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Boundedness criterion of Journé's class of singular integrals on multiparameter Hardy spaces ☆

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Abstract

This article concerns nonconvolutional type operators (also known as Journé's type operators) associated with a multiparameter family of dilations given by $(x_1, x_2, \dots, x_m) \to (\delta_1 x_1, \delta_2 x_2, \dots, \delta_m x_m)$ where $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, \dots, x_m \in \mathbb{R}^{n_m}$ and $m \geqslant 3$. We are especially interested in the boundedness of such operators on the multiparameter Hardy spaces. This work is motivated by Pipher's result on the boundedness of these operators from the multiparameter H^p spaces to L^p spaces for $0 < p \leqslant 1$, and Journé's counterexample which shows that the number of parameter plays a crucial role in boundedness of singular integral operators on multiparameter Hardy spaces. Journé's work shows that there is a sharp difference between the situations for two and three or more parameters. We establish in this paper the necessary and sufficient conditions under which the singular integral operators in Journé's class are bounded on multiparameter H^p spaces $(0 < p \leqslant 1)$ with arbitrary number of parameters.

Keywords: Journé's class of singular integrals; Calderón–Zygmund operators; Multiparameter Hardy spaces; Almost orthogonality estimates; Atomic decompositions

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1. Introduction and statement of results

In their well-known theory of singular integral operators, Calderón and Zygmund in [2] generalized the Hilbert transform on $\mathbb{R}^1$ and obtained the $L^p, 1 < p < \infty$, boundedness of certain convolution singular integral operators on $\mathbb{R}^n$. C. Fefferman and E.M. Stein in [10] proved the H^p, $0 < p \leqslant 1$, boundedness of such convolution operators. The theory has been generalized in two ways. First, the convolution singular integral operators were replaced by non-convolution singular integral operators associated with a kernel in the following sense. Let $\mathcal{K}$ be a locally integrable function defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\}$. The non-convolution singular integral operator $T : C_0^\infty(\mathbb{R}^n) \to (C_0^\infty(\mathbb{R}^n))^*$ is a linear operator defined by $\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x)\mathcal{K}(x, y) f(y)\,dx\,dy$ for $f, g \in C_0^\infty(\mathbb{R}^n)$ with disjoint supports. Moreover, $\mathcal{K}$, the kernel of T, satisfies some size and smoothness properties analogous to those enjoyed by the Calderón–Zygmund convolution operators. Of course, in general, one cannot conclude that such an operator T is bounded on $L^2(\mathbb{R}^n)$ because Plancherel's theorem doesn't work for non-convolution operators. However, if one assumes that T is bounded on $L^2(\mathbb{R}^n)$, then the L^p, $1 < p < \infty$, boundedness follows from Calderón–Zygmund's real variable method. Moreover, using atomic decomposition and molecular theory of the classical Hardy spaces, Coifman and Weiss in [7] proved the $H^p \to L^p$ boundedness of T and the $H^p \to H^p$ boundedness of T provided some assumptions of certain vanishing moments. The L^2 boundedness of non-convolution singular integral operators was finally proved by the remarkable $T1$ theorem of David and Journé in [8].

Secondly, by taking the space $\mathbb{R}^n \times \mathbb{R}^m$ along with two-parameter family of dilations $(x, y) \to (\delta_1 x, \delta_2 y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\delta_i > 0$, $i = 1, 2$, instead of the classical one-parameter dilation, R. Fefferman and Stein studied the product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on $\mathbb{R} \times \mathbb{R}$. In [11] the $L^p, 1 < p < \infty$, boundedness of such product convolution operators is obtained. Journé in [18] introduced a non-convolution product singular integral operators and provided the $T1$ theorem. Moreover, Journé proved the $L^\infty \to BMO$ boundedness for such operators, which opened the door for proving the product H^p boundedness of operators in Journé's class. The product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ was first introduced by Gundy and Stein [13]. Chang and Fefferman [5,6] developed the theory of atomic decomposition and established the dual space of Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$, namely the product $BMO(\mathbb{R}^n \times \mathbb{R}^m)$ space. Another characterization of such product BMO space was given by Ferguson and Lacey [12]. However, the atomic decomposition of the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ given in [6] is more complicated than the classical $H^p(\mathbb{R}^n)$ and cannot be directly used to provide the $H^p \to L^p$ and $H^p \to H^p$ boundedness of operators in Journé's class. Indeed it was conjectured that the product atomic Hardy space on $\mathbb{R}^n \times \mathbb{R}^m$ could be defined by rectangle atoms. However, this conjecture was disproved by a counter-example constructed by Carleson [4]. This leads to the fact that the role of cubes in the classical atomic decomposition of $H^p(\mathbb{R}^n)$ was replaced by arbitrary open sets of finite measures in the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. It is surprising that despite of its complexity of multiparameter atoms, R. Fefferman in [9] proved the criterion of the $H^p \to L^p$ boundedness of singular integral operators in Journé's class by considering its action only on rectangle atoms. One of the key tools used in [9] is Journé's covering lemma. R. Fefferman's theorem is stated as follows.

Theorem A. *Let T be a bounded linear operator on $L^2(\mathbb{R}^{n+m})$ and $0 < p \leqslant 1$. Suppose that for any $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ rectangle atom a supported on the rectangle R we have*

$$\int_{(\gamma R)^c} \left|T(a)\right|^p dx_1\, dx_2 \leqslant c\gamma^{-\delta} \tag{1.1}$$

for some fixed $\delta > 0$ and for every $\gamma \geqslant 2$. Then T is a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^{n+m})$.

However, Theorem A, in general, cannot be extended to three or more factors without further assumptions on the nature of T as demonstrated by Journé [19]. In fact, Journé provided a counter-example in the three-parameter setting of singular integral operators such that Theorem A does not hold. Moreover, he showed in [19] that if T is a convolution operator such that (1.1) holds with some $\delta > \frac{1}{8}$, then Theorem A can be extended to three-parameter case. Journé's work shows a sharp difference between the two and more parameters settings. On the other hand, Carbery and Seeger [3] showed that Fefferman's criterion remains valid for arbitrary number of parameters with a different interpretation of rectangle atoms, namely, vector-valued rectangle atoms. Moreover, the H^p to L^p boundedness for Journé's class of singular integral operators with arbitrary number of parameters was established by J. Pipher [21] by using atomic decomposition and an extension of Journé's geometric lemma to higher dimensions. More recently, the first author and Lee, Lin and Lin [15] have established the necessary and sufficient conditions of the $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of Journé's product singular integrals in the two-parameter case. Nevertheless, the boundedness from H^p to H^p for Journé's class of singular integrals with three or more parameters still remains open. This is the main motivation for us to study the boundedness of this class of singular integrals on multiparameter Hardy spaces H^p with three or more parameters in this paper.

The main goal of this paper is to establish the necessary and sufficient conditions for Journé's class of singular integral operators to be bounded on multiparameter Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$ (Theorem 1.4). We like to remark that our results proved in this paper can be extended for more than three parameters with similar arguments.

We first recall the definitions of the multiparameter Hardy spaces H^p and H^p-atoms on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$. Given $0 < p \leqslant 1$, let

$$C^\infty_{0,0}(\mathbb{R}^n) = \left\{\psi \in C^\infty(\mathbb{R}^n)\text{: } \psi \text{ has a compact support and } \int_{\mathbb{R}^n} \psi(x)x^\alpha\, dx = 0 \right.$$
$$\left. \text{for } 0 \leqslant |\alpha| \leqslant N_{p,n}\right\},$$

where $N_{p,n}$ is a large integer depending on p and n. Let $\psi \in C^\infty_{0,0}(\mathbb{R}^n)$ satisfy the condition

$$\int_0^\infty \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t} = 1 \quad \text{for all } \xi \in \mathbb{R}^n/\{0\}. \tag{1.2}$$

For $t > 0$ and $x \in \mathbb{R}^n$, set $\psi_t(x) = t^{-n}\psi(\frac{x}{t})$. Let $\psi^i \in C^\infty_{0,0}(\mathbb{R}^{n_i})$ be supported in the unit ball of $\mathbb{R}^{n_i}$ with condition (1.2), $i = 1, 2, 3$. For $t_i > 0$ and $(x_1, x_2, x_3) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$, set

$\psi^i_{t_i}(x_i) = t_i^{-n_i}\psi(\frac{x_i}{t_i})$ and $\psi_{t_1,t_2,t_3}(x_1,x_2,x_3) = \psi^1_{t_1}(x_1)\psi^2_{t_2}(x_2)\psi^3_{t_3}(x_3)$. The product Littlewood–Paley square function of $f \in \mathcal{S}'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ is defined by

$$\mathcal{G}(f)(x_1,x_2,x_3) = \left\{\int_0^\infty\int_0^\infty\int_0^\infty \left|\psi_{t_1,t_2,t_3} * f(x_1,x_2,x_3)\right|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\right\}^{1/2}.$$

For $0 < p \leqslant 1$, the multiparameter Hardy space H^p on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$ can be defined by

$$H^p\left(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}\right) = \left\{f \in \mathcal{S}'\left(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}\right):\ \mathcal{G}(f) \in L^p\left(\mathbb{R}^{n_1+n_2+n_3}\right)\right\}$$

with the norm $\|f\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})} \approx \|\mathcal{G}(f)\|_{L^p(\mathbb{R}^{n_1+n_2+n_3})}$.

