of this or a similar $g$-function. Crucial for this is the use of a space of test functions arising from the lifting technique in [Müller et al. 1995], and a "wavelet" Calderón reproducing formula adapted to these test functions. Here is the order in which we implement these ideas.

(1) We first use the $L^p$ theory of Littlewood–Paley square functions $g_{\text{flag}}$ as in [Müller et al. 1996] to develop a Plancherel–Pólya-type inequality.

(2) We next define the flag Hardy spaces $H^p_{\text{flag}}$ using the flag $g$-function $g_{\text{flag}}$ together with a space of test functions that is motivated by the lifting technique in [Müller et al. 1995]. We then develop the theory of Hardy spaces $H^p_{\text{flag}}$ associated to the two-parameter flag structures and the boundedness of flag singular integrals on these spaces. We also establish the boundedness of flag singular integrals from $H^p_{\text{flag}}$ to $L^p$.

(3) We then turn to duality theory for the flag Hardy space $H^p_{\text{flag}}$ and introduce the dual space $\text{CMO}^p_{\text{flag}}$. In particular we establish the duality between $H^1_{\text{flag}}$ and the space $\text{BMO}_{\text{flag}}$. We then establish the

boundedness of flag singular integrals on $\mathrm{BMO}_{\mathrm{flag}}$. It is worthwhile to point out that in the classical one-parameter or pure product case, BMO is related to the concept of Carleson measure. The space $\mathrm{CMO}^p_{\mathrm{flag}}$ for all $0 < p \le 1$, as the dual space of $H^p_{\mathrm{flag}}$ introduced in this paper, is then defined by a generalized Carleson measure condition.

(4) We finally establish a Calderón–Zygmund decomposition lemma for any $H^p_{\mathrm{flag}}$ function ($0 < p < \infty$) in terms of functions in $H^{p_1}_{\mathrm{flag}}$ and $H^{p_2}_{\mathrm{flag}}$ with $0 < p_1 < p < p_2 < \infty$. This gives rise to an interpolation theorem between $H^{p_1}_{\mathrm{flag}}$ and $H^{p_2}_{\mathrm{flag}}$ for any $0 < p_2 < p_1 < \infty$ ($H^p_{\mathrm{flag}} = L^p$ for $1 < p < \infty$).

We now describe our approach and results in more detail. Proofs will be given in subsequent parts of the paper.

## 2. The square function on the Heisenberg group

We begin with an *implicit* two-parameter continuous variant of the Littlewood–Paley square function that is introduced in [Müller et al. 1996]. For this we need the standard Calderón reproducing formula on the Heisenberg group. Note that spectral theory was used in place of the Calderón reproducing formula in [Müller et al. 1996].

**Theorem 2** [Geller and Mayeli 2006, Corollary 1]. *There is $\psi \in C^\infty(\mathbb{H}^n)$ satisfying either*

$$\psi \in \mathcal{S}(\mathbb{H}^n) \text{ and all moments of } \psi \text{ vanish, or}$$

$$\psi \in C_c^\infty(\mathbb{H}^n) \text{ and all arbitrarily large moments of } \psi \text{ vanish,}$$

*such that the following Calderón reproducing formula holds*:

$$f = \int_0^\infty \psi_s^\vee * \psi_s * f \frac{ds}{s}, \quad f \in L^2(\mathbb{H}^n),$$

*where $*$ is Heisenberg convolution, $\psi^\vee(\zeta) = \overline{\psi(\zeta^{-1})}$, and $\psi_s(z,t) = s^{-2n-2}\psi(z/s, u/s^2)$ for $s > 0$.*

**Remark 3.** We will usually assume that $\psi$ above has compact support. However, it will sometimes be convenient for us if the component functions $\psi_s$ have *infinitely* many vanishing moments. In particular we can then use the *same* component functions to define the flag Hardy spaces for *all* $0 < p < \infty$ (the smaller $p$ is, the more vanishing moments are required to obtain necessary decay of singular integrals). Thus we will sometimes sacrifice the property of having compactly supported component functions.

We now wish to extend this formula to encompass the flag structure on the Heisenberg group $\mathbb{H}^n$.

**2.1.** ***The component functions.*** Following [Müller et al. 1996], we construct a Littlewood–Paley *component* function $\psi$ defined on $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$, given by the partial convolution $*_2$ in the second variable only:

$$\psi(z,u) = \psi^{(1)} *_2 \psi^{(2)}(z,u) = \int_{\mathbb{R}} \psi^{(1)}(z, u-v)\psi^{(2)}(v)\, dv, \quad (z,u) \in \mathbb{C}^n \times \mathbb{R},$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{H}^n)$ is as in Theorem 2, and $\psi^{(2)} \in \mathcal{S}(\mathbb{R})$ satisfies

$$\int_0^\infty |\widehat{\psi^{(2)}}(t\eta)|^2 \frac{dt}{t} = 1$$

for all $\eta \in \mathbb{R}\backslash\{0\}$, along with the moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta \psi^{(1)}(z,u)\,dz\,du = 0, \quad |\alpha| + 2\beta \le M,$$
$$\int_{\mathbb{R}} v^\gamma \psi^{(2)}(v)dv = 0, \quad \gamma \ge 0.$$

Here the positive integer $M$ may be taken arbitrarily large when the support of $\psi^{(1)}$ is compact, and may be infinite otherwise.

Thus we have

$$f(z,u) = \int_0^\infty \int_0^\infty \check{\psi}_{s,t} * \psi_{s,t} * f(z,u) \frac{ds}{s}\frac{dt}{t}, \quad f \in L^2(\mathbb{H}^n), \tag{2-1}$$

where the functions $\psi_{s,t}$ are given by

$$\psi_{s,t}(z,u) = \psi_s^{(1)} *_2 \psi_t^{(2)}(z,u), \tag{2-2}$$

with

$$\psi_s^{(1)}(z,u) = s^{-2n-2}\psi^{(1)}\left(\frac{z}{s}, \frac{u}{s^2}\right) \quad \text{and} \quad \psi_t^{(2)}(v) = t^{-1}\psi^{(2)}\left(\frac{v}{t}\right),$$

and where the integrals in (2-1) converge in $L^2(\mathbb{H}^n)$. Indeed,

$$\begin{aligned}\check{\psi}_{s,t} *_{\mathbb{H}^n} \psi_{s,t} *_{\mathbb{H}^n} f(z,u) &= (\check{\psi}_s^{(1)} *_2 \psi_t^{(2)}) *_{\mathbb{H}^n} (\check{\psi}_s^{(1)} *_2 \psi_t^{(2)}) *_{\mathbb{H}^n} f(z,u)\\ &= (\check{\psi}_s^{(1)} *_{\mathbb{H}^n} \check{\psi}_s^{(1)}) *_{\mathbb{H}^n} (\psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)}) *_2 f(z,u)\end{aligned}$$

yields (2-1) upon invoking the standard Calderón reproducing formula on $\mathbb{R}$ and then Theorem 2 on $\mathbb{H}^n$:

$$\begin{aligned}\int_0^\infty \int_0^\infty \check{\psi}_{s,t} *_{\mathbb{H}^n} \psi_{s,t} *_{\mathbb{H}^n} f(z,u)\frac{ds}{s}\frac{dt}{t} &= \int_0^\infty \check{\psi}_s^{(1)} *_{\mathbb{H}^n} \check{\psi}_s^{(1)} *_{\mathbb{H}^n} \left\{\int_0^\infty \psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)} *_2 f(z,u)\frac{dt}{t}\right\}\frac{ds}{s}\\ &= \int_0^\infty \check{\psi}_s^{(1)} *_{\mathbb{H}^n} \check{\psi}_s^{(1)} *_{\mathbb{H}^n} f(z,u)\frac{ds}{s} = f(z,u).\end{aligned}$$

For $f \in L^p$, $1 < p < \infty$, the *continuous* Littlewood–Paley square function $g_{\text{flag}}(f)$ of $f$ is defined by

$$g_{\text{flag}}(f)(z,u) = \left\{\int_0^\infty \int_0^\infty |\psi_{s,t} * f(z,u)|^2 \frac{ds}{s}\frac{dt}{t}\right\}^{\frac{1}{2}},$$

Note that we have the *flag* moment conditions, so called because they include only half of the product moment conditions associated with the product $\mathbb{C}^n \times \mathbb{R}$:

$$\int_{\mathbb{R}} u^\alpha \psi(z,u)\,du = 0 \quad \text{for all } \alpha \in \mathbb{Z}_+ \text{ and } z \in \mathbb{C}^n. \tag{2-3}$$

Indeed, with the change of variable $u' = u - v$ and the binomial theorem

$$(u'+v)^\beta = \sum_{\beta=\gamma+\delta} c_{\gamma,\delta}(u')^\gamma v^\delta,$$

we have

$$\begin{aligned}\int_{\mathbb{R}} u^\alpha \psi(z,u)\,du &= \int_{\mathbb{R}} u^\alpha \left\{ \int_{\mathbb{R}} \psi^{(2)}(u-v)\psi^{(1)}(z,v)\,dv \right\} du \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} (u'+v)^\alpha \psi^{(2)}(u')\,du \right\} \psi^{(1)}(z,v)\,dv \\ &= \sum_{\alpha=\gamma+\delta} c_{\gamma,\delta} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} (u')^\gamma \psi^{(2)}(u')\,du' \right\} v^\delta \psi^{(1)}(z,v)\,dv \\ &= \sum_{\alpha=\gamma+\delta} c_{\gamma,\delta} \int_{\mathbb{R}} \{0\} v^\delta \psi^{(1)}(z,v)\,dv = 0\end{aligned}$$

for all $\alpha \in \mathbb{Z}_+$ and each $z \in \mathbb{C}^n$. Note that, as a consequence, the *full* moments $\int_{\mathbb{H}^n} z^\alpha u^\beta \psi(z,u)\,dz\,du$ all vanish, but that, in general, the partial moments $\int_{\mathbb{C}^n} z^\alpha \psi(z,u)\,dz$ do *not* vanish.

**Remark 4.** As observed in [Nagel et al. 2012], there is a weak cancellation substitute for this failure to vanish, namely an *estimate* for $\int_{\mathbb{C}^n} z^\alpha \psi(z,u)\,dz$ that is derived from the vanishing moments of $\psi^{(1)}(z,v)$ and the smoothness of $\psi^{(2)}(u)$ via the identity

$$\begin{aligned}\int_{\mathbb{C}^n} z^\alpha \psi(z,u)\,dz &= \int_{\mathbb{C}^n}\int_{\mathbb{R}} z^\alpha \psi^{(1)}(z,v)\psi^{(2)}(u-v)\,dz\,dv \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{R}} z^\alpha \psi^{(1)}(z,v)[\psi^{(2)}(u-v)-\psi^{(2)}(u)]\,dz\,dv.\end{aligned}$$

We will not pursue this further here.

We will also consider the associated *sequence* of component functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, where the functions $\psi_{j,k}$ are given by

$$\psi_{j,k}(z,u) = \psi_j^{(1)} *_2 \psi_k^{(2)}(z,u), \tag{2-4}$$

with

$$\psi_j^{(1)}(z,u) = 2^{\alpha j(2n+2)}\psi^{(1)}(2^{\alpha j}z, 2^{2\alpha j}u) \quad \text{and} \quad \psi_k^{(2)}(v) = 2^{2\alpha k}\psi^{(2)}(2^{2\alpha k}v),$$

and $\psi^{(1)}$ and $\psi^{(2)}$ as above. Here $\alpha$ is a small positive constant that will be fixed in Theorem 17 below, where we establish a *wavelet* Calderón reproducing formula using this sequence of component functions for small $\alpha$. We then have a corresponding *discrete* (convolution) Littlewood–Paley square function $g_{\text{flag}}(f)$ defined by

$$g_{\text{flag}}(f)(z,u) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(z,u)|^2 \right\}^{\frac{1}{2}}.$$

This should be compared with the analogous square function in [Müller et al. 1996].

**Remark 5.** The terminology "implicit two-parameter structure" is inspired by the fact that the functions $\psi_{s,t}(z,u)$ and $\psi_{j,k}(z,u)$ are not dilated directly from $\psi(z,u)$, but rather from a lifting of $\psi$ to a product function. It is the subtle convolution $*_2$ that facilitates a passage from one-parameter "cubes" to two-parameter "rectangles" as dictated by the geometry of the kernels considered.

**2.2.** ***Square function inequalities.*** Altogether, we have from above that

$$f(z,u)=\int_0^\infty\int_0^\infty \psi_{s,t}*\psi_{s,t}*f(z,u)\frac{ds}{s}\frac{dt}{t},\qquad f\in L^2(\mathbb{H}^n). \tag{2-5}$$

Note that if one considers the integral on the right-hand side as an operator, then, by the construction of the function $\psi$, it is a flag singular integral operator and has the implicit multiparameter structure mentioned above. Using iteration and the vector-valued Littlewood–Paley estimate together with the Calderón reproducing formula on $L^2$ allows us to obtain $L^p$ estimates for $g_{\text{flag}}$, $1<p<\infty$, in Theorem 6 below. This should be compared to the variant in [Müller et al. 1996, Proposition 4.1] for $g$-functions constructed from spectral theory for $\mathcal{L}$ and $T$.

**Theorem 6.** *Let $1<p<\infty$. There exist constants $C_1$ and $C_2$ depending on $n$ and $p$ such that*

$$C_1\|f\|_p\le\|g_{\text{flag}}(f)\|_p\le C_2\|f\|_p,\qquad f\in L^p(\mathbb{H}^n).$$

In order to state our results for flag singular integrals on $\mathbb{H}^n$, we need to recall some definitions given in [Nagel et al. 2001]. We begin with the definition of a class of distributions on Euclidean space $\mathbb{R}^N$. A $k$-normalized bump function on a space $\mathbb{R}^N$ is a $C^k$-function supported on the unit ball with $C^k$ norm bounded by 1. As pointed out in [Nagel et al. 2001], the definitions given below are independent of the choices of $k\ge 1$, and thus we will simply refer to a "normalized bump function" without specifying the index $k$.

We will rephrase Definition 2.1.1 in [Nagel et al. 2001] of a flag kernel in the case of the Heisenberg group as follows.

**Definition 7.** A flag convolution kernel on $\mathbb{H}^n=\mathbb{C}^n\times\mathbb{R}$ is a distribution $K$ on $\mathbb{R}^{2n+1}$ which coincides with a $C^\infty$ function away from the coordinate subspace $\{(0,u)\}\subset\mathbb{H}^n$, where $0\in\mathbb{C}^n$ and $u\in\mathbb{R}$, and satisfies the following:

(1) (differential inequalities) For any multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_m)$,

$$|\partial_z^\alpha\partial_u^\beta K(z,u)|\le C_{\alpha,\beta}|z|^{-2n-|\alpha|}\cdot(|z|^2+|u|)^{-1-|\beta|}$$

for all $(z,u)\in\mathbb{H}^n$ with $z\ne 0$.

(2) (cancellation condition) For every multi-index $\alpha$ and every normalized bump function $\phi_1$ on $\mathbb{R}$ and every $\delta>0$,

$$\left|\int_{\mathbb{R}}\partial_z^\alpha K(z,u)\phi_1(\delta u)\,du\right|\le C_\alpha|z|^{-2n-|\alpha|};$$

for every multi-index $\beta$ and every normalized bump function $\phi_2$ on $\mathbb{C}^n$ and every $\delta>0$,

$$\left|\int_{\mathbb{C}^n}\partial_u^\beta K(z,u)\phi_2(\delta z)\,dz\right|\le C_\gamma|u|^{-1-|\beta|};$$

and for every normalized bump function $\phi_3$ on $\mathbb{H}^n$ and every $\delta_1>0$ and $\delta_2>0$,

$$\left|\int_{\mathbb{H}^n}K(z,u)\phi_3(\delta_1 z,\delta_2 u)\,dz\,du\right|\le C.$$

As in [Müller et al. 1995], we may always assume that a flag kernel $K(z,u)$ is integrable on $\mathbb{H}^n$ by using a smooth truncation argument.

Informally, we can now define the flag Hardy space $H^p_{\text{flag}}(\mathbb{H}^n)$ on the Heisenberg group for $0<p\le 1$ by

$$H^p_{\text{flag}}(\mathbb{H}^n)=\{f \text{ a } \textit{distribution} \text{ on } \mathbb{H}^n : g_{\text{flag}}(f)\in L^p(\mathbb{H}^n)\},$$

and, for $f\in H^p_{\text{flag}}(\mathbb{H}^n)$, define

$$\|f\|_{H^p_{\text{flag}}}=\|g_{\text{flag}}(f)\|_p.$$

Of course we need to give a precise definition of *distribution* in this context, and a natural question then arises as to whether or not the resulting definition is independent of the choice of component functions $\psi_{j,k}$ in the definition of the square function $g_{\text{flag}}$. Moreover, to study the $H^p_{\text{flag}}$-boundedness of flag singular integrals and establish the duality theory of $H^p_{\text{flag}}$, this definition is difficult to use when $0<p\le 1$. We need to approximately discretize the quasinorm of $H^p_{\text{flag}}$. In order to obtain this discrete $H^p_{\text{flag}}$ quasinorm we will prove certain Plancherel–Pólya-type inequalities, and the main tool used in proving such inequalities will be the wavelet Calderón reproducing formula that we define below. To be more specific, we will prove that the formula (2-5) converges in certain spaces of test functions $\mathcal{M}^M_{\text{flag}}(\mathbb{H}^n)$ adapted to the flag structure, and thus also in the dual spaces $\mathcal{M}^M_{\text{flag}}(\mathbb{H}^n)'$ (see Theorem 17 below). Furthermore, using an approximation procedure and an almost-orthogonality argument, we prove in Theorem 17 below a *wavelet* Calderón reproducing formula which expresses $f$ as a Fourier-like *series* of molecules or "wavelets" $(z,u)\to\tilde{\psi}_{j,k}(z,u,z_I,u_J)$ with coefficients $\psi_{j,k}*f(z_I,u_J)$.

In order to describe this formula explicitly in Section 3 below, we will use the *flag* dyadic decomposition

$$\mathbb{H}^n=\dot{\bigcup_{(\alpha,\tau)\in K_j}}\mathcal{S}_{j,\alpha,\tau}$$

of the Heisenberg group given in Theorem 68 below (this is a "hands on" variant of the tiling construction in [Strichartz 1992]), as well as the notion of Heisenberg rectangles

$$\mathcal{R}^{\mathcal{S}_{k,\beta,\upsilon}}_{\mathcal{S}_{j,\alpha,\tau}}(\text{ver}) \quad\text{and}\quad \mathcal{R}^{\mathcal{S}_{k,\beta,\upsilon}}_{\mathcal{S}_{j,\alpha,\tau}}(\text{hor})$$

given in Definition 69 below when $j\le k$ and $\mathcal{S}_{j,\alpha,\tau}$ and $\mathcal{S}_{k,\beta,\upsilon}$ are dyadic cubes in $\mathbb{H}^n$ with $\mathcal{S}_{j,\alpha,\tau}\subset\mathcal{S}_{k,\beta,\upsilon}$. Recall that

$$\{I\}_{I \text{ dyadic}}=\{I^j_\alpha\}_{j\in\mathbb{Z} \text{ and } \alpha\in 2^j\mathbb{Z}^{2n}}$$

is the usual dyadic grid in $\mathbb{C}^n$ and that

$$\{J\}_{J \text{ dyadic}}=\{J^k_\tau\}_{k\in\mathbb{Z} \text{ and } \tau\in 2^k\mathbb{Z}}$$

is the usual dyadic grid in $\mathbb{R}$. The projection of the dyadic cube $\mathcal{S}_{j,\alpha,\tau}$ onto $\mathbb{C}^n$ is the dyadic cube $I^j_\alpha$, and

$$\mathcal{R}^{\mathcal{S}_{k,\beta,\upsilon}}_{\mathcal{S}_{j,\alpha,\tau}}(\text{ver}) \quad (\text{respectively } \mathcal{R}^{\mathcal{S}_{k,\beta,\upsilon}}_{\mathcal{S}_{j,\alpha,\tau}}(\text{hor}))$$

plays the role of the dyadic rectangle $I_\alpha^j \times J_\upsilon^{2k}$ (respectively $I_\beta^k \times J_\tau^{2j}$). In the Heisenberg group, these rectangles necessarily "rotate" with the group structure.

**Notation 8.** It will be convenient to use the suggestive, if somewhat imprecise, notation

$$\mathcal{R} = I \times J = I_\alpha^j \times J_\upsilon^{2k}$$

for the dyadic rectangle $\mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{ver})$, etc. It should be emphasized that $\mathcal{R} = I \times J$ is *not* a product set, but rather a dyadic Heisenberg rectangle $\mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{ver})$ that serves as a Heisenberg substitute for the actual product set $I_\alpha^j$ times $J_\upsilon^{2k}$. Thus we will say that the dyadic rectangle $\mathcal{R} = I \times J$ has *side lengths* $\ell(I) = 2^j$ and $\ell(J) = 2^{2k}$. For $j \le k$, the collection of all dyadic Heisenberg rectangles $\mathcal{R} = I \times J$ with side lengths $2^j$ and $2^{2k}$ will be denoted by

$$\mathcal{R}(2^j \times 2^{2k}) \equiv \left\{\mathcal{R} = I \times J = I_\alpha^j \times J_\upsilon^{2k} = \mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{ver}) : \mathcal{S}_{j,\alpha,\tau} \subset \mathcal{S}_{k,\beta,\upsilon}\right\}.$$

*Caution*: For $k \le j$, the support of the component function $\psi_{j,k}$ defined in (2-4) is essentially a vertical Heisenberg rectangle $I \times J$ having side lengths $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-2k}$. Note the passage from $j, k$ to $-j, -k$.

**2.3. *Standard test functions.*** We now describe the features inherent in giving a precise definition of the flag Hardy space $H^p_{\text{flag}}(\mathbb{H}^n)$ as elements in the dual of familiar test spaces. We begin by introducing the test spaces $\mathcal{M}^M_{\text{flag}}(\mathbb{H}^n)$ associated with the flag structure on $\mathbb{H}^n$ that are obtained by *projecting* the corresponding product test spaces $\mathcal{M}^M_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$ onto $\mathbb{H}^n$. Our definitions here will encompass the entire range $0 < p \le 1$. For this we use the projection of functions $F$ defined on $\mathbb{H}^n \times \mathbb{R}$ to functions $f = \pi F$ defined on $\mathbb{H}^n$ as introduced in [Müller et al. 1995]:

$$f(z, u) = (\pi F)(z, u) \equiv \int_{\mathbb{R}} F((z, u - v), v)\, dv. \tag{2-6}$$

We will also use the notation $\pi F = F_\flat$ as in [Müller et al. 1995]. Recall that $2n + 1$ is the Euclidean dimension of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and that $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$.

In this notation, the component function $\psi(z, u)$ in Subsection 2.1 above is given by $\pi\Psi(z, u)$, where

$$\Psi(z, u, v) \equiv \psi^{(1)}(z, u)\psi^{(2)}(v). \tag{2-7}$$

We now define an appropriate *product* molecular space $\mathcal{M}^{M_1,M_2,M}_{\text{product}}$ on $\mathbb{H}^n \times \mathbb{R}$ with three parameters $M_1, M_2, M$.

**Remark 9.** Note that, in the definition below, we require *equally* many moments and derivatives in each of the $u$ and $v$ variables, and exactly *twice* as many moments and derivatives in the $z$ variable. The integer $M$ controls the decay of the function, the integer $M_1$ controls the total number of moments, and the integer $M_2$ controls the total weighted number of derivatives permitted.

**Definition 10.** Let $M, M_1, M_2 \in \mathbb{N}$ be positive integers and let $0 < \delta \le 1$. The *product* molecular space $\mathcal{M}^{M+\delta,M_1,M_2}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$ consists of all functions $F((z, u), v)$ on $\mathbb{H}^n \times \mathbb{R}$ satisfying the product moment conditions

$$\begin{aligned} \int_{\mathbb{H}^n} z^\alpha u^\beta F((z,u),v)\,dz\,du = 0 \quad &\text{for all } |\alpha| + 2\beta \le M_1, \\ \int_{\mathbb{R}} v^\gamma F((z,u),v)\,dv = 0 \quad &\text{for all } 2\gamma \le M_1, \end{aligned} \tag{2-8}$$

and such that there is a nonnegative constant $A$ satisfying the four differential inequalities

$$|\partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z,u),v)| \le A \frac{1}{(1+|z|^2+|u|)^{(Q+M+|\alpha|+2\beta+\delta)/2}} \frac{1}{(1+|v|)^{1+M+\gamma+\delta}} \quad \text{for all } |\alpha|+2\beta \le M_2 \text{ and } 2\gamma \le M_2, \tag{2-9}$$

$$\begin{aligned} &|\partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z,u),v) - \partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z',u'),v)| \\ &\qquad \le A \frac{|(z,u)\circ(z',u')^{-1}|^\delta}{(1+|z|^2+|u|)^{(Q+M+M_2+2\delta)/2}} \frac{1}{(1+|v|)^{1+M+\gamma+\delta}} \\ &\qquad\qquad \text{for all } |\alpha|+2\beta = M_2 \text{ and } |(z,u)\circ(z',u')^{-1}| \le \tfrac{1}{2}(1+|z|^2+|u|)^{\frac{1}{2}}, \end{aligned} \tag{2-10}$$

$$\begin{aligned} &|\partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z,u),v) - \partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z,u),v')| \\ &\qquad \le A \frac{1}{(1+|z|^2+|u|)^{(Q+M+|\alpha|+2\beta+\delta)/2}} \frac{|v-v'|^\delta}{(1+|v|)^{1+M+M_2/2+2\delta}}, \\ &\qquad\qquad \text{for all } |\alpha|+2\beta \le M_2,\ 2\gamma = M_2, \text{ and } |v-v'| \le \tfrac{1}{2}(1+|v|), \end{aligned} \tag{2-11}$$

$$\begin{aligned} &\big|[\partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z,u),v) - \partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z',u'),v)] - [\partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z,u),v') - \partial_z^\alpha \partial_u^\beta \partial_v^\gamma F((z',u'),v')]\big| \\ &\qquad \le A \frac{|(z,u)\circ(z',u')^{-1}|^\delta}{(1+|z|^2+|u|)^{(Q+M+M_2+2\delta)/2}} \frac{|v-v'|^\delta}{(1+|v|)^{1+M+M_2/2+2\delta}} \\ &\text{for all } |\alpha|+2\beta = M_2,\ 2\gamma = M_2,\ |(z,u)\circ(z',u')^{-1}| \le \tfrac{1}{2}(1+|z|^2+|u|)^{\frac{1}{2}}, \text{ and } |v-v'| \le \tfrac{1}{2}(1+|v|). \end{aligned} \tag{2-12}$$

The space $\mathcal{M}_{\text{product}}^{M+\delta,M_1,M_2}(\mathbb{H}^n \times \mathbb{R})$ becomes a Banach space under the norm defined by the least nonnegative number $A$ for which the above four inequalities hold.

Now we define the *flag* molecular space $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ as the projection of $\mathcal{M}_{\text{product}}^{M+\delta,M_1,M_2}(\mathbb{H}^n \times \mathbb{R})$ under the map $\pi$ given in (2-6).

**Definition 11.** Let $M, M_1, M_2 \in \mathbb{N}$ be positive integers and $0 < \delta \le 1$. The *flag* molecular space $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ consists of all functions $f$ on $\mathbb{H}^n$ such that there is $F \in \mathcal{M}_{\text{product}}^{M+\delta,M_1,M_2}(\mathbb{H}^n \times \mathbb{R})$ with $f = \pi F = F_\flat$. Define a norm on $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ by

$$\|f\|_{\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)} \equiv \inf_{F: f=\pi F} \|F\|_{\mathcal{M}_{\text{product}}^{M+\delta,M_1,M_2}(\mathbb{H}^n\times\mathbb{R})}.$$

Thus the norm on $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ is the quotient norm

$$\|f\|_{\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)} = \mathcal{M}_{\text{product}}^{M+\delta,M_1,M_2}(\mathbb{H}^n \times \mathbb{R})/\pi^{-1}(\{0\}),$$

and $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ is a Banach space.

We record here an intertwining formula for $\pi$ and a convolution operator $T$ on $\mathbb{H}^n$. Let

$$Tf(z,u) = K *_{\mathbb{H}^n} f(z,u) = \int_{\mathbb{H}^n} K((z,u)\circ(z',u')^{-1}) f(z',u')\,dz'\,du'.$$

Extend $T$ to an operator $\widetilde{T} = T \otimes \delta_0$ on the group $\mathbb{H}^n \times \mathbb{R}$ by acting $T$ in the $\mathbb{H}^n$ factor only:

$$\widetilde{T}F((z,u),v) = \int_{\mathbb{H}^n} K((z,u)\circ(z',u')^{-1}) F(z',u',v)\,dz'\,du'.$$

**Lemma 12.** *Let $T$ be a convolution operator on $\mathbb{H}^n$ and let $\widetilde{T} = T \otimes \delta_0$ be its extension to $\mathbb{H}^n \times \mathbb{R}$ defined above. Then*

$$T(\pi F)(z,u) = \pi(\widetilde{T}F)(z,u).$$

*Proof.* Formally we have

$$\begin{aligned} T(\pi F)(z,u) &= \int_{\mathbb{H}^n} K((z,u)\circ(z',u')^{-1})(\pi F)(z',u')\,dz'\,du' \\ &= \int_{\mathbb{H}^n} K((z,u)\circ(z',u')^{-1}) \left\{ \int_{\mathbb{R}} F(z',u'-v,v)\,dv \right\} dz'\,du' \\ &= \int_{\mathbb{H}^n} \int_{\mathbb{R}} K(z-z', u-u'+2\operatorname{Im}\overline{z}'z) F(z',u'-v,v)\,dv\,dz'\,du'. \end{aligned}$$

Now make the change of variable $w' = u' - v$ to get

$$\begin{aligned} T(\pi F)(z,u) &= \int_{\mathbb{H}^n} \int_{\mathbb{R}} K(z-z', u-w'-v+2\operatorname{Im}\overline{z}'z) F(z',w',v)\,dv\,dz'\,dw' \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{H}^n} K((z,u-v)\circ(z',w')^{-1}) F(z',w',v)\,dz'\,dw' \right\} dv \\ &= \int_{\mathbb{R}} \{\widetilde{T}F(z,u-v,v)\}\,dv = \pi(\widetilde{T}F)(z,u). \end{aligned}$$ □

Later in the paper we will fix $M_1 = M_2 = M$ and denote $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ simply by $\mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)$, but for now we will allow $M_1$ and $M_2$ to remain independent of $M$ in order to further analyze the space $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$.