A function a defined in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$ is called an H^p atom (i.e. H^p-$(p,2)$-atom) if a is supported in an open set $\Omega \subset \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$ with finite measure and satisfies the following conditions:

(i) $\|a\|_{L^2} \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}}$;

(ii) a can be decomposed as $a(x_1,x_2,x_3) = \sum_{R\in\mathcal{M}(\Omega)} a_R(x_1,x_2,x_3)$, where a_R are supported on the double of $R = I_1\times I_2\times I_3$, I_i are dyadic cubes in $\mathbb{R}^{n_i}$, $i = 1,2,3$, and $\mathcal{M}(\Omega)$ is the collection of all maximal dyadic sub-rectangles contained in Ω and

$$\left\{\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_2^2\right\}^{\frac{1}{2}} \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}};$$

(iii) $\int_{2I_1} a_R(x_1,x_2,x_3)x_1^\alpha\,dx_1 = 0$ for all $x_2 \in \mathbb{R}^{n_2}$, $x_3 \in \mathbb{R}^{n_3}$ and $0 \leqslant |\alpha| \leqslant N_{p,n_1}$,
$\int_{2I_2} a_R(x_1,x_2,x_3)x_2^\beta\,dx_2 = 0$ for all $x_1 \in \mathbb{R}^{n_1}$, $x_3 \in \mathbb{R}^{n_3}$ and $0 \leqslant |\beta| \leqslant N_{p,n_2}$,
$\int_{2I_3} a_R(x_1,x_2,x_3)x_3^\mu\,dx_3 = 0$ for all $x_1 \in \mathbb{R}^{n_2}$, $x_2 \in \mathbb{R}^{n_2}$ and $0 \leqslant |\mu| \leqslant N_{p,n_3}$.
We remark in passing that multiparameter (p,q)-atoms for any $1 < q < \infty$ has been derived in [16].

Now we recall the classical definition of the Calderón–Zygmund operator on $\mathbb{R}^n$.

Definition 1.1. A continuous complex-valued function $\mathcal{K}$ defined on $D = \mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y):\ x = y\}$ is called a one-parameter Calderón–Zygmund kernel with regularity exponent $\epsilon > 0$ $(0 < \epsilon \leqslant 1)$ if there exists a constant $C > 0$ such that

(i) $|\mathcal{K}(x,y)| \leqslant C|x-y|^{-n}$;

(ii) $|\mathcal{K}(x,y) - \mathcal{K}(x',y)| \leqslant C|x-x'|^\epsilon|x-y|^{-n-\epsilon}$ if $|x-x'| \leqslant \frac{1}{2}|x-y|$;

(iii) $|\mathcal{K}(x,y) - \mathcal{K}(x,y')| \leqslant C|y-y'|^\epsilon|x-y|^{-n-\epsilon}$ if $|y-y'| \leqslant \frac{1}{2}|x-y|$.

The smallest such constant C is denoted by $\|\mathcal{K}\|_{CZ}$.

We call an operator T is a Calderón–Zygmund operator if T is a singular integral operator associated with a Calderón–Zygmund kernel $\mathcal{K}(x,y)$ given by

$$\langle T(f), g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x)\mathcal{K}(x,y)f(y)\,dy\,dx$$

for all $f, g \in C_0^\infty(\mathbb{R}^n)$ with disjoint supports and T extends to be a bounded operator on $L^2(\mathbb{R}^n)$. The norm of the Calderón–Zygmund operator T is defined by $\|T\|_{CZ} = \|T\|_{L^2->L^2} + \|\mathcal{K}\|_{CZ}$.

We remark that if T is bounded on $L^2(\mathbb{R}^n)$ then T is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $\frac{n}{n+\epsilon} < p \leqslant 1$, provided only the condition (iii), that is, the regularity with respect to the variable y.

Next, we recall the definition of the singular integral operator of two parameters due to Journé [18].

Definition 1.2. A singular integral operator T is said to be in Journé's class on $\mathbb{R}^n \times \mathbb{R}^m$ with regularity exponent $\epsilon > 0$ if

$$T(f)(x,y) = \int_{\mathbb{R}^n\times\mathbb{R}^m} \mathcal{K}(x,y,u,v) f(u,v)\,du\,dv,$$

where the kernel $\mathcal{K}$ satisfies the following conditions: for each fixed $x_1, y_1 \in \mathbb{R}^n$, set $\widetilde{\mathcal{K}}^1(x_1,y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^m$ with the kernel $\widetilde{\mathcal{K}}^1(x_1,y_1)(x_2,y_2) = \mathcal{K}(x_1,x_2,y_1,y_2)$, and similarly, $\widetilde{\mathcal{K}}^2(x_2,y_2)(x_1,y_1) = \mathcal{K}(x_1,x_2,y_1,y_2)$ for each fixed $x_2, y_2 \in \mathbb{R}^m$, then there exist a constant $C > 0$ and $0 < \varepsilon \leqslant 1$ such that

(i) T is bounded on $L^2(\mathbb{R}^{n+m})$;
(ii) $\|\widetilde{\mathcal{K}}^1(x_1,y_1)\|_{CZ} \leqslant C|x_1-y_1|^{-n}$,
$\|\widetilde{\mathcal{K}}^1(x_1,y_1) - \widetilde{\mathcal{K}}^1(x_1,y_1')\|_{CZ} \leqslant C|y_1-y_1'|^\varepsilon |x_1-y_1|^{-n-\varepsilon}$ for $|y_1-y_1'| \leqslant \frac{1}{2}|x_1-y_1|$,
$\|\widetilde{\mathcal{K}}^1(x_1,y_1) - \widetilde{\mathcal{K}}^1(x_1',y_1)\|_{CZ} \leqslant C|x_1-x_1'|^\varepsilon |x_1-y_1|^{-n-\varepsilon}$ for $|x_1-x_1'| \leqslant \frac{1}{2}|x_1-y_1|$;
(iii) $\|\widetilde{\mathcal{K}}^2(x_2,y_2)\|_{CZ} \leqslant C|x_2-y_2|^{-m}$,
$\|\widetilde{\mathcal{K}}^2(x_2,y_2) - \widetilde{\mathcal{K}}^2(x_2,y_2')\|_{CZ} \leqslant C|y_2-y_2'|^\varepsilon\ |x_2-y_2|^{-m-\varepsilon}$ for $|y_2-y_2'| \leqslant \frac{1}{2}|x_2-y_2|$,
$\|\widetilde{\mathcal{K}}^2(x_2,y_2) - \widetilde{\mathcal{K}}^2(x_2',y_2)\|_{CZ} \leqslant C|x_2-x_2'|^\varepsilon\ |x_2-y_2|^{-m-\varepsilon}$ for $|x_2-x_2'| \leqslant \frac{1}{2}|x_2-y_2|$.

The smallest such constant C is denoted by $\|\mathcal{K}\|_{CZ}$. The norm of the operator T in Journé's class is defined by $\|T\| = \|T\|_{L^2->L^2} + \|\mathcal{K}\|_{CZ}$.

By Definition 1.2, we can extend the singular integral operator in Journé's class from two-parameter case to arbitrary number of parameters setting, especially for three-parameter case.

Definition 1.3. A singular integral operator T with the kernel $\mathcal{K}$ is said to be in Journé's class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ with regularity exponent $\epsilon > 0$ if for $f \in C_0^\infty(\mathbb{R}^{n_1+n_2+n_3})$

$$T(f)(x_1,x_2,x_3) = \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}} \mathcal{K}(x_1,x_2,x_3,y_1,y_2,y_3) f(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3,$$

where (x_1,x_2,x_3) is not in the support of f and the kernel $\mathcal{K}(x_1,x_2,x_3,y_1,y_2,y_3)$ satisfies the following conditions: for each $x_1, y_1 \in \mathbb{R}^{n_1}$, set $\widetilde{K}^1(x_1,y_1)$ to be the singular integral operator acting on $\mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ with the kernel

$$\widetilde{\mathcal{K}}^1(x_1,y_1)(x_2,x_3,y_2,y_3) = \mathcal{K}(x_1,x_2,x_3,y_1,y_2,y_3).$$

Similarly, $\widetilde{\mathcal{K}}^2(x_2,y_2)$ and $\widetilde{\mathcal{K}}^3(x_3,y_3)$ can be defined in the same way.

We also set $\widetilde{\mathcal{K}}^{1,2}(x_1, y_1, x_2, y_2)$ to be the singular integral operator acting on $\mathbb{R}^{n_3}$ with the kernel

$$\widetilde{\mathcal{K}}^{1,2}(x_1, y_1, x_2, y_2)(x_3, y_3) = \mathcal{K}(x_1, x_2, x_3, y_1, y_2, y_3).$$

We can similarly set $\widetilde{\mathcal{K}}^{1,3}(x_1, y_1, x_3, y_3)$, $\widetilde{\mathcal{K}}^{2,1}(x_2, y_2, x_1, y_1)$, $\widetilde{\mathcal{K}}^{2,3}(x_2, y_2, x_3, y_3)$, $\widetilde{\mathcal{K}}^{3,1}(x_3, y_3, x_1, y_1)$, and $\widetilde{\mathcal{K}}^{3,2}(x_3, y_3, x_2, y_2)$. There exist constants $C > 0$ and $0 < \varepsilon \leqslant 1$ such that

(A_1) T is bounded on $L^2(\mathbb{R}^{n_1+n_2+n_3})$;
(A_2) For any $i = 1, 2, 3$,

$$\big\|\widetilde{\mathcal{K}}^i(x_i, y_i)\big\|_{CZ} \leqslant C|x_i - y_i|^{-n_i},$$

$$\big\|\widetilde{\mathcal{K}}^i(x_i, y_i) - \widetilde{\mathcal{K}}^i\big(x_i, y_i'\big)\big\|_{CZ} \leqslant C\big|y_i - y_i'\big|^{\varepsilon}|x_i - y_i|^{-n_i-\varepsilon}, \quad \text{if } \big|y_i - y_i'\big| \leqslant \frac{1}{2}|x_i - y_i|,$$

$$\big\|\widetilde{\mathcal{K}}^i(x_i, y_i) - \widetilde{\mathcal{K}}^i\big(x_i', y_i\big)\big\|_{CZ} \leqslant C\big|x_i - x_i'\big|^{\varepsilon}|x_i - y_i|^{-n_i-\varepsilon}, \quad \text{if } \big|x_i - x_i'\big| \leqslant \frac{1}{2}|x_i - y_i|;$$