**2.3.1.** *An analysis of the projected flag molecular space.* Lemma 14 below shows that functions $f(z,u)$ in the "projected" flag molecular space $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ have moments in the $u$ variable alone, as well as *more* moments in the $(z,u)$ variable than we might expect. We refer loosely to this situation as having *half-product* moments. There is a more familiar space of test functions $\mathsf{M}_F^{M+\delta,M_1,M_2}(\mathbb{H}^n)$, defined below with half-product moments, that *avoids* the operation of projection, and that is closely related to the projected test space $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$. While we do not know if the spaces $\mathcal{M}_{\text{flag}}^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ and $\mathsf{M}_F^{M+\delta,M_1,M_2}(\mathbb{H}^n)$ coincide, the embeddings in Lemma 14 below are enough for our purposes.

**Definition 13.** Let $M$, $M_1$, $M_2 \in \mathbb{N}$ be positive integers and $0 < \delta \le 1$. Define the "moment" *flag* molecular space $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ to consist of all functions $f$ on $\mathbb{H}^n$ satisfying the moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta f(z,u)\,dz\,du = 0 \quad \text{for all } |\alpha| \le M_1,\ |\alpha| + 2\beta \le 2M_1 + 2,$$

$$\int_{\mathbb{R}} u^\gamma f(z,u)\,du = 0 \quad \text{for all } \gamma \le M_1,$$

and such that there is a nonnegative constant $A$ satisfying the differential inequalities

$$|\partial_z^\alpha \partial_u^\beta f(z,u)| \le A \frac{1}{(1+|z|^2+|u|)^{(Q+M+|\alpha|+2\beta)/2}} \quad \text{for all } |\alpha| + 2\beta \le M_2,$$

$$|\partial_z^\alpha \partial_u^\beta f(z,u) - \partial_z^\alpha \partial_u^\beta f(z',u')| \le A \frac{|(z,u)\circ(z',u')^{-1}|^\delta}{(1+|z|^2+|u|)^{(Q+M+\delta+M_2)/2}}$$

$$\text{for all } |\alpha| + 2\beta = M_2 \text{ and } |(z,u)\circ(z',u')^{-1}| \le \tfrac{1}{2}(1+|z|^2+|u|)^{\frac{1}{2}}.$$

Note that the moment conditions in the definition of $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ permit larger values of $\beta$ depending on $|\alpha|$ than in the definition of $\mathcal{M}_{\text{flag}}^{M+\delta, M_1, M_2}(\mathbb{H}^n)$. The space $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ becomes a Banach space under the norm defined by the least nonnegative number $A$ for which the above two inequalities hold.

**Lemma 14.** *The spaces* $\mathcal{M}_{\text{flag}}^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ *and* $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ *satisfy the containments*

$$\mathsf{M}_F^{3M+\delta+M_2, M_1, 2M_2+4}(\mathbb{H}^n) \subset \mathcal{M}_{\text{flag}}^{M+\delta, M_1, M_2}(\mathbb{H}^n) \subset \mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n),$$

*which are continuous*:

$$\|f\|_{\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)} \lesssim \|f\|_{\mathcal{M}_{\text{flag}}^{M+\delta, M_1, M_2}(\mathbb{H}^n)} \lesssim \|f\|_{\mathsf{M}_F^{3M+\delta+M_2, M_1, 2M_2+4}(\mathbb{H}^n)}.$$

**Remark 15.** The importance of the "projected" flag molecular space $\mathcal{M}_{\text{flag}}^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ lies in the existence of a *wavelet* Calderón reproducing formula for this space of test functions; see Theorem 17 below. We do not know if such a reproducing formula holds for the "moment" flag space $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$, but the embeddings in Lemma 14 will prove important in identifying the distributions in the dual space $\mathcal{M}_{\text{flag}}^{M+\delta, M_1, M_2}(\mathbb{H}^n)'$ as being "roughly" those in a dual space $\mathsf{M}_F^{M'+\delta, M_1', M_2'}(\mathbb{H}^n)'$.

**Remark 16.** The integer $M_1$ that controls the number of moments in $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$ remains the same in both the smaller space $\mathsf{M}_F^{3M+\delta+M_2, M_1, 2M_2+4}(\mathbb{H}^n)$ and the larger space $\mathsf{M}_F^{M+\delta, M_1, M_2}(\mathbb{H}^n)$. However, we lose both derivatives and decay in passing from the smaller to the larger space.

While we cannot say that $H_{\text{flag}}^p(\mathbb{H}^n)$ is a *subspace* of the more familiar one-parameter Hardy space $H^p(\mathbb{H}^n)$, we can show that the quotient space

$$Q_{\text{flag}}^p(\mathbb{H}^n) \equiv H_{\text{flag}}^p(\mathbb{H}^n) / \mathsf{M}_F^{M'+\delta, M_1', M_2'}(\mathbb{H}^n)^\perp$$

of $H_{\text{flag}}^p(\mathbb{H}^n)$ can be identified with a closed subspace of the corresponding quotient space

$$Q^p(\mathbb{H}^n) \equiv H^p(\mathbb{H}^n) / \mathsf{M}_F^{M'+\delta, M_1', M_2'}(\mathbb{H}^n)^\perp$$

of $H^p(\mathbb{H}^n)$, thus giving a sense in which the distributions we use to define $H^p_{\text{flag}}(\mathbb{H}^n)$ are "roughly" the same as those used to define $H^p(\mathbb{H}^n)$. See [Han et al. 2012] for details.

## 3. The wavelet Calderón reproducing formula

We can now state our *wavelet* Calderón reproducing formula for the flag structure in terms of the projected product test spaces

$$\mathcal{M}^{M+\delta}_{\text{flag}}(\mathbb{H}^n) \equiv \mathcal{M}^{M+\delta,M.M}_{\text{flag}}(\mathbb{H}^n),$$

defined by projecting the product test spaces

$$\mathcal{M}^{M+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R}) \equiv \mathcal{M}^{M+\delta,M,M}_{\text{product}}(\mathbb{H}^n \times \mathbb{R}).$$

We remind the reader that Euclidean versions of such reproducing formulas were obtained by Frazier and Jawerth [1990] using the Fourier transform together with the very special property that $\mathbb{R}^n$ is *tiled* by the compact abelian torus $\mathbb{T}^n$ and its discrete dual group, the lattice $\mathbb{Z}^n$.

It is convenient to introduce some new notation for the dyadic rectangles defined in Notation 8. Given $0 < \alpha < 1$ and a positive integer $N$, we write

$$\mathsf{R}(j,k) \equiv \mathcal{R}(2^{-\alpha(j+N)} \times 2^{-2\alpha(k+N)}),$$
$$\mathsf{Q}(j) \equiv \mathcal{R}(2^{-\alpha(j+N)} \times 2^{-2\alpha(j+N)}).$$

Now, for $\mathcal{Q} \in \mathsf{Q}(j)$, let $(z_{\mathcal{Q}}, u_{\mathcal{Q}})$ be any *fixed* point in the cube $\mathcal{Q}$, and for $\mathcal{R} \in \mathsf{R}(j,k)$ with $k < j$, let $(z_{\mathcal{R}}, u_{\mathcal{R}})$ be any *fixed* point in the rectangle $\mathcal{R}$. Let us write the collection of *all* dyadic cubes as

$$\mathsf{Q} \equiv \bigcup_{j\in\mathbb{Z}} \mathsf{Q}(j),$$

and the collection of *all strictly vertical* dyadic rectangles as

$$\mathsf{R}_{\text{vert}} \equiv \bigcup_{j>k} \mathsf{R}(j,k).$$

We now set

$$\psi'_{\mathcal{Q}} = \psi^{(1)}_j \qquad \text{if } \mathcal{Q} \in \mathsf{Q}(j),$$
$$\psi'_{\mathcal{R}} = \psi_{j,k} = \psi^{(1)}_j *_2 \psi^{(2)}_k \qquad \text{if } \mathcal{R} \in \mathsf{R}(j,k),$$

where the $\psi_{j,k}$ are as in (2-4). Given an appropriate distribution $f$ on $\mathbb{H}^n$, we define its *wavelet coefficients* $f_{\mathcal{Q}}$ and $f_{\mathcal{R}}$ by

$$f_{\mathcal{Q}} = \psi'_{\mathcal{Q}} * f(z_{\mathcal{Q}}, u_{\mathcal{Q}}) \qquad \text{if } \mathcal{Q} \in \mathsf{Q},$$
$$f_{\mathcal{R}} = \psi'_{\mathcal{R}} * f(z_{\mathcal{R}}, u_{\mathcal{R}}) \qquad \text{if } \mathcal{R} \in \mathsf{R}_{\text{vert}},\ \text{that is, when } j > k.$$

Below is the *wavelet* Calderón reproducing formula.

**Theorem 17.** *Suppose the notation is as above. Then there are associated functions* $\tilde{\psi}_{\mathcal{Q}}, \tilde{\psi}_{\mathcal{R}} \in \mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)$ *for* $\mathcal{Q} \in \mathsf{Q}$ *and* $\mathcal{R} \in \mathsf{R}_{\text{vert}}$ *satisfying*

$$\|\tilde{\psi}_{\mathcal{Q}}\|_{\mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)} \lesssim \|\psi'_{\mathcal{Q}}\|_{\mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)}, \quad \mathcal{Q} \in \mathsf{Q},$$
$$\|\tilde{\psi}_{\mathcal{R}}\|_{\mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)} \lesssim \|\psi'_{\mathcal{R}}\|_{\mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)}, \quad \mathcal{R} \in \mathsf{R}_{\text{vert}},$$

*and*

$$f(z,u) = \sum_{\mathcal{Q}\in\mathsf{Q}} f_{\mathcal{Q}} \tilde{\psi}_{\mathcal{Q}}(z,u) + \sum_{\mathcal{R}\in\mathsf{R}_{\text{vert}}} f_{\mathcal{R}} \tilde{\psi}_{\mathcal{R}}(z,u), \quad (z,u) \in \mathbb{H}^n, \tag{3-1}$$

*where the series in (3-1) converges in three spaces*:

(1) *in* $L^p(\mathbb{H}^n)$ *for* $1 < p < \infty$,

(2) *in the Banach space* $\mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n)$ *for* $M'$ *large enough*,

(3) *and in the corresponding dual space* $\mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n)'$ *for* $M'$ *large enough.*

**Remark 18.** Note that only *half* of the collection of dyadic rectangles, namely the vertical ones $\mathsf{R}_{\text{vert}}$, are used in the wavelet Calderón reproducing formula. This is a reflection of the implicit product structure inherent in the Heisenberg group $\mathbb{H}^n$.

## 3.1. *Plancherel–Pólya inequalities and flag Hardy spaces.*

The wavelet Calderón reproducing formula (3-1) yields the following Plancherel–Pólya-type inequalities; cf. [Pólya 1936; Plancherel and Pólya 1937]. We use the notation $A \approx B$ to indicate that two quantities $A$ and $B$ are comparable.

**Theorem 19.** *Suppose* $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{C}^n)$ *and* $\psi^{(2)}, \phi^{(2)} \in \mathcal{S}(\mathbb{R})$, *and let*

$$\psi(z,u) = \int_{\mathbb{R}} \psi^{(1)}(z, u-v)\psi^{(2)}(v)\,dv,$$
$$\phi(z,u) = \int_{\mathbb{R}} \phi^{(1)}(z, u-v)\psi^{(2)}(v)\,dv$$

*be two component functions that each satisfies the conditions in Section 2.1. Then with* $\mathsf{Q}$, $\mathsf{R}_{\text{vert}}$, $\psi'_{\mathcal{Q}}$, *and* $\psi'_{\mathcal{R}}$ *as above, and for* $f \in \mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)'$, $0 < p < \infty$, *and* $M$ *chosen large enough depending on* $n$ *and* $p$,

$$\left\| \left\{ \sum_{\mathcal{Q}\in\mathsf{Q}} \sup_{(z',u')\in\mathcal{Q}} |\psi'_{\mathcal{Q}} * f(z',u')|^2 \chi_{\mathcal{Q}}(z,u) + \sum_{\mathcal{R}\in\mathsf{R}_{\text{vert}}} \sup_{(z',u')\in\mathcal{R}} |\psi'_{\mathcal{R}} * f(z',u')|^2 \chi_{\mathcal{R}}(z,u) \right\}^{\frac12} \right\|_{L^p(\mathbb{H}^n)}$$
$$\approx \left\| \left\{ \sum_{\mathcal{Q}\in\mathsf{Q}} \inf_{(z',u')\in\mathcal{Q}} |\psi'_{\mathcal{Q}} * f(z',u')|^2 \chi_{\mathcal{Q}}(z,u) + \sum_{\mathcal{R}\in\mathsf{R}_{\text{vert}}} \inf_{(z',u')\in\mathcal{R}} |\psi'_{\mathcal{R}} * f(z',u')|^2 \chi_{\mathcal{R}}(z,u) \right\}^{\frac12} \right\|_{L^p(\mathbb{H}^n)}.$$

The Plancherel–Pólya-type inequalities in Theorem 19 will prove useful in establishing properties of the *wavelet* Littlewood–Paley $g$-function

$$g_{\text{flag}}(f)(z,u) = \left\{ \sum_{\mathcal{Q}\in\mathsf{Q}} |\psi'_{\mathcal{Q}} * f(z_{\mathcal{Q}}, u_{\mathcal{Q}})|^2 \chi_{\mathcal{Q}}(z,u) + \sum_{\mathcal{R}\in\mathsf{R}_{\text{vert}}} |\psi'_{\mathcal{R}} * f(z_{\mathcal{R}}, u_{\mathcal{R}})|^2 \chi_{\mathcal{R}}(z,u) \right\}^{\frac12},$$

where we are using the notation of Theorems 17 and 19.

We can now give a precise definition of the *flag* Hardy spaces.

**Definition 20.** Let $0 < p < \infty$. Then, for $M$ sufficiently large depending on $n$ and $p$, we define the *flag* Hardy space $H^p_{\mathrm{flag}}(\mathbb{H}^n)$ on the Heisenberg group by

$$H^p_{\mathrm{flag}}(\mathbb{H}^n) = \left\{ f \in \mathcal{M}^{M+\delta}_{\mathrm{flag}}(\mathbb{H}^n)' : g_{\mathrm{flag}}(f) \in L^p(\mathbb{H}^n) \right\},$$

and, for $f \in H^p_{\mathrm{flag}}(\mathbb{H}^n)$, we set

$$\|f\|_{H^p_{\mathrm{flag}}} = \|g_{\mathrm{flag}}(f)\|_p. \tag{3-2}$$

**Remark 21.** We can take $M$ in Definition 20 to satisfy

$$M \geq M_{n,p} \equiv (2n+2)\left[\frac{2}{p} - 1\right] + 1.$$

We have not computed the optimal value of $M_{n,p}$.

It is easy to see using Theorem 19 that the Hardy space $H^p_{\mathrm{flag}}$ in Definition 20 is well defined and that the $H^p_{\mathrm{flag}}$ norm of $f$ is equivalent to the $L^p$ norm of $g_{\mathrm{flag}}$. By use of the Plancherel–Pólya-type inequalities, we will prove the boundedness of flag singular integrals on $H^p_{\mathrm{flag}}$ below.

**3.2.** ***Boundedness of singular integrals and Marcinkiewicz multipliers.*** Our main result is the $H^p_{\mathrm{flag}} \to H^p_{\mathrm{flag}}$ boundedness of flag singular integrals.

**Theorem 22.** *Suppose that $T$ is a flag singular integral with kernel $K(z,u)$ as in Definition 7. Then $T$ is bounded on $H^p_{\mathrm{flag}}$ for $0 < p \leq 1$. Namely, for all $0 < p \leq 1$ there exists a constant $C_{p,n}$ such that*

$$\|Tf\|_{H^p_{\mathrm{flag}}} \leq C_{p,n}\|f\|_{H^p_{\mathrm{flag}}}.$$

To obtain the $H^p_{\mathrm{flag}} \to L^p$ boundedness of flag singular integrals, we prove the following general result:

**Theorem 23.** *Let $0 < p \leq 1$. If $T$ is a linear operator which is bounded simultaneously on $L^2(\mathbb{R}^{2n+1})$ and $H^p_{\mathrm{flag}}(\mathbb{H}^n)$, then $T$ can be extended to a bounded operator from $H^p_{\mathrm{flag}}(\mathbb{H}^n)$ to $L^p(\mathbb{R}^{2n+1})$.*

**Remark 24.** From the proof given in the next part of the paper, we see that this result holds in a larger setting, which includes the classical one-parameter and product Hardy spaces and the Hardy spaces on spaces of homogeneous type. Thus this provides an alternative approach to using Fefferman's criterion on boundedness of a singular integral operator by restricting its action on rectangle atoms [Fefferman 1986], and then combining this with Journé's geometric lemma; see [Journé 1985; 1986; Pipher 1986].

In particular, for flag singular integrals we can deduce the following.

**Corollary 25.** *Let $T$ be a flag singular integral as in Theorem 23. Then $T$ is bounded from $H^p_{\mathrm{flag}}(\mathbb{H}^n)$ to $L^p(\mathbb{R}^{2n+1})$ for $0 < p \leq 1$.*

**Remark 26.** The conclusions of both Theorem 22 and Corollary 25 persist if we only require the moment and smoothness conditions on the flag kernel in Definition 7 to hold for $|\alpha|, \beta \leq N_{n,p}$, where $N_{n,p} < \infty$ is taken sufficiently large.

As a consequence, we can extend the Marcinkiewicz multiplier theorem in [Müller et al. 1995] (see Lemma 2.1 there) to flag Hardy spaces for $0 < p \leq 1$. To describe this extension, recall the standard sub-Laplacian $\mathscr{L}$ on the Heisenberg group

$$\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} = \{(z,t) : z = (z_j)_{j=1}^n,\ z_j = x_j + i y_j \in \mathbb{C},\ t \in \mathbb{R}\},$$

defined by

$$\mathscr{L} \equiv -\sum_{j=1}^{n} (X_j^2 + Y_j^2), \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.$$

The operators $\mathscr{L}$ and $T = \partial/(\partial t)$ commute, and so do their spectral measures $dE_1(\xi)$ and $dE_2(\eta)$. Given a bounded function $m(\xi,\eta)$ on $\mathbb{R}_+ \times \mathbb{R}$, define the multiplier operator $m(\mathscr{L}, iT)$ on $L^2(\mathbb{H}^n)$ by

$$m(\mathscr{L}, iT) = \iint_{\mathbb{R}_+ \times \mathbb{R}} m(\xi,\eta)\, dE_1(\xi)\, dE_2(\eta).$$

Then $m(\mathscr{L}, iT)$ is automatically bounded on $L^2(\mathbb{H}^n)$, and if we impose Marcinkiewicz conditions on the multiplier, we obtain boundedness on flag Hardy spaces; this despite the fact that $m$ is invariant under a two-parameter family of dilations $\delta_{(s,t)}$ which are group automorphisms only when $t = s^2$.

**Corollary 27.** *Let $0 < p \leq 1$, and suppose that $m(\xi,\eta)$ is a bounded function defined on $\mathbb{R}_+ \times \mathbb{R}$ that satisfies the Marcinkiewicz conditions*

$$|(\xi \partial_\xi)^\alpha (\eta \partial_\eta)^\beta m(\xi,\eta)| \leq C_{\alpha,\beta}$$

*for all $|\alpha|, \beta \leq N_{n,p}$, where $N_{n,p} < \infty$ is taken sufficiently large. Then $m(\mathscr{L}, iT)$ is a bounded operator on $H^p_{\text{flag}}(\mathbb{H}^n)$ for $0 < p \leq 1$.*

The corollary follows from the results above together with [Müller et al. 1995, Theorem 3.1], which shows that the kernel $K(z,u)$ of a Marcinkiewicz multiplier $m(\mathscr{L}, iT)$ satisfies the conditions defining a flag convolution kernel in Definition 7.

**3.3.** ***Carleson measures and duality.*** To study the dual space of $H^p_{\text{flag}}$, we introduce the Carleson measure space $\mathrm{CMO}^p_{\text{flag}}$.

**Notation 28.** It will often be convenient from now on to bundle the set $\mathsf{Q}$ of all dyadic cubes and the set $\mathsf{R}_{\text{vert}}$ of all vertical dyadic rectangles into a single set

$$\mathsf{R}_+ = \mathsf{Q} \cup \mathsf{R}_{\text{vert}}$$

consisting of all dyadic cubes and all vertical dyadic rectangles. We also write

$$\psi_{\mathcal{R}} = \begin{cases} \psi'_{\mathcal{Q}} & \text{if } \mathcal{R} = \mathcal{Q} \in \mathsf{Q}, \\ \psi'_{\mathcal{R}} & \text{if } \mathcal{R} \in \mathsf{R}_{\text{vert}}. \end{cases}$$

**Definition 29.** Let $\psi_{j,k}$ be as in (2-4) with notation as above. We say that $f \in \mathrm{CMO}^p_{\mathrm{flag}}$ if $f \in \mathcal{M}^{M+\delta}_{\mathrm{flag}}(\mathbb{H}^n)'$ and the norm $\|f\|_{\mathrm{CMO}^p_{\mathrm{flag}}}$ is finite, where

$$\|f\|_{\mathrm{CMO}^p_{\mathrm{flag}}} \equiv \sup_{\Omega}\left\{\frac{1}{|\Omega|^{2/p-1}} \sum_{\mathcal{R}\in \mathrm{R}_+} \int_{\Omega} \sum_{\mathcal{R}\subset\Omega} |\psi_{\mathcal{R}} * f(z,u)|^2 \chi_{\mathcal{R}}(z,u)\, dz\, du\right\}^{\frac{1}{2}}$$

for all open sets $\Omega$ in $\mathbb{H}^n$ with finite measure.

Note that the Carleson measure condition is used with the implicit multiparameter structure in $\mathrm{CMO}^p_{\mathrm{flag}}$. When $p=1$, we denote the space $\mathrm{CMO}^1_{\mathrm{flag}}$ as usual by $\mathrm{BMO}_{\mathrm{flag}}$. To see that the space $\mathrm{CMO}^p_{\mathrm{flag}}$ is well defined, one needs to show that the definition of $\mathrm{CMO}^p_{\mathrm{flag}}$ is independent of the choice of the component functions $\psi_{j,k}$. This can be proved just as for the Hardy space $H^p_{\mathrm{flag}}$, using the following Plancherel–Pólya-type inequality.

**Theorem 30.** *Suppose $\psi, \phi$ satisfy the conditions as in Theorem 19. Then, for $f \in \mathcal{M}^{M+\delta}_{\mathrm{flag}}(\mathbb{H}^n)'$,*

$$\sup_{\Omega}\left\{\frac{1}{|\Omega|^{2/p-1}} \sum_{\mathcal{R}\in \mathrm{R}_+} \sum_{\mathcal{R}\subset\Omega} \sup_{(z,u)\in\mathcal{R}} |\psi_{\mathcal{R}} * f(z,u)|^2 |\mathcal{R}|\right\}^{\frac{1}{2}} \approx \sup_{\Omega}\left\{\frac{1}{|\Omega|^{2/p-1}} \sum_{\mathcal{R}\in \mathrm{R}_+} \sum_{\mathcal{R}\subset\Omega} \inf_{(z,u)\in\mathcal{R}} |\psi_{\mathcal{R}} * f(z,u)|^2 |\mathcal{R}|\right\}^{\frac{1}{2}},$$

*where $\Omega$ ranges over all open sets in $\mathbb{H}^n$ with finite measure.*

To show that $\mathrm{CMO}^p_{\mathrm{flag}}$ is the dual of $H^p_{\mathrm{flag}}$, we introduce appropriate sequence spaces.

**Definition 31.** Let $s^p$ be the collection of all sequences $s = \{s_{\mathcal{R}}\}_{\mathcal{R}\in \mathrm{R}_+}$ such that

$$\|s\|_{s^p} = \left\|\left\{\sum_{\mathcal{R}\in \mathrm{R}_+} |s_{\mathcal{R}}|^2 |\mathcal{R}|^{-1} \chi_{\mathcal{R}}\right\}^{\frac{1}{2}}\right\|_{L^p(\mathbb{H}^n)} < \infty.$$

Let $c^p$ be the collection of all sequences $s = \{s_{\mathcal{R}}\}$ such that

$$\|s\|_{c^p} = \sup_{\Omega}\left\{\frac{1}{|\Omega|^{2/p-1}} \sum_{\mathcal{R}\in \mathrm{R}_+} \sum_{\mathcal{R}\subset\Omega} |s_{\mathcal{R}}|^2\right\}^{\frac{1}{2}} < \infty,$$

where $\Omega$ ranges over all open sets in $\mathbb{H}^n$ with finite measure.

We point out that only certain of the dyadic rectangles are used in $s^p$ and $c^p$ and these choices reflect the implicit multiparameter structure. Moreover, the Carleson measure condition is used in the definition of $c^p$. Next, we obtain the following duality theorem for sequence spaces.

**Theorem 32.** *Let $0 < p \le 1$. Then we have $(s^p)^* = c^p$. More precisely, the map which sends $s = \{s_{\mathcal{R}}\}$ to $\langle s, t\rangle \equiv \sum_{\mathcal{R}} s_{\mathcal{R}} \bar{t}_{\mathcal{R}}$ defines a continuous linear functional on $s^p$ with operator norm $\|t\|_{(s^p)^*} \approx \|t\|_{c^p}$, and, moreover, every $\ell \in (s^p)^*$ is of this form for some $t \in c^p$.*

When $p=1$, this theorem in the one-parameter setting on $\mathbb{R}^n$ was proved in [Frazier and Jawerth 1990]. The proof given in [Frazier and Jawerth 1990] depends on estimates of certain distribution functions, which seem to be difficult to apply to the multiparameter case. For all $0 < p \le 1$, we give a simple and more constructive proof of Theorem 32, which uses a stopping time argument for sequence spaces.

Theorem 32 together with the discrete Calderón reproducing formula and the Plancherel–Pólya-type inequalities yield the duality of $H^p_{\text{flag}}$.

**Theorem 33.** *Let* $0 < p \leq 1$. *Then*

$$(H^p_{\text{flag}})^* = \text{CMO}^p_F.$$

*More precisely, if* $g \in \text{CMO}^p_{\text{flag}}$, *the map* $\ell_g$ *given by* $\ell_g(f) = \langle f, g\rangle$, *defined initially for* $f \in \mathcal{M}^{M+\delta}_{\text{flag}}(\mathbb{H}^n)$, *extends to a continuous linear functional on* $H^p_{\text{flag}}$ *with* $\|\ell_g\| \approx \|g\|_{\text{CMO}^p_{\text{flag}}}$. *Conversely, for every* $\ell \in (H^p_{\text{flag}})^*$, *there exists some* $g \in \text{CMO}^p_{\text{flag}}$ *so that* $\ell = \ell_g$. *In particular*, $(H^1_{\text{flag}})^* = \text{BMO}_{\text{flag}}$.

As a consequence of the duality of $H^1_{\text{flag}}$ and $\text{BMO}_{\text{flag}}$, together with the $H^1_{\text{flag}}$-boundedness of flag singular integrals, we obtain the $\text{BMO}_{\text{flag}}$-boundedness of flag singular integrals. Furthermore, we will see that $L^\infty \subseteq \text{BMO}_{\text{flag}}$ and hence the $L^\infty \to \text{BMO}_{\text{flag}}$ boundedness of flag singular integrals is also obtained. These provide the endpoint results of [Müller et al. 1995; Nagel et al. 2001], and can be summarized as follows.

**Theorem 34.** *Suppose that* $T$ *is a flag singular integral with kernel as in Definition 7. Then* $T$ *is bounded on* $\text{BMO}_{\text{flag}}$. *Moreover, there exists a constant* $C$ *such that*

$$\|T(f)\|_{\text{BMO}_{\text{flag}}} \leq C\|f\|_{\text{BMO}_{\text{flag}}}.$$

**3.4.** ***Calderón–Zygmund decompositions and interpolation.*** Now we give the Calderón–Zygmund decomposition and interpolation theorems for flag Hardy spaces. We note that $H^p_{\text{flag}}(\mathbb{H}^n) = L^p(\mathbb{R}^{2n+1})$ for $1 < p < \infty$ by Theorem 6.

**Theorem 35** (Calderón–Zygmund decomposition for flag Hardy spaces). *Let* $0 < p_2 \leq 1$, $p_2 < p < p_1 < \infty$, *let* $\alpha > 0$ *be given, and suppose* $f \in H^p_{\text{flag}}(\mathbb{H}^n)$. *Then we can write*

$$f = g + b,$$

*where* $g \in H^{p_1}_{\text{flag}}(\mathbb{H}^n)$ *with* $p < p_1 < \infty$ *and* $b \in H^{p_2}_{\text{flag}}(\mathbb{H}^n)$ *with* $0 < p_2 < p$, *such that*

$$\|g\|^{p_1}_{H^{p_1}_{\text{flag}}} \leq C\alpha^{p_1-p}\|f\|^p_{H^p_{\text{flag}}} \quad \textit{and} \quad \|b\|^{p_2}_{H^{p_2}_{\text{flag}}} \leq C\alpha^{p_2-p}\|f\|^p_{H^p_{\text{flag}}},$$

*where* $C$ *is an absolute constant.*

**Theorem 36** (interpolation theorem on flag Hardy spaces). *Let* $0 < p_2 < p_1 < \infty$ *and let* $T$ *be a linear operator which is bounded from* $H^{p_2}_{\text{flag}}$ *to* $L^{p_2}$ *and bounded from* $H^{p_1}_{\text{flag}}$ *to* $L^{p_1}$. *Then* $T$ *is bounded from* $H^p_{\text{flag}}$ *to* $L^p$ *for all* $p_2 < p < p_1$. *Similarly, if* $T$ *is bounded on* $H^{p_2}_{\text{flag}}$ *and* $H^{p_1}_{\text{flag}}$, *then* $T$ *is bounded on* $H^p_{\text{flag}}$ *for all* $p_2 < p < p_1$.

**Remark 37.** Combining Theorem 36 with Corollary 27 recovers the $L^p$ boundedness of Marcinkiewicz multipliers in [Müller et al. 1995] (but not the sharp versions in [Müller et al. 1996]).

We point out that the Calderón–Zygmund decomposition in pure product domains for all $L^p$ functions $(1 < p < 2)$ into $H^1$ and $L^2$ functions, as well as the corresponding interpolation theorem, was established by Chang and Fefferman [1985; 1982].

# Part II. Proofs of results

Part II of this paper contains the proofs of the results stated in Part I, and is organized as follows.

(1) In Section 4, we establish $L^p$ estimates for the multiparameter Littlewood–Paley $g$-function when $1 < p < \infty$, and prove Theorems 6 and 38.

(2) In Section 5, we show that the Calderón reproducing formula holds on the flag molecular test function space $\mathcal{M}_{\text{flag}}^{M+\delta}$ and its dual space $(\mathcal{M}_{\text{flag}}^{M+\delta})'$. Then we prove the almost-orthogonality estimates and establish the wavelet Calderón reproducing formula on $\mathcal{M}_{\text{flag}}^{M+\delta}$ and $(\mathcal{M}_{\text{flag}}^{M+\delta})'$ in Theorem 17. Some estimates are established for the strong maximal function, and together with the wavelet Calderón reproducing formula, we then derive the Plancherel–Pólya-type inequalities in Theorem 19.

(3) In Section 6, we give a general result for bounding the $L^p$ norm of the function by its $H^p_{\text{flag}}$ norm (Theorem 56). We then prove the $H^p_{\text{flag}}$ boundedness of flag singular integrals for all $0 < p \le 1$ in Theorem 22. The boundedness from $H^p_{\text{flag}}$ to $L^p$ for all $0 < p \le 1$ for the flag singular integral operators, Theorem 23, is thus a consequence of Theorem 22 and Theorem 56.

(4) Duality theory for the Hardy space $H^p_{\text{flag}}$ is then established in Section 7 along with the boundedness of flag singular integral operators on $\text{BMO}_{\text{flag}}$. The proofs of Theorems 30, 32, 33, and 34 will all be given in Section 7.

(5) In Section 8, we prove the Calderón–Zygmund decomposition in the flag two-parameter setting (Theorem 35) and then derive an interpolation result, Theorem 36.

(6) In Section 9, we show that flag singular integrals are not in general bounded from the classical one-parameter Hardy space $H^1(\mathbb{H}^n)$ on the Heisenberg group to $L^1(\mathbb{H}^n)$.

## 4. $\mathbf{L}^p$ estimates for the Littlewood–Paley square function

The purpose of this section is to show that the $L^p$ norm of $f$ is equivalent to the $L^p$ norm of $g_{\text{flag}}(f)$ when $1 < p < \infty$. This was shown in [Müller et al. 1996, Proposition 4.1] for a function $g_{\text{flag}}(f)$ only slightly different than that used here. Our proof is similar in spirit to that work.

*Proof of Theorem 6.* The proof is similar to that in the pure product case given in [Fefferman and Stein 1982], and follows from iteration and standard vector-valued Littlewood–Paley inequalities. To see this, define

$$L^p(\mathbb{H}^n) \ni f \to F \in H = \ell^2$$

by $F(z,u) = \{\psi_j^{(1)} * f(z,u)\}$, so that

$$\|F\|_H = \left\{\sum_j |\psi_j^{(1)} * f(z,u)|^2\right\}^{\frac{1}{2}}.$$

For $z$ fixed, set

$$\tilde{g}(F)(z,u) = \left\{\sum_k \|\psi_k^{(2)} *_2 F(z,\cdot)(y)\|_H^2\right\}^{\frac{1}{2}}.$$

It is then easy to see that $\tilde{g}(F)(z,u) = g_{\text{flag}}(f)(z,u)$. For $z$ fixed, by the vector-valued Littlewood–Paley inequality,

$$\int_{\mathbb{H}^n} \tilde{g}(F)^p(z,u)\,dz\,du \le C \int_{\mathbb{H}^n} \|F\|_H^p\,dz\,du.$$

However, $\|F\|_H^p = \big\{\sum_j : \psi_j^{(1)} * f(z,y)|^2\big\}^{p/2}$, so integrating with respect to $z$ together with the standard Littlewood–Paley inequality yields

$$\int_{\mathbb{C}^n}\int_{\mathbb{R}} g_{\text{flag}}(f)^p(z,u)\,dz\,du \le C \int_{\mathbb{C}^n}\int_{\mathbb{R}} \Big\{\sum_j |\psi_j^{(1)} * f(z,u)|^2\Big\}^{p/2} dz\,du \le C\|f\|^p_{L^p(\mathbb{H}^n)},$$

which shows that $\|g_{\text{flag}}(f)\|_p \le C\|f\|_p$.

The proof of the estimate $\|f\|_p \le C\|g_{\text{flag}}(f)\|_p$ is a routine duality argument using the Calderón reproducing formula on $L^2(\mathbb{H}^n)$, for all $f \in L^2 \cap L^p$, $g \in L^2 \cap L^{p'}$ and $1/p + 1/p' = 1$, and the inequality $\|g_{\text{flag}}(f)\|_p \le C\|f\|_p$, which was just proved. This completes the proof of Theorem 6. □

As in Theorem 6, let $\psi^{(1)} \in \mathcal{S}(\mathbb{H}^n)$ be supported in the unit ball in $\mathbb{H}^n$ and $\psi^{(2)} \in \mathcal{S}(\mathbb{R})$ be supported in the unit ball of $\mathbb{R}$ and satisfy

$$\int_0^\infty |\widehat{\psi^{(2)}}(t\eta)|^4 \frac{dt}{t} = 1$$

for all $\eta \in \mathbb{R}\backslash\{0\}$. We define $\psi^\natural(z,u,v) = \psi^{(1)}(z,u)\psi^{(2)}(v)$. Set $\psi_s^{(1)}(z,u) = s^{-n-2}\psi^{(1)}(z/s, u/s^2)$, $\psi_t^{(2)}(v) = t^{-1}\psi(z/t)$ and

$$\psi_{s,t}(z,u) = \int_{\mathbb{R}} \psi_s^{(1)}(z,u-v)\psi_t^{(2)}(v)\,dv.$$

Repeating the proof of Theorem 6, we get, for $1 < p < \infty$,

$$\left\| \left\{ \int_0^\infty\int_0^\infty |\psi_{s,t} * f(z,u)|^2 \frac{dt}{t}\frac{ds}{s} \right\}^{\frac12} \right\|_p \le C\|f\|_p$$

and

$$\|f\|_p \approx \left\| \left\{ \int_0^\infty\int_0^\infty |\psi_{s,t} * \psi_{s,t} * f(z,y)|^2 \frac{dt}{t}\frac{ds}{s} \right\}^{\frac12} \right\|_p. \tag{4-1}$$

The $L^p$ boundedness of flag singular integrals for $1 < p < \infty$ is then an easy consequence of Theorem 6. This theorem was originally obtained in [Müller et al. 1995] using a different proof that involved the method of transference.

**Theorem 38.** *Suppose that $T$ is a flag singular integral defined on $\mathbb{H}^n$ with flag kernel $K(z,u)$ as in Definition 7 above. Then $T$ is bounded on $L^p$ for $1 < p < \infty$. Moreover, there exists a constant $C$ depending on $p$ such that, for $f \in L^p$,*

$$\|Tf\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

*Proof.* We may first assume that $K$ is an integrable function and then prove the $L^p$ boundedness of $T$ is independent of the $L^1$ norm of $K$. The conclusion for general $K$ then follows by an argument used in [Müller et al. 1995]. For all $f \in L^p$, by (4-1),

$$\|T(f)\|_p \le C \left\| \left\{ \int_0^\infty \int_0^\infty |\psi_{s,t} * \psi_{s,t} * K * f|^2 \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_p. \tag{4-2}$$

Now we claim the following estimate: for $f \in L^p$,

$$|\psi_{s,t} * K * f(z,u)| \le C M_S(f)(z,u), \tag{4-3}$$

where $C$ is a constant which is independent of the $L^1$ norm of $K$ and $M_S(f)$ is the strong maximal function of $f$ defined in (1-1).

Assuming (4-3) for the moment, we obtain from (4-2) that

$$\|Tf\|_p \le C \left\| \left\{ \int_0^\infty \int_0^\infty (M_S(\psi_{s,t} * f))^2 \frac{dt}{t} \frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_p \le C\|f\|_p,$$

where the last inequality follows from the Fefferman–Stein vector-valued maximal inequality.

We now turn to the claim (4-3). This follows from dominating $|\psi_{s,t} * K * f|$ by a product Poisson integral $\mathbb{P}_{\text{prod}} f$, and then dominating the product Poisson integral $\mathbb{P}_{\text{prod}} f$ by the strong maximal function $M_S f$. The arguments are familiar and we leave them to the reader. □

## 5. Developing the wavelet Calderón reproducing formula

In this section, we develop the wavelet Calderón reproducing formula and the Plancherel–Pólya-type inequalities on test function spaces. These are the main tools used in establishing the theory of Hardy spaces associated with the flag dilation structure. In order to establish the wavelet Calderón reproducing formula and the Plancherel–Pólya-type inequalities, we use the continuous version of the Calderón reproducing formula on $L^2(\mathbb{H}^n)$ and the almost-orthogonality estimates.

We now start the relatively long proof of Theorem 17, beginning with the Calderón reproducing formula in (2-1) that holds for $f \in L^2(\mathbb{H}^n)$ and converges in $L^2(\mathbb{H}^n)$. For any given $\alpha > 0$, we discretize it as

$$\begin{aligned} f(z,u) &= \int_0^\infty \int_0^\infty \check{\psi}_{s,t} *_{\mathbb{H}^n} \psi_{s,t} *_{\mathbb{H}^n} f(z,u) \frac{ds}{s} \frac{dt}{t} \\ &= \sum_{j,k\in\mathbb{Z}} \int_{2^{-\alpha(j+1)}}^{2^{-\alpha j}} \int_{2^{-2\alpha(k+1)}}^{2^{-2\alpha k}} \check{\psi}_{s,t} * \psi_{s,t} * f(z,u) \frac{dt}{t} \frac{ds}{s} \\ &= c_\alpha \sum_{j\le k} \check{\psi}_{j,k} * \psi_{j,k} * f(z,u) + c_\alpha \sum_{j>k} \check{\psi}_{j,k} * \psi_{j,k} * f(z,u) \\ &\qquad + \sum_{j,k\in\mathbb{Z}} \int_{2^{-\alpha(j+1)}}^{2^{-\alpha j}} \int_{2^{-2\alpha(k+1)}}^{2^{-2\alpha k}} \{\check{\psi}_{s,t} * \psi_{s,t} - \check{\psi}_{j,k} * \psi_{j,k}\} * f(z,u) \frac{dt}{t} \frac{ds}{s} \\ &= T_\alpha^{(1)} f(z,u) + T_\alpha^{(2)} f(z,u) + R_\alpha f(z,u), \end{aligned}$$

where

$$\psi_{j,k} = \psi_{2^{-\alpha j}, 2^{-2\alpha k}}, \qquad c_\alpha = \int_{2^{-\alpha(j+1)}}^{2^{-\alpha j}} \int_{2^{-2\alpha(k+1)}}^{2^{-2\alpha k}} \frac{dt}{t}\frac{ds}{s} = \ln \frac{2^{-\alpha j}}{2^{-\alpha(j+1)}} \ln \frac{2^{-2\alpha k}}{2^{-2\alpha(k+1)}} = 2(\alpha \ln 2)^2.$$

**Notation 39.** We have relabeled $\psi_{2^{-\alpha j}, 2^{-2\alpha k}}$ as simply $\psi_{j,k}$ when we replace integrals $\int_0^\infty \int_0^\infty (ds/s)(dt/t)$ by sums $\sum_{j,k\in\mathbb{Z}}$. This abuse of notation should not cause confusion as we will always use $j, k, j', k'$ as subscripts for the discrete components $\psi_{j,k}$, while we always use $s, t, s', t'$ as subscripts for the continuous components $\psi_{s,t}$. Note however that directions are *reversed* in passing from $s, t \in (0, \infty)$ to $j, k \in \mathbb{Z}$, in the sense that $s = 2^{-\alpha j}$ and $t = 2^{-2\alpha k}$ decrease as $j$ and $k$ increase.

To continue we choose a large positive integer $N$ to be fixed later. We decompose the first term $T_\alpha^{(1)} f(z, u)$ by writing the Heisenberg group $\mathbb{H}^n$ as a pairwise disjoint union of dyadic cubes $\mathcal{Q}$ of side length $2^{-\alpha(j+N)}$, that is,

$$\mathcal{Q} \in \mathcal{R}(2^{-\alpha(j+N)} \times 2^{-2\alpha(j+N)}).$$

We decompose the second term $T_\alpha^{(2)} f(z, u)$ by writing the Heisenberg group $\mathbb{H}^n$ as a pairwise disjoint union of dyadic rectangles $\mathcal{R}$ of dimension $2^{-\alpha(j+N)} \times 2^{-2\alpha(k+N)}$, that is, $\mathcal{R} \in \mathcal{R}(2^{-\alpha(j+N)} \times 2^{-2\alpha(k+N)})$. Recall that

$$\mathsf{R}(j, k) \equiv \mathcal{R}(2^{-\alpha(j+N)} \times 2^{-2\alpha(k+N)}),$$
$$\mathsf{Q}(j) \equiv \mathcal{R}(2^{-\alpha(j+N)} \times 2^{-2\alpha(j+N)}),$$

and that $(z_\mathcal{Q}, u_\mathcal{Q})$ is any *fixed* point in the cube $\mathcal{Q} \in \mathsf{Q}(j)$, and that $(z_\mathcal{R}, u_\mathcal{R})$ is any *fixed* point in the rectangle $\mathcal{R} \in \mathsf{R}(j, k)$.

We further discretize the terms $T_\alpha^{(1)} f(z, u)$ and $T_\alpha^{(2)} f(z, u)$ in different ways, exploiting the one-parameter structure of the Heisenberg group for $T_\alpha^{(1)}$, and exploiting the implicit product structure for $T_\alpha^{(2)}$. We rewrite $T_\alpha^{(1)} f(z, u)$ as

$$\begin{aligned}
T_\alpha^{(1)} f(z, u) &= c_\alpha \sum_{j\le k} \check{\psi}_{j,k} * \psi_{j,k} * f(z, u) \\
&= c_\alpha \sum_{j\le k} (\check{\psi}_j^{(1)} *_2 \check{\psi}_k^{(2)}) * (\psi_j^{(1)} *_2 \psi_k^{(2)}) * f(z, u) \\
&= c_\alpha \sum_{j\le k} (\check{\psi}_j^{(1)} *_2 \check{\psi}_k^{(2)} *_2 \psi_k^{(2)}) * \psi_j^{(1)} * f(z, u) \\
&= c_\alpha \sum_{j\in\mathbb{Z}} \left( \check{\psi}_j^{(1)} *_2 \left( \sum_{k\ge j} \check{\psi}_k^{(2)} *_2 \psi_k^{(2)} \right) \right) * \psi_j^{(1)} * f(z, u) \\
&= c_\alpha \sum_{j\in\mathbb{Z}} \check{\psi}_j * \psi_j * f(z, u),
\end{aligned}$$

where

$$\psi_j \equiv \psi_j^{(1)} \quad \text{and} \quad \check{\psi}_j \equiv \check{\psi}_j^{(1)} *_2 \left( \sum_{k\ge j} \check{\psi}_k^{(2)} *_2 \psi_k^{(2)} \right). \tag{5-1}$$

**Remark 40.** It is a standard exercise to prove that $\check{\psi}_j$ satisfies the same type of estimates as does $\psi_j^{(1)}$ on the Heisenberg group $\mathbb{H}^n$.

Now we write

$$T_\alpha^{(1)} f(z,u) = \sum_{j\le k} \sum_{\mathcal{Q}\in \mathsf{Q}(j)} f_{\mathcal{Q}} \psi_{\mathcal{Q}}(z,u) + R_{\alpha,N}^{(1)} f(z,u),$$
$$T_\alpha^{(2)} f(z,u) = \sum_{j> k} \sum_{\mathcal{R}\in \mathsf{R}(j,k)} f_{\mathcal{R}} \psi_{\mathcal{R}}(z,u) + R_{\alpha,N}^{(2)} f(z,u),$$

where

$$\begin{aligned}
f_{\mathcal{Q}} &\equiv c_\alpha |\mathcal{Q}| \psi_{j,k} * f(z_{\mathcal{Q}}, u_{\mathcal{Q}}) && \text{for } \mathcal{Q}\in \mathsf{Q}(j) \text{ and } k\ge j,\\
f_{\mathcal{R}} &\equiv c_\alpha |\mathcal{R}| \psi_{j,k} * f(z_{\mathcal{R}}, u_{\mathcal{R}}) && \text{for } \mathcal{R}\in \mathsf{R}(j,k) \text{ and } k< j,\\
\psi_{\mathcal{Q}}(z,u) &= \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} \check{\psi}_{j,k}((z,u)\circ(z',u')^{-1})\,dz'\,du' && \text{for } \mathcal{Q}\in \mathsf{Q}(j) \text{ and } k\ge j,\\
\psi_{\mathcal{R}}(z,u) &= \frac{1}{|\mathcal{R}|}\int_{\mathcal{R}} \check{\psi}_{j,k}((z,u)\circ(z',u')^{-1})\,dz'\,du' && \text{for } \mathcal{R}\in \mathsf{R}(j,k) \text{ and } k< j,
\end{aligned}$$

and

$$R_{\alpha,N}^{(1)} f(z,u) = c_\alpha \sum_{j\le k} \sum_{\mathcal{Q}\in \mathsf{Q}(j)} \int_{\mathcal{Q}} \check{\psi}_{j,k}((z,u)\circ(z',u')^{-1}) \times [\psi_{j,k} * f(z',u') - \psi_{j,k} * f(z_{\mathcal{Q}}, u_{\mathcal{Q}})]\,dz'\,du',$$
$$R_{\alpha,N}^{(2)} f(z,u) = c_\alpha \sum_{j> k} \sum_{\mathcal{R}\in \mathsf{R}(j,k)} \int_{\mathcal{R}} \check{\psi}_{j,k}((z,u)\circ(z',u')^{-1}) \times [\psi_{j,k} * f(z',u') - \psi_{j,k} * f(z_{\mathcal{R}}, u_{\mathcal{R}})]\,dz'\,du'.$$

Altogether we have

$$f(z,u) = \sum_{j\in\mathbb{Z}} \sum_{\mathcal{Q}\in \mathsf{Q}(j)} f_{\mathcal{Q}} \psi_{\mathcal{Q}}(z,u) + \sum_{j>k} \sum_{\mathcal{R}\in \mathsf{R}(j,k)} f_{\mathcal{R}} \psi_{\mathcal{R}}(z,u) + \{R_\alpha f(z,u) + R_{\alpha,N}^{(1)} f(z,u) + R_{\alpha,N}^{(2)} f(z,u)\}. \tag{5-2}$$

Recall that we denote by $\mathsf{Q} \equiv \bigcup_{j\in\mathbb{Z}} \mathsf{Q}(j)$ the collection of *all* dyadic cubes, and by $\mathsf{R}_{\text{vert}} \equiv \bigcup_{j>k} \mathsf{R}(j,k)$ the collection of *all strictly vertical* dyadic rectangles. Then we can rewrite (5-2) as

$$f(z,u) = \sum_{\mathcal{Q}\in \mathsf{Q}} f_{\mathcal{Q}} \psi_{\mathcal{Q}}(z,u) + \sum_{\mathcal{R}\in \mathsf{R}_{\text{vert}}} f_{\mathcal{R}} \psi_{\mathcal{R}}(z,u) + \{R_\alpha + R_{\alpha,N}^{(1)} + R_{\alpha,N}^{(2)}\} f(z,u), \tag{5-3}$$

which is a precursor to the *wavelet* form of the Calderón reproducing formula given in the statement of Theorem 17.

The following theorem is the analogue of [Han 1994, Theorem 1.19] for the operators $R_\alpha$, $R_{\alpha,N}^{(1)}$, and $R_{\alpha,N}^{(2)}$.

**Theorem 41.** *For fixed $M$ and $0<\delta<1$, we can choose $M'$ and $0<\alpha<\varepsilon$ sufficiently small, and then choose $N$ sufficiently large, so that the operators $R_\alpha$, $R_\alpha^{(1)}$, and $R_\alpha^{(2)}$ satisfy*

$$\begin{aligned}
\|R_\alpha f\|_{L^p(\mathbb{H}^n)} + \|R_{\alpha,N}^{(1)} f\|_{L^p(\mathbb{H}^n)} + \|R_{\alpha,N}^{(2)} f\|_{L^p(\mathbb{H}^n)} &\le \tfrac12 \|f\|_{L^p(\mathbb{H}^n)}, \quad f\in L^p(\mathbb{H}^n),\ 1<p<\infty,\\
\|R_\alpha f\|_{\mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n)} + \|R_{\alpha,N}^{(1)} f\|_{\mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n)} + \|R_{\alpha,N}^{(2)} f\|_{\mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n)} &\le \tfrac12 \|f\|_{\mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n)}, \quad f\in \mathcal{M}_{\text{flag}}^{M'+\delta}(\mathbb{H}^n).
\end{aligned} \tag{5-4}$$

With Theorem 41 in hand, we obtain that the operator

$$S_{\alpha,N} \equiv I - R_\alpha - R^{(1)}_{\alpha,N} f - R^{(2)}_{\alpha,N}$$

is bounded and invertible on $\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)$. It follows that, with $\tilde{\psi}_{\mathcal{Q}} \equiv S^{-1}_{\alpha,N}\psi_{\mathcal{Q}}$ and $\tilde{\psi}_{\mathcal{R}} \equiv S^{-1}_{\alpha,N}\psi_{\mathcal{R}}$,

$$f(z,u) = \sum_{\mathcal{Q}\in\mathsf{Q}} f_{\mathcal{Q}}\tilde{\psi}_{\mathcal{Q}}(z,u) + \sum_{\mathcal{R}\in\mathsf{R}_{\text{vert}}} f_{\mathcal{R}}\tilde{\psi}_{\mathcal{R}}(z,u), \quad f \in \mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n), \tag{5-5}$$

where $\tilde{\psi}_{\mathcal{Q}}$ and $\tilde{\psi}_{\mathcal{R}}$ are in $\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)$, and the convergence in (5-5) is in both $L^p(\mathbb{H}^n)$ and in the Banach space $\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)$. This finally is the *wavelet* form of the Calderón reproducing formula given in the statement of Theorem 17. The same argument shows that (5-5) holds for $f \in L^p(\mathbb{H}^n)$ with convergence in $L^p(\mathbb{H}^n)$, provided $1 < p < \infty$. In fact we obtain that (5-5) holds for $f$ in any Banach space $\mathcal{X}(\mathbb{H}^n)$ with convergence in $\mathcal{X}(\mathbb{H}^n)$, provided we have operator bounds

$$\|R_\alpha f\|_{\mathcal{X}(\mathbb{H}^n)} + \|R^{(1)}_{\alpha,N} f\|_{\mathcal{X}(\mathbb{H}^n)} + \|R^{(2)}_{\alpha,N} f\|_{\mathcal{X}(\mathbb{H}^n)} \le \tfrac{1}{2}\|f\|_{\mathcal{X}(\mathbb{H}^n)}, \quad f \in \mathcal{X}(\mathbb{H}^n).$$

We turn first to proving the molecular estimates in (5-4), but only for

$$\|R^{(1)}_{\alpha,N} f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)} \quad \text{and} \quad \|R^{(2)}_{\alpha,N} f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)},$$

as the estimate for $\|R_\alpha f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)}$ is similar, but easier. We will use the following special $T1$-type theorem on the Heisenberg group $\mathbb{H}^n$ (see [Han 1998; 1994] for the Euclidean case) to prove a corresponding product version below. Recall the definition of the one-parameter molecular space $\mathcal{M}^{M'+\delta}(\mathbb{H}^n)$.

**Definition 42.** Let $M' \in \mathbb{N}$ be a positive integer, $0 < \delta \le 1$, and let $Q = 2n+2$ denote the homogeneous dimension of $\mathbb{H}^n$. The *one-parameter* molecular space $\mathcal{M}^{M'+\delta}(\mathbb{H}^n)$ consists of all functions $f(z,u)$ on $\mathbb{H}^n$ satisfying the moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta f(z,u)\,dz\,du = 0 \quad \text{for all } |\alpha| + 2|\beta| \le M',$$

and such that there is a nonnegative constant $A$ satisfying the differential inequalities

$$|\partial_z^\alpha \partial_u^\beta f(z,u)| \le A \frac{1}{(1+|z|^2+|u|)^{(Q+M'+|\alpha|+2|\beta|+\delta)/2}} \quad \text{for all } |\alpha| + 2|\beta| \le M'$$

and

$$|\partial_z^\alpha \partial_u^\beta f(z,u) - \partial_z^\alpha \partial_u^\beta f(z',u')| \le A \frac{|(z,u)\circ(z',u')^{-1}|^\delta}{(1+|z|^2+|u|)^{(Q+M+\delta+M'+2\delta)/2}}$$
$$\text{for all } |\alpha| + 2|\beta| = M' \text{ and } |(z,u)\circ(z',u')^{-1}| \le \tfrac{1}{2}(1+|z|^2+|u|)^{\frac{1}{2}}.$$

**Theorem 43.** *Suppose $T : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ is a bounded linear operator with kernel $K((z,u),(z',u'))$; that is,*

$$Tf(z,u) = \int_{\mathbb{H}^n} K((z,u),(z',u'))\,f(z',u')\,dz'\,du'.$$

*Suppose furthermore that* $K$ *satisfies*

$$\int_{\mathbb{H}^n} z^\alpha u^\beta K((z,u),(z',u'))\,dz\,du = 0,$$
$$\int_{\mathbb{H}^n} (z')^\alpha (u')^\beta K((z,u),(z',u'))\,dz'\,du' = 0$$

*for all* $0 \le |\alpha|, \beta$, *and*

$$|\partial_z^\alpha \partial_u^\beta \partial_{z'}^{\alpha'} \partial_{u'}^{\beta'} K((z,u),(z',u'))| \le A \frac{1}{|(z,u)\circ(z',u')^{-1}|^{Q+|\alpha|+2\beta+|\alpha'|+2\beta'}}$$

*for all* $0 \le |\alpha|, \beta, |\alpha'|, \beta'$. *Then*

$$T : L^p(\mathbb{H}^n) \to L^p(\mathbb{H}^n) \qquad \textit{for } 1 < p < \infty,$$
$$T : \mathcal{M}^{M'+\delta}(\mathbb{H}^n) \to \mathcal{M}^{M'+\delta}(\mathbb{H}^n) \quad \textit{for all } M' \textit{ and } 0 < \delta < 1,$$

*and*, *moreover*, *the operator norms satisfy*

$$\|T\|_{L^p(\mathbb{H}^n)} \le C_p A \quad \textit{and} \quad \|T\|_{\mathcal{M}^{M'+\delta}(\mathbb{H}^n)} \le C_{M',\delta} A.$$

We will use the technique of lifting to the product space $\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$ together with the following special product $T1$-type theorem on the product group $\mathbb{H}^n \times \mathbb{R}$.

**Theorem 44.** *Suppose that* $T : L^2(\mathbb{H}^n \times \mathbb{R}) \to L^2(\mathbb{H}^n \times \mathbb{R})$ *is a bounded linear operator with kernel* $K([(z,u),v],[(z',u'),v'])$; *that is*,

$$Tf((z,u),v) = \int_{\mathbb{H}^n \times \mathbb{R}} K([(z,u),v],[(z',u'),v'])\, f((z',u'),v')\,dz'\,du'\,dv'.$$

*Suppose furthermore that* $K$ *satisfies*

$$\int_{\mathbb{H}^n} z^\alpha u^\beta K([(z,u),v],[(z',u'),v'])\,dz\,du = 0,$$
$$\int_{\mathbb{H}^n} (z')^\alpha (u')^\beta K([(z,u),v],[(z',u'),v'])\,dz'\,du' = 0,$$
$$\int_{\mathbb{R}} v^\gamma K([(z,u),v],[(z',u'),v'])\,dv = 0,$$
$$\int_{\mathbb{R}} (v')^\gamma K([(z,u),v],[(z',u'),v'])\,dv' = 0$$

*for all* $0 \le |\alpha|, \beta, \gamma$, *and*

$$|\partial_z^\alpha \partial_u^\beta \partial_v^\gamma \partial_{z'}^{\alpha'} \partial_{u'}^{\beta'} \partial_{v'}^{\gamma'} K([(z,u),v],[(z',u'),v'])| \le A \frac{1}{|(z,u)\circ(z',u')^{-1}|^{Q+|\alpha|+2\beta+|\alpha'|+2\beta'}} \frac{1}{|v-v'|^{1+\gamma_1+\gamma_2}}$$

*for all* $0 \le |\alpha|, \beta, \gamma, |\alpha'|, \beta', \gamma'$. *Then*

$$T : L^p(\mathbb{H}^n \times \mathbb{R}) \to L^p(\mathbb{H}^n \times \mathbb{R}) \qquad \textit{for } 1 < p < \infty,$$
$$T : \mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R}) \to \mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R}) \quad \textit{for all } M' \textit{ and } 0 < \delta < 1,$$

*and*, *moreover*, *the operator norms satisfy*

$$\|T\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \le C_p A \quad \textit{and} \quad \|T\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})} \le C_{M',\alpha} A.$$

We postpone the proofs of these $T1$-type theorems, and turn now to using them to complete the proof of Theorem 41, which in turn completes the proof of Theorem 17.

**5.1.** ***Boundedness on the flag molecular space.*** We prove the estimates for the operators $R^{(1)}_{\alpha,N}$ and $R^{(2)}_{\alpha,N}$ in Theorem 41 separately, beginning with $R^{(2)}_{\alpha,N}$.