(A_3) For any $1 \leqslant i, j \leqslant 3$ and $i \neq j$,

$$\big\|\widetilde{\mathcal{K}}^{i,j}(x_i, y_i, x_j, y_j)\big\|_{CZ} \leqslant C|x_i - y_i|^{-n_i}|x_j - y_j|^{-n_j},$$

$$\big\|\widetilde{\mathcal{K}}^{i,j}(x_i, y_i, x_j, y_j) - \widetilde{\mathcal{K}}^{i,j}\big(x_i', y_i, x_j, y_j\big)\big\|_{CZ}$$
$$\leqslant C\big|x_i - x_i'\big|^{\varepsilon}|x_i - y_i|^{-n_i-\epsilon}|x_j - y_j|^{-n_j} \quad \text{if } \big|x_i - x_i'\big| \leqslant \frac{1}{2}|x_i - y_i|.$$

The same estimate still holds if the difference is taken for y_i, x_j, and y_j, respectively.
(A_4) For any $1 \leqslant i, j \leqslant 3$ and $i \neq j$,

$$\big\|\widetilde{\mathcal{K}}^{i,j}(x_i, y_i, x_j, y_j) - \widetilde{\mathcal{K}}^{i,j}\big(x_i, y_i', x_j, y_j\big) - \widetilde{\mathcal{K}}^{i,j}\big(x_i, y_i, x_j, y_j'\big) + \widetilde{\mathcal{K}}^{i,j}\big(x_i, y_i', x_j, y_j'\big)\big\|_{CZ}$$
$$\leqslant C\frac{|y_i - y_i'|^{\varepsilon}}{|x_i - y_i|^{n_i+\varepsilon}}\frac{|y_j - y_j'|^{\varepsilon}}{|x_j - y_j|^{n_j+\varepsilon}}, \quad \text{if } \big|y_i - y_i'\big| \leqslant \frac{1}{2}|x_i - y_i| \text{ and } \big|y_j - y_j'\big| \leqslant \frac{1}{2}|x_j - y_j|.$$

The same estimate still holds if the double difference is taken for x_i, y_i; x_i, y_j; y_i, x_j; x_i, x_j; and x_j, y_j, respectively.

The smallest such constant C is denoted by $\|\mathcal{K}\|_{CZ}$. We define the norm of the operator T in Journé's class by $\|T\| = \|T\|_{L^2->L^2} + \|\mathcal{K}\|_{CZ}$.

From a result in [21], we have that the singular integral T in Journé's class is bounded from $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$ to $L^1(\mathbb{R}^{n_1+n_2+n_3})$. Note that if $\varphi^1 \in C_{0,0}^{\infty}(\mathbb{R}^{n_1})$, $\varphi^2 \in C_{0,0}^{\infty}(\mathbb{R}^{n_2})$ and $\varphi^3 \in C_{0,0}^{\infty}(\mathbb{R}^{n_3})$, then $\varphi^1\varphi^2\varphi^3 \in H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$. Thus, $T(\varphi^1\varphi^2\varphi^3) \in L^1(\mathbb{R}^{n_1+n_2+n_3})$. Hence, for any $1 \leqslant i \leqslant 3$, $T(\varphi^1\varphi^2\varphi^3)$, as a function of x_i is integrable on $\mathbb{R}^{n_i}$, and then we can define $T_i^*(1) = 0$ by

$$\int\limits_{\mathbb{R}^{n_i}} \int\limits_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}} \mathcal{K}(x_1, x_2, x_3, y_1, y_2, y_3)\varphi^1(y_1)\varphi^2(y_2)\varphi^3(y_3)\,dy_1\,dy_2\,dy_3\,dx_i = 0,$$

for all $\varphi^1 \in C_{0,0}^{\infty}(\mathbb{R}^{n_1})$, $\varphi^2 \in C_{0,0}^{\infty}(\mathbb{R}^{n_2})$, $\varphi^3 \in C_{0,0}^{\infty}(\mathbb{R}^{n_3})$.

Our main result in this article is the following theorem which provides the necessary and sufficient conditions for the operator in Journé's class to be bounded on $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$.

Theorem 1.4. *Let T be a singular integral operator in Journé's class on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$ with regularity exponent ε. Then T is bounded on $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ for $\max\{\frac{n_i}{n_i+\varepsilon}\colon 1\leqslant i\leqslant 3\}<p\leqslant 1$ if and only if $T_i^*(1)=0$, $1\leqslant i\leqslant 3$.*

The necessary conditions of Theorem 1.4 can be obtained from the cancellation conditions of functions in the classical Hardy spaces. Indeed, by the maximal characterization of Hardy spaces (see [13]), for $f\in H^p\cap L^2$ defined on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$, f^*, the maximal function of f, belongs to $L^p\cap L^2$ on $\mathbb{R}^{n_1+n_2+n_3}$. This implies that $f^*(x_1,x_2,x_3)$ belongs to $L^1(\mathbb{R}^{n_1+n_2+n_3})$. Denote $f_1^*(x_1,x_2,x_3)$ by the maximal function of $f(x_1,x_2,x_3)$, as the function of x_1 variable when x_2 and x_3 are fixed. Then for any fixed x_2 and x_3, $f_1^*(x_1,x_2,x_3)\leqslant f^*(x_1,x_2,x_3)$, which means that $f(x_1,x_2,x_3)$ as a function of x_1 belongs to $H^1(\mathbb{R}^{n_1})$. By a classical result of functions in $H^1(\mathbb{R}^{n_1})$, we have

$$\int_{\mathbb{R}^{n_1}} f(x_1,x_2,x_3)\,dx_1=0\quad \text{for any fixed } x_2 \text{ and } x_3.$$

Now for $\varphi^i\in C_{0,0}^\infty(\mathbb{R}^{n_i})$, $1\leqslant i\leqslant 3$, let $g(x_1,x_2,x_3)=\varphi^1(x_1)\varphi^2(x_2)\varphi^3(x_3)$, then $g\in H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$. Thus, by the L^2 boundedness and H^p boundedness of T, $T(g)\in L^2\cap H^p$, and we obtain

$$\int_{\mathbb{R}^{n_1}}\int \mathcal{K}(x_1,x_2,x_3,y_1,y_2,y_3)\varphi^1(y_1)\varphi^2(y_2)\varphi^3(y_3)\,dy_1\,dy_2\,dy_3\,dx_1=0,$$

for any fixed x_2 and x_3.

This implies that $T_1^*(1)=0$. Similarly, $T_2^*(1)=0$ and $T_3^*(1)=0$ can be obtained in the same way.

We prove the sufficiency of the main Theorem 1.4 by the following two steps.

Step 1. Reduce the H^p boundedness of operator T to $H^p-L^p_{\mathcal{H}}$ boundedness of $\mathcal{H}$-valued operator $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$, where $L^p_{\mathcal{H}}$ is the $\mathcal{H}$-valued L^p space and $\mathcal{H}$ is the Hilbert space defined by

$$\mathcal{H}=\left\{\{h_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}\colon \big\|\{h_{t_1,t_2,t_3}\}\big\|_{\mathcal{H}}=\left(\int_0^\infty\int_0^\infty\int_0^\infty |h_{t_1,t_2,t_3}|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\right)^{1/2}<\infty\right\}$$

and $T_{t_1,t_2,t_3}(f)=\psi_{t_1,t_2,t_3}*(T(f))$.

It is easy to see that the H^p norm of $T(f)$, by the Littlewood–Paley characterization, is equivalent to the L^p norm of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$ in this Hilbert space $\mathcal{H}$. This means that the boundedness of T on $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ is equivalent to the boundedness of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$ from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$.

The main step is

Step 2. Reduce to $H^p-L^p_{\mathcal{H}}$ boundedness of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$ via H^p-atoms.

To carry out step 2, in Section 2, we will first establish the following criterion of the boundedness of $\mathcal{H}$-valued operator $\mathcal{L}$ from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$.

Lemma 1.5. *Let $\mathcal{L}$ be a bounded $\mathcal{H}$-valued operator from $L^2(\mathbb{R}^{n_1+n_2+n_3})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$. Then, for $0<p\leqslant 1$, $\mathcal{L}$ can be extended to be bounded from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$ if and only if $\|\mathcal{L}(a)\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})}\leqslant C$ for all $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$-atoms a, where the constant C is independent of a.*

By the Littlewood–Paley theory, it is easy to see that the operator $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$ is bounded from $L^2(\mathbb{R}^{n_1+n_2+n_3})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$. Therefore, to show that $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$ is bounded from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$, by Lemma 1.5, it suffices to check the fact that $\|T_{t_1,t_2,t_3}(a)\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})}\leqslant C$ for every $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$-atom a. This will follow from the following theorem.

Theorem 1.6. *For every $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ atom a,*

$$\bigl\|\bigl\{T_{t_1,t_2,t_3}(a)\bigr\}_{t_1,t_2,t_3>0}\bigr\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})}\leqslant C, \tag{1.3}$$

where the constant C is independent of a.

The sufficiency of Theorem 1.4 then follows from Theorem 1.6 and Lemma 1.5.