**5.1.1.** *The operator* $R^{(2)}_{\alpha,N}$. Here we prove the boundedness of the error operator

$$R^{(2)}_{\alpha,N} f(z,u) = c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in\mathsf{R}(j,k)} \int_{\mathcal{R}} \check{\psi}_{j,k}((z,u)\circ(z',u')^{-1}) \times [\psi_{j,k} * f(z',u') - \psi_{j,k} * f(z_{\mathcal{R}},u_{\mathcal{R}})]\,dz'\,du'$$

on the flag molecular space $\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)$, where $M'$ is taken sufficiently small compared to $M$ as in the component functions. We begin by lifting the desired inequality to the product group $\mathbb{H}^n\times\mathbb{R}$ and reducing matters to Theorem 44. So we begin by writing

$$\begin{aligned}
&R^{(2)}_{\alpha,N} f(z,u)\\
&\quad= c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in\mathsf{R}(j,k)} \int_{\mathcal{R}} \check{\psi}_{j,k}((z,u)\circ(z',u')^{-1})\\
&\qquad\times \int [\psi_{j,k}((z',u')\circ(z'',u'')^{-1}) - \psi_{j,k} * f((z_{\mathcal{R}},u_{\mathcal{R}})\circ(z'',u'')^{-1})] f(z'',u'')\,dz''\,du''\,dz'\,du'\\
&\quad= c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in\mathsf{R}(j,k)} \int_{\mathcal{R}} \left\{\int \check{\psi}^{(1)}_j(z-z',u-u'+\operatorname{Im} z\overline{z'}-w)\check{\psi}^{(2)}_k(w)\,dw\right\}\\
&\qquad\times \int\left\{\int \psi^{(1)}_j(z'-z'',u'-u''+\operatorname{Im} z'\overline{z''}-w')\check{\psi}^{(2)}_k(w')\right.\\
&\qquad\qquad \left. - \int \psi^{(1)}_j(z_{\mathcal{R}}-z'',u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''}-w')\psi^{(2)}_k(w')\right\} dw' \int F(z'',u''-w'',w'')\,dw'',
\end{aligned}$$

where

$$f(z,u) = \pi F(z,u) = \int F((z,u-w),w)\,dw$$

and $F((z,u),w)\in\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})$. We continue with

$$\begin{aligned}
R^{(2)}_{\alpha,N} f(z,u) = c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in\mathsf{R}(j,k)} \int_{\mathcal{R}} \iiiint \check{\psi}^{(1)}_j(z-z',u-u'+\operatorname{Im} z\overline{z'}-w)\check{\psi}^{(2)}_k(w)&\\
\times\{\psi^{(1)}_j(z'-z'',u'-u''+\operatorname{Im} z'\overline{z''}-w') - \psi^{(1)}_j(z_{\mathcal{R}}-z'',u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''}-w')\}&\\
\times\check{\psi}^{(2)}_k(w')F(z'',u''-w'',w'')\,dz''\,du''\,dw''\,dw'\,dw\,dz'\,du'.&
\end{aligned}$$

Now for fixed $w''$ make the change of variable $u'' \to u'' + w''$ (in the sense that $u'' \to \tilde{u}'' + w''$ and we then rewrite $\tilde{u}''$ as $u''$) to obtain

$$R^{(2)}_{\alpha,N} f(z,u) = c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in \mathrm{R}(j,k)} \int_{\mathcal{R}} \iiiint \check{\psi}^{(1)}_j(z-z', u-u'+\operatorname{Im} z\overline{z'} - w)\check{\psi}^{(2)}_k(w) \times \{\psi^{(1)}_j(z'-z'', u'-u''-w''+\operatorname{Im} z'\overline{z''} - w') - \psi^{(1)}_j(z_{\mathcal{R}}-z'', u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''} - w'-w'')\} \times \check{\psi}^{(2)}_k(w') F(z'',u'',w'')\,dz''\,du''\,dw''\,dw'\,dw\,dz'\,du'.$$

Then, making a change of variable $w' \to w' - w''$ (in the sense of the previous change of variable), we get

$$R^{(2)}_{\alpha,N} f(z,u) = c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in \mathrm{R}(j,k)} \int_{\mathcal{R}} \iiiint \check{\psi}^{(1)}_j(z-z', u-u'+\operatorname{Im} z\overline{z'} - w)\check{\psi}^{(2)}_k(w) \times \{\psi^{(1)}_j(z'-z'', u'-u''+\operatorname{Im} z'\overline{z''} - w') - \psi^{(1)}_j(z_{\mathcal{R}}-z'', u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''} - w')\} \times \check{\psi}^{(2)}_k(w'-w'') F(z'',u'',w'')\,dz''\,du''\,dw''\,dw'\,dw\,dz'\,du'.$$

Finally, making the change of variable $w \to w - w'$, we get

$$\begin{aligned} &R^{(2)}_{\alpha,N} f(z,u) \\ &= c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in \mathrm{R}(j,k)} \int_{\mathcal{R}} \iiiint \check{\psi}^{(1)}_j(z-z', u-u'+\operatorname{Im} z\overline{z'} - w + w')\check{\psi}^{(2)}_k(w-w') \\ &\quad \times \{\psi^{(1)}_j(z'-z'', u'-u''+\operatorname{Im} z'\overline{z''} - w') - \psi^{(1)}_j(z_{\mathcal{R}}-z'', u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''} - w')\}\check{\psi}^{(2)}_k(w'-w'') \\ &\quad \times F(z'',u'',w'')\,dz''\,du''\,dw''\,dw'\,dw\,dz'\,du' \\ &= \int \widetilde{R}^{(2)}_{\alpha,N} F((z,u-w),w)\,dw, \end{aligned}$$

where the kernel of $\widetilde{R}^{(2)}_{\alpha,N}$ is given by

$$\widetilde{R}^{(2)}_{\alpha,N}[((z,u),w),((z'',u''),w'')] = c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in \mathrm{R}(j,k)} \int_{\mathcal{R}} \int \check{\psi}^{(1)}_j(z-z', u-u'+\operatorname{Im} z\overline{z'} + w')\check{\psi}^{(2)}_k(w-w') \times\{\psi^{(1)}_j(z'-z'', u'-u''+\operatorname{Im} z'\overline{z''} - w') - \psi^{(1)}_j(z_{\mathcal{R}}-z'', u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''} - w')\}\check{\psi}^{(2)}_k(w'-w'')\,dz'\,du'\,dw'.$$

Now it suffices to show that

$$\widetilde{R}^{(2)}_{\alpha,N} F \in \mathcal{M}^{M'+\delta}_{\mathrm{product}}(\mathbb{H}^n \times \mathbb{R})$$

with small norm, since we then conclude that

$$R^{(2)}_{\alpha,N} f \in \mathcal{M}^{M'+\delta}_{\mathrm{flag}}(\mathbb{H}^n)$$

with small norm. To do this we need only check that the kernel of $\widetilde{R}^{(2)}_{\alpha,N}$ satisfies the conditions of Theorem 44 with small bounds.

For this we rewrite the kernel in terms of Heisenberg group multiplication as

$$\widetilde{R}^{(2)}_{\alpha,N}[((z,u),w),((z'',u''),w'')] = c_\alpha \sum_{j>k} \sum_{\mathcal{R}\in \mathsf{R}(j,k)} \int_{\mathcal{R}} \int \check{\psi}^{(1)}_j((z,u)\circ(z',u'-w')^{-1})\check{\psi}^{(2)}_k(w-w') \\ \times \{\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1}) - \psi^{(1)}_j((z_{\mathcal{R}},u_{\mathcal{R}}-w')\circ(z'',u'')^{-1})\}\psi^{(2)}_k(w'-w'')\,dz'\,du'\,dw'.$$

By construction we have

$$\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1}) - \psi^{(1)}_j((z_{\mathcal{R}},u_{\mathcal{R}}-w')\circ(z'',u'')^{-1}) \sim 2^{-N}\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1}),$$

in the sense that the left side satisfies the same moment, size and smoothness conditions as the right side. Thus we have

$$\sum_{\mathcal{R}\in \mathsf{R}(j,k)} \int_{\mathcal{R}} \int \check{\psi}^{(1)}_j((z,u)\circ(z',u'-w')^{-1}) \\ \times \{\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1}) - \psi^{(1)}_j((z_{\mathcal{R}},u_{\mathcal{R}}-w')\circ(z'',u'')^{-1})\}\,dz'\,du' \\ \sim \sum_{\mathcal{R}\in \mathsf{R}(j,k)} \int_{\mathcal{R}} \int \check{\psi}^{(1)}_j((z,u)\circ(z',u'-w')^{-1})2^{-N}\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1})\,dz'\,du' \\ \sim 2^{-N}\psi^{(1)}_j((z,u)\circ(z'',u'')^{-1}). \tag{5-6}$$

We also have

$$\int \check{\psi}^{(2)}_k(w-w')\psi^{(2)}_k(w'-w'')\,dw' \sim \psi^{(2)}_k(w-w'').$$

So altogether we obtain

$$\widetilde{R}^{(2)}_{\alpha,N}[((z,u),w),((z'',u''),w'')] \sim 2^{-N}\sum_{j>k}\psi^{(1)}_j((z,u)\circ(z'',u'')^{-1})\psi^{(2)}_k(w-w''),$$

which satisfies the hypotheses of Theorem 44 with bounds roughly $2^{-N}$, since $\psi^{(1)}\in\mathcal{S}(\mathbb{H}^n)$ and $\psi^{(2)}\in\mathcal{S}(\mathbb{R})$. Here we are using the well-known fact that the partial sums $\sum_{j<M}\psi_j$ of an approximate identity satisfy Calderón–Zygmund kernel conditions of infinite order *uniformly* in $M$.

**5.1.2.** *The operator $R^{(1)}_{\alpha,N}$.* Now we turn to boundedness of the error operator

$$R^{(1)}_{\alpha,N}f(z,u) = c_\alpha\sum_{j\le k}\sum_{\mathcal{Q}\in \mathsf{Q}(j)}\int_{\mathcal{Q}}\check{\psi}_{j,k}((z,u)\circ(z',u')^{-1})[\psi_{j,k}*f(z',u')-\psi_{j,k}*f(z_{\mathcal{Q}},u_{\mathcal{Q}})]\,dz'\,du',$$

on the flag molecular space $\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)$, where $M'$ is taken sufficiently small compared to $M$ as in the component functions. Applying the calculation used for the term $R^{(2)}_{\alpha,N}$ above, we can obtain

$$R^{(1)}_{\alpha,N}f(z,u) = \int \widetilde{R}^{(1)}_{\alpha,N}F((z,u-w),w)\,dw,$$

where the kernel of $\widetilde{R}^{(1)}_{\alpha,N}$ is given by

$$\widetilde{R}^{(1)}_{\alpha,N}[((z,u),w),((z'',u''),w'')] = c_\alpha \sum_{j\le k}\sum_{\mathcal{Q}\in\mathsf{Q}(j)} \int_{\mathcal{Q}}\int \check{\psi}^{(1)}_j(z-z',u-u'+\operatorname{Im} z\overline{z'}+w')\check{\psi}^{(2)}_k(w-w')$$
$$\times\{\psi^{(1)}_j(z'-z'',u'-u''+\operatorname{Im} z'\overline{z''}-w')-\psi^{(1)}_j(z_{\mathcal{R}}-z'',u_{\mathcal{R}}-u''+\operatorname{Im} z_{\mathcal{R}}\overline{z''}-w')\}\check{\psi}^{(2)}_k(w'-w'')\,dz'\,du'\,dw'.$$

By construction we have

$$\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1})-\psi^{(1)}_j((z_{\mathcal{R}},u_{\mathcal{R}}-w')\circ(z'',u'')^{-1})\sim 2^{-N}\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1}),$$

in the sense that the left side satisfies the same moment, size, and smoothness conditions as the right side. Thus we have

$$\begin{aligned}\sum_{\mathcal{Q}\in\mathsf{Q}(j)}\int_{\mathcal{Q}}\int &\check{\psi}^{(1)}_j((z,u)\circ(z',u'-w')^{-1})\\ &\times\{\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1})-\psi^{(1)}_j((z_{\mathcal{R}},u_{\mathcal{R}}-w')\circ(z'',u'')^{-1})\}\,dz'\,du'\\ &\sim \sum_{\mathcal{Q}\in\mathsf{Q}(j)}\int_{\mathcal{R}}\int \check{\psi}^{(1)}_j((z,u)\circ(z',u'-w')^{-1})2^{-N}\psi^{(1)}_j((z',u'-w')\circ(z'',u'')^{-1})\,dz'\,du'\\ &\sim 2^{-N}\psi^{(1)}_j((z,u)\circ(z'',u'')^{-1}).\end{aligned}$$

We also have

$$\int \check{\psi}^{(2)}_k(w-w')\psi^{(2)}_k(w'-w'')\,dw'\sim\psi^{(2)}_k(w-w'').$$

So altogether we obtain

$$\widetilde{R}^{(1)}_{\alpha,N}[((z,u),w),((z'',u''),w'')]\sim 2^{-N}\sum_{j\le k}\psi^{(1)}_j((z,u)\circ(z'',u'')^{-1})\psi^{(2)}_k(w-w''),$$

which satisfies the hypotheses of Theorem 44 with bounds roughly $2^{-N}$, since $\psi^{(1)}\in\mathcal{S}(\mathbb{H}^n)$ and $\psi^{(2)}\in\mathcal{S}(\mathbb{R})$.

It now follows that the kernels of both $\widetilde{R}^{(1)}_{\alpha,N}$ and $\widetilde{R}^{(2)}_{\alpha,N}$ satisfy the hypotheses of Theorem 44 with bounds roughly $2^{-N}$, and we conclude that

$$\|\widetilde{R}^{(i)}_{\alpha,N}F\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})}\lesssim 2^{-N}\|F\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})},\qquad i=1,2.$$

Thus we obtain, for each $i=1,2$,

$$\|R^{(i)}_{\alpha,N}f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)}\le\inf_{f=\pi F}\|\widetilde{R}^{(i)}_{\alpha,N}F\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})}\lesssim 2^{-N}\inf_{f=\pi F}\|F\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})}=2^{-N}\|f\|_{\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)},$$

and taking $N$ sufficiently large completes the proof of the molecular estimates in (5-4).

**5.1.3.** *The $L^p$ estimates.* Finally, we turn to proving the $L^p$ estimates in (5-4) for $1<p<\infty$,

$$\|R_\alpha f\|_{L^p(\mathbb{H}^n)}+\|R^{(1)}_{\alpha,N}f\|_{L^p(\mathbb{H}^n)}+\|R^{(2)}_{\alpha,N}f\|_{L^p(\mathbb{H}^n)}\le\tfrac{1}{2}\|f\|_{L^p(\mathbb{H}^n)}.$$

The estimates for $R^{(1)}_{\alpha,N}$ and $R^{(2)}_{\alpha,N}$ follow from the estimates established above for the kernels of the lifted operators $\widetilde{R}^{(1)}_{\alpha,N}$ and $\widetilde{R}^{(2)}_{\alpha,N}$. Indeed, for $f \in L^p(\mathbb{H}^n)$, we can use a result in [Müller et al. 1995] to find $F \in L^p(\mathbb{H}^n \times \mathbb{R})$ with $f = \pi F$ and $\|F\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \le C\|f\|_{L^p(\mathbb{H}^n)}$. Then we have

$$\|R^{(i)}_{\alpha,N} f\|_{L^p(\mathbb{H}^n)} \le \|\widetilde{R}^{(i)}_{\alpha,N} F\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \lesssim 2^{-N}\|F\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \le C2^{-N}\|f\|_{L^p(\mathbb{H}^n)}.$$

In similar fashion, the kernel of the lifted operator $\widetilde{R}_\alpha$ can be shown to satisfy product kernel estimates with constant $A$ that is a multiple of $1-2^{-\alpha}$, and so we obtain from Theorem 44 that

$$\|\widetilde{R}_\alpha F\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \lesssim (1-2^{-\alpha})\|F\|_{L^p(\mathbb{H}^n\times\mathbb{R})},$$

and hence, with $f = \pi F$ as above,

$$\|R_\alpha f\|_{L^p(\mathbb{H}^n)} \le \|\widetilde{R}_\alpha F\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \lesssim (1-2^{-\alpha})\|F\|_{L^p(\mathbb{H}^n\times\mathbb{R})} \le C(1-2^{-\alpha})\|f\|_{L^p(\mathbb{H}^n)}.$$

If we now take $0<\alpha<1$ sufficiently small, and then $N$ sufficiently large, we obtain the $L^p$ estimates in (5-4). This concludes our proof of Theorem 41.

**5.2. *The $T1$-type theorems.*** The proof of Theorem 43 in the one-parameter case follows the argument in [Gilbert et al. 2002], where the same result is proved in the Euclidean setting. For this we will need an extension to the Heisenberg group of the generalization of Meyer's lemma by Torres [1991].

**Lemma 45.** *Suppose $T : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ is a bounded linear operator with kernel $K((z,u),(z',u'))$ satisfying the kernel conditions in the hypotheses of Theorem 43. Suppose that $M \ge 0$ and that $T((z,u)^{(\alpha'',\beta'')}) = 0$ for all multi-indices $(\alpha'',\beta'')$ with $|\alpha''|+2\beta'' \le M$. Then, for any two points $(z,u),(z'',u'') \in \mathbb{H}^n$ and any smooth $\varphi$ on $\mathbb{H}^n$ with compact support, and any multi-index $(\alpha',\beta')$ with $|\alpha'|+2\beta' = M' \le M$, we have the identity*

$$\begin{aligned}
&\partial_z^{\alpha'}\partial_u^{\beta'} T\varphi(z,u) - \partial_z^{\alpha'}\partial_u^{\beta'} T\varphi(z'',u'')\\
&= \int \partial_z^{\alpha'}\partial_u^{\beta'} K((z,u),(z',u'))\\
&\qquad\times\Big\{\varphi(z',u') - \sum_{|\alpha''|+2\beta''\le M'} c_{\alpha'',\beta''}\partial_z^{\alpha''}\partial_u^{\beta''}\varphi(z,u)[(z',u')\circ(z,u)^{-1}]^{(\alpha'',\beta'')}\Big\}\tilde\theta(z',u')\,dz'\,du'\\
&\quad - \int \partial_z^{\alpha'}\partial_u^{\beta'} K((z'',u''),(z',u'))\\
&\qquad\times\Big\{\varphi(z',u') - \sum_{|\alpha''|+2\beta''\le M'} c_{\alpha'',\beta''}\partial_z^{\alpha''}\partial_u^{\beta''}\varphi(z'',u'')[(z',u')\circ(z'',u'')^{-1}]^{(\alpha'',\beta'')}\Big\}\tilde\theta(z',u')\,dz'\,du'\\
&\quad + \int \{\partial_z^{\alpha'}\partial_u^{\beta'} K((z,u),(z',u')) - \partial_z^{\alpha'}\partial_u^{\beta'} K((z'',u''),(z',u'))\}\\
&\qquad\times\Big\{\varphi(z',u') - \sum_{|\alpha''|+2\beta''\le M'} c_{\alpha'',\beta''}\partial_z^{\alpha''}\partial_u^{\beta''}\varphi(z'',u'')[(z',u')\circ(z'',u'')^{-1}]^{(\alpha'',\beta'')}\Big\}(1-\tilde\theta(z',u'))\,dz'\,du'\\
&\quad + \sum_{|\alpha''|+2\beta''\le M'} \Big\{c_{\alpha'',\beta''}\partial_z^{\alpha''}\partial_u^{\beta''}\varphi(z,u) - \sum_{|\alpha'''|+2\beta'''\le M'-|\alpha''|-2\beta''} c_{\alpha''',\beta'''}\partial_z^{\alpha'''+\alpha''}\partial_u^{2\beta'''+2\beta''}\\
&\qquad\qquad \times\varphi(z'',u'')[(z,u)\circ(z'',u'')^{-1}]^{(\alpha''',\beta''')}\Big\} T_{(\alpha'',\beta''),(\alpha',\beta')}\tilde\theta(z,u).
\end{aligned}$$

The proof of this lemma follows verbatim that of [Torres 1991, Lemma 3.1.22, page 62].

With this result in hand, the proof of Theorem 43 follows closely the argument in the Euclidean case in [Gilbert et al. 2002], and the reader can find complete details in [Han et al. 2012].

*Proof of Theorem 44.* To prove the product version we note that the above one-parameter proof extends virtually verbatim to establish a *vector-valued* version in a Banach space. Indeed, all the main tools, such as integration, differentiation, and Taylor's formula, carry over to the Banach space setting. First we will define the $X$-valued molecular space $\mathcal{M}^{M+\delta,M_1,M_2}(\mathbb{H}^n;X)$, and then we will give the extension of Theorem 43 to this space.

**Definition 46.** Let $X$ be a Banach space with norm $|x|$ for $x\in X$. Let $M, M_1, M_2\in\mathbb{N}$ be positive integers, $0<\delta\le 1$, and let $Q=2n+2$ denote the homogeneous dimension of $\mathbb{H}^n$. The *one-parameter* molecular space $\mathcal{M}^{M+\delta,M_1,M_2}(\mathbb{H}^n;X)$ consists of all $X$-valued functions $f:\mathbb{H}^n\to X$ satisfying the moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta f(z,u)\,dz\,du=0\quad\text{for all } |\alpha|+2|\beta|\le M_1,$$

and such that there is a nonnegative constant $A$ satisfying the differential inequalities

$$|\partial_z^\alpha\partial_u^\beta f(z,u)|_X\le A\frac{1}{(1+|z|^2+|u|)^{(Q+M+|\alpha|+2|\beta|+\delta)/2}}\quad\text{for all } |\alpha|+2|\beta|\le M_2$$

and

$$|\partial_z^\alpha\partial_u^\beta f(z,u)-\partial_z^\alpha\partial_u^\beta f(z',u')|_X\le A\frac{|(z,u)\circ(z',u')^{-1}|^\delta}{(1+|z|^2+|u|)^{(Q+M+\delta+M_2+2\delta)/2}}$$

$$\text{for all } |\alpha|+2|\beta|=M_2 \text{ and } |(z,u)\circ(z',u')^{-1}|\le\tfrac12(1+|z|^2+|u|)^{\frac12}.$$

We have the following extension of Theorem 43 to $X$-valued functions for an arbitrary Banach space $X$.

**Theorem 47.** *Suppose $T:L^2(\mathbb{H}^n)\to L^2(\mathbb{H}^n)$ is a bounded linear operator with kernel $K((z,u),(z',u'))$; that is,*

$$Tf(z,u)=\int_{\mathbb{H}^n}K((z,u),(z',u'))f(z',u')\,dz'\,du',\qquad f\in L^2(\mathbb{H}^n).$$

*Suppose furthermore that $K$ satisfies*

$$\int_{\mathbb{H}^n} z^\alpha u^\beta K((z,u),(z',u'))\,dz\,du=0,$$

$$\int_{\mathbb{H}^n}(z')^\alpha(u')^\beta K((z,u),(z',u'))\,dz'\,du'=0$$

*for all $0\le|\alpha|,\beta$, and*

$$|\partial_z^\alpha\partial_u^\beta\partial_{z'}^{\alpha'}\partial_{u'}^{\beta'}K((z,u),(z',u'))|\lesssim\frac{1}{|(z,u)\circ(z',u')^{-1}|^{Q+|\alpha|+2\beta+|\alpha'|+2\beta'}}$$

*for all $0\le|\alpha|,\beta,|\alpha'|,\beta'$. For $f:\mathbb{H}^n\to X$, we define $Tf$ by the Banach-space-valued integrals*

$$Tf(z,u)=\int_{\mathbb{H}^n}K((z,u),(z',u'))f(z',u')\,dz'\,du'.$$

*Then*

$$T : \mathcal{M}^{M'+\delta}(\mathbb{H}^n; X) \to \mathcal{M}^{M'+\delta}(\mathbb{H}^n; X)$$

*is bounded for all* $M'$ *and* $0 < \delta < 1$. *Moreover, the operator norm satisfies*

$$\|T\|_{\mathcal{M}^{M'+\delta}(\mathbb{H}^n; X)} \le C_{M',\delta}.$$

*Proof.* We simply repeat the scalar proof of Theorem 43 but replace $|\partial_z^\alpha \partial_u^\beta f(z,u)|$ by $|\partial_z^\alpha \partial_u^\beta f(z,u)|_X$ throughout and use Banach space analogues of Taylor's theorem and the identities of [Torres 1991]. □

Now we can quickly finish the proof of Theorem 44. We take $X = \mathcal{M}^{M'+\delta}(\mathbb{R})$ and note that the identification of product and iterated molecular spaces, namely,

$$\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R}) = \mathcal{M}^{M'+\delta}\big(\mathbb{H}^n; \mathcal{M}^{M'+\delta}(\mathbb{R})\big) = \mathcal{M}^{M'+\delta}(\mathbb{H}^n; X), \tag{5-7}$$

follows immediately from the definitions of the spaces involved; see Definitions 42 and 10 and the definition of $\mathcal{M}^{M+\delta,M_1,M_2}(\mathbb{R})$, which we recall here.

**Definition 48.** Let $M \in \mathbb{N}$ be a positive integer and $0 < \delta \le 1$. The *one-parameter* molecular space $\mathcal{M}^{M+\delta,M_1,M_2}(\mathbb{R})$ consists of all functions $f(v)$ on $\mathbb{R}$ satisfying the moment conditions

$$\int_{\mathbb{R}} v^\gamma f(v)\,dv = 0 \quad \text{for all } 2\gamma \le M_1,$$

and such that there is a nonnegative constant $A$ satisfying the differential inequalities

$$|\partial_v^\gamma f(v)| \le A \frac{1}{(1+|v|)^{1+M+\gamma+\delta}} \qquad \text{for all } 2\gamma \le M_2,$$

$$|\partial_v^{M_2} f(v) - \partial_v^{M_2} f(v')| \le A \frac{|v-v'|^\delta}{(1+|v|)^{1+(3/2)M+\gamma+2\delta}} \quad \text{for all } |v-v'| \le \tfrac{1}{2}(1+|v|).$$

For $f \in \mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$, denote the realization of $f$ as an $X$-valued map by $\tilde{f} : \mathbb{H}^n \to \mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{R})$. Then, from (5-7) and Theorem 47, we have

$$\|Tf\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})} = \|T\tilde{f}\|_{\mathcal{M}^{M'+\delta}(\mathbb{H}^n;\mathcal{M}^{M'+\delta}(\mathbb{R}))} \le C\|\tilde{f}\|_{\mathcal{M}^{M'+\delta}(\mathbb{H}^n;\mathcal{M}^{M'+\delta}(\mathbb{R}))} = C\|f\|_{\mathcal{M}^{M'+\delta}_{\text{product}}(\mathbb{H}^n\times\mathbb{R})}.$$

This completes the proof of Theorem 44. □

**5.3.** ***Orthogonality estimates and the proof of the Plancherel–Pólya inequalities.*** We will need *almost-orthogonality estimates* in order to prove both the Plancherel–Pólya inequalities and the boundedness of flag singular integrals on $H^p_{\text{flag}}(\mathbb{H}^n)$. Recall from (2-2) the definition of the components $\psi_{t,s}$ of the continuous decomposition of the identity adapted to the Heisenberg group:

$$\psi(z,u) = \psi^{(1)} *_2 \psi^{(2)}(z,u) = \int_{\mathbb{R}} \psi^{(1)}(z,u-v)\psi^{(2)}(v)\,dv, \quad (z,u) \in \mathbb{C}^n \times \mathbb{R},$$

and

$$\psi_{t,s}(z,u) = \psi_t^{(1)} *_2 \psi_s^{(2)}(z,u) = \int_{\mathbb{R}} \psi_t^{(1)}(z,u-v)\psi_s^{(2)}(v)\,dv = \int_{\mathbb{R}} t^{-2n-2}\psi^{(1)}\left(\frac{z}{t}, \frac{u-v}{t^2}\right) s^{-1}\psi^{(2)}\left(\frac{v}{s}\right) dv.$$

Here $\psi^{(1)} \in \mathcal{S}(\mathbb{H}^n)$ is as in Theorem 2, and $\psi^{(2)} \in \mathcal{S}(\mathbb{R})$ satisfies

$$\int_0^\infty |\widehat{\psi^{(2)}}(t\eta)|^2 \frac{dt}{t} = 1$$

for all $\eta \in \mathbb{R}\backslash\{0\}$, along with the moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta \psi^{(1)}(z,u)\,dz\,du = 0, \quad |\alpha| + 2\beta \le M,$$

$$\int_{\mathbb{R}} v^\gamma \psi^{(2)}(v)\,dv = 0, \quad \gamma \ge 0,$$

where $M$ may be fixed arbitrarily large.

In particular, the collection of component functions $\{\psi_{t,s}\}_{t,s>0}$ satisfies

$$\begin{aligned} \psi_{t,s} &= \psi_t^{(1)} *_2 \psi_s^{(2)}, \\ \psi_t^{(1)}(z,u) &= t^{-2n-2}\psi^{(1)}\left(\frac{z}{t}, \frac{u-v}{t^2}\right), \\ \psi_s^{(2)}(v) &= s^{-1}\psi^{(2)}\left(\frac{v}{s}\right), \\ \psi^{(1)}(z,u)\psi^{(2)}(v) &\in \mathcal{M}^{M+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R}). \end{aligned} \tag{5-8}$$

Of course the conditions in (5-8) imply that $\psi_{t,s} \in \mathcal{M}^M_{\text{flag}}(\mathbb{H}^n)$ for all $t, s > 0$, but (5-8) also contains the implicit dilation information that cannot be expressed solely in terms of $\psi_{1,1}$. Motivated by these considerations we make the following definition.

**Definition 49.** To each function $\Psi \in \mathcal{M}^{M+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$ we associate a collection of *product dilations* $\{\Psi_{t,s}\}_{t,s>0}$ defined by

$$\Psi_{t,s}(z,u,v) = t^{-2n-2}s^{-1}\Psi\left(\left(\frac{z}{t}, \frac{u}{t^2}\right), \frac{v}{s}\right),$$

and a collection of *component functions* $\{\psi_{t,s}\}_{t,s>0}$ defined by

$$\psi_{t,s}(z,u) = \pi\Psi_{t,s}(z,u) = \int_{\mathbb{R}} t^{-2n-2}s^{-1}\Psi\left(\left(\frac{z}{t}, \frac{u-v}{t^2}\right), \frac{v}{s}\right)dv, \quad t, s > 0.$$

Given two functions in $\mathcal{M}^{M+\delta}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$ and their corresponding collections of component functions we have the *almost-orthogonality* estimates given below. We use $*_{\mathbb{H}^n}$ to denote convolution on the Heisenberg group $\mathbb{H}^n$, and $*_{\mathbb{H}^n\times\mathbb{R}}$ to denote convolution on the product group $\mathbb{H}^n \times \mathbb{R}$. From Lemma 12 we obtain that $\pi$ intertwines these two convolutions, which we record here.

**Lemma 50.** *For $\psi_{t,s}$, $\Psi_{t,s}$, $\phi_{t',s'}$, $\Phi_{t',s'}$ as above, we have*

$$\psi_{t,s} *_{\mathbb{H}^n} \phi_{t',s'} = \pi\{\Psi_{t,s} *_{\mathbb{H}^n\times\mathbb{R}} \Phi_{t',s'}\}. \tag{5-9}$$

We now give the orthogonality estimates, first in the product case and then in the flag case. The product case in Lemma 51 will prove crucial in establishing Theorem 41 for the flag molecular space $\mathcal{M}^{M'+\delta}_{\text{flag}}(\mathbb{H}^n)$.

For convenience, we give the almost orthogonal estimates only for the case $\mathcal{M}^{4M+2,2M,2M}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$.

**Lemma 51.** *Suppose* $\Psi, \Phi \in \mathcal{M}^{4M+2,2M,2M}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$. *Then there exists a constant* $C = C(M)$ *depending only on* $M$ *such that*

$$|\Psi_{t,s} *_{\mathbb{H}^n\times\mathbb{R}} \Phi_{t',s'}((z,u),v)| \leq C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^{2M+1}\left(\frac{s}{s'} \wedge \frac{s'}{s}\right)^{M+1} \frac{(t\vee t')^{2(4M+2)/2}}{((t\vee t')^2+|z|^2+|u|)^{(Q+4M+2)/2}} \frac{(s\vee s')^{4M+2}}{(s\vee s'+|v|)^{1+4M+2}}. \tag{5-10}$$

The proof of Lemma 51 uses a standard orthogonality argument on the integral

$$\Psi_{t,s} *_{\mathbb{H}^n\times\mathbb{R}} \Phi_{t',s'}((z,u),v) = \int_{\mathbb{H}^n\times\mathbb{R}} \Psi_{t,s}((z,u)\circ(z',u')^{-1}, v-v')\Phi_{t',s'}((z',u'),v')\,dz'\,du'\,dv', \tag{5-11}$$

and we refer the reader to [Han et al. 2012] for details.

There are corresponding orthogonality estimates for component functions on $\mathbb{H}^n$.

**Lemma 52.** *Suppose* $\Psi, \Phi \in \mathcal{M}^{2M}_{\text{product}}(\mathbb{H}^n \times \mathbb{R})$ *and let* $\{\psi_{t,s}\}_{t,s>0}$ *and* $\{\phi_{t,s}\}_{t,s>0}$ *be the associated collections of component functions as defined in* (2-7) *above. Then there exists a constant* $C = C(M)$ *depending only on* $M$ *such that, if* $(t\vee t')^2 \leq s\vee s'$, *then*

$$|\psi_{t,s} *_{\mathbb{H}^n} \phi_{t',s'}(z,u)| \leq C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^{2M}\left(\frac{s}{s'} \wedge \frac{s'}{s}\right)^{M} \times \frac{(t\vee t')^{2M}}{(t\vee t'+|z|)^{2n+2M}} \frac{(s\vee s')^{M}}{(s\vee s'+|u|)^{1+M}}, \tag{5-12}$$

*and if* $(t\vee t')^2 \geq s\vee s'$, *then*

$$|\psi_{t,s} * \phi_{t',s'}(z,u)| \leq C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^{M}\left(\frac{s}{s'} \wedge \frac{s'}{s}\right)^{M} \times \frac{(t\vee t')^{M}}{(t\vee t'+|z|)^{2n+M}} \frac{(t\vee t')^{M}}{(t\vee t'+\sqrt{|u|})^{2+2M}}. \tag{5-13}$$

Roughly speaking, $\psi_{t,s} * \phi_{t',s'}(z,u)$ satisfies the *product multiparameter* almost-orthogonality when $(t\vee t')^2 \leq s\vee s'$ and the *one-parameter* almost-orthogonality when $(t\vee t')^2 \geq s\vee s'$.

*Proof of Lemma 52.* We will use Lemma 50 to pass from the orthogonality estimates for the product dilations $\{\Psi_{t,s}\}_{t,s>0}$ and $\{\Phi_{t,s}\}_{t,s>0}$ in Lemma 51 to the estimates for the component functions $\{\psi_{t,s}\}_{t,s>0}$ and $\{\phi_{t,s}\}_{t,s>0}$ in Lemma 52.

From (5-10) and (5-9) we obtain

$$\begin{aligned}|\psi_{t,s} * \phi_{t',s'}(z,u)| &= \left|\int_{\mathbb{R}} \Psi_{t,s} *_{\mathbb{H}^n\times\mathbb{R}} \Phi_{t',s'}((z,u-v),v)\,dv\right| \\ &\lesssim C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^{2M}\left(\frac{s}{s'} \wedge \frac{s'}{s}\right)^{M} \times \int_{\mathbb{R}} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u-v|)^{n+1+2M}} \frac{(s\vee s')^{2M}}{(s\vee s'+|v|)^{1+2M}}\,dv.\end{aligned} \tag{5-14}$$

Now we consider four cases separately.

<u>*Case* 1</u>: $(t\vee t')^2 \leq s\vee s'$ *and* $|u| \geq s\vee s'$. In this case we use the fact that

$$\frac{(s\vee s')^{2M}}{(s\vee s'+|v|)^{1+2M}} = \frac{1}{s\vee s'} \frac{1}{(1+|v/(s\vee s')|)^{1+2M}} \tag{5-15}$$

has integral roughly 1, with essential support $[-s\vee s', s\vee s']$, to obtain

$$\int_{\mathbb{R}} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u-v|)^{n+1+2M}} \frac{(s\vee s')^{2M}}{(s\vee s'+|v|)^{1+2M}}\,dv$$
$$\approx \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u|)^{n+1+2M}} \le \frac{(t\vee t')^{2M}}{((t\vee t')^2+|z|^2)^{n+M}} \frac{(t\vee t')^{2M}}{((t\vee t')^2+|u|)^{1+M}}$$
$$\le \frac{(t\vee t')^{2M}}{((t\vee t')+|z|)^{2n+2M}} \frac{(s\vee s')^{M}}{(s\vee s'+|u|)^{1+M}}.$$

Plugging this estimate into the right side of (5-14) leads to the correct product estimate (5-12) for this case.

<u>*Case 2*</u>: $(t\vee t')^2\le s\vee s'$ *and* $|u|\le s\vee s'$. In this case we bound the left side of (5-15) by $1/(s\vee s')$ to obtain

$$\int_{\mathbb{R}} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u-v|)^{n+1+2M}} \frac{(s\vee s')^{2M}}{(s\vee s'+|v|)^{1+2M}}\,dv$$
$$\lesssim \frac{1}{s\vee s'}\int_{\mathbb{R}} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u-v|)^{n+1+2M}}\,dv$$
$$\lesssim \frac{1}{s\vee s'} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2)^{n+2M}} \le \frac{(t\vee t')^{2M}}{((t\vee t')+|z|)^{2n+2M}} \frac{(s\vee s')^{M}}{(s\vee s'+|u|)^{1+M}}.$$

Plugging this estimate into the right side of (5-14) again leads to the correct product estimate (5-12) for this case.

<u>*Case 3*</u>: $(t\vee t')^2\ge s\vee s'$ *and* $|u|\le (t\vee t')^2$. In this case we have

$$\int_{\mathbb{R}} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u-v|)^{n+1+2M}} \frac{(s\vee s')^{2M}}{(s\vee s'+|v|)^{1+2M}}\,dv$$
$$\lesssim \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2)^{n+1+2M}} \lesssim \frac{(t\vee t')^{2M}}{((t\vee t')^2+|z|^2)^{n+M}} \frac{(t\vee t')^{2M}}{((t\vee t')^2+|u|)^{1+M}}$$
$$\approx \frac{(t\vee t')^{2M}}{(t\vee t'+|z|)^{2n+2M}} \frac{(t\vee t')^{2M}}{(t\vee t'+\sqrt{|u|})^{2+2M}}.$$

Plugging this estimate into the right side of (5-14) leads to the correct one-parameter estimate (5-13) for this case.

<u>*Case 4*</u>: $(t\vee t')^2\ge s\vee s'$ *and* $|u|\ge (t\vee t')^2$. In this case we have

$$\int_{\mathbb{R}} \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u-v|)^{n+1+2M}} \frac{(s\vee s')^{2M}}{(s\vee s'+|v|)^{1+2M}}\,dv$$
$$\lesssim \frac{(t\vee t')^{4M}}{((t\vee t')^2+|z|^2+|u|)^{n+1+2M}} \lesssim \frac{(t\vee t')^{2M}}{((t\vee t')^2+|z|^2)^{n+M}} \frac{(t\vee t')^{2M}}{((t\vee t')^2+|u|)^{1+M}}$$
$$\approx \frac{(t\vee t')^{2M}}{(t\vee t'+|z|)^{2n+2M}} \frac{(t\vee t')^{2M}}{(t\vee t'+\sqrt{|u|})^{2+2M}}.$$

Plugging this estimate into the right side of (5-14) again leads to the correct one-parameter estimate (5-13). □

**5.3.1.** *Proof of the Plancherel–Pólya inequalities.* Before we prove the Plancherel–Pólya-type inequality in Theorem 19, we first prove the following lemma. We will often use the notation $(x_I, y_J)$ in place of $(z_I, u_J)$ for the center of the dyadic rectangle $I \times J$ in $\mathbb{H}^n$; that is, we write $x$ in place of $z$, and $y$ in place of $u$.

**Lemma 53.** *Let $I \times J$ and $I' \times J'$ be dyadic rectangles in $\mathbb{H}^n$ such that*

$$\ell(I) = 2^{-j-N}, \quad \ell(J) = 2^{-j-N} + 2^{-k-N}, \quad \ell(I') = 2^{-j'-N}, \quad \text{and } \ell(J') = 2^{-j'-N} + 2^{-k'-N}.$$

*Thus, for any $(u, v)$ and $(u^*, v^*)$ in $\mathbb{H}^n$, we have, when $j \wedge j' \ge k \wedge k'$,*

$$\sum_{I',J'} \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(k\wedge k')K_2} |I'||J'|}{(2^{-j\wedge j'} + |u - x_{I'}|)^{2n+K_1} (2^{-k\wedge k'} + |v - y_{J'}|)^{1+K_2}} \cdot |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$
$$\le C_1(N, r, j, j', k, k') 2^{-|j-j'|L_1} \times 2^{-|k-k'|L_2} \Big\{ M_S \Big[ \Big( \sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \Big)^r \Big] \Big\}^{\frac{1}{r}} (u^*, v^*),$$

*and when $j \wedge j' \le k \wedge k'$,*

$$\sum_{I',J'} \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(j\wedge j')K_2} |I'||J'|}{(2^{-j\wedge j'} + |u - x_{I'}|)^{2n+K_1} (2^{-j\wedge j'} + |v - y_{J'}|)^{1+K_2}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$
$$\le C_2(N, r, j, j', k, k') 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} \times \Big\{ M \Big[ \Big( \sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \Big)^r \Big] \Big\}^{\frac{1}{r}} (u^*, v^*),$$

*where $M$ is the Hardy–Littlewood maximal function on $\mathbb{H}^n$, $M_S$ is the strong maximal function on $\mathbb{H}^n$ as defined in (1-1),* $\max\{2n/(2n + K_1), 1/(1 + K_2)\} < r$ *and*

$$C_1(N, r, j, j', k, k') = 2^{(1/r-1)N(2n+1)} \cdot 2^{[2n(j\wedge j'-j')+(k\wedge k'-k')](1-1/r)},$$
$$C_2(N, r, j, j', k, k') = 2^{(1/r-1)N(2n+1)} \cdot 2^{[2n(j\wedge j'-j')+(j\wedge j'-j'\wedge k')](1-1/r)}.$$

*Proof.* We set

$$A_0 = \Big\{ I' : \ell(I') = 2^{-j'-N}, \ \frac{|u - x_{I'}|}{2^{-j\wedge j'}} \le 1 \Big\},$$
$$B_0 = \Big\{ J' : \ell(J') = 2^{-j'-N} + 2^{-k'-N}, \ \frac{|v - y_{J'}|}{2^{-k\wedge k'}} \le 1 \Big\},$$

where $x_{I'} \in I'$ and $y_{J'} \in J'$, and where, for $\ell \ge 1$, $i \ge 1$,

$$A_\ell = \Big\{ I' : \ell(I') = 2^{-j'-N}, \ 2^{\ell-1} < \frac{|u - x_{I'}|}{2^{-j\wedge j'}} \le 2^\ell \Big\}.$$
$$B_i = \Big\{ J' : \ell(J') = 2^{-j'-N} + 2^{-k'-N}, \ 2^{i-1} < \frac{|v - y_{J'}|}{2^{-k\wedge k'}} \le 2^i \Big\}.$$

We first consider the case when $j \wedge j' \geq k \wedge k'$, and let

$$\tau = [2n(j \wedge j' - j') + (k \wedge k' - k')]\Big(1 - \frac{1}{r}\Big).$$

Then

$$\begin{aligned}
&\sum_{I',J'} \frac{2^{-(j\wedge j')K_1-(k\wedge k')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{2n+K_1}(2^{-k\wedge k'}+|v-y_{J'}|)^{1+K_2}} \cdot |\phi_{j',k'} * f(x_{I'}, y_{J'})| \\
&\quad \leq \sum_{\ell,i\geq 0} 2^{-\ell(2n+K_1)} 2^{-i(1+K_2)} 2^{-N(2n+1)} 2^{(j\wedge j'-j')n+(k\wedge k'-k')m} \sum_{I'\in A_\ell, J'\in B_i} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \\
&\quad \leq \sum_{\ell,i\geq 0} 2^{-\ell(n+K_1)} 2^{-i(m+K_2)} 2^{-N(n+m)} 2^{(j\wedge j'-j')2n+(k\wedge k'-k')} \Bigg( \sum_{I'\in A_\ell, J'\in B_i} (|\phi_{j',k'} * f(x_{I'}, y_{J'})|)^r \Bigg)^{1/r} \\
&\quad = \sum_{\ell,i\geq 0} 2^{-\ell(2n+K_1)-i(1+K_2)-N(2n+1)} 2^{(j\wedge j'-j')2n+(k\wedge k'-k')} \\
&\qquad\qquad \times \Bigg( \int_{\mathbb{H}^n} |I'|^{-1}|J'|^{-1} \sum_{I'\in A_\ell, J'\in B_i} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^r \chi_{I'}\chi_{J'} \Bigg)^{1/r} \\
&\quad \leq \sum_{\ell,i\geq 0} 2^{-\ell(2n+K_1-2n/r)-i(1+K_2-1/r)+(1/r-1)N(2n+1)} \\
&\qquad\qquad \times 2^{\tau} \Bigg( M_S \Bigg( \sum_{I'\in A_\ell, J'\in B_i} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^r \chi_{I'}\chi_{J'} \Bigg)(u^*, v^*) \Bigg)^{1/r} \\
&\quad \leq C_1(N, r, j, k, j', k') \Bigg( M_S \Bigg( \sum_{I',J'} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^r \chi_{I'}\chi_{J'} \Bigg)(u^*, v^*) \Bigg)^{1/r}
\end{aligned}$$

The last inequality follows from the assumption that $r > \max\{2n/(2n+K_1), 1/(1+K_2)\}$, which can be achieved by choosing $K_1, K_2$ large enough. The second inequality can be proved similarly. □

We are now ready to give the proof of the Plancherel–Pólya inequality.

*Proof of Theorem 19.* By Theorem 17, $f \in \mathcal{M}_{\text{flag}}^{M+\delta}(\mathbb{H}^n)'$ can be represented by

$$f(z,u) = \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |J'||I'| \tilde{\phi}_{j',k'}((z,u) \circ (x_{I'}, y_{J'})^{-1})(\phi_{j',k'} * f)(x_{I'}, y_{J'}).$$

We write

$$(\psi_{j,k} * f)(u,v) = \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |I'||J'|(\psi_{j,k} * \tilde{\phi}_{j',k'}((\cdot,\cdot) \circ (x_{I'}, y_{J'})^{-1}))(u,v)(\phi_{j',k'} * f)(x_{I'}, y_{J'}).$$

By the almost-orthogonality estimates in Lemma 52, and by choosing $t = 2^{-j}$, $s = 2^{-k}$, $t' = 2^{-j'}$, $s' = 2^{-k'}$, and for any given positive integers $L_1, L_2, K_1, K_2$, we have, if $j \wedge j' \geq k \wedge k'$,

$$\begin{aligned}
&|(\psi_{j,k} * \tilde{\phi}_{j',k'}((\cdot,\cdot) \circ (x_{I'}, y_{J'})^{-1}))(u,v)| \\
&\qquad\qquad \leq \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(k\wedge k')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{2n+K_1}(2^{-k\wedge k'}+|v-y_{J'}|)^{1+K_2}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|,
\end{aligned}$$

and when $j \wedge j' \le k \wedge k'$, we have

$$|(\psi_{j,k} * \tilde{\phi}_{j',k'}((\cdot,\cdot)\circ(x_{I'}, y_{J'})^{-1}))(u,v)| \le \frac{2^{-|j-j'|L_1-|k-k'|L_2}2^{-(j\wedge j')K_1-(j\wedge j')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{2n+K_1}(2^{-j\wedge j'}+|v-y_{J'}|)^{1+K_2}}|\phi_{j',k'} * f(x_{I'}, y_{J'})|.$$

Using Lemma 53, for any $u, u^* \in I$, $x_{I'} \in I'$, $v, v^* \in J$, and $y_{J'} \in J'$, we have

$$\begin{aligned}
|\psi_{j,k} * f(u,v)| &\le C_1 \sum_{j',k':j\wedge j'\ge k\wedge k'} 2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2} \times \Bigl\{M_S\Bigl[\Bigl(\sum_{J'}\sum_{I'}|\phi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'}\Bigr)^r\Bigr]\Bigr\}^{1/r}(u^*, v^*) \\
&\quad + C_2 \sum_{j',k':j\wedge j'\le k\wedge k'} 2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2} \times \Bigl\{M\Bigl[\Bigl(\sum_{J'}\sum_{I'}|\phi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'}\Bigr)^r\Bigr]\Bigr\}^{1/r}(u^*, v^*) \\
&\le C \sum_{j',k'} 2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2} \times \Bigl\{M_S\Bigl[\Bigl(\sum_{J'}\sum_{I'}|\phi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'}\Bigr)^r\Bigr]\Bigr\}^{1/r}(u^*, v^*),
\end{aligned}$$

where $M$ is the Hardy–Littlewood maximal function on $\mathbb{H}^n$, $M_S$ is the strong maximal function on $\mathbb{H}^n$, and $\max\{2n/(2n+K_1), 1/(1+K_2)\} < r < p$.

Applying Hölder's inequality and summing over $j, k, I, J$ yields

$$\Bigl\{\sum_{j,k}\sum_{I,J}\sup_{u\in I, v\in J}|\psi_{j,k} * f(u,v)|^2\chi_I\chi_J\Bigr\}^{\frac12} \le C\Bigl\{\sum_{j',k'}\Bigl\{M_S\Bigl(\sum_{I',J'}|\phi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{I'}\chi_{J'}\Bigr)^r\Bigr\}^{2/r}\Bigr\}^{\frac12}.$$

Since $x_{I'}$ and $y_{J'}$ are arbitrary points in $I'$ and $J'$, respectively, we have

$$\Bigl\{\sum_{j,k}\sum_{I,J}\sup_{u\in I, v\in J}|\psi_{j,k} * f(u,v)|^2\chi_I\chi_J\Bigr\}^{\frac12} \le C\Bigl\{\sum_{j',k'}\Bigl\{M_S\Bigl(\sum_{I',J'}\inf_{\substack{u\in I'\\ v\in J'}}|\phi_{j',k'} * f(u,v)|\chi_{I'}\chi_{J'}\Bigr)^r\Bigr\}^{2/r}\Bigr\}^{\frac12},$$

and hence, by the Fefferman–Stein vector-valued maximal function inequality [Fefferman and Stein 1982] with $r < p$, we get

$$\Bigl\|\Bigl\{\sum_j\sum_k\sum_J\sum_I\sup_{\substack{u\in I\\ v\in J}}|\psi_{j,k} * f(u,v)|^2\chi_I\chi_J\Bigr\}^{\frac12}\Bigr\|_p \le C\Bigl\|\Bigl\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}\inf_{\substack{u\in I'\\ v\in J'}}|\phi_{j',k'} * f(u,v)|^2\chi_{I'}\chi_{J'}\Bigr\}^{\frac12}\Bigr\|_p.$$

This completes the proof of Theorem 19. □

## 6. Boundedness of flag singular integrals

As a consequence of Theorem 19, it is easy to see that the Hardy space $H^p_{\text{flag}}$ is independent of the choice of the functions $\psi$. Moreover, we have the following characterization of $H^p_{\text{flag}}$ using the wavelet norm.

**Proposition 54.** *Let* $0 < p \le 1$. *Then we have*

$$\|f\|_{H^p_{\rm flag}} \approx \Big\| \Big\{ \sum_j \sum_k \sum_J \sum_I |\psi_{j,k} * f(x_I, y_J)|^2 \chi_I(x) \chi_J(y) \Big\}^{\frac{1}{2}} \Big\|_p,$$

*where* $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ *are as in Theorem 19.*

Before we give the proof of the boundedness of flag singular integrals on $H^p_{\rm flag}$, we demonstrate several properties of $H^p_{\rm flag}$.

**Proposition 55.** $\mathcal{M}^{M+\delta}_{\rm flag}(\mathbb{H}^n)$ *is dense in* $H^p_{\rm flag}(\mathbb{H}^n)$ *for* $M$ *large enough.*

*Proof.* Suppose $f \in H^p_{\rm flag}$, and set $W = \{(j, k, I, J) : |j| \le L, |k| \le M, I \times J \subseteq B(0, r)\}$, where $I \times J$ is a dyadic rectangle in $\mathbb{H}^n$ with $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$, and where $B(0, r)$ is the ball in $\mathbb{H}^n$ centered at the origin with radius $r$. It is easy to see that

$$\sum_{(j,k,I,J)\in W} |I||J|\tilde{\psi}_{j,k}((z, y) \circ (x_I, y_J)^{-1})\psi_{j,k} * f(x_I, y_J)$$

is a test function in $\mathcal{M}^{M+\delta}_{\rm flag}(\mathbb{H}^n)$ for any fixed $L, M, r$. To obtain the proposition, it suffices to prove

$$\sum_{(j,k,I,J)\in W^c} |I||J|\tilde{\psi}_{j,k}((z, y) \circ (x_I, y_J)^{-1})\psi_{j,k} * f(x_I, y_J)$$

tends to zero in the $H^p_{\rm flag}$ norm as $L, M, r$ tend to infinity. This follows from an argument similar to that in the proof of Theorem 19. In fact, repeating the argument in Theorem 19 yields

$$\Big\| \sum_{(j,k,I,J)\in W^c} |I||J|\tilde{\psi}_{j,k}((z, y) \circ (x_I, y_J)^{-1})\psi_{j,k} * f(x_I, y_J) \Big\|_{H^p_{\rm flag}} \le C \Big\| \Big\{ \sum_{(j,k,I,J)\in W^c} |\psi_{j,k} * f(x_I, y_J)|^2 \chi_I \chi_J \Big\}^{\frac{1}{2}} \Big\|_p,$$

where the last term tends to zero as $L, M, r$ tend to infinity whenever $f \in H^p_{\rm flag}$. □

As a consequence of Proposition 55, $L^2(\mathbb{H}^n) \cap H^p_{\rm flag}(\mathbb{H}^n)$ is dense in $H^p_{\rm flag}(\mathbb{H}^n)$. Furthermore, we have the following theorem.

**Theorem 56.** *If* $f \in L^2(\mathbb{H}^n) \cap H^p_{\rm flag}(\mathbb{H}^n)$, $0 < p \le 1$, *then* $f \in L^p(\mathbb{H}^n)$ *and there is a constant* $C_p > 0$ *which is independent of the* $L^2$ *norm of* $f$ *such that*

$$\|f\|_p \le C\|f\|_{H^p_{\rm flag}}.$$

To prove Theorem 56, we need a discrete Calderón reproducing formula on $L^2(\mathbb{H}^n)$. To be more precise, take $\phi^{(1)} \in C^\infty_0(\mathbb{H}^n)$ as in Theorem 2 with

$$\int_{\mathbb{H}^n} \phi^{(1)}(z, u) z^\alpha u^\beta dz\, du = 0 \quad \text{for all } \alpha, \beta \text{ satisfying } 0 \le |\alpha| \le M_0,\ 0 \le |\beta| \le M_0,$$

and take $\phi^{(2)} \in C_0^\infty(\mathbb{R})$ with

$$\int_{\mathbb{R}} \phi^{(2)}(v) z^\gamma \, dv = 0 \quad \text{for all } 0 \le |\gamma| \le M_0,$$

and $\sum_k |\widehat{\phi^{(2)}}(2^{-k}\xi_2)^2 = 1$ for all $\xi_2 \in \mathbb{R}\backslash\{0\}$.

Furthermore, we may assume that $\phi^{(1)}$ and $\phi^{(2)}$ are radial functions and supported in the unit balls of $\mathbb{H}^n$ and $\mathbb{R}$, respectively. Set

$$\phi_{jk}(z,u) = \int_{\mathbb{R}} \phi_j^{(1)}(z, u-v)\phi_k^{(2)}(v)\, dv.$$

By Theorem 2 we have the following continuous version of the Calderón reproducing formula on $L^2$: for $f \in L^2(\mathbb{H}^n)$,

$$f(z,u) = \sum_j \sum_k \phi_{jk} * \phi_{jk} * f(z,u).$$

For our purposes, we need a discrete version of the above reproducing formula.

**Theorem 57.** *There exist functions $\tilde{\phi}_{jk}$ and an operator $T_N^{-1}$ such that*

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J| \tilde{\phi}_{j,k}((x,y)\circ(x_I,y_J)^{-1})\phi_{j,k} * (T_N^{-1}(f))(x_I,y_J),$$

*where the functions $\tilde{\phi}_{jk}((x,y)\circ(x_I,y_J)^{-1})$ satisfy the conditions in Theorem 17 with $\alpha_1, \beta_1, \gamma_1, N, M$ depending on $M_0$. Moreover, $T_N^{-1}$ is bounded on both $L^2(\mathbb{H}^n)$ and $H^p_{\text{flag}}(\mathbb{H}^n)$, and the series converges in $L^2(\mathbb{H}^n)$.*

**Remark 58.** The difference between Theorems 57 and 17 is that the $\tilde{\phi}_{jk}$ in Theorem 57 have compact support. The price we pay here is that $\tilde{\phi}_{jk}$ only satisfies moment conditions of finite order, unlike in Theorem 17, where moment conditions of infinite order are satisfied. Moreover, the formula in Theorem 57 only holds on $L^2(\mathbb{H}^n)$ while the formula in Theorem 17 holds in both the test function space $\mathcal{M}^{M+\delta}_{\text{flag}}$ and its dual space $(\mathcal{M}^{M+\delta}_{\text{flag}})'$.

*Proof of Theorem 57.* Following the proof of Theorem 17, we have

$$f(z,u) = \sum_j \sum_k \sum_J \sum_I \left[\int_J \int_I \phi_{j,k}((z,u)\circ(u,v)^{-1})\, du\, dv\right](\phi_{j,k} * f)(x_I,y_J) + \mathcal{R}f(x,y),$$

where $I$, $J$, $j$, $k$, and $\mathcal{R}$ are as in Theorem 17.

We need the following lemma to handle the remainder term $\mathcal{R}$.

**Lemma 59.** *Let $0 < p \le 1$. Then the operator $\mathcal{R}$ is bounded on $L^2(\mathbb{H}^n)$ and $H^p_{\text{flag}}(\mathbb{H}^n)$ whenever $M_0$ is chosen to be a large positive integer. Moreover, there exists a constant $C > 0$ such that*

$$\|\mathcal{R}f\|_2 \le C2^{-N}\|f\|_2 \quad \textit{and} \quad \|\mathcal{R}f\|_{H^p_{\text{flag}}(\mathbb{H}^n)} \le C2^{-N}\|f\|_{H^p_{\text{flag}}(\mathbb{H}^n)}.$$

*Proof.* Following the proofs of Theorems 17 and 19 and using the wavelet Calderón reproducing formula for $f \in L^2(\mathbb{H}^n)$, we have

$$\begin{aligned}
&\|g_{\text{flag}}(\mathcal{R}f)\|_p \\
&\quad \le \Big\| \Big\{ \sum_j \sum_k \sum_J \sum_I |(\psi_{j,k} * \mathcal{R}f)|^2 \chi_I \chi_J \Big\}^{\frac12} \Big\|_p \\
&\quad = \Big\| \Big\{ \sum_{j,k,J,I} \sum_{j',k',J',I'} |J'||I'||(\psi_{j,k} * \mathcal{R}\tilde{\psi}_{j',k'}((\cdot,\cdot)\circ(x_{I'},y_{J'})^{-1}) \cdot \psi_{j'k'} * f(x_{I'},y_{J'}))|^2 \chi_I \chi_J \Big\}^{\frac12} \Big\|_p,
\end{aligned}$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are as in Theorem 19.