We would like to mention again that in the classical case, if T is bounded on $L^2(\mathbb{R}^n)$ then $\|T(a)\|_{L^p(\mathbb{R}^n)}\leqslant C$ for every $H^p(\mathbb{R}^n)$ atom a provided that T satisfies only the regularity with respect to the second variable. Similarly, Theorem 1.6 will follow from the regularities of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$ with respect to y_1, y_2 and y_3 together with variants of covering lemma due to J. Pipher [21]. To obtain such regularities, we will employ a new unified approach. In order to see how this new approach works, we describe this approach first for the classical case with one parameter. Let $\psi\in C^\infty_{0,0}(\mathbb{R}^n)$ satisfy the condition in (1.2). Suppose that T is a Calderón–Zygmund operator as in Definition 1.1. By Calderón's identity on $L^2(\mathbb{R}^n)$

$$f(x)=\int_0^\infty \psi_s*\psi_s*f(x)\,\frac{ds}{s},$$

we can write $T_t(x,y)$, the kernel of $T_t=\psi_t*T$, by

$$T_t(x,y)=\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi_t(x-u)\mathcal{K}(u,v)(\psi_s*\psi_s)(v-y)\,du\,dv\,\frac{ds}{s}.$$

If T satisfies the cancellation conditions $T(1)=T^*(1)=0$, then the regularity of T_t follows from the almost orthogonal argument. Here the almost orthogonal argument means that for such given function ψ there exists a constant C such that

$$\bigl|\psi_t*\psi_s(x)\bigr|\leqslant C\biggl(\frac{t}{s}\wedge\frac{s}{t}\biggr)\frac{(t\vee s)}{(t\vee s+|x|)^{(n+1)}},$$

where $a\wedge b=\min(a,b)$ and $a\vee b=\max(a,b)$.

If T satisfies the cancellation conditions $T(1) = T^*(1) = 0$, then one still has the following almost orthogonal argument:

$$\left| \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \psi_t(x-u)\mathcal{K}(u,v)\psi_s(v-y)\,du\,dv \right| \leqslant C\left(\frac{t}{s} \wedge \frac{s}{t}\right)^{\epsilon'} \frac{(t \vee s)^{\epsilon'}}{(t \vee s + |x-y|)^{(n+\epsilon')}},$$

where $0 < \epsilon' < \epsilon$ and ϵ is the regularity exponent of the kernel $\mathcal{K}$. However, if T satisfies only the condition $T^*(1) = 0$, in general, the above almost orthogonal argument holds only for the case when $s \leqslant t$. To deal with the case for $t \leqslant s$, we decompose $\psi_t(x-u)\mathcal{K}(u,v)\psi_s(v-y)$ by

$$\psi_t(x-u)\mathcal{K}(u,v)\psi_s(v-y) = \psi_t(x-u)\mathcal{K}(u,v)\big[\psi_s(v-y) - \psi_s(x-y)\big] + \psi_t(x-u)\mathcal{K}(u,v)\psi_s(x-y).$$

The first term above satisfies the almost orthogonal argument and hence the regularity of this part follows. The second term leads to a para-product-type operator, namely the kernel of this para-product-type operator is given by

$$\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \psi_t(x-u)\mathcal{K}(u,v)\,du\,dv \left(\int\limits_t^\infty \psi_s * \psi_s(x-y)\,\frac{ds}{s} \right) = \big(\psi_t * T(1)\big)(x)\phi_t(x-y),$$

where $\phi_t = \int_t^\infty \psi_s * \psi_s \frac{ds}{s}$ and ϕ satisfies the same conditions as ψ does but except $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. By the fact $T(1) \in BMO(\mathbb{R}^n)$ and the dual of $H^1(\mathbb{R}^n)$ is given by $BMO(\mathbb{R}^n)$, it is easy to check that this para-product-type operator satisfies the required regularities. In Section 3, we will carry out this approach to multiparameter case and prove the desired regularities of the operator $T_{t_1.t_2,t_3}$.

We finally mention that our results in this paper have been extended to the weighted case. Namely, we have proved in [17] the Journé type operators are bounded on weighted multiparameter $H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ when the weight function w is in some class of product Muckenhoupt weights. More precisely, suppose that T is a singular integral operator in Journé's class with regularity exponent ε on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$, then we have shown in [17]:

(i) Whenever its kernel $\mathcal{K}$ has infinite order of smoothness and cancellation conditions, we have that the operator T is bounded on $H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$, where $w \in A_\infty(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$;
(ii) If the operator T has kth order cancellation and smoothness condition, then we have that T is bounded on $H^p_w(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$, where $w \in A_r(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$, $r \geqslant 1$, and $\max\{\frac{n_i r}{n_i + k + \varepsilon}, 1 \leqslant i \leqslant k\} < p \leqslant 1$.

Furthermore, the dual spaces of such weighted multiparameter Hardy spaces $H^p_w(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ have also been identified in [20].

Throughout this paper, we will denote $\int$ by $\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}}$.

2. The proof of the sufficiency of Theorem 1.4

As mentioned in Section 1, Lemma 1.5 can ensure that step 2 is feasible. We now give the proof of Lemma 1.5.

Proof of Lemma 1.5. Since the necessary conditions are obvious, we only need to prove the sufficiency. The argument is similar to that in [16], where a unified boundedness criterion on atoms has been established in the multiparameter setting. For the sake of completeness, we provide the details in our $\mathcal{H}$-valued operator setting. The main idea is to obtain a new atomic decomposition for $\mathcal{H}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})\cap\mathcal{L}^2(\mathbb{R}^{n_1+n_2+n_3})$ instead of a classical atomic decomposition for $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$. The new feature of such a decomposition is that the series in the decomposition converges in both $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ and $L^2(\mathbb{R}^{n_1+n_2+n_3})$ while the convergence takes only in the sense of distribution in the classical case. To be more precise, given $f\in H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})\cap L^2(\mathbb{R}^{n_1+n_2+n_3})$, set $\Omega_k=\{(x_1,x_2,x_3)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}\colon \mathcal{G}(f)(x_1,x_2,x_3)>2^k\}$, where $\mathcal{G}(f)$ is the $\mathcal{G}$-function defined in Section 1. Let $B_k=\{R=I_1\times I_2\times I_3\colon |R\cap\Omega_k|>\frac{1}{2}|R|,\ |R\cap\Omega_{k+1}|\leqslant\frac{1}{2}|R|\}$, where I_i are dyadic cubes in $\mathbb{R}^{n_i}$, $i=1,2,3$. By Calderón's identity on $L^2(\mathbb{R}^{n_1+n_2+n_3})$,

$$\begin{aligned} f(x_1,x_2,x_3) &= \int_0^\infty\int_0^\infty\int_0^\infty \psi_{t_1,t_2,t_3}*\psi_{t_1,t_2,t_3}*f(x_1,x_2,x_3)\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\\ &= \sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}}\psi_{t_1,t_2,t_3}(x_1-y_1,x_2-y_2,x_3-y_3)\\ &\quad\times\psi_{t_1,t_2,t_3}*f(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}, \end{aligned}\tag{2.1}$$

where $\widehat{R}=\{(y_1,y_2,y_3,t_1,t_2,t_3)\colon y_i\in I_i,\ \frac{\ell(I_i)}{2}\leqslant t_i<\ell(I_i),\ 1\leqslant i\leqslant 3\}$.

Then (2.1) provides an atomic decomposition such that the series converges in the sense of distribution [5]. It can be further shown that the series in (2.1) also converges in $L^2(\mathbb{R}^{n_1+n_2+n_3})$ (see also [16]). In fact, by the duality argument, for $g\in L^2$ with $\|g\|_{L^2}=1$, by Schwartz's inequality and L^2 boundedness of the Littlewood–Paley square function,

$$\begin{aligned} &\left|\left\langle\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}}\psi_{t_1,t_2,t_3}(\cdot-y_1,\cdot-y_2,\cdot-y_3)\psi_{t_1,t_2,t_3}*f(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3},g\right\rangle\right|\\ &\quad=\left|\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}}\tilde{\psi}_{t_1,t_2,t_3}*g(y_1,y_2,y_3)\psi_{t_1,t_2,t_3}*f(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\right|\\ &\quad\leqslant C\left(\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}}\left|\psi_{t_1,t_2,t_3}*f(y_1,y_2,y_3)\right|^2dy_1\,dy_2\,dy_3\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\right)^{1/2}\\ &\qquad\times\left(\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}}\left|\tilde{\psi}_{t_1,t_2,t_3}*g(y_1,y_2,y_3)\right|^2dy_1\,dy_2\,dy_3\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\right)^{1/2}\\ &\quad\leqslant C\|f\|_{L^2}\|g\|_{L^2}\leqslant C\|f\|_{L^2}, \end{aligned}$$

where in the first equality $\tilde{\psi}_{t_1,t_2,t_3}(x_1,x_2,x_3)=\tilde{\psi}_{t_1,t_2,t_3}(-x_1,-x_2,-x_3)$. Thus, the series in (2.1) converges in L^2. As in [5], for $f\in L^2\cap H^p$, $f(x_1,x_2,x_3)=\sum_j\lambda_j a_j(x_1,x_2,x_3)$, where (a_j) are H^p-atoms and $\sum_j|\lambda_j|^p\leqslant C\|f\|_{H^p}^p$. Since $\mathcal{L}$ is bounded from L^2 to $L^2_{\mathcal{H}}$ and the series in the atomic decomposition of f converges in L^2, we have $\mathcal{L}(f)(x_1,x_2,x_3) = \sum_j \lambda_j \mathcal{L}(a_j)(x_1,x_2,x_3)$. Moreover, this series also converges in L^2 and then a subsequence converges almost everywhere. Hence

$$\begin{aligned}\|\mathcal{L}(f)\|^p_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})} &\leqslant \sum_j |\lambda_j|^p \|\mathcal{L}(a_j)\|^p_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})} \\ &\leqslant C\sum_j |\lambda_j|^p \leqslant C\|f\|^p_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})}.\end{aligned}$$

Since $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}) \cap L^2(\mathbb{R}^{n_1+n_2+n_3})$ is dense in $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$, $\mathcal{L}$ can be extended to a bounded operator from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$ by limiting argument. Therefore, the proof of Lemma 1.5 is complete. □

The sufficiency of Theorem 1.4 will follow by combining Lemma 1.5 and Theorem 1.6 which is actually the main result of this section. Now we return to the proof of Theorem 1.6.