**Claim.** *We have*

$$\begin{aligned}
&|(\psi_{j,k} * \mathcal{R}(\tilde{\psi}_{j',k'}((\cdot,\cdot)\circ(x_{I'},y_{J'})^{-1})))(z,u)| \\
&\quad \le C 2^{-N} 2^{-|j-j'|K} 2^{-|k-k'|K} \int_{\mathbb{R}} \frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')} + |z - x_{I'}| + |u - v - y_{J'})^{2n+1+K}} \frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')} + |v|)^{1+K}}\, dv,
\end{aligned}$$

*where, for simplicity, we have chosen*

$$L_1 = L_2 = K_1 = K_2 = K < M_0, \quad \max\Big(\frac{2n}{2n+K}, \frac{1}{1+K}\Big) < p,$$

*and $M_0$ is chosen to be a larger integer later.*

Assuming the claim for the moment, we can repeat an argument used in Lemma 53, and then use Theorem 19 to obtain

$$\begin{aligned}
\||g_{\text{flag}}(\mathcal{R}f)\|_p &\le C 2^{-N} \Big\| \Big\{ \sum_{j'} \sum_{k'} \Big[ M_S \Big( \sum_{J'} \sum_{I'} |\psi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \Big)^r \Big]^{2/r} \Big\}^{\frac12} \Big\|_p \\
&\le C 2^{-N} \Big\| \Big\{ \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |\psi_{j',k'} * f(x_{I'}, y_{J'})|^2 \chi_{I'} \chi_{J'} \Big\}^{\frac12} \Big\|_p \le C 2^{-N} \|f\|_{H^p_{\text{flag}}(\mathbb{H}^n)}.
\end{aligned}$$

It is clear that the above estimates continue to hold when $p$ is replaced by 2. This completes the proof of Lemma 59 modulo the claim.

In order to prove the claim made above, we note that Theorem 41 shows that the functions

$$\mathcal{R}(\tilde{\psi}_{j',k'}((\cdot,\cdot)\circ(x_{I'},y_{J'})^{-1}))(z,u)$$

are flag molecules. Then the claim follows from Lemma 53, and this completes the proof of Lemma 59. □

We now return to the proof of Theorem 57. Let $(T_N)^{-1} = \sum_{i=1}^{\infty} \mathcal{R}^i$, where

$$T_N f = \sum_j \sum_k \sum_J \sum_I \Big( \frac{1}{|I||J|} \int_J \int_I \phi_{j,k}((x,y)\circ(u,v)^{-1})\, du\, dv \Big) |I||J| (\phi_{j,k} * f)(x_I, y_J).$$

Lemma 59 shows that if $N$ is large enough, then both $T_N$ and $(T_N)^{-1}$ are bounded on $L^2(\mathbb{H}^n)\cap H^p_{\text{flag}}(\mathbb{H}^n)$. Hence, we can get the reproducing formula

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I |I||J|\tilde{\phi}_{j,k}((x,y)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}f)(x_I,y_J),$$

where the functions $\tilde{\phi}_{jk}((x,y)\circ(x_I,y_J)^{-1})$ are flag molecules, and the series converges in $L^2(\mathbb{H}^n)$. This completes the proof of Theorem 57. □

As a consequence of Theorem 57, we obtain the following corollary.

**Corollary 60.** *If $f\in L^2(\mathbb{H}^n)\cap H^p_{\text{flag}}(\mathbb{H}^n)$ and $0<p\le 1$, then*

$$\|f\|_{H^p_{\text{flag}}}\approx\bigg\|\bigg\{\sum_j\sum_k\sum_J\sum_I|\phi_{jk}*(T_N^{-1}f)(x_I,y_J)|^2\chi_I(z)\chi_J(u)\bigg\}^{\frac{1}{2}}\bigg\|_p,$$

*where the constants are independent of the $L^2$ norm of $f$.*

*Proof.* Note that if $f\in L^2(\mathbb{H}^n)$, we can apply the Calderón reproducing formula in Theorem 57 and then repeat the proof of Theorem 19. We leave the details to the reader. □

We now start the proof of Theorem 56. We define a square function by

$$\tilde{g}(f)(z,u)=\bigg\{\sum_j\sum_k\sum_J\sum_I|\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)|^2\chi_I(z)\chi_J(u)\bigg\}^{\frac{1}{2}},$$

where the $\phi_{jk}$ are as in Theorem 57. By Corollary 60, for $f\in L^2(\mathbb{H}^n)\cap H^p_{\text{flag}}(\mathbb{H}^n)$, we have

$$\|\tilde{g}(f)\|_{L^p(\mathbb{H}^n)}\le C\|f\|_{H^p_{\text{flag}}(\mathbb{H}^n)}.$$

To complete the proof of Theorem 56, let $f\in L^2(\mathbb{H}^n)\cap H^p_{\text{flag}}(\mathbb{H}^n)$. Set

$$\Omega_i=\{(z,u)\in\mathbb{H}^n:\tilde{g}(f)(z,u)>2^i\}.$$

Let

$$\mathcal{B}_i=\{(j,k,I,J):|(I\times J)\cap\Omega_i|>\tfrac{1}{2}|I\times J|,\ |(I\times J)\cap\Omega_{i+1}|\le\tfrac{1}{2}|I\times J|\},$$

where $I\times J$ are rectangles in $\mathbb{H}^n$ with side lengths $\ell(I)=2^{-j-N}$ and $\ell(J)=2^{-k-N}+2^{-j-N}$. Since $f\in L^2(\mathbb{H}^n)$, the discrete Calderón reproducing formula in Theorem 57 gives

$$\begin{aligned}f(z,u)&=\sum_j\sum_k\sum_J\sum_I\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})|I||J|\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)\\&=\sum_i\sum_{(j,k,I,J)\in\mathcal{B}_i}|I||J|\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J),\end{aligned}$$

where the series converges rapidly in $L^2$ norm, and hence almost everywhere.

**Claim.** *We have*

$$\Big\| \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)\Big\|_p^p \le C2^{ip}|\Omega_i|,$$

*which together with the fact that* $0<p\le 1$ *yields*

$$\begin{aligned}\|f\|_p^p &\le \sum_i \Big\| \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)\Big\|_p^p \\ &\le C\sum_i 2^{ip}|\Omega_i| \le C\|\tilde{g}(f)\|_p^p \le C\|f\|_{H^p_{\text{flag}}}^p.\end{aligned}$$

To obtain the claim, note that $\phi^{(1)}$ and $\psi^{(2)}$ are radial functions supported in unit balls in $\mathbb{H}^n$ and $\mathbb{R}$, respectively. Hence, if $(j,k,I,J)\in\mathcal{B}_i$, then $\phi_{j,k}((z,u)\circ(x_I,y_J)^{-1})$ is supported in

$$\widetilde{\Omega_i} = \big\{(z,u) : M_S(\chi_{\Omega_i})(z,u) > \tfrac{1}{100}\big\}.$$

Thus, by Hölder's inequality,

$$\begin{aligned}&\Big\| \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)\Big\|_p^p \\ &\qquad \le |\widetilde{\Omega_i}|^{1-p/2}\Big\| \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)\Big\|_2^p.\end{aligned}$$

By duality, for all $g\in L^2$ with $\|g\|_2\le 1$,

$$\begin{aligned}&\Big|\Big\langle \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\tilde{\phi}_{j,k}((z,u)\circ(x_I,y_J)^{-1})\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J), g\Big\rangle\Big| \\ &\qquad = \Big| \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\tilde{\phi}_{j,k}*g(x_I,y_J)\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)\Big| \\ &\qquad \le C\Big( \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\phi_{j,k}*(T_N^{-1}(f))(x_I,y_J)|^2\Big)^{\frac12}\cdot\Big( \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\tilde{\phi}_{j,k}*g(x_I,y_J)|^2\Big)^{\frac12}.\end{aligned}$$

Since

$$\begin{aligned}\Big( \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\tilde{\phi}_{j,k}*g(x_I,y_J)|^2\Big)^{\frac12} &\le \Big( \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|(M_S(\tilde{\phi}_{j,k}*g)(z,u)\chi_I(z)\chi_J(u))^2\Big)^{\frac12} \\ &\le C\Big(\sum_{j,k}\int_{\mathbb{C}^n}\int_{\mathbb{R}} (M_S(\tilde{\phi}_{j,k}*g)^2(z,u)\,dz\,du)\Big)^{\frac12} \le C\|g\|_2,\end{aligned}$$

the claim now follows from the fact that $|\tilde{\Omega}_i| \le C|\Omega_i|$ and the estimate

$$C2^{2i}|\Omega_i| \ge \int_{\widetilde{\Omega_i}\setminus\Omega_{i+1}} \tilde{g}^2(f)(z,u)\,dz\,du \ge \sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * (T_N^{-1}(f))(x_I,y_J)|^2 |(I\times J)\cap\widetilde{\Omega_i}\setminus\Omega_{i+1}|$$
$$\ge \frac{1}{2}\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\phi_{j,k} * (T_N^{-1}(f))(x_I,y_J)|^2,$$

where the fact that $|(I\times J)\cap\widetilde{\Omega_i}\setminus\Omega_{i+1}| > \frac{1}{2}|I\times J|$ when $(j,k,I,J)\in\mathcal{B}_i$ is used in the last inequality. This finishes the proof of Theorem 56.

As a consequence of Theorem 56, we have the following corollary.

**Corollary 61.** *$H^1_{\text{flag}}(\mathbb{H}^n)$ is a subspace of $L^1(\mathbb{H}^n)$.*

*Proof.* Given $f \in H^1_{\text{flag}}(\mathbb{H}^n)$, by Proposition 55, there is a sequence $\{f_n\}$ such that $f_n \in L^2(\mathbb{H}^n)\cap H^1_{\text{flag}}(\mathbb{H}^n)$ and $f_n$ converges to $f$ in the norm of $H^1_{\text{flag}}(\mathbb{H}^n)$. By Theorem 56, $f_n$ converges to $g$ in $L^1(\mathbb{H}^n)$ for some $g \in L^1(\mathbb{H}^n)$. Therefore, $f = g$ in $(\mathcal{M}^{M+\delta}_{\text{flag}})'$. □

*Proof of Theorem 22.* We assume that $K$ is the kernel of $T$. Applying the discrete Calderón reproducing formula in Theorem 57 implies that, for $f \in L^2(\mathbb{H}^n)\cap H^p_{\text{flag}}(\mathbb{H}^n)$,

$$\Big\|\Big\{\sum_{j,k}\sum_{I,J}|\phi_{j,k}*K*f(z,u)|^2\chi_I(x)\chi_J(y)\Big\}^{\frac{1}{2}}\Big\|_p$$
$$= \Big\|\Big\{\sum_{j,k}\sum_{I,J}\Big|\sum_{j',k'}\sum_{I',J'}|J'||I'|\phi_{j,k}*K*\tilde{\phi}_{j',k'}((\cdot,\cdot)\circ(x_I,y_J)^{-1})(z,u)$$
$$\times\phi_{j',k'}*(T_N^{-1}(f))(x_{I'},y_{J'})\Big|^2\chi_I(x)\chi_J(y)\Big\}^{\frac{1}{2}}\Big\|_p,$$

where the discrete Calderón reproducing formula in $L^2(\mathbb{H}^n)$ is used.

Note that the $\phi_{jk}$ are dilations of bump functions, and by estimates similar to those in (5-10), one can easily check that

$$|\phi_{j,k}*K*\tilde{\phi}_{j',k'}((\cdot,\cdot)\circ(x_{I'},y_{J'})^{-1})(z,u)|$$
$$\le C2^{-|j-j'|K}2^{-|k-k'|K}\int_{\mathbb{R}}\frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')}+|z-x_{I'}|+|u-v-y_{J'}|)^{2n+1+K}}\cdot\frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')}+|v|)^{1+K}}\,dv,$$

where $K$ depends on $M_0$ given in Theorem 22, and $M_0$ is chosen large enough.

Repeating an argument similar to that in the proof of Theorem 19, together with Corollary 60, we obtain

$$\|Tf\|_{H^p_{\text{flag}}} \le C\Big\|\Big\{\sum_{j'}\sum_{k'}\Big\{M_S\Big(\sum_{J'}\sum_{I'}|\phi_{j',k'}*(T_N^{-1}(f))(x_{I'},y_{J'})|\chi_{J'}\chi_{I'}\Big)^r\Big\}^{2/r}(z,u)\Big\}^{\frac{1}{2}}\Big\|_p$$
$$\le C\Big\|\Big\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|\phi_{j',k'}*(T_N^{-1}(f))(x_{I'},y_{J'})|^2\chi_{J'}(y)\chi_{I'}(x)\Big\}^{\frac{1}{2}}\Big\|_p \le C\|f\|_{H^p_{\text{flag}}},$$

where the last inequality follows from Corollary 60.

Since $L^2(\mathbb{H}^n)\cap H^p_{\mathrm{flag}}(\mathbb{H}^n)$ is dense in $H^p_{\mathrm{flag}}(\mathbb{H}^n)$, $T$ can be extended to a bounded operator on $H^p_{\mathrm{flag}}(\mathbb{H}^n)$, and this ends the proof of Theorem 22. □

*Proof of Theorem 23.* We note that $H^p_{\mathrm{flag}}\cap L^2$ is dense in $H^p_{\mathrm{flag}}$, so we only have to obtain the required inequality for $f\in H^p_{\mathrm{flag}}\cap L^2$. Thus Theorem 23 follows immediately from Theorems 22 and 56. □

## 7. Duality of Hardy spaces $H^p_{\mathrm{flag}}$

Chang and Fefferman [1985] established that the dual space of $H^1(\mathbb{H}^n)$ is $\mathrm{BMO}(\mathbb{H}^n)$ by using the bi-Hilbert transform, and, consequently, their method is not directly applicable to the implicit two-parameter structure associated to flag singular integrals. In order to deal with the duality theory of $H^p_{\mathrm{flag}}(\mathbb{H}^n)$ for all $0<p\le 1$, we proceed differently, and first prove Theorem 30, the Plancherel–Pólya inequalities for the Carleson space $\mathrm{CMO}^p_{\mathrm{flag}}$. This theorem implies that the function space $\mathrm{CMO}^p_{\mathrm{flag}}$ is well defined.

*Proof of Theorem 30.* The idea of the proof of this theorem is, as in the proof of Theorem 19, to use the wavelet Calderón reproducing formula and the almost-orthogonality estimate. For convenience, we prove Theorem 30 for the smallest Heisenberg group $\mathbb{H}^1=\mathbb{C}\times\mathbb{R}$. However, it will be clear from the proof that its extension to general $\mathbb{H}^n$ is straightforward. Moreover, to simplify notation, we denote $f_{j,k}=f_R$, where $R=I\times J\subset\mathbb{H}^1$, $\ell(I)=2^{-j-N}$, $\ell(J)=2^{-k-N}+2^{-j-N}$, $I$ is a dyadic cube in $\mathbb{R}^2$ and $J$ is an interval in $\mathbb{R}$. Here $N$ is the same as in Theorem 17. We also denote by $\mathrm{dist}(I,I')$ the distance between intervals $I$ and $I'$,

$$S_R=\sup_{\substack{u\in I\\ v\in J}}|\psi_R*f(u,v)|^2,\qquad T_R=\inf_{\substack{u\in I\\ v\in J}}|\phi_R*f(u,v)|^2.$$

With this notation, we can rewrite the wavelet Calderón reproducing formula in Theorem 17 as

$$f(z,u)=\sum_{R=I\times J}|I||J|\tilde{\phi}_R(z,u)\phi_R*f(x_I,y_J),$$

where the sum runs over all rectangles $R=I\times J$. Let

$$R'=I'\times J',\quad |I'|=2^{-j'-N},\quad |J'|=2^{-j'-N}+2^{-k'-N}.$$

Applying the above wavelet Calderón reproducing formula and the almost-orthogonality estimates in Section 5.3 yields, for all $(u,v)\in R$,

$$\begin{aligned}|\psi_R*f(u,v)|^2\le C&\sum_{\substack{R'=I'\times J'\\ j'>k'}}\Big(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big)^L\Big(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big)^L\\ &\times\frac{|I'|^K}{(|I'|+|u-x_{I'}|)^{(1+K)}}\frac{|J'|^K}{(\,|J'|+|v-y_{J'}|)^{(1+K)}}|I'||J'||\phi_{R'}*f(x_{I'},y_{J'})|^2\\ &+C\sum_{\substack{R'=I'J'\\ j'\le k'}}\Big(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big)^L\Big(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big)^L\\ &\times\frac{|I'|^K}{(|I'|+|u-x_{I'}|)^{(1+K)}}\frac{|I'|^K}{(|I'|+|v-y_{J'}|)^{(1+K)}}|I'||J'||\phi_{R'}*f(x_{I'},y_{J'})|^2,\end{aligned}$$

where $K$, $L$ are any positive integers which can be chosen such that $L, K > 2/p-1$ (for general $\mathbb{H}^n$, $K$ can be chosen greater than $(2n+2)(2/p-1)$), the constant $C$ depends only on $K$, $L$, and the functions $\psi$ and $\phi$, where $x_{I'}$ and $y_{J'}$, are any fixed points in $I'$ and $J'$, respectively.

Adding up over $R \subseteq \Omega$, we obtain

$$\sum_{R\subseteq\Omega} |I||J|S_R \leq C \sum_{R\subseteq\Omega}\sum_{R'} |I'||J'| r(R,R')P(R,R')T_{R'}, \tag{7-1}$$

where

$$r(R,R') = \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L-1}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L-1}$$

and

$$P(R,R') = \frac{1}{(1+\operatorname{dist}(I,I')/|I'|)^{1+K}(1+\operatorname{dist}(J,J')/|J'|)^{1+K}}$$

if $j' > k'$, and

$$P(R,R') = \frac{1}{(1+\operatorname{dist}(I,I')/|I'|)^{1+K}(1+\operatorname{dist}(J,J')/|I'|)^{1+K}}$$

if $j' \leq k'$.

We estimate the right-hand side in the above inequality, where we first consider

$$R' = I'\times J', \quad |I'| = 2^{-j'-N}, \quad |J'| = 2^{-j'-N}+2^{-k'-N}, \quad j' > k'.$$

Define

$$\Omega^{i,\ell} = \bigcup_{I\times J\subset\Omega} 3(2^i I\times 2^\ell J) \quad \text{for } i,\ell\geq 0.$$

Let $B_{i,\ell}$ be a collection of dyadic rectangles $R'$ so that, for $i,\ell \geq 1$,

$$\begin{aligned}
B_{i,\ell} &= \{R' = I'\times J' : 3(2^i I'\times 2^\ell J')\cap\Omega^{i,\ell}\neq\varnothing \text{ and } 3(2^{i-1}I'\times 2^{\ell-1}J')\cap\Omega^{i,\ell}=\varnothing\},\\
B_{0,\ell} &= \{R' = I'\times J' : 3(I'\times 2^\ell J')\cap\Omega^{0,\ell}\neq\varnothing \text{ and } 3(I'\times 2^{\ell-1}J')\cap\Omega^{0,\ell}=\varnothing\} \qquad \text{for } \ell\geq 1,\\
B_{i,0} &= \{R' = I'\times J' : 3(2^i I'\times J')\cap\Omega^{i,0}\neq\varnothing \text{ and } 3(2^{i-1}I'\times J')\cap\Omega^{i,0}=\varnothing\} \qquad \text{for } i\geq 1,\\
B_{0,0} &= \{R' = I'\times J' : 3(I'\times J')\cap\Omega^{0,0}\neq\varnothing\}.
\end{aligned}$$

We write

$$\sum_{R\subseteq\Omega}\sum_{R'}|I'||J'|r(R,R')P(R,R')T_{R'} = \sum_{\substack{i\geq 0\\ \ell\geq 0}}\sum_{R'\in B_{i,\ell}}\sum_{R\subseteq\Omega}|I'||J'|r(R,R')P(R,R')T_{R'}.$$

To estimate the right-hand side of the above equality, we first consider the case when $i=\ell=0$. Note that when $R'\in B_{0,0}$, $3R'\cap\Omega^{0,0}\neq\varnothing$. For each integer $h\geq 1$, let

$$\mathcal{F}_h = \left\{R' = I'\times J'\in B_{0,0} : |(3I'\times 3J')\cap\Omega^{0,0}| \geq \frac{1}{2^h}|3I'\times 3J'|\right\}.$$

Let $\mathcal{D}_h = \mathcal{F}_h \backslash \mathcal{F}_{h-1}$, and $\Omega_h = \bigcup_{R' \in \mathcal{D}_h} R'$. Finally, assume that, for any open set $\Omega \subset R^2$,

$$\sum_{R = I \times J \subseteq \Omega} |I||J| T_R \le C |\Omega|^{2/p-1}.$$

Since $B_{0,0} = \bigcup_{h \ge 1} \mathcal{D}_h$ and for each $R' \in B_{0,0}$, $P(R, R') \le 1$, we have

$$\sum_{R' \in B_{0,0}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'} \le \sum_{h \ge 1} \sum_{R' \subseteq \Omega_h} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') T_{R'}.$$

For each $h \ge 1$ and $R' \subseteq \Omega_h$, we decompose $\{R : R \subseteq \Omega\}$ into

$$A_{0,0}(R') = \big\{R = I \times J \subseteq \Omega : \operatorname{dist}(I, I') \le |I| \vee |I'|,\ \operatorname{dist}(J, J') \le |J| \vee |J'|\big\},$$
$$A_{i',0}(R') = \big\{R = I \times J \subseteq \Omega :\ 2^{i'-1}(|I| \vee |I'|) < \operatorname{dist}(I, I') \le 2^{i'}(|I| \vee |I'|),\ \operatorname{dist}(J, J') \le |J| \vee |J'|\big\},$$
$$A_{0,\ell'}(R') = \big\{R = I \times J \subseteq \Omega : \operatorname{dist}(I, I') \le |I| \vee |I'|,\ 2^{\ell'-1}(|J| \vee |J'|) < \operatorname{dist}(J, J') \le 2^{\ell'}(|J| \vee |J'|)\big\},$$
$$A_{i',\ell'}(R') = \big\{R = I \times J \subseteq \Omega :\ 2^{i'-1}(|I| \vee |I'|) < \operatorname{dist}(I, I') \le 2^{i'}(|I| \vee |I'|),$$
$$2^{\ell'-1}(|J| \vee |J'|) < \operatorname{dist}(J, J') \le 2^{\ell'}(|J| \vee |J'|)\big\},$$

where $i', \ell' \ge 1$.

Now we split $\sum_{h \ge 1} \sum_{R' \subseteq \Omega_h} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}$ into

$$\sum_{h \ge 1} \sum_{R' \in \Omega_h} \Bigg( \sum_{R \in A_{0,0}(R')} + \sum_{i' \ge 1} \sum_{R \in A_{i',0}(R')} + \sum_{\ell' \ge 1} \sum_{R \in A_{0,\ell'}(R')} + \sum_{i',\ell' \ge 1} \sum_{R \in A_{i',\ell'}(R')} \Bigg) |I'||J'| r(R, R') P(R, R') T_{R'}$$
$$=: I_1 + I_2 + I_3 + I_4.$$

To estimate the term $I_1$, we only need to estimate $\sum_{R \in A_{0,0}(R')} r(R, R')$, since $P(R, R') \le 1$ in this case.

Note that $R \in A_{0,0}(R')$ implies $3R \cap 3R' \ne \varnothing$. For such $R$, there are four cases:

*Case* 1: $|I'| \ge |I|$, $|J'| \le |J|$.

*Case* 2: $|I'| \le |I|$, $|J'| \ge |J|$.

*Case* 3: $|I'| \ge |I|$, $|J'| \ge |J|$.

*Case* 4: $|I'| \le |I|$, $|J'| \le |J|$.

In each case, we can estimate $\sum_{R \in A_{0,0}} r(R, R') \le C 2^{-hL}$ by using a simple geometric argument similar to that in [Chang and Fefferman 1980]. This implies that $I_1$ is bounded by

$$\sum_{h \ge 1} 2^{-hL} |\Omega_h|^{2/p-1} \le C \sum_{h \ge 1} h^{2/p-1} 2^{-h(L-2/p+1)} |\Omega^{0,0}|^{2/p-1} \le C |\Omega|^{2/p-1},$$

since $|\Omega_h| \le C h 2^h |\Omega^{0,0}|$ and $|\Omega^{0,0}| \le C |\Omega|$.

Thus it remains to estimate term $I_4$, since estimates of $I_2$ and $I_3$ can be derived using the same techniques as for $I_1$ and $I_4$. The estimate for this term is more complicated than that for term $I_1$.

As in estimating term $I_1$, we only need to estimate the sum $\sum_{R \in A_{i',\ell'}(R')} r(R, R')$, since $P(R, R') \le 2^{-i(1+K)} 2^{-i'(1+K)}$. Note that $R \in A_{i',\ell'}(R')$ implies $3(2^{i'} I \times 2^{\ell'} J) \cap 3(2^{i'} I' \times 2^{\ell'} J') \ne \varnothing$. We also split our estimate into four cases.

Case 1. In this case, $|2^i I'| \geq |2^{i'} I|$, $|2^{\ell'} J'| \leq |2^{\ell'} J|$. Then

$$\frac{|2^{i'} I|}{|3 \cdot 2^{i'} I'|} |3(2^{i'} I' \times 2^{\ell'} J')| \leq |3(2^{i'} I' \times 2^{\ell'} J')| \wedge |3(2^i I \times 2^{\ell'} J)|$$

$$\leq C 2^{i'} 2^{\ell'} |3R' \cap \Omega^{0,0}| \leq C 2^{i'} 2^{\ell'} \frac{1}{2^{h-1}} |3R'| \leq C \frac{1}{2^{h-1}} |3(2^{i'} I' \times 2^{\ell'} J')|.$$

Thus $|2^{i'} I'| = \sum^{h-1+n} |2^i I|$ for some $n \geq 0$. For each fixed $n$, the number of such $2^{i'} I$ must be $\leq 2^n \cdot 5$. As for $|2^{\ell'} J| = 2^m |2^{\ell'} J'|$, for some $m \geq 0$, for each fixed $m$, $3 \cdot 2^{\ell'} J \cap 3 \cdot 2^{\ell'} J' \neq \varnothing$ implies that the number of such $2^{\ell'} J'$ is less than 5. Thus

$$\sum_{R \in \text{Case 1}} r(R, R') \leq \sum_{m,n \geq 0} \left( \frac{1}{2^{n+m+h-1}} \right)^L 2^n \cdot 5^2 \leq C 2^{-hL}.$$

We can handle the other three cases similarly. Combining the four cases, we have

$$\sum_{R \in A_{i',\ell'}(R')} r(R, R') \leq C 2^{-hL},$$

which, together with the estimate for $P(R, R')$, imply that

$$I_4 \leq C \sum_{h \geq 1} \sum_{i',\ell' \geq 1} \sum_{R' \subseteq \Omega_h} 2^{-hL} 2^{-i'(1+K)} 2^{-\ell'(1+K)} |I'||J'| T_{R'}.$$

Hence $I_4$ is bounded by

$$\sum_{h \geq 1} 2^{-hL} |\Omega_h|^{2/p-1} \leq C \sum_{h \geq 1} h^{2/p-1} 2^{-h(L-2/p+1)} |\Omega^{0,0}|^{2/p-1} \leq C |\Omega|^{2/p-1},$$

since $\sum_{R' \subset \Omega_h} |I'||J'| T_{R'} \leq C |\Omega_h|^{2/p-1}$ and $|\Omega_h| \leq C h 2^h |\Omega^{0,0}|$ and $|\Omega^{0,0}| \leq C |\Omega|$. Combining $I_1$, $I_2$, $I_3$, and $I_4$, we have

$$\frac{1}{|\Omega|^{2/p-1}} \sum_{R' \in B_{0,0}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'} \leq C \sup_{\bar{\Omega}} 1/|\bar{\Omega}|^{2/p-1} \sum_{R' \subseteq \bar{\Omega}} |I'||J'| T_{R'}.$$

Now we consider

$$\sum_{i,\ell \geq 1} \sum_{R' \in B_{i,\ell}} \sum_{R \subseteq \Omega} |I'||J'| r(R, R') P(R, R') T_{R'}.$$

Note that for $R' \in B_{i,\ell}$, $3(2^i I' \times 2^\ell J) \cap \Omega_{i,\ell} \neq \varnothing$. Let

$$\mathscr{F}_h^{i,\ell} = \left\{ R' \in B_{i,\ell} : |3(2^i I' \times 2^\ell J') \cap \Omega^{i,\ell}| \geq \frac{1}{2^h} |3(2^i I' \times 2^\ell J')| \right\},$$

$$\mathscr{D}_h^{i,\ell} = \mathscr{F}_h^{i,\ell} \setminus \mathscr{F}_{h-1}^{i,\ell},$$

and

$$\Omega_h^{i,\ell} = \bigcup_{R' \in \mathscr{D}_h^{i,\ell}} R'.$$

Since $B_{i,\ell} = \bigcup_{h\ge 1} \mathscr{D}_h^{i,\ell}$, we first estimate

$$\sum_{R'\in\mathscr{D}_h^{i,\ell}} \sum_{R\subseteq\Omega} |I'||J'|r(R, R')P(R, R')T_{R'}$$

for some $i, \ell, h \ge 1$.

Note that for each $R' \in \mathscr{D}_h^{i,\ell}$, $3(2^i I' \times 2^\ell J') \cap \Omega^{i-1,\ell-1} = \varnothing$. So, for any $R \subseteq \Omega$, we have $2^i(|I| \vee |I'|) \le \operatorname{dist}(I, I')$ and $2^\ell(|J| \vee |J'|) \le \operatorname{dist}(J, J')$. We decompose $\{R : R \subseteq \Omega\}$ as

$$A_{i',\ell'}(R') = \Big\{ R \subseteq \Omega : 2^{i'-1}2^i(|I| \vee |I'|) \le \operatorname{dist}(I, I') \le 2^{i'}2^i(|I| \vee |I'|),\\ 2^{\ell'-1}2^\ell(|J| \vee |J'|) \le \operatorname{dist}(J, J') \le 2^{\ell'}2^\ell(|J| \vee |J'|) \Big\},$$

where $i', \ell' \ge 1$. Then we write

$$\sum_{R'\in\mathscr{D}_h^{i,\ell}} \sum_{R\subseteq\Omega} |I'||J'|r(R, R')P(R, R')T_{R'} = \sum_{i',\ell'\ge 1} \sum_{R'\in\mathscr{D}_h^{i,\ell}} \sum_{R\in A_{i',\ell'}(R')} |I'||J'|r(R, R')P(R, R')T_{R'}\,.$$

Since

$$P(R, R') \le C2^{-i(1+K)}2^{-\ell(1+K)}2^{-i'(1+K)}2^{-\ell'(1+K)}$$

for $R' \in B_{i,\ell}$ and $R \in A_{i',\ell'}(R')$, repeating the same proof with $B_{0,0}$ replaced by $B_{i,\ell}$ and using the fact that

$$|\Omega_h^{i,\ell}| \le C2^h|\Omega^{i,\ell}|, \quad |\Omega^{i,\ell}| \le C(i2^i)(\ell 2^\ell)|\Omega^{0,0}|, \quad |\Omega^{0,0}| \le C|\Omega|,$$

yield

$$\begin{aligned}
&\sum_{R'\in\mathscr{D}_h^{i,\ell}} \sum_{R\in A_{i',\ell'}(R')} |I'||J'|r(R, R')P(R, R')T_{R'}\\
&\le C2^{-i(1+K)}2^{-\ell(1+K)}2^{-i'(1+K)}2^{-\ell'(1+K)}|\Omega_h^{i,\ell}|^{2/p-1}\bigg(\frac{1}{|\Omega_h^{i,\ell}|^{2/p-1}} \sum_{R'\subset\Omega_h^{i,\ell}} |I'||J'|T_{R'}\bigg)\\
&\le C2^{-i(1+K)}2^{-\ell(1+K)}2^{-i'(1+K)}2^{-\ell'(1+K)} i^{2/p-1}2^{i(2/p-1)}\ell^{2/p-1}2^{\ell(2/p-1)}h^{2/p-1}2^{-h(L-2/p+1)}|\Omega|^{2/p-1}\\
&\qquad\times \sup_{\bar\Omega} \frac{1}{|\bar\Omega|^{2/p-1}} \sum_{R'\subset\bar\Omega} |I'||J'|T_{R'}.
\end{aligned}$$

Adding over all $i, \ell, i', \ell', h \ge 1$, we get

$$\frac{1}{|\Omega|^{2/p-1}} \sum_{i,\ell\ge 1} \sum_{R'\in B_{i,\ell}} \sum_{R\subseteq\Omega} |I'||J'|r(R, R')P(R, R')T_{R'} \le C \sup_{\bar\Omega} \frac{1}{|\bar\Omega|^{2/p-1}} \sum_{R'\subseteq\bar\Omega} |I'||J'|T_{R'}.$$

Similar estimates, which we leave to the reader, hold for

$$\sum_{i\ge 1} \sum_{R'\in B_{i,0}} \sum_{R\subseteq\Omega} |I'||J'|r(R, R')P(R, R')T_{R'} \quad \text{and} \quad \sum_{\ell\ge 1} \sum_{R'\in B_{0,\ell}} \sum_{R\subseteq\Omega} |I'||J'|r(R, R')P(R, R')T_{R'},$$

which, after adding over all $i, \ell \ge 0$, completes the proof of Theorem 30. □

As a consequence of Theorem 30, it is easy to see that the space $\mathrm{CMO}_F^p$ is well defined. In particular, we have the following:

**Corollary 62.** *We have*

$$\|f\|_{\mathrm{CMO}_F^p} \approx \sup_{\Omega}\left\{\frac{1}{|\Omega|^{2/p-1}}\sum_j\sum_k\sum_{I\times J\subseteq\Omega}|\psi_{j,k}*f(x_I,y_J)|^2|I||J|\right\}^{\frac{1}{2}},$$

*where $I\times J$ is a dyadic rectangle in $\mathbb{H}^n$ with $\ell(I)=2^{-j-N}$ and $\ell(J)=2^{-j-N}+2^{-k-N}$, and where $x_I$, $y_J$ are any fixed points in $I$, $J$, respectively.*

*Proof of Theorem 32.* We first prove $c^p\subseteq(s^p)^*$. Applying the proof in Theorem 56, set

$$s(z,u)=\left\{\sum_{I\times J}|s_{I\times J}|^2|I|^{-1}|J|^{-1}\chi_I(z)\chi_J(u)\right\}^{\frac{1}{2}}$$

and

$$\Omega_i=\{(z,u)\in\mathbb{H}^n: s(z,u)>2^i\}.$$

Let

$$\mathcal{B}_i=\left\{(I\times J):|(I\times J)\cap\Omega_i|>\tfrac{1}{2}|I\times J|,\ |(I\times J)\cap\Omega_{i+1}|\le\tfrac{1}{2}|I\times J|\right\},$$

where $I\times J$ is a dyadic rectangle in $\mathbb{H}^n$ with $\ell(I)=2^{-j-N}$ and $\ell(J)=2^{-j-N}+2^{-k-N}$. Suppose $t=\{t_{I\times J}\}\in c^p$, and write

$$\begin{aligned}\left|\sum_{I\times J}s_{I\times J}\bar t_{I\times J}\right| &= \left|\sum_i\sum_{(I\times J)\in\mathcal{B}_i}s_{I\times J}\bar t_{I\times J}\right|\\ &\le\left\{\sum_i\left[\sum_{(I\times J)\in\mathcal{B}_i}|s_{I\times J}|^2\right]^{p/2}\left[\sum_{(I\times J)\in\mathcal{B}_i}|t_{I\times J}|^2\right]^{p/2}\right\}^{1/p}\\ &\le C\|t\|_{c^p}\left\{\sum_i|\Omega_i|^{1-p/2}\left[\sum_{(I\times J)\in\mathcal{B}_i}|s_{I\times J}|^2\right]^{p/2}\right\}^{1/p},\end{aligned}\tag{7-2}$$

since if $I\times J\in\mathcal{B}_i$, then

$$\begin{aligned}I\times J\subseteq\widetilde{\Omega_i}&=\left\{(z,u):M_S(\chi_{\Omega_i})(z,u)>\tfrac{1}{2}\right\},\\ |\widetilde{\Omega_i}|&\le C|\Omega_i|,\end{aligned}$$

and $\{t_{I\times J}\}\in c^p$ yield

$$\left\{\sum_{(I\times J)\in\mathcal{B}_i}|t_{I\times J}|^2\right\}^{\frac{1}{2}}\le C\|t\|_{c^p}|\Omega_i|^{1/p-1/2}.$$

The same proof as in the claim of Theorem 56 implies

$$\sum_{(I\times J)\in\mathcal{B}_i}|s_{I\times J}|^2\le C2^{2i}|\Omega_i|.$$

Substituting the above term back into the last term in (7-2) gives $c^p\subseteq(s^p)^*$.

The proof of the converse is simple and is similar to the one given in [Frazier and Jawerth 1990] for $p=1$ in the one-parameter setting on $R^n$. If $\ell\in(s^p)^*$, then it is clear that $\ell(s)=\sum_{I\times J}s_{I\times J}\bar t_{I\times J}$ for some $t = \{t_{I\times J}\}$. Now fix an open set $\Omega \subset \mathbb{H}^n$ and let $S$ be the sequence space of all $s = \{s_{I\times J}\}$ such that $I \times J \subseteq \Omega$. Finally, let $\mu$ be a measure on $S$ so that the $\mu$-measure of the "point" $I \times J$ is $1/|\Omega|^{2/p-1}$. Then,

$$\left\{\frac{1}{|\Omega|^{2/p-1}} \sum_{I\times J\subseteq\Omega} |t_{I\times J}|^2\right\}^{\frac{1}{2}} = \|t_{I\times J}\|_{\ell^2(S,d\mu)}$$

$$= \sup_{\|s\|_{\ell^2(S,d\mu)}\le 1} \left|\frac{1}{|\Omega|^{2/p-1}} \sum_{I\times J\subseteq\Omega} s_{I\times J}\bar{t}_{I\times J}\right|$$

$$\le \|t\|_{(s^p)^*} \sup_{\|s\|_{\ell^2(S,d\mu)}\le 1} \left\| s_{I\times J} \frac{1}{|\Omega|^{2/p-1}}\right\|_{s^p}.$$

By Hölder's inequality,

$$\left\| s_{I\times J} \frac{1}{|\Omega|^{2/p-1}}\right\|_{s^p} = \frac{1}{|\Omega|^{2/p-1}} \left\{\int_\Omega \left(\sum_{I\times J\subseteq\Omega} |s_{I\times J}|^2 |I\times J|^{-1}\chi_I(x)\chi_J(y)\right)^{p/2} dz\,du\right\}^{1/p}$$

$$\le \left\{\frac{1}{|\Omega|^{2/p-1}} \int_\Omega \sum_{I\times J\subseteq\Omega} |s_{I\times J}|^2 |I\times J|^{-1}\chi_I(x)\chi_J(y)\,dz\,du\right\}^{1/2} = \|s\|_{\ell^2(S,d\mu)} \le 1,$$

which shows $\|t\|_{c^p} \le \|t\|_{(s^p)^*}$. □

In order to use Theorem 32 to obtain Theorem 33, we introduce a map $S$ which takes $f \in (\mathcal{M}_{\text{flag}}^{M+\delta})'$ to the sequence of coefficients

$$Sf \equiv \{s_{I\times J}\} = \left\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\psi_{j,k} * f(x_I, y_J)\right\},$$

where $I \times J$ is a dyadic rectangle in $\mathbb{H}^n$ with $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-j-N} + 2^{-k-N}$, and where $x_I, y_J$ are any fixed points in $I, J$, respectively. For any sequence $s = \{s_{I\times J}\}$, we define a map $T$ which takes $s$ to

$$T(s) = \sum_j \sum_k \sum_J \sum_I |I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\tilde{\psi}_{j,k}(z,u)s_{I\times J},$$

where the $\tilde{\psi}_{j,k}$ are as in (3-1).

The following result together with Theorem 32 will give Theorem 33.

**Theorem 63.** *The maps $S : H^p_{\text{flag}} \to s^p$ and $S : \text{CMO}^p_{\text{flag}} \to c^p$, as well as the maps $T : s^p \to H^p_{\text{flag}}$ and $T : c^p \to \text{CMO}^p_{\text{flag}}$, are bounded. Moreover, $T \circ S$ is the identity on both $H^p_{\text{flag}}$ and $\text{CMO}^p_{\text{flag}}$.*

*Proof.* The boundedness of $S$ on $H^p_{\text{flag}}$ and $\text{CMO}^p_{\text{flag}}$ follows directly from the Plancherel–Pólya inequalities, Theorems 19 and 30. The boundedness of $T$ also follows from the arguments in Theorems 19 and 30. Indeed, to see that $T$ is bounded from $s^p$ to $H^p_{\text{flag}}$, let $s = \{s_{I\times J}\}$. Then, by Proposition 54,

$$\|T(s)\|_{H^p_{\text{flag}}} \le C\left\|\left\{\sum_j\sum_k\sum_J\sum_I |\psi_{j,k} * T(s)(z,u)|^2\chi_I(x)\chi_J(y)\right\}^{\frac{1}{2}}\right\|_p.$$

By adapting an argument similar to that in the proof of Theorem 19, we have, for some $0 < r < p$,

$$
\begin{aligned}
&|\psi_{j,k} * T(s)(z,u)\chi_I(x)\chi_J(y)|^2 \\
&\quad = \Big| \sum_{j',k'} \sum_{I',J'} |I'||I'| \psi_{j,k} * \tilde{\psi}_{j',k'}(\,\cdot\,,\,\cdot\,)(z,u) s_{I'\times J'} |I'|^{-\frac{1}{2}} |J'|^{-\frac{1}{2}} \chi_I(x)\chi_J(y) \Big|^2 \\
&\quad \le C \sum_{k\wedge k' \le j \wedge j'} 2^{-|j-j'|K} 2^{-|k-k'|K} \Big\{ M_S\Big( \sum_{I',J'} |s_{I'\times J'}| |I'|^{-1} |J'|^{-1} \chi_{J'} \chi_{I'} \Big)^r \Big\}^{2/r} (z,u)\chi_I(x)\chi_J(y) \\
&\qquad\quad + \sum_{k\wedge k' > j\wedge j'} 2^{-|j-j'|K} 2^{-|k-k'|K} \Big\{ M\Big( \sum_{I',J'} |s_{I'\times J'}| |I'|^{-1} |J'|^{-1} \chi_{J'} \chi_{I'} \Big)^r \Big\}^{2/r} (z,u)\chi_I(x)\chi_J(y).
\end{aligned}
$$

Repeating the argument in Theorem 19 gives the boundedness of $T$ from $s^p$ to $H^p_{\text{flag}}$. A similar adaptation of the argument in the proof of Theorem 30 applies to yield the boundedness of $T$ from $c^p$ to $\text{CMO}^p_{\text{flag}}$. We leave the details to the reader. The discrete Calderón reproducing formula and Theorems 17 and 30 show that $T \circ S$ is the identity on both $H^p_{\text{flag}}$ and $\text{CMO}^p_{\text{flag}}$. □

We are now ready to give the proofs of Theorems 33 and 34.

*Proof of Theorem 33.* If $f \in \mathcal{M}^{M+\delta}_{\text{flag}}$ and $g \in \text{CMO}^p_{\text{flag}}$, let $\ell_g = \langle f, g\rangle$. Then the discrete Calderón reproducing formula and Theorems 30 and 32 imply

$$|\ell_g| = |\langle f, g\rangle| = \Big| \sum_{R=I\times J} |I||J| \psi_R * f(x_I, y_J) \tilde{\psi}_R(g)(x_I, y_J) \Big| \le C \|f\|_{H^p_{\text{flag}}} \|g\|_{\text{CMO}^p_F}.$$

Because $\mathcal{M}^{M+\delta}_{\text{flag}}$ is dense in $H^p_{\text{flag}}$, this shows that the map $\ell_g = \langle f, g\rangle$, defined initially for $f \in \mathcal{M}^{M+\delta}_{\text{flag}}$, can be extended to a continuous linear functional on $H^p_{\text{flag}}$ with $\|\ell_g\| \le C\|g\|_{\text{CMO}^p_{\text{flag}}}$.

Conversely, let $\ell \in (H^p_{\text{flag}})^*$ and set $\ell_1 = \ell \circ T$, where $T$ is defined as in Theorem 32. Then, by Theorem 32, $\ell_1 \in (s^p)^*$, so by Theorem 30, there exists $t = \{t_{I\times J}\}$ such that $\ell_1(s) = \sum_{I\times J} s_{I\times J} \bar{t}_{I\times J}$ for all $s = \{s_{I\times J}\}$, and where

$$\|t\|_{c^p} \approx \|\ell_1\| \le C\|\ell\|,$$

because $T$ is bounded. Again by Theorem 32, $\ell = \ell \circ T \circ S = \ell_1 \circ S$. Hence, with

$$f \in \mathcal{M}^{M+\delta}_{\text{flag}} \quad \text{and} \quad g = \sum_{I\times J} t_{I\times J} \psi_R((z,u)\circ(x_I, y_J)^{-1}),$$

and where without loss the generality we may assume that $\psi$ is a radial function, we have

$$\ell(f) = \ell_1(S(f)) = \langle S(f), t\rangle = \langle f, g\rangle.$$

This proves $\ell = \ell_g$, and by Theorem 32 we have

$$\|g\|_{\text{CMO}^p_{\text{flag}}} \le C\|t\|_{c^p} \le C\|\ell_g\|. \qquad □$$

*Proof of Theorem 34.* As mentioned earlier, $H^1_{\text{flag}}$ is a subspace of $L^1$. By the duality of $H^1_{\text{flag}}$ and $\text{BMO}_{\text{flag}}$, we now conclude that $L^\infty$ is a subspace of $\text{BMO}_{\text{flag}}$, and from the boundedness of flag singular integrals on $H^1_{\text{flag}}$, we get that flag singular integrals are bounded on $\text{BMO}_{\text{flag}}$ and also from $L^\infty$ to $\text{BMO}_{\text{flag}}$. □

## 8. Calderón–Zygmund decomposition and interpolation decomposition

In this section we derive a Calderón–Zygmund decomposition using functions in flag Hardy spaces. As an application, we prove an interpolation theorem for the spaces $H^p_{\text{flag}}(\mathbb{H}^n)$.

We first recall that Chang and Fefferman [1982] established the following Calderón–Zygmund decomposition on the pure product domain $\mathbb{R}^2_+ \times \mathbb{R}^2_+$.

**Lemma 64** (Calderón–Zygmund lemma). *Let $\alpha > 0$ be given and $f \in L^p(\mathbb{R}^2)$, $1 < p < 2$. Then we may write $f = g + b$, where $g \in L^2(\mathbb{R}^2)$ and $b \in H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ with $\|g\|_2^2 \le \alpha^{2-p}\|f\|_p^p$ and $\|b\|_{H^1(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \le C\alpha^{1-p}\|f\|_p^p$, where c is an absolute constant.*

We now prove the Calderón–Zygmund decomposition in the setting of flag Hardy spaces on the Heisenberg group.

*Proof of Theorem 35.* We first assume $f \in L^2(\mathbb{H}^n) \cap H^p_{\text{flag}}(\mathbb{H}^n)$. Let $\alpha > 0$ and

$$\Omega_\ell = \{(z,u) \in \mathbb{H}^n : S(f)(z,u) > \alpha 2^\ell\},$$

where, as in Corollary 60,

$$S(f)(z,u) = \left\{ \sum_{j,k} \sum_{I,J} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}}.$$

It was shown in Corollary 60 that for $f \in L^2(\mathbb{H}^n) \cap H^p_{\text{flag}}(\mathbb{H}^n)$, we have $\|f\|_{H^p_{\text{flag}}} \approx \|S(f)\|_p$.

In the following, we denote dyadic rectangles in $\mathbb{H}^n$ by $R = I \times J$ with $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-j-N} + 2^{-k-N}$, where $j, k$ are integers and $N$ is sufficiently large. Let

$$\mathcal{R}_0 = \left\{ R = I \times J : |R \cap \Omega_0| < \tfrac{1}{2}|R| \right\}$$

and, for $\ell \ge 1$,

$$\mathcal{R}_\ell = \left\{ R = I \times J : |R \cap \Omega_{\ell-1}| \ge \tfrac{1}{2}|R| \text{ but } |R \cap \Omega_\ell| < \tfrac{1}{2}|R| \right\}.$$

By the discrete Calderón reproducing formula in Theorem 57,

$$\begin{aligned} f(z,u) &= \sum_{j,k} \sum_{I,J} |I||J| \tilde{\phi}_{jk}((z,u) \circ (x_I, y_J)^{-1}) \phi_{jk} * (T_N^{-1}(f))(x_I, y_J) \\ &= \sum_{\ell \ge 1} \sum_{I \times J \in \mathcal{R}_\ell} |I||J| \tilde{\phi}_{jk}((z,u) \circ (x_I, y_J)^{-1}) \phi_{jk} * (T_N^{-1}(f))(x_I, y_J) \\ &\qquad\qquad + \sum_{I \times J \in \mathcal{R}_0} |I||J| \tilde{\phi}_{jk}((z,u) \circ (x_I, y_J)^{-1}) \phi_{jk} * (T_N^{-1}(f))(x_I, y_J) \\ &= b(z,u) + g(z,u) \end{aligned}$$

When $p_1 > 1$, using a duality argument it is easy to show

$$\|g\|_{p_1} \le C \left\| \left\{ \sum_{R = I \times J \in \mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right\|_{p_1}.$$

Next, we estimate $\|g\|_{H^{p_1}_{\text{flag}}}$ when $0 < p_1 \le 1$. Clearly, the duality argument will not work here. Nevertheless, we can estimate the $H^{p_1}_{\text{flag}}$ norm directly by using the discrete Calderón reproducing formula in Theorem 57. To this end, we note that

$$\|g\|_{H^{p_1}_{\text{flag}}} \le \Big\| \Big\{ \sum_{j',k'} \sum_{I',J'} |(\psi_{j'k'} * g)(x_{I'}, y_{J'})|^2 \chi_{I'}(z)\chi_{J'}(u) \Big\}^{\frac{1}{2}} \Big\|_{L^{p_1}}.$$

Since

$$(\psi_{j',k'} * g)(x_{I'}, y_{J'}) = \sum_{I\times J\in\mathcal{R}_0} |I||J|(\psi_{j'k'} * \tilde{\phi}_{jk})((x_{I'}, y_{J'}) \circ (x_I, y_J)^{-1})\phi_{jk} * (T_N^{-1}(f))(x_I, y_J),$$

we can repeat the argument in the proof of Theorem 56 to obtain

$$\Big\| \Big\{ \sum_{j',k'} \sum_{I',J'} |(\psi_{j'k'}*g)(x_{I'}, y_{J'})|^2 \chi_{I'}(z)\chi_{J'}(u) \Big\}^{\frac{1}{2}} \Big\|_{L^{p_1}} \le C \Big\| \Big\{ \sum_{R=I\times J\in\mathcal{R}_0} |\phi_{jk}*(T_N^{-1}(f))(x_I, y_J)|^2 \chi_I\chi_J \Big\}^{\frac{1}{2}} \Big\|_{p_1}.$$

This shows that for all $0 < p_1 < \infty$,

$$\|g\|_{H^{p_1}_{\text{flag}}} \le C \Big\| \Big\{ \sum_{R=I\times J\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I\chi_J \Big\}^{\frac{1}{2}} \Big\|_{p_1}.$$

**Claim 65.** *We have*

$$\int_{S(f)(z,u)\le\alpha} S^{p_1}(f)(z,u)\,dz\,du \ge C \Big\| \Big\{ \sum_{R=I\times J\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I\chi_J \Big\}^{\frac{1}{2}} \Big\|_{p_1}.$$

This claim implies

$$\|g\|_{p_1} \le C \int_{S(f)(z,u)\le\alpha} S^{p_1}(f)(z,u)\,dz\,du \le C\alpha^{p_1-p} \int_{S(f)(z,u)\le\alpha} S^{p}(f)(z,u)\,dz\,du \le C\alpha^{p_1-p}\|f\|^p_{H^p_{\text{flag}}(\mathbb{H}^n)}.$$

To prove Claim 65, we let $R = I \times J \in \mathcal{R}_0$. Choose $0 < q < p_1$ and note that

$$\begin{aligned}
&\int_{S(f)(z,u)\le\alpha} S^{p_1}(f)(z,u)\,dz\,du \\
&\quad = \int_{S(f)(z,u)\le\alpha} \Big\{ \sum_{j,k} \sum_{I,J} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I(x)\chi_J(y) \Big\}^{p_1/2} dz\,du \\
&\quad \ge C \int_{\Omega_0^c} \Big\{ \sum_{R\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_I\chi_J \Big\}^{p_1/2} dz\,du \\
&\quad = C \int_{\mathbb{H}^n} \Big\{ \sum_{R\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_{R\cap\Omega_0^c}(z,u) \Big\}^{p_1/2} dz\,du \\
&\quad \ge C \int_{\mathbb{H}^n} \Big\{ \Big\{ \sum_{R\in\mathcal{R}_0} (M_S(|\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^q \chi_{R\cap\Omega_0^c})(z,u))^{2/q} \Big\}^{q/2} \Big\}^{p_1/q} dz\,du \\
&\quad \ge C \int_{\mathbb{H}^n} \Big\{ \sum_{R\in\mathcal{R}_0} |\phi_{jk} * (T_N^{-1}(f))(x_I, y_J)|^2 \chi_R(z,u) \Big\}^{p_1/2} dz\,du.
\end{aligned}$$

In the last inequality above we have used the fact that $|\Omega_0^c \cap (I \times J)| \geq \frac{1}{2}|I \times J|$ for $I \times J \in \mathcal{R}_0$, and thus

$$\chi_R(z,u) \leq 2^{1/q} M_S(\chi_{R\cap\Omega_0^c})^{1/q}(z,u).$$

In the second-to-last inequality above we have used the vector-valued Fefferman–Stein inequality for the strong maximal function

$$\Big\| \Big( \sum_{k=1}^{\infty} (M_S f_k)^r \Big)^{1/r} \Big\|_p \leq C \Big\| \Big( \sum_{k=1}^{\infty} |f_k|^r \Big)^{1/r} \Big\|_p,$$

with the exponents $r = 2/q > 1$ and $p = p_1/q > 1$. Thus Claim 65 follows.

We now recall that $\widetilde{\Omega}_\ell = \big\{(z,u) \in \mathbb{H}^n : M_S(\chi_{\Omega_\ell}) > \frac{1}{2}\big\}$.

**Claim 66.** *For $p_2 \leq 1$,*

$$\Big\| \sum_{I\times J\in\mathcal{R}_\ell} |I||J| \widetilde{\phi}_{jk}((x,y)\circ(x_I,y_J)^{-1}) \phi_{jk} * (T_N^{-1} f)(x_I,y_J) \Big\|_{H^{p_2}_{\text{flag}}}^{p_2} \leq C(2^\ell\alpha)^{p_2} |\widetilde{\Omega_{\ell-1}}|.$$

Claim 66 implies

$$\begin{aligned} \|b\|_{H^{p_2}_{\text{flag}}}^{p_2} &\leq \sum_{\ell\geq 1} (2^\ell\alpha)^{p_2} |\widetilde{\Omega_{\ell-1}}| \\ &\leq C \sum_{\ell\geq 1} (2^\ell\alpha)^{p_2} |\Omega_{\ell-1}| \\ &\leq C \int_{S(f)(z,u)>\alpha} S^{p_2} f(z,u)\, dz\, du \\ &\leq C\alpha^{p_2-p} \int_{S(f)(z,u)>\alpha} S^{p} f(z,u)\, dz\, du \leq C\alpha^{p_2-p} \|f\|_{H^p_{\text{flag}}}^p. \end{aligned}$$

To prove Claim 66, we again have

$$\begin{aligned} &\Big\| \sum_{I\times J\in\mathcal{R}_\ell} |I||J| \widetilde{\phi}_{jk}((x,y)\circ(x_I,y_J)^{-1}) \phi_{jk} * (T_N^{-1} f)(x_I,y_J) \Big\|_{H^{p_2}_{\text{flag}}}^{p_2} \\ &\quad \leq C \Big\| \Big\{ \sum_{j'k'} \sum_{I',J'} \Big| \sum_{I\times J\in\mathcal{R}_\ell} |I||J| (\psi_{j'k'} * \tilde{\phi}_{jk})((x_{I'},y_{J'})\circ(x_I,y_J)^{-1}) \phi_{jk} * (T_N^{-1} f)(x_I,y_J) \Big|^2 \Big\}^{\frac{1}{2}} \Big\|_{L^{p_2}} \\ &\quad \leq C \Big\| \Big\{ \sum_{R=I\times J\in\mathcal{R}_\ell} |\phi_{jk} * (T_N^{-1} f)(x_I,y_J)|^2 \chi_I \chi_J \Big\}^{\frac{1}{2}} \Big\|_{p_2}, \end{aligned}$$

where we can use an argument similar to that in the proof of Theorem 56 to prove the last inequality.

However,

$$\begin{aligned}\sum_{\ell=1}^{\infty}(2^{\ell}\alpha)^{p_2}|\widetilde{\Omega}_{\ell-1}| &\geq \int_{\widetilde{\Omega}_{\ell-1}\setminus\Omega_\ell} S(f)^{p_2}(z,u)\,dz\,du\\ &= \int_{\widetilde{\Omega}_{\ell-1}\setminus\Omega_\ell}\Big\{\sum_{j,k}\sum_{I,J}|\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2\chi_I(z)\chi_J(u)\Big\}^{p_2/2}dz\,du\\ &= \int_{\mathbb{H}^n}\Big\{\sum_{j,k}\sum_{I,J}|\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\setminus\Omega_\ell)}(z,u)\Big\}^{p_2/2}dz\,du\\ &\geq \int_{\mathbb{H}^n}\Big\{\sum_{I\times J\in\mathcal{R}_\ell}|\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\setminus\Omega_\ell)}(z,u)\Big\}^{p_2/2}dz\,du\\ &\geq \int_{\mathbb{H}^n}\Big\{\sum_{I\times J\in\mathcal{R}_\ell}|\phi_{jk}*(T_N^{-1}(f))(x_I,y_J)|^2\chi_I(z)\chi_J(u)\Big\}^{p_2/2}dz\,du.\end{aligned}$$

In the above string of inequalities, we have used the fact that, for $R\in\mathcal{R}_\ell$, we have

$$|R\cap\Omega_{\ell-1}|>\tfrac{1}{2}|R| \quad\text{and}\quad |R\cap\Omega_\ell|\leq\tfrac{1}{2}|R|,$$

and, consequently, $R\subset\widetilde{\Omega}_{\ell-1}$. Therefore

$$|R\cap(\widetilde{\Omega}_{\ell-1}\setminus\Omega_\ell)|>\tfrac{1}{2}|R|.$$

Thus the same argument applies here to conclude the last inequality above. Finally, since

$$L^2(\mathbb{H}^n)\cap H^p_{\text{flag}}(\mathbb{H}^n)$$

is dense in $H^p_{\text{flag}}(\mathbb{H}^n)$, Theorem 35 is proved. □

We are now ready to prove the interpolation theorem on Hardy spaces $H^p_{\text{flag}}$ for all $0<p<\infty$.

*Proof of Theorem 36.* Suppose that $T$ is bounded from $H^{p_2}_{\text{flag}}$ to $L^{p_2}$ and from $H^{p_1}_{\text{flag}}$ to $L^{p_1}$. For any given $\lambda>0$ and $f\in H^p_{\text{flag}}$, by the Calderón–Zygmund decomposition,

$$f(z,u)=g(z,u)+b(z,u)$$

with

$$\|g\|^{p_1}_{H^{p_1}_{\text{flag}}}\leq C\lambda^{p_1-p}\|f\|^p_{H^p_{\text{flag}}} \quad\text{and}\quad \|b\|^{p_2}_{H^{p_2}_{\text{flag}}}\leq C\lambda^{p_2-p}\|f\|^p_{H^p_{\text{flag}}}.$$

Moreover, we have proved the estimates

$$\|g\|^{p_1}_{H^{p_1}_{\text{flag}}}\leq C\int_{S(f)(z,u)\leq\alpha}S(f)^{p_1}(z,u)\,dz\,du \quad\text{and}\quad \|b\|^{p_2}_{H^{p_2}_{\text{flag}}}\leq C\int_{S(f)(z,u)>\alpha}S(f)^{p_2}(z,u)\,dz\,du,$$

which imply that

$$\begin{aligned}\|Tf\|_p^p &= p\int_0^\infty \alpha^{p-1}\big|\{(z,u):|Tf(z,u)|>\lambda\}\big|\,d\alpha\\ &\le p\int_0^\infty \alpha^{p-1}\Big|\Big\{(z,u):|Tg(z,u)|>\tfrac{1}{2}\lambda\Big\}\Big|\,d\alpha + p\int_0^\infty \alpha^{p-1}\Big|\Big\{(z,u):|Tb(z,u)|>\tfrac{1}{2}\lambda\Big\}\Big|\,d\alpha\\ &\le p\int_0^\infty \alpha^{p-1}\int_{S(f)(z,u)\le\alpha} S(f)^{p_1}(z,u)\,dz\,du\,d\alpha + p\int_0^\infty \alpha^{p-1}\int_{S(f)(z,u)>\alpha} S(f)^{p_2}(z,u)\,dz\,du\,d\alpha\\ &\le C\|f\|^p_{H^p_{\text{flag}}}.\end{aligned}$$

Thus,

$$\|Tf\|_p \le C\|f\|_{H^p_{\text{flag}}}$$

for any $p_2 < p < p_1$. Hence $T$ is bounded from $H^p_{\text{flag}}$ to $L^p$.