Proof of Theorem 1.6. Suppose a is an $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$ atom supported on an open set $\Omega \subseteq \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$ with finite measure, furthermore, a can be decomposed as $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, where $\mathcal{M}(\Omega)$ is the collection of all maximal dyadic sub-rectangles contained in Ω, each a_R is supported on $2R = 2I_1\times 2I_2\times 2I_3$, $\int_{2I_i} a_R(x_1,x_2,x_3)\,dx_i = 0$, here I_i are dyadic cubes in $\mathbb{R}^{n_i}$, $1 \leqslant i \leqslant 3$. Moreover,

$$\|a\|_2 \leqslant |\Omega|^{\frac12-\frac1p} \quad \text{and} \quad \Bigl(\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_2^2\Bigr)^{\frac12} \leqslant |\Omega|^{\frac12-\frac1p}.$$

Let $D_1 = \{(x_1,x_2,x_3)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}\colon M_s(\chi_\Omega)(x_1,x_2,x_3) > 4^{-n_1-n_2-n_3} n_1^{-n_1/2} n_2^{-n_2/2} \times n_3^{-n_3/2}\}$, where M_s is the strong maximal function defined by

$$M_s(f)(x_1,x_2,x_3) = \sup_{(x_1,x_2,x_3)\in R} \frac{1}{|R|}\int_R \bigl|f(y_1,y_2,y_3)\bigr|\,dy_1\,dy_2\,dy_3,$$

where the supremum is taken over all rectangles $R = I_1\times I_2\times I_3$ with sides parallel to the axis and containing the point (x_1,x_2,x_3).

And for any $\ell \geqslant 1$, $\ell\in\mathbb{N}$, let

$$\begin{aligned} D_{\ell+1} = \bigl\{&(x_1,x_2,x_3)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}\colon \\ &M_s(\chi_{D_\ell})(x_1,x_2,x_3) > 4^{-n_1-n_2-n_3} n_1^{-n_1/2} n_2^{-n_2/2} n_3^{-n_3/2}\bigr\}.\end{aligned}$$

It follows from the $L^p (1 < p < \infty)$ boundedness of the strong maximal operator M_s that $|D_{\ell+1}| \leqslant C|D_\ell| \leqslant C|\Omega|$. Note that $\mathcal{M}(\Omega) \subseteq \mathcal{M}_i(\Omega)$, where $\mathcal{M}_i(\Omega)$ is the collection of all dyadic rectangles $R\subseteq\Omega$ that are maximal in the x_i direction, $i = 1,2,3$. For each $R = I_1\times I_2\times I_3 \in \mathcal{M}(\Omega)$, we choose $\widehat{I_1}$ to be the largest dyadic cube containing I_1 such that $\widehat{I_1}\times I_2\times I_3 \in \mathcal{M}_1(D_1)$. Similarly, we can choose $\widehat{I_2}$ and $\widehat{I_3}$ to be the largest dyadic cubes containing I_2 and I_3 respectively such that $\widehat{I_1} \times \widehat{I_2} \times I_3 \in \mathcal{M}_2(D_2)$ and $\widehat{I_1} \times \widehat{I_2} \times \widehat{I_3} \in \mathcal{M}_3(D_3)$. Note that $4\sqrt{n_1}\,\widehat{I_1} \times 4\sqrt{n_2}\,\widehat{I_2} \times 4\sqrt{n_3}\,\widehat{I_3} \subseteq D_4$.

We now estimate the $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2+n_3})$ norm of $\{T_{t_1,t_2,t_3}(a)\}_{t_1,t_2,t_3>0}$ as follows. First we write

$$\int \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3 = \int\limits_{D_4} \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3 + \int\limits_{(D_4)^c} \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3.$$

By Hölder's inequality and the $L^2 - L^2_{\mathcal{H}}$ boundedness of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$, we have

$$\begin{aligned}
&\int\limits_{D_4} \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3 \\
&\quad\leqslant \bigg(\int\limits_{D_4} \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^2_{\mathcal{H}}\,dx_1\,dx_2\,dx_3\bigg)^{\frac{p}{2}} |D_4|^{1-\frac{p}{2}} \\
&\quad\leqslant C\|a\|_2^p|\Omega|^{1-\frac{p}{2}} \leqslant C.
\end{aligned} \tag{2.2}$$

It remains to estimate $\int_{(D_4)^c} \|\{T_{t_1,t_2,t_3}(a)\}(x_1,x_2,x_3)\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3$. Since

$$\begin{aligned}
(D_4)^c &\subseteq \big(4\sqrt{n_1}\,\widehat{I_1} \times 4\sqrt{n_2}\,\widehat{I_2} \times 4\sqrt{n_3}\,\widehat{I_3}\big)^c \\
&\subseteq \big((4\sqrt{n_1}\,\widehat{I_1})^c \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}\big) \cup \big(\mathbb{R}^{n_1} \times (4\sqrt{n_2}\,\widehat{I_2})^c \times \mathbb{R}^{n_3}\big) \cup \big(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times (4\sqrt{n_3}\,\widehat{I_3})^c\big) \\
&:= E_1 \cup E_2 \cup E_3,
\end{aligned}$$

we have that

$$\int\limits_{(D_4)^c} \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3 \leqslant U_1 + U_2 + U_3,$$

where

$$U_i = \int\limits_{E_i} \big\|\big\{T_{t_1,t_2,t_3}(a)\big\}(x_1,x_2,x_3)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2\,dx_3, \quad i = 1,2,3.$$

We first estimate U_1. Denote $A_{I_1,k} = \{S\colon\ I_1 \times S \in \mathcal{M}_3(\Omega) \text{ and } \widehat{I_1} = 2^k I_1\}$. Note that if $I_1 \times S \in \mathcal{M}_3(\Omega)$ and $\widehat{I_1} = 2^k I_1$, then $S \in \mathcal{M}_2(A_{I_1,k})$. Thus,

$$U_1 \leqslant \sum_{I_1} \sum_{k=0}^{\infty} \int\limits_{E_1} \bigg\|\bigg\{T_{t_1,t_2,t_3}\bigg(\sum_{\substack{S:\ \widehat{I_1}=2^k I_1\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\bigg)\bigg\}\bigg\|^p_{\mathcal{H}} \chi_{(4\sqrt{n_1}\,2^k I_1)^c}\,dx$$

$$\leqslant \sum_{I_1} \sum_{k=0}^{\infty} \int_{x:\ x_1 \notin 4\sqrt{n_1}2^k I_1} \left\| \left\{ T_{t_1,t_2,t_3}\Big(\sum_{\substack{S \in \mathcal{M}_2(A_{I_1,k}) \\ I_1 \times S \in \mathcal{M}_3(\Omega)}} a_{I_1 \times S} \Big) \right\} \right\|_{\mathcal{H}}^p dx. \tag{2.3}$$

Denote $\widetilde{A}_{I_1,k} = \{(x_2,x_3) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}\colon\ M_s(\chi_{A_{I_1,k}}) > 4^{-n_2-n_3} n_2^{-n_2/2} n_3^{-n_3/2}\}$, and $\widetilde{\widetilde{A}}_{I_1,k} = \{(x_2,x_3) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}\colon\ M_s(\chi_{\widetilde{A}_{I_1,k}}) > 4^{-n_2-n_3} n_2^{-n_2/2} n_3^{-n_3/2}\}$. Obviously, $A_{I_1,k} \subset \widetilde{A}_{I_1,k} \subset \widetilde{\widetilde{A}}_{I_1,k}$. The strong maximal theorem about M_s yields that $|\widetilde{\widetilde{A}}_{I_1,k}| \leqslant C|\widetilde{A}_{I_1,k}| \leqslant C|A_{I_1,k}|$. For each $S = I_2 \times I_3 \in \mathcal{M}_2(A_{I_1,k})$, let $\bar{I}_2$ and $\bar{I}_3$ be the largest dyadic cubes containing I_2 and I_3 respectively such that $\bar{I}_2 \times I_3 \in \mathcal{M}_1(\widetilde{A}_{I_1,k})$ and $\bar{I}_2 \times \bar{I}_3 \in \bar{\bar{S}}_k := \mathcal{M}_2(\widetilde{\widetilde{A}}_{I_1,k})$. Then

$$\begin{aligned} U_1 &\leqslant \sum_{I_1} \sum_{k=0}^{\infty} \int_{4\bar{\bar{S}}_k} \int_{(4\sqrt{n_1}2^k I_1)^c} \left\| \left\{ T_{t_1,t_2,t_3}\Big(\sum_{\substack{S \in \mathcal{M}_2(A_{I_1,k}) \\ I_1 \times S \in \mathcal{M}_3(\Omega)}} a_{I_1 \times S} \Big) \right\} \right\|_{\mathcal{H}}^p dx_1\, dx_2\, dx_3 \\ &\quad + \sum_{I_1} \sum_{k=0}^{\infty} \int_{(4\bar{\bar{S}}_k)^c} \int_{(4\sqrt{n_1}2^k I_1)^c} \left\| \left\{ T_{t_1,t_2,t_3}\Big(\sum_{\substack{S \in \mathcal{M}_2(A_{I_1,k}) \\ I_1 \times S \in \mathcal{M}_3(\Omega)}} a_{I_1 \times S} \Big) \right\} \right\|_{\mathcal{H}}^p dx_1\, dx_2\, dx_3 \\ &:= U_{1,1} + U_{1,2}. \end{aligned} \tag{2.4}$$