Now we prove the second assertion, that $T$ is bounded on $H^p_{\text{flag}}$ for $p_2 < p < p_1$. For any given $\lambda > 0$ and $f \in H^p_{\text{flag}}$, we have, again by the Calderón–Zygmund decomposition,

$$\begin{aligned}&\big|\{(z,u):|g(Tf)(z,u)|>\alpha\}\big|\\ &\qquad\le \Big|\Big\{(z,u):|g(Tg)(z,u)|>\tfrac{1}{2}\alpha\Big\}\Big| + \Big|\Big\{(z,u):|g(Tb)(z,u)|>\tfrac{1}{2}\alpha\Big\}\Big|\\ &\qquad\le C\alpha^{-p_1}\|Tg\|^{p_1}_{H^{p_1}_{\text{flag}}} + C\alpha^{-p_2}\|Tb\|^{p_2}_{H^{p_2}_{\text{flag}}}\\ &\qquad\le C\alpha^{-p_1}\|g\|^{p_1}_{H^{p_1}_{\text{flag}}} + C\alpha^{-p_2}\|b\|^{p_2}_{H^{p_2}_{\text{flag}}}\\ &\qquad\le C\alpha^{-p_1}\int_{S(f)(z,u)\le\alpha}(Sf)^{p_1}(z,u)\,dz\,du + C\alpha^{-p_2}\int_{S(f)(z,u)>\alpha}(Sf)^{p_2}(z,u)\,dz\,du,\end{aligned}$$

which, as above, shows that $\|Tf\|_{H^p_{\text{flag}}} \le C\|g(TF)\|_p \le C\|f\|_{H^p_{\text{flag}}}$ for any $p_2 < p < p_1$. □

## 9. A counterexample for the one-parameter Hardy space

Recall that $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is the Heisenberg group with group multiplication

$$(\zeta,t)\cdot(\eta,s) = (\zeta+\eta,\, t+s+2\,\mathrm{Im}(\zeta\cdot\bar{\eta})), \quad (\zeta,t),(\eta,s)\in\mathbb{C}^n\times\mathbb{R},$$

and that $(\eta,s)^{-1} = (-\eta,-s)$. Consider the mixed kernel $K(z,t) = K_1(z)K_2(z,t)$ for $(z,t)\in\mathbb{H}^n = \mathbb{C}^n\times\mathbb{R}$ given by

$$K_1(z) = \frac{\Omega(z)}{|z|^{2n}} \quad \text{and} \quad K_2(z,t) = \frac{1}{|z|^2+it},$$

where $\Omega$ is smooth with mean zero on the unit sphere in $\mathbb{C}^n$. We show in the subsection below that $K$ satisfies the smoothness and cancellation conditions required of a flag kernel. It then follows from [Müller et al. 1995] that there is an operator $T$ having kernel $K$ such that, for each $1 < p < \infty$,

$$\|Tf\|_{L^p(\mathbb{H}^n)} \le C_{p,n}\|f\|_{L^p(\mathbb{H}^n)}, \quad f\in L^p(\mathbb{H}^n).$$

The action of the corresponding singular integral operator $Tf = K * f$ is given by

$$\begin{aligned} Tf(\zeta,t) = K *_{\mathbb{H}^n} f(\zeta,t) &= \int_{\mathbb{H}^n} K((\zeta,t)\circ(\eta,s)^{-1}) f(\eta,s)\,d\eta\,ds \\ &= \int_{\mathbb{H}^n} f(\eta,s) K(\zeta-\eta, t-s-2\operatorname{Im}(\zeta\cdot\bar\eta))\,d\eta\,ds \\ &= \int_{\mathbb{H}^n} f(\eta,s)\frac{\Omega(\zeta-\eta)}{|\zeta-\eta|^{2n}}\frac{1}{|\zeta-\eta|^2+i(t-s-2\operatorname{Im}(\zeta\cdot\bar\eta))}\,d\eta\,ds. \end{aligned}$$

**Theorem 67.** *There is a smooth function $\Omega$ with mean zero on the unit sphere in $\mathbb{C}^n$ such that there is* no *operator $T$ having kernel $K$ that is bounded from $H^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$.*

To prove the theorem, we fix $f(z,u) = \psi(z)\varphi(u)$, where

(1) $\psi$ is smooth with support in the unit ball of $\mathbb{C}^n$,

(2) $\varphi$ is smooth with support in $(-1, 1)$,

(3) $\int_{\mathbb{C}^n} \psi(z)\,dz = 0$ and $\int_{\mathbb{R}} \varphi(u)\,du = 1$.

Such a function $f$ is clearly in $H^1(\mathbb{H}^n)$ since $f$ is smooth, compactly supported, and has mean zero:

$$\int_{\mathbb{H}^n} f(z,u)\,dz\,du = \int_{\mathbb{R}}\left\{\int_{\mathbb{C}^n}\psi(z)\,dz\right\}\varphi(u)\,du = \int_{\mathbb{R}}\{0\}\varphi(u)\,du = 0.$$

We next show that $T$ fails to be bounded from $H^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$, and then that $T$ is a flag singular integral.

**9.1.** ***Failure of boundedness of T.*** For

$$\zeta \in B((100,\mathbf{0}),0) = \{(\zeta_1,\zeta') \in \mathbb{R}\times\mathbb{C}^{n-1} : (\zeta_1-100)^2+|\zeta'|^2 < 1\}, \quad |t| > 10^6,$$

we have

$$|Tf(\zeta,t)| \approx \int \psi(\eta)\varphi(s)\frac{\Omega(\zeta-\eta)}{|\zeta|^{2n}}\frac{1}{|\zeta|^2+i(t-2|\zeta|^2)}\,d\eta\,ds \approx \frac{1}{|\zeta|^{2n}|t|},$$

since, for $\zeta \in B((100,\mathbf{0}),0)$, we have

$$\left|\int \psi(\eta)\Omega(\zeta-\eta)\,d\eta\right| \geq c > 0,$$

for an appropriately chosen $\Omega$ with mean zero on the sphere. The point is that both functions $\psi$ and $\Omega$ have mean zero on their respective domains, but the product can destroy enough of the cancellation. For example, when $n = 1$, we can take

$$\Omega(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$
$$\psi(x,y) = y\psi_1(x)\psi_2(y),$$

where $\psi_i$ is an even function identically one on $\big(-1/2, 1/2\big)$ and supported in $(-1/\sqrt{2}, 1/\sqrt{2})$. Then, for

$$\zeta = (100+\nu,\omega), \quad |\nu|^2+|\omega|^2 \leq 1,$$

we have

$$\begin{aligned}\int \psi(\eta)\Omega(\zeta-\eta)\,d\eta &= \int y\psi_1(x)\psi_2(y)\Omega(100+\nu-x,\omega-y)\\ &= \int y\psi_1(x)\psi_2(y)\frac{\omega-y}{\sqrt{(100+\nu-x)^2+(\omega-y)^2}}\\ &= \omega\int \frac{y\psi_1(x)\psi_2(y)}{\sqrt{(100+\nu-x)^2+(\omega-y)^2}} - \int \frac{y^2\psi_1(x)\psi_2(y)}{\sqrt{(100+\nu-x)^2+(\omega-y)^2}}\\ &\approx -\frac{1}{100}.\end{aligned}$$

We conclude from the above that

$$\int_{\mathbb{H}^n} |Tf(\zeta,t)|\,d\zeta\,dt \gtrsim \int_{\{\zeta\in B((100,\mathbf{0}),0)\text{ and }|t|>10^6\}} \frac{1}{|\zeta|^{2n}|t|}\,d\zeta\,dt = \infty.$$

**9.2.** ***T is a flag singular integral.*** Let $K$ be the kernel

$$K(z,t) = \frac{\Omega(z)}{|z|^{2n}}\frac{1}{|z|^2+it}, \qquad (z,t)\in\mathbb{H}^n.$$

In order to show that $K$ is a flag kernel, we must establish the following smoothness and cancellation conditions.

(1) (differential inequalities) For any multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_m)$,

$$|\partial_z^\alpha\partial_u^\beta K(z,u)| \le C_{\alpha,\beta}|z|^{-2n-|\alpha|}\cdot(|z|^2+|u|)^{-1-|\beta|}$$

for all $(z,u)\in\mathbb{H}^n$ with $z\ne 0$.

(2) (cancellation condition) For every multi-index $\alpha$, every normalized bump function $\phi_1$ on $\mathbb{R}$, and every $\delta>0$,

$$\left|\int_{\mathbb{R}} \partial_z^\alpha K(z,u)\phi_1(\delta u)\,du\right| \le C_\alpha|z|^{-2n-|\alpha|};$$

for every multi-index $\beta$, every normalized bump function $\phi_2$ on $\mathbb{C}^n$, and every $\delta>0$,

$$\left|\int_{\mathbb{C}^n} \partial_u^\beta K(z,u)\phi_2(\delta z)\,dz\right| \le C_\gamma|u|^{-1-|\beta|};$$

and for every normalized bump function $\phi_3$ on $\mathbb{H}^n$ and every $\delta_1>0$ and $\delta_2>0$,

$$\left|\int_{\mathbb{H}^n} K(z,u)\phi_3(\delta_1 z,\delta_2 u)\,dz\,du\right| \le C.$$

The differential inequalities in (1) follow immediately from the definition of $K$.

The first cancellation condition in (2) exploits the fact that $t$ is an odd function. For convenience we assume $\alpha = 0$. We then have

$$\begin{aligned}\left|\int_{\mathbb{R}} K(z,t)\phi_1(\delta t)\,dt\right| &= \left|\int_{\mathbb{R}} \frac{\Omega(z)}{|z|^{2n}}\left\{\frac{|z|^2}{|z|^4+t^2} - \frac{it}{|z|^4+t^2}\right\}\phi_1(\delta t)\,dt\right| \\ &\le \int_{\mathbb{R}} \frac{1}{|z|^{2n}}\frac{|z|^2}{|z|^4+t^2}|\phi_1(\delta t)|\,dt + \left|\int_{\mathbb{R}} \frac{\Omega(z)}{|z|^{2n}}\frac{it}{|z|^4+t^2}\{\phi_1(\delta t)-\phi_1(0)\}\,dt\right| \\ &\lesssim \frac{1}{|z|^{2n-2}}\int_0^\infty \frac{1}{|z|^4+t^2}\,dt + \frac{1}{|z|^{2n}}\int_0^{1/\delta}\frac{\delta t^2}{|z|^4+t^2}\,dt.\end{aligned}$$

Now

$$\frac{1}{|z|^{2n-2}}\int_0^\infty \frac{1}{|z|^4+t^2}\,dt \lesssim \frac{1}{|z|^{2n-2}}\left(\int_0^{|z|^2}\frac{1}{|z|^4}\,dt + \int_{|z|^2}^\infty \frac{1}{t^2}\,dt\right) \lesssim \frac{1}{|z|^{2n}},$$

and, for $|z|^2 \le 1/\delta$, we have

$$\int_0^{1/\delta}\frac{\delta t^2}{|z|^4+t^2}\,dt \lesssim \int_0^{|z|^2}\frac{\delta t^2}{|z|^4}\,dt + \int_{|z|^2}^{1/\delta}\frac{\delta t^2}{t^2}\,dt \lesssim \delta\frac{|z|^6}{|z|^4} + 1 \lesssim 1,$$

while for $|z|^2 > 1/\delta$, we have

$$\int_0^{1/\delta}\frac{\delta t^2}{|z|^4+t^2}\,dt \lesssim \int_0^{1/\delta}\frac{\delta t^2}{|z|^4}\,dt \lesssim \delta\frac{(1/\delta)^3}{|z|^4} \lesssim 1.$$

Altogether we have $\left|\int_{\mathbb{R}} K(z,t)\phi_1(\delta t)\,dt\right| \lesssim |z|^{-2n}$ as required.

The second cancellation condition in (2) uses the assumption that $\Omega$ has mean zero on the sphere. For convenience we take $\beta = 0$. Then we have

$$\begin{aligned}\left|\int_{\mathbb{C}^n} K(z,t)\phi_2(\delta z)\,dz\right| &= \left|\int_{\mathbb{C}^n}\frac{\Omega(z)}{|z|^{2n}}\frac{1}{|z|^2+it}\{\phi_2(\delta z)-\phi_2(0)\}\,dz\right| \\ &\lesssim \delta\int_{\{|z|\le 1/\delta\}}\frac{1}{|z|^{2n}}\frac{1}{|z|^2+|t|}|z|\,dz \\ &\lesssim \frac{\delta}{|t|}\int_0^{1/\delta}\frac{1}{r^{2n}}r(r^{2n-1}\,dr) \approx |t|^{-1},\end{aligned}$$

as required.

The third cancellation condition in (2) is handled similarly. We have

$$\begin{aligned}&\int_{\mathbb{H}^n} K(z,t)\phi_3(\delta_1 z,\delta_2 t)\,dz\,dt \\ &\quad= \int_{\mathbb{H}^n}\frac{\Omega(z)}{|z|^{2n}}\left\{\frac{|z|^2}{|z|^4+t^2} - \frac{it}{|z|^4+t^2}\right\}\{\phi_3(\delta_1 z,\delta_2 t)-\phi_3(0,\delta_2 t)\}\,dz\,dt \\ &\quad= \int_{\mathbb{H}^n}\frac{\Omega(z)}{|z|^{2n}}\frac{|z|^2}{|z|^4+t^2}\{\phi_3(\delta_1 z,\delta_2 t)-\phi_3(0,\delta_2 t)\}\,dz\,dt \\ &\qquad - \int_{\mathbb{H}^n}\frac{\Omega(z)}{|z|^{2n}}\frac{it}{|z|^4+t^2}\{\phi_3(\delta_1 z,\delta_2 t)-\phi_3(0,\delta_2 t)-\phi_3(\delta_1 z,0)+\phi_3(0,0)\}\,dz\,dt,\end{aligned}$$

and so

$$\left|\int_{\mathbb{H}^n} K(z,t)\phi_3(\delta_1 z, \delta_2 t)\,dz\,dt\right|$$
$$\lesssim \int_{|t|\le 1/\delta_2}\int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n}}\frac{|z|^2}{|z|^4+t^2}\delta_1|z|\,dz\,dt + \int_{|t|\le 1/\delta_2}\int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n}}\frac{|t|}{|z|^4+t^2}\delta_1|z|\delta_2|t|\,dz\,dt = \mathrm{I}+\mathrm{II}.$$

Now if $1/\delta_2 \le |z|^2$, then

$$\mathrm{I} \lesssim \delta_1 \int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n-3}}\left\{\int_0^{1/\delta_2} \frac{1}{|z|^4}\,dt\right\} dz \lesssim \delta_1 \int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n-1}}\,dz \approx \delta_1 \int_0^{1/\delta_1} dr = 1,$$

while if $1/\delta_2 > |z|^2$, then

$$\mathrm{I} \lesssim \delta_1 \int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n-3}}\left\{\int_0^{|z|^2} \frac{1}{|z|^4}\,dt + \int_{|z|^2}^{1/\delta_2} \frac{1}{t^2}\,dt\right\} dz \lesssim \delta_1 \int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n-1}}\,dz \approx 1.$$

Finally, we have

$$\mathrm{II} \lesssim \delta_1 \int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n-1}}\left\{\delta_2 \int_{|t|\le 1/\delta_2} \frac{t^2}{|z|^4+t^2}\,dt\right\} dz \lesssim \delta_1 \int_{|z|\le 1/\delta_1} \frac{1}{|z|^{2n-1}}\,dz \approx 1.$$

# Part III. Appendix

Here in the appendix, we construct a *flag* dyadic decompositon of the Heisenberg group using the tiling theorem of Strichartz. See [Han et al. 2012] for an approach that generalizes to certain products of spaces of homogeneous type.

## 10. The Heisenberg grid

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with group multiplication

$$(\zeta,t)\cdot(\eta,s) = (\zeta+\eta, t+s+2\,\mathrm{Im}(\zeta\cdot\bar{\eta})), \quad (\zeta,t),(\eta,s)\in\mathbb{C}^n\times\mathbb{R}.$$

Note that $(\eta,s)^{-1} = (-\eta,-s)$. Relative to this multiplication, we define the dilation

$$\delta_\lambda(\zeta,t) = (\lambda\zeta, \lambda^2 t),$$

and its corresponding “norm” on $\mathbb{H}^n$ by

$$\rho(\zeta,t) = \sqrt[4]{|\zeta|^4+t^2}.$$

Then we define a symmetric quasimetric $d$ on $\mathbb{H}^n$ by

$$d((\zeta,t),(\eta,s)) = \rho((\zeta,t)\cdot(\eta,s)^{-1}),$$

and note that

$$d(\delta_\lambda(\zeta,t),\delta_\lambda(\eta,s)) = \lambda d((\zeta,t),(\eta,s)).$$

The center of the group $\mathbb{H}^n$ is

$$\mathscr{Z}^n = \{(\zeta, t) \in \mathbb{H}^n : \zeta = 0\},$$

which is isomorphic to the abelian group $\mathbb{R}$. The quotient group $\mathbb{Q}^n = \mathbb{H}^n / \mathscr{Z}^n$ consists of equivalence classes $[(\zeta, t)]$ such that $[(\zeta, t)] = [(\eta, s)]$ if and only if

$$(\zeta, t) \cdot (\eta, s)^{-1} \in \mathscr{Z}^n, \quad \text{that is, } \zeta = \eta.$$

Thus we may identify $\mathbb{Q}^n$ with $\mathbb{C}^n$ as abelian groups. Thus we see that $\mathbb{H}^n = \mathbb{C}^n \otimes_{\text{twist}} \mathbb{R}$ is a twisted group product of the abelian groups $\mathbb{C}^n$ and $\mathbb{R}$ with bihomomorphism $\beta(z, w) = 2\,\text{Im}(z \cdot \bar{w})$. See the appendix for a discussion of this notion of twisted group product.

Now we apply the usual dyadic decomposition to the quotient metric space $\mathbb{Q}^n = \mathbb{C}^n$ to obtain a grid of "almost balls" (which are actually cubes here)

$$\{I\}_{I \text{ dyadic}} = \{I_\alpha^j\}_{j \in \mathbb{Z} \text{ and } \alpha \in 2^j \mathbb{Z}^{2n}},$$

where $I_0^j = [0, 2^j)^{2n}$ and $I_\alpha^j = I_0^j + \alpha$ for $j \in \mathbb{Z}$ and $\alpha \in 2^j \mathbb{Z}^{2n}$, so that $\ell(I_\alpha^j) = 2^j$. By a grid of almost balls we mean that the sets $I_\alpha^j$ decompose $\mathbb{C}^n$ at each scale $2^j$, are almost balls, and are nested at differing scales; that is, there are positive constants $C_1, C_2$ and points $c_{I_\alpha^j} \in I_\alpha^j$ such that

$$\begin{aligned}
&\mathbb{C}^n = \dot{\bigcup} I_\alpha^j && j \in \mathbb{Z}, \\
&B(c_{I_\alpha^j}, C_1 2^j) \subset I_\alpha^j \subset B(c_{I_\alpha^j}, C_2 2^j) && j \in \mathbb{Z}, \alpha \in 2^j \mathbb{Z}^{2n}, \\
&I_{\alpha'}^{j'} \subset I_\alpha^j, \quad I_\alpha^j \subset I_{\alpha'}^{j'} \quad \text{or} \quad I_{\alpha'}^{j'} = I_\alpha^j.
\end{aligned} \tag{10-1}$$

Here we can take $c_I$ to be the center of the cube $I$, and $C_1 = 1/2$, $C_2 = \sqrt{2n}/2 = \sqrt{n/2}$. We also have the usual dyadic grid $\{J_\tau^k\}_{k \in \mathbb{Z} \text{ and } \tau \in 2^k \mathbb{Z}}$ for $\mathbb{R}$, where $J_0^k = [0, 2^k)$ and $I_\tau^k = I_0^k + \tau$ for $k \in \mathbb{Z}$ and $\tau \in 2^k \mathbb{Z}$.

In order to use these grids to construct a "product-like" grid for $\mathbb{H}^n$, we must take into account the twisted structure of the product $\mathbb{H}^n = \mathbb{C}^n \otimes_{\text{twist}} \mathbb{R}$. Here is our theorem on the existence of a twisted grid for $\mathbb{H}^n$.

**Theorem 68.** *There is a positive integer $m$ and positive constants $C_1, C_2$, such that, for each $j \in m\mathbb{Z}$ and*

$$(\alpha, \tau) \in K_j \equiv 2^j \mathbb{Z}^{2n} \times 2^{2j} \mathbb{Z},$$

*there are subsets $\mathscr{S}_{j,\alpha,\tau}$ of $\mathbb{H}^n$ satisfying*

$$\begin{aligned}
&\mathbb{H}^n = \dot{\bigcup_{(\alpha,\tau) \in K_j}} \mathscr{S}_{j,\alpha,\tau}, && \textit{for each } j \in m\mathbb{Z}, \\
&P_{\mathbb{C}^n} \mathscr{S}_{j,\alpha,\tau} = I_\alpha^j, && j \in m\mathbb{Z}, (\alpha, \tau) \in K_j, \\
&B_d(c_{j,\alpha,\tau}, C_1 2^j) \subset \mathscr{S}_{j,\alpha,\tau} \subset B_d(c_{j,\alpha,\tau}, C_2 2^j), && j \in m\mathbb{Z}, (\alpha, \tau) \in K_j, \\
&\mathscr{S}_{j,\alpha,\tau} \subset \mathscr{S}_{j',\alpha',\tau'}, \quad \mathscr{S}_{j',\alpha',\tau'} \subset \mathscr{S}_{j,\alpha,\tau} \quad \textit{or} \quad \mathscr{S}_{j,\alpha,\tau} \cap \mathscr{S}_{j',\alpha',\tau'} = \phi, \\
&c_{j,\alpha,\tau} = \big(P_{j,\alpha}, \tau + \tfrac{1}{2} 2^{2j}\big),
\end{aligned} \tag{10-2}$$

*where $P_{j,\alpha} = c_{I_\alpha^j}$ and $P_{\mathbb{C}^n}$ denotes orthogonal projection of $\mathbb{H}^n$ onto $\mathbb{C}^n$.*

Thus at each dyadic scale $2^j$ with $j \in m\mathbb{Z}$, we have a pairwise disjoint decomposition of $\mathbb{H}^n$ into sets $\mathcal{S}_{j,\alpha,\tau}$ that are almost Heisenberg balls of radius $2^j$. These decompositions are nested, and moreover are *product-like* in the sense that the sets $\mathcal{S}_{j,\alpha,\tau}$ project onto the usual dyadic grid in the factor $\mathbb{C}^n$, and have centers $c_{j,\alpha,\tau} = \left(P_{j,\alpha}, \tau + \frac{1}{2}2^{2j}\right)$ that for each $j$ form a product set indexed by $K_j \equiv 2^j\mathbb{Z}^{2n} \times 2^{2j}\mathbb{Z}$ and satisfy

$$|c_{j,\alpha,\tau} - c_{j,\alpha',\tau}| = 2^j \quad \text{and} \quad |c_{j,\alpha,\tau} - c_{j,\alpha,\tau'}| = 2^{2j},$$

if $\alpha$ and $\alpha'$ are neighbors in $2^j\mathbb{Z}^{2n}$ and if $\tau$ and $\tau'$ are neighbors in $2^{2j}\mathbb{Z}$.

Theorem 68 follows easily from the theory of self-similar tilings (neatly stacked over dyadic cubes) in [Strichartz 1992]. An excellent source for this material is [Tyson 2008, pp. 39–42]. See [Han et al. 2012] for more detail.

## 11. Rectangles in the Heisenberg group

Recall from Theorem 68 that at each dyadic scale $2^j$ with $j \in m\mathbb{Z}$ there is a pairwise disjoint decomposition of $\mathbb{H}^n$ into sets $\mathcal{S}_{j,\alpha,\tau}$ that are "almost Heisenberg ball" of radius $2^j$. We will refer to these sets as dyadic cubes at scale $2^j$. These decompositions are nested, and moreover are *product-like* in the sense that the cubes $\mathcal{S}_{j,\alpha,\tau}$ project onto $I_\alpha^j$ in the usual dyadic grid in the factor $\mathbb{C}^n$, and have centers $c_{j,\alpha,\tau} = \left(P_{j,\alpha}, \tau + \frac{1}{2}2^{2j}\right)$ that, for each $j$, form a product set indexed by $K_j \equiv 2^j\mathbb{Z}^{2n} \times 2^{2j}\mathbb{Z}$ and satisfy

$$|c_{j,\alpha,\tau} - c_{j,\alpha',\tau}| = 2^j \quad \text{and} \quad |c_{j,\alpha,\tau} - c_{j,\alpha,\tau'}| = 2^{2j},$$

if $\alpha$ and $\alpha'$ are neighbors in $2^j\mathbb{Z}^{2n}$ and if $\tau$ and $\tau'$ are neighbors in $2^{2j}\mathbb{Z}$.

We now define vertical and horizontal dyadic rectangles relative to this decomposition into dyadic cubes. The analogy with dyadic rectangles in the plane $\mathbb{R}^2$ that we are pursuing here is that a dyadic rectangle $I = I_1 \times I_2$ in the plane is vertical if $|I_2| \geq |I_1|$, and is horizontal if $|I_1| \geq |I_2|$ (and both if and only if $I$ is a dyadic square). If we consider the grid of dyadic cubes $\{\mathcal{S}_{j,\alpha,\tau}\}$ in $\mathbb{H}^n$ in place of the grid of dyadic squares in $\mathbb{R}^2$, we are led to the following definition.

**Definition 69.** Let $j, k \in m\mathbb{Z}$, with $j \leq k$, and let $\mathcal{S}_{j,\alpha,\tau}$ and $\mathcal{S}_{k,\beta,\upsilon}$ be dyadic cubes in $\mathbb{H}^n$ with $\mathcal{S}_{j,\alpha,\tau} \subset \mathcal{S}_{k,\beta,\upsilon}$. The set

$$\mathcal{R}(\text{ver}) = \mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{ver}) = \bigcup\{\mathcal{S}_{j,\alpha,\tau'} : \mathcal{S}_{j,\alpha,\tau'} \subset \mathcal{S}_{k,\beta,\upsilon}\}$$

will be referred to as a *vertical dyadic rectangle*, or, more precisely, the vertical dyadic rectangle in $\mathcal{S}_{k,\beta,\upsilon}$ containing $\mathcal{S}_{j,\alpha,\tau}$. We define the *base* of the rectangle $\mathcal{R}(\text{ver})$ to be the dyadic cube $I_\alpha^j$ in $\mathbb{C}^n$, and we define the *cobase* of the rectangle $\mathcal{R}(\text{ver})$ to be the dyadic interval $J_\upsilon^{2k}$ in $\mathbb{R}$. We say the rectangle $\mathcal{R}(\text{ver})$ has *width* $2^j$ and *height* $2^{2k}$. Similarly, the set

$$\mathcal{R}(\text{hor}) = \mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{hor}) = \bigcup\{\mathcal{S}_{j,\alpha',\tau} : \mathcal{S}_{j,\alpha',\tau} \subset \mathcal{S}_{k,\beta,\upsilon}\}$$

will be referred to as a *horizontal dyadic rectangle*, or, more precisely, the horizontal dyadic rectangle in $\mathcal{S}_{k,\beta,\upsilon}$ containing $\mathcal{S}_{j,\alpha,\tau}$. We define the *base* of the rectangle $\mathcal{R}(\text{hor})$ to be the dyadic cube $I_\beta^k$ in $\mathbb{C}^n$, and we define the *cobase* of the rectangle $\mathcal{R}(\text{ver})$ to be the dyadic interval $J_\tau^{2j}$ in $\mathbb{R}$. We say the rectangle $\mathcal{R}(\text{hor})$ has *width* $2^k$ and *height* $2^{2j}$.

We will usually write just $\mathcal{R}$ to denote a dyadic rectangle that is either vertical or horizontal. Note that a dyadic rectangle $\mathcal{R}$ is both vertical and horizontal if and only if $\mathcal{R}$ is a dyadic cube $\mathcal{S}_{j,\alpha,\tau}$. Finally note that $\mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{ver})$ can be thought of as a Heisenberg substitute for the Euclidean rectangle $I_\alpha^j \times J_\upsilon^{2k}$ in $\mathbb{H}^n$ with width $2^j$ and height $2^{2k}$, and that $\mathcal{R}_{\mathcal{S}_{j,\alpha,\tau}}^{\mathcal{S}_{k,\beta,\upsilon}}(\text{hor})$ can be thought of as a Heisenberg substitute for the Euclidean rectangle $I_\beta^k \times J_\tau^{2j}$ in $\mathbb{H}^n$ with width $2^k$ and height $2^{2j}$. The vertical Heisenberg rectangles are constructed by stacking Heisenberg cubes neatly on top of each other, while the horizontal Heisenberg rectangles are constructed by placing Heisenberg cubes next to each other, although the placement is far from neat.

**Remark 70.** In applications to operators with flag kernels, or more generally a semiproduct structure, it is appropriate to restrict attention to the set of *vertical* dyadic rectangles.

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