Since $|\bar{\bar{S}}_k| \leqslant C|\widetilde{\widetilde{A}}_{I_1,k}| \leqslant C|A_{I_1,k}|$, applying Hölder's inequality, we yield

$$\begin{aligned} U_{1,1} &\leqslant C \sum_{I_1} \sum_{k=0}^{\infty} |A_{I_1,k}|^{1-\frac{p}{2}} \\ &\quad \times \int_{(4\sqrt{n_1}2^k I_1)^c} \left(\int_{4\bar{\bar{S}}_k} \left\| \left\{ T_{t_1,t_2,t_3}\Big(\sum_{\substack{S \in \mathcal{M}_2(A_{I_1,k}) \\ I_1 \times S \in \mathcal{M}_3(\Omega)}} a_{I_1 \times S} \Big) \right\} \right\|_{\mathcal{H}}^2 dx_2\, dx_3 \right)^{\frac{p}{2}} dx_1. \end{aligned}$$

Using the cancellation condition of $a_{I_1 \times S}$ and then applying Schwartz's and Minkowski's inequalities we obtain

$$\begin{aligned} &\left\| \left\{ T_{t_1,t_2,t_3}\Big(\sum_{\substack{S \in \mathcal{M}_2(A_{I_1,k}) \\ I_1 \times S \in \mathcal{M}_3(\Omega)}} a_{I_1 \times S} \Big) \right\}(x_1,x_2,x_3) \right\|_{\mathcal{H}}^2 \\ &\quad \leqslant C|I_1| \int_{2I_1} \left\| \left\{ \int_{\mathbb{R}^{n_2} \times \mathbb{R}^{n_3}} \big[T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}(x_1,x_2,x_3,x_{I_1},y_2,y_3) \big] \right.\right. \\ &\quad\quad \left.\left. \times \Big(\sum_{\substack{S \in \mathcal{M}_2(A_{I_1,k}) \\ I_1 \times S \in \mathcal{M}_3(\Omega)}} a_{I_1 \times S} \Big)(y_1,y_2,y_3)\, dy_2\, dy_3 \right\} \right\|_{\mathcal{H}}^2 dy_1. \end{aligned}$$

Note that if $y_1 \in 2I_1$ and $x_1 \in (4\sqrt{n_1}\,2^k I_1)^c$, $k \geqslant 0$, then $|y_1 - x_{I_1}| \leqslant \frac{1}{2}|x_1 - x_{I_1}|$, where x_{I_1} is the center of I_1. Thus for $x_1 \in (4\sqrt{n_1}\,2^k I_1)^c$, by the regularity of T_{t_1,t_2,t_3} given in Corollary 3.2(i), we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}} \left\| \left\{ T_{t_1,t_2,t_3}\Bigg(\sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\Bigg)\right\}(x_1,x_2,x_3)\right\|_{\mathcal{H}}^2 dx_2\,dx_3 \\
&\quad\leqslant C|I_1| \int_{2I_1}\int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}} \Bigg\| \Bigg\{ \int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}} \big[T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}(x_1,x_2,x_3,x_{I_1},y_2,y_3)\big] \\
&\qquad\times \Bigg(\sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\Bigg)(y_1,y_2,y_3)\,dy_2\,dy_3 \Bigg\} \Bigg\|_{\mathcal{H}}^2 dx_2\,dx_3\,dy_1 \\
&\quad\leqslant C|I_1|\,\frac{\ell(I_1)^{2\varepsilon'}}{|x_1 - x_{I_1}|^{2(n_1+\varepsilon')}} \Bigg\| \sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\Bigg\|_{L^2(\mathbb{R}^{n_1+n_2+n_3})}^2,
\end{aligned}$$

which together with Hölder's inequality implies that for any $\frac{n_1}{n_1+\varepsilon'} < p \leqslant 1$,

$$\begin{aligned}
U_{1,1} &\leqslant C\sum_{I_1}\sum_{k=0}^{\infty} |A_{I_1,k}|^{1-\frac{p}{2}} |I_1|^{\frac{p}{2}} \Bigg\| \sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\Bigg\|_{L^2}^p \int_{(4\sqrt{n_1}\,2^k I_1)^c} \left(\frac{\ell(I_1)^{\varepsilon'}}{|x_1 - x_{I_1}|^{n_1+\varepsilon'}}\right)^p dx_1 \\
&\leqslant C\sum_{I_1}\sum_{k=0}^{\infty} |A_{I_1,k}|^{1-\frac{p}{2}} |I_1|^{1-\frac{p}{2}} 2^{k[n_1 - p(n_1+\varepsilon')]} \sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} \|a_{I_1\times S}\|_{L^2}^p \\
&\leqslant C\Bigg(\sum_{I_1}\sum_{k=0}^{\infty} |A_{I_1,k}||I_1| 2^{k[n_1 - p(n_1+\varepsilon')]\frac{2}{2-p}}\Bigg)^{1-\frac{p}{2}} \Bigg(\sum_{I_1}\sum_{k=0}^{\infty} \sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} \|a_{I_1\times S}\|_{L^2}^2\Bigg)^{\frac{p}{2}} \\
&\leqslant C|\Omega|^{1-\frac{p}{2}}\,|\Omega|^{\frac{p}{2}-1} = C, \qquad (2.5)
\end{aligned}$$

where in the last inequality we use the following variant of covering lemma due to J. Pipher [21]

$$\sum_{I_1}\sum_{k=0}^{\infty} |A_{I_1,k}||I_1| 2^{-k\delta} \leqslant C|\Omega| \quad \text{for any } \delta > 0. \qquad (2.6)$$

Now we estimate $U_{1,2}$. For $\bar{x} = (x_2, x_3) \in (4\bar{\bar{S}}_k)^c \subseteq ((4\bar{I}_2)^c \times \mathbb{R}^{n_2}) \cup (\mathbb{R}^{n_2} \times (4\bar{I}_3)^c)$,

$$U_{1,2} \leqslant \sum_{I_1}\sum_{k=0}^{\infty} \int_{(4\bar{\bar{S}}_k)^c} \int_{(4\sqrt{n_1}\,2^k I_1)^c} \left\| \left\{ T_{t_1,t_2,t_3}\Bigg(\sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\Bigg)\right\}\right\|_{\mathcal{H}}^p \chi_{(4\bar{I}_2)^c}\,dx_1\,d\bar{x}$$

$$+\sum_{I_1}\sum_{k=0}^{\infty}\int_{(4\bar{\bar{S}}_k)^c}\int_{(4\sqrt{n_1}2^k I_1)^c}\left\|\left\{T_{t_1,t_2,t_3}\left(\sum_{\substack{S\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times S}\right)\right\}\right\|_{\mathcal{H}}^p \chi_{(4\bar{I}_3)^c}\,dx_1\,d\bar{x}$$

$$:= W_1 + W_2. \tag{2.7}$$

We first estimate W_1. Denote

$$B_{I_1,I_2,k,j}=\left\{Q:\ I_2\times Q\in\mathcal{M}_2(A_{I_1,k}),\ \bar{I}_2=2^j I_2\right\}.$$

Note that if $I_2\times Q\in\mathcal{M}_2(A_{I_1,k})$ and $\bar{I}_2=2^j I_2$, then $Q\in\mathcal{M}(B_{I_1,I_2,k,j})$ and

$$W_1\leqslant\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}\int_{(4\bar{\bar{S}}_k)^c}\int_{(4\sqrt{n_1}2^k I_1)^c}\left\|\left\{T_{t_1,t_2,t_3}\left(\sum_{\substack{Q:\ \bar{I}_2=2^j I_2\\ S=I_2\times Q\in\mathcal{M}_2(A_{I_1,k})\\ I_1\times S\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\right)\right\}\right\|_{\mathcal{H}}^p \chi_{(4\bar{I}_2)^c}\,dx_1\,d\bar{x}$$

$$\leqslant\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}\int_{(4\bar{\bar{S}}_k)^c}\int_{(4\sqrt{n_1}2^k I_1)^c}\left\|\left\{T_{t_1,t_2,t_3}\left(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\right)\right\}\right\|_{\mathcal{H}}^p \chi_{(2^{j+2}I_2)^c}\,dx_1\,d\bar{x}.$$

For each $Q\in\mathcal{M}(B_{I_1,I_2,k,j})$, let $\bar{\bar{Q}}$ be the largest dyadic cube containing Q such that $|\bar{\bar{Q}}\cap B_{I_1,I_2,k,j}|>4^{-n_3}n_3^{-n_3/2}|\bar{\bar{Q}}|$ and denote by $\Theta_{k,j}$ the collection of such $\bar{\bar{Q}}$. The weak L^1 boundedness of the Hardy–Littlewood maximal operator implies that $|\Theta_{k,j}|\leqslant C|B_{I_1,I_2,k,j}|$. Thus,

$$W_1\leqslant\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}\int_{4\Theta_{k,j}}\int_{(2^{j+2}I_2)^c}\int_{(4\sqrt{n_1}2^k I_1)^c}\left\|\left\{T_{t_1,t_2,t_3}\left(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\right)\right\}\right\|_{\mathcal{H}}^p dx_1\,dx_2\,dx_3$$

$$+\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}\int_{(4\Theta_{k,j})^c}\int_{(2^{j+2}I_2)^c}\int_{(4\sqrt{n_1}2^k I_1)^c}\left\|\left\{T_{t_1,t_2,t_3}\right.\right.$$

$$\left.\left.\times\left(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\right)\right\}\right\|_{\mathcal{H}}^p dx_1\,dx_2\,dx_3$$

$$:= W_{1,1}+W_{1,2}. \tag{2.8}$$

We first estimate the term $W_{1,1}$. By Hölder's inequality,

$$W_{1,1}\leqslant C\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}|B_{I_1,I_2,k,j}|^{1-\frac{p}{2}}$$

$$\times\int_{(4\sqrt{n_1}2^k I_1)^c}\int_{(2^{j+2}I_2)^c}\left(\int_{4\Theta_{k,j}}\left\|\left\{T_{t_1,t_2,t_3}\left(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\right)\right\}\right\|_{\mathcal{H}}^2 dx_3\right)^{\frac{p}{2}} dx_1\,dx_2.$$

From cancellation conditions of $a_{I_1\times I_2\times Q}$, we can write

$$T_{t_1,t_2,t_3}\Big(\sum_{Q\in\mathcal{M}(B_{I_1,I_2,k,j})} a_{I_1\times I_2\times Q}\Big)(x_1,x_2,x_3)$$
$$= \sum_{Q\in\mathcal{M}(B_{I_1,I_2,k,j})} \int_{2(I_1\times I_2\times Q)} \big[T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,y_1,y_2,y_3) - T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,x_{I_1},y_2,y_3)$$
$$- T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,y_1,x_{I_2},y_3) + T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,x_{I_1},x_{I_2},y_3)\big](x_1,x_2,x_3)$$
$$\times (a_{I_1\times I_2\times Q})(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3,$$

where x_{I_i} is the center of I_i, $i=1,2$. For $x_1\in(4\sqrt{n_1}2^kI_1)^c$, $x_2\in(2^{j+2}I_2)^c$, $y_1\in 2I_1$ and $y_2\in 2I_2$, we have $|y_1-x_{I_1}|\leqslant\frac{1}{2}|x_1-x_{I_1}|$ and $|y_2-x_{I_2}|\leqslant\frac{1}{2}|x_2-x_{I_2}|$.

Applying Schwartz's and Minkowski's inequalities, we get that

$$\Bigg\|\Bigg\{T_{t_1,t_2,t_3}\Big(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\Big)\Bigg\}(x_1,x_2,x_3)\Bigg\|_{\mathcal{H}}^2$$
$$\leqslant C|I_1||I_2|\int_{2I_1\times 2I_2}\Bigg\|\Bigg\{\int_{\mathbb{R}^{n_3}}\big[T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}(x_1,x_2,x_3,x_{I_1},y_2,y_3)$$
$$- T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,x_{I_2},y_3) + T_{t_1,t_2,t_3}(x_1,x_2,x_3,x_{I_1},x_{I_2},y_3)\big]$$
$$\times\Big(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\Big)(y_1,y_2,y_3)\,dy_3\Bigg\}\Bigg\|_{\mathcal{H}}^2 dy_1\,dy_2,$$

which together with the regularity of T_{t_1,t_2,t_3} given in Corollary 3.2(iv) implies that

$$\int_{\mathbb{R}^{n_3}}\Bigg\|\Bigg\{T_{t_1,t_2,t_3}\Big(\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\Big)\Bigg\}(x_1,x_2,x_3)\Bigg\|_{\mathcal{H}}^2 dx_3$$
$$\leqslant C|I_1||I_2|\bigg(\frac{\ell(I_1)^{\varepsilon'}}{|x_1-x_{n_1}|^{n_1+\varepsilon'}}\bigg)^2\bigg(\frac{\ell(I_2)^{\varepsilon'}}{|x_2-x_{n_2}|^{n_2+\varepsilon'}}\bigg)^2$$
$$\times\Bigg\|\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\Bigg\|_{L^2}^2\big(\mathbb{R}^{n_1+n_2+n_3}\big).$$

Then for $\frac{n_i}{n_i+\varepsilon'}<p\leqslant 1$, $i=1,2$,

$$W_{1,1}\leqslant C\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}|B_{I_1,I_2,k,j}|^{1-\frac{p}{2}}|I_1|^{\frac{p}{2}}|I_2|^{\frac{p}{2}}\Bigg\|\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\Bigg\|_{L^2}^p$$

$$\times \int_{(4\sqrt{n_1}\,2^k I_1)^c\times(2^{j+2}I_2)^c} \left(\frac{\ell(I_1)^{\varepsilon'}}{|x_1-x_{n_1}|^{n_1+\varepsilon'}}\frac{\ell(I_2)^{\varepsilon'}}{|x_2-x_{n_2}|^{n_2+\varepsilon'}}\right)^p dx_1\,dx_2$$

$$\leqslant C\sum_{I_1,I_2}\sum_{k,j=0}^{\infty} 2^{k[n_1-p(n_1+\varepsilon')]}2^{j[n_2-p(n_2+\varepsilon')]}|I_1|^{1-\frac{p}{2}}|I_2|^{1-\frac{p}{2}}|B_{I_1,I_2,k,j}|^{1-\frac{p}{2}}$$

$$\times\Bigg\|\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}} a_{I_1\times I_2\times Q}\Bigg\|_{L^2}^p$$

$$\leqslant C\left(\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}|B_{I_1,I_2,k,j}||I_1||I_2|2^{k[n_1-p(n_1+\varepsilon')]\frac{2}{2-p}}2^{j[n_2-p(n_2+\varepsilon')]\frac{2}{2-p}}\right)^{1-\frac{p}{2}}$$

$$\times\left(\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}\sum_{\substack{Q\in\mathcal{M}(B_{I_1,I_2,k,j})\\ I_1\times I_2\times Q\in\mathcal{M}_3(\Omega)}}\|a_{I_1\times I_2\times Q}\|_2^2\right)^{\frac{p}{2}}$$

$$\leqslant C|\Omega|^{1-\frac{p}{2}}|\Omega|^{\frac{p}{2}-1}=C, \tag{2.9}$$

where in the last inequality we again apply the following variant of covering lemma due to J. Pipher [21] (see also [1])

$$\sum_{I_1,I_2}\sum_{k,j=0}^{\infty}|B_{I_1,I_2,k,j}||I_1||I_2|2^{-k\delta}2^{-j\delta}\leqslant C|\Omega|\quad\text{for any }\delta>0. \tag{2.10}$$

Now we estimate $W_{1,2}$. For any $Q\in\mathcal{M}(B_{I_1,I_2,k,j})$, we have from the cancellation conditions of $a_{I_1\times I_2\times Q}$,

$$\begin{aligned}
&T_{t_1,t_2,t_3}(a_{I_1\times I_2\times Q})(x_1,x_2,x_3)\\
&\quad=\int_{2(I_1\times I_2\times Q)}\big[T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,y_1,y_2,y_3)-T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,x_{I_1},y_2,y_3)-T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,y_1,x_{I_2},y_3)\\
&\qquad+T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,x_{I_1},x_{I_2},y_3)-T_{t_1,t_2,t_3}(x,y_1,y_2,x_{I_3})+T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,x_{I_1},y_2,x_{I_3})\\
&\qquad+T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,y_1,x_{I_2},x_{I_3})-T_{t_1,t_2,t_3}(\cdot,\cdot,\cdot,x_{I_1},x_{I_2},x_{I_3})\big](x_1,x_2,x_3)\\
&\qquad\times a_{I_1\times I_2\times Q}(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3,
\end{aligned}$$

where x_{I_i} is the center of I_i, $i=1,2,3$. For any $Q\in\mathcal{M}(B_{I_1,I_2,k,j})$, if $x_1\in(4\sqrt{n_1}\,2^kI_1)^c$, $x_2\in(2^{j+2}I_2)^c$, $x_3\in(4\Theta_{k,j})^c$, $y_1\in 2I_1$, $y_2\in 2I_2$, and $y_3\in 2Q$, then $|y_i-x_i|\leqslant\frac{1}{2}|x_i-x_{I_i}|$, $1\leqslant i\leqslant 3$. Thus Minkowski's and Hölder's inequalities together with Theorem 3.1(B_3) imply that

$$\Bigg\|\Bigg\{T_{t_1,t_2,t_3}\Bigg(\sum_{Q\in\mathcal{M}(B_{I_1,I_2,k,j})}a_{I_1\times I_2\times Q}\Bigg)\Bigg\}(x_1,x_2,x_3)\Bigg\|_{\mathcal{H}}$$

$$\leqslant C|I_1 \times I_2 \times B_{I_1,I_2,k,j}|^{\frac{1}{2}} \frac{\ell(I_1)^{\varepsilon'}}{|x_1 - x_{I_1}|^{n_1+\varepsilon'}} \frac{\ell(I_2)^{\varepsilon'}}{|x_2 - x_{I_2}|^{n_2+\varepsilon'}} \frac{\ell(B_{I_1,I_2,k,j})^{\varepsilon'}}{|x_3 - x_{I_3}|^{n_3+\varepsilon'}} \times \Big\| \sum_{Q \in \mathcal{M}(B_{I_1,I_2,k,j})} a_{I_1 \times I_2 \times Q} \Big\|_{L^2}.$$

Hence for $\frac{n_i}{n_i+\varepsilon} < p \leqslant 1$, $i = 1, 2, 3$,

$$\begin{aligned}
W_{1,2} &\leqslant C \sum_{I_1,I_2} \sum_{k,j=0}^{\infty} |I_1 \times I_2 \times B_{I_1,I_2,k,j}|^{\frac{p}{2}} \Big\| \sum_{Q \in \mathcal{M}(B_{I_1,I_2,k,j})} a_{I_1 \times I_2 \times Q} \Big\|_{L^2}^p \\
&\quad \times \int_{(4\sqrt{n_1}2^k I_1)^c \times (2^{j+2} I_2)^c \times (4\Theta_{k,j})^c} \frac{\ell(I_1)^{p\varepsilon'}}{|x_1 - x_{n_1}|^{p(n_1+\varepsilon')}} \frac{\ell(I_2)^{p\varepsilon'}}{|x_2 - x_{n_2}|^{p(n_2+\varepsilon')}} \frac{\ell(B_{I_1,I_2,k,j})^{p\varepsilon'}}{|x_3 - x_{n_3}|^{p(n_3+\varepsilon')}} \\
&\leqslant C \sum_{I_1,I_2} \sum_{k,j=0}^{\infty} |I_1 \times I_2 \times B_{I_1,I_2,k,j}|^{1-\frac{p}{2}} 2^{k[n_1 - p(n_1+\varepsilon')]} 2^{j[n_2 - p(n_2+\varepsilon')]} \\
&\quad \times \Big\| \sum_{Q \in \mathcal{M}(B_{I_1,I_2,k,j})} a_{I_1 \times I_2 \times Q} \Big\|_{L^2}^p \\
&\leqslant C \Bigg(\sum_{I_1,I_2} \sum_{k,j=0}^{\infty} |I_1 \times I_2 \times B_{I_1,I_2,k,j}| 2^{k[n_1 - p(n_1+\varepsilon')]\frac{2}{2-p}} 2^{j[n_2 - p(n_2+\varepsilon')]\frac{2}{2-p}} \Bigg)^{1-\frac{p}{2}} \\
&\quad \times \Bigg(\sum_{I_1,I_2} \sum_{k,j=0}^{\infty} \sum_{Q \in \mathcal{M}(B_{I_1,I_2,k,j})} \|a_{I_1 \times I_2 \times Q}\|_{L^2}^2 \Bigg)^{\frac{p}{2}} \\
&\leqslant C|\Omega|^{1-\frac{p}{2}} |\Omega|^{\frac{p}{2}-1} = C,
\end{aligned} \tag{2.11}$$

where in the last inequality we use the estimate in (2.10). From (2.9) and (2.11), we see that $|W_1| \leqslant C$ for some positive constant C under the condition $\frac{n_i}{n_i+\varepsilon} < p \leqslant 1$, $i = 1, 2, 3$. Following similar arguments, we can derive that $|W_2| \leqslant C$ by applying the regularity of T_{t_1,t_2,t_3} given in Corollary 3.2(v) instead of Corollary 3.2(iv). Thus, $|U_{1,2}| \leqslant C$ and hence, $|U_1| \leqslant C$.

Similarly, by applying Corollary 3.2(ii) and (iv) and Theorem 3.1(B_3), we can conclude $|U_2| \leqslant C$. Using Corollary 3.2(iii) and (v) and Theorem 3.1(B_3), we can yield that $|U_3| \leqslant C$. We leave the details to the reader. Finally we obtain

$$\int_{(D_4)^c} \big\| \big\{ T_{t_1,t_2,t_3}(a) \big\}(x_1, x_2, x_3) \big\|_{\mathcal{H}}^p \, dx_1 \, dx_2 \, dx_3 \leqslant C.$$

The proof of Theorem 1.6 is complete. □

Combining Lemma 1.5 and Theorem 1.6, we conclude Theorem 1.4. □

3. The regularities of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$

The main purpose of this section is to show the regularities of $\{T_{t_1,t_2,t_3}\}_{t_1,t_2,t_3>0}$.

We first write out the kernel of the operator T_{t_1,t_2,t_3} for any $t_1, t_2, t_3 > 0$. To do this, for $f \in L^2$, by the L^2 boundedness of the operator T and Calderón's identity on L^2, we obtain

$$\begin{aligned}
&T_{t_1,t_2,t_3}(f)(x_1,x_2,x_3)\\
&\quad = (\psi_{t_1,t_2,t_3} * Tf)(x_1,x_2,x_3)\\
&\quad = \left[\psi_{t_1,t_2,t_3} * T\left(\int_0^\infty\int_0^\infty\int_0^\infty \psi_{s_1,s_2,s_3} * \psi_{s_1,s_2,s_3} * f(\cdot,\cdot,\cdot)\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\frac{ds_3}{s_3}\right)\right](x_1,x_2,x_3).
\end{aligned}$$

This implies that the kernel of T_{t_1,t_2,t_3} can be expressed by

$$\begin{aligned}
&T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int\int \psi_{t_1,t_2,t_3}(x_1-u_1,x_2-u_2,x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3)\\
&\quad \times \psi_{s_1,s_2,s_3} * \psi_{s_1,s_2,s_3}(v_1-y_1,v_2-y_2,v_3-y_3)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\frac{ds_3}{s_3}.
\end{aligned}$$

The main result of this section is the following

Theorem 3.1. *Under the assumption $T_1^*(1) = T_2^*(1) = T_3^*(1) = 0$ for the Journé class of singular operator T given in Definition* 1.3*, $T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)$, the kernel of the operator $T_{t_1,t_2,t_3} = \psi_{t_1,t_2,t_3} * T$, satisfies the following estimates*:

(B_1) *For* $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned}
&\big\|\big\{T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big)\big\}\big\|_{\mathcal{H}}\\
&\quad \leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n_1+\varepsilon'}}|x_2-y_2|^{-n_2}|x_3-y_3|^{-n_3} \quad \text{if } \big|y_1-y_1'\big| \leqslant \frac{1}{2}|x_1-y_1|.
\end{aligned}$$

The same difference estimates hold for y_2 and y_3, respectively.
(B_2) *For* $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned}
&\big\|\big\{T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big)\\
&\qquad - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3\big) + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3\big)\big\}\big\|_{\mathcal{H}}\\
&\quad \leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n_1+\varepsilon'}}\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{n_2+\varepsilon'}}|x_3-y_3|^{-n_3} \quad \text{if } \big|y_i-y_i'\big| \leqslant \frac{1}{2}|x_i-y_i|,\ i=1,2.
\end{aligned}$$

The same double difference estimates hold for y_1, y_3 and y_2, y_3, respectively.

(B_3) *For* $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned}
&\big\|\big\{T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3\big)\\
&\quad + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3\big) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2,y_3'\big)\\
&\quad + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3'\big) + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3'\big)\\
&\quad - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3'\big)\big\}\big\|_{\mathcal{H}}\\
&\leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n_1+\varepsilon'}}\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{n_2+\varepsilon'}}\frac{|y_3-y_3'|^{\varepsilon'}}{|x_3-y_3|^{n_3+\varepsilon'}}\quad \text{if } \big|y_i-y_i'\big|\leqslant\frac{1}{2}|x_i-y_i|,\ i=1,2,3.
\end{aligned}$$

As a consequence of Theorem 3.1, we directly obtain the following regularities of the operator T_{t_1,t_2,t_3}.

Corollary 3.2. (i) *Suppose* $f\in L^2(\mathbb{R}^{n_2+n_3})$. *If* $|y_1-x_{I_1}|\leqslant\frac{1}{2}|x_1-x_{I_1}|$, *then*

$$\begin{aligned}
&\bigg\|\bigg\{\int\limits_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}}\big[T_{t_1,t_2,t_3}(x_1,\cdot,\cdot,y_1,y_2,y_3)\\
&\qquad - T_{t_1,t_2,t_3}(x_1,\cdot,\cdot,x_{I_1},y_2,y_3)\big]f(y_2,y_3)\,dy_2\,dy_3\bigg\}\bigg\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_2+n_3})}\\
&\leqslant C\frac{|y_1-x_{I_1}|^{\varepsilon'}}{|x_1-x_{I_1}|^{n_1+\varepsilon'}}\|f\|_{L^2(\mathbb{R}^{n_2+n_3})};
\end{aligned}$$

(ii) *Suppose* $f\in L^2(\mathbb{R}^{n_1+n_3})$. *If* $|y_2-x_{I_2}|\leqslant\frac{1}{2}|x_2-x_{I_2}|$, *then*

$$\begin{aligned}
&\bigg\|\bigg\{\int\limits_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_3}}\big[T_{t_1,t_2,t_3}(\cdot,x_2,\cdot,y_1,y_2,y_3)\\
&\qquad - T_{t_1,t_2,t_3}(\cdot,x_2,\cdot,y_1,x_{I_2},y_3)\big]f(y_1,y_3)\,dy_1\,dy_3\bigg\}\bigg\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_3})}\\
&\leqslant C\frac{|y_2-x_{I_2}|^{\varepsilon'}}{|x_2-x_{I_2}|^{n_2+\varepsilon'}}\|f\|_{L^2(\mathbb{R}^{n_1+n_3})};
\end{aligned}$$

(iii) *Suppose* $f\in L^2(\mathbb{R}^{n_1+n_2})$. *If* $|y_3-x_{I_3}|\leqslant\frac{1}{2}|x_3-x_{I_3}|$, *then*

$$\begin{aligned}
&\bigg\|\bigg\{\int\limits_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\big[T_{t_1,t_2,t_3}(\cdot,\cdot,x_3,y_1,y_2,y_3)\\
&\qquad - T_{t_1,t_2,t_3}(\cdot,\cdot,x_3,y_1,y_2,x_{I_3})\big]f(y_1,y_2)\,dy_1\,dy_2\bigg\}\bigg\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})}\\
&\leqslant C\frac{|y_3-x_{I_3}|^{\varepsilon'}}{|x_3-x_{I_3}|^{n_3+\varepsilon'}}\|f\|_{L^2(\mathbb{R}^{n_1+n_2})};
\end{aligned}$$

(iv) *Suppose* $f \in L^2(\mathbb{R}^{n_3})$. *If* $|y_1 - x_{I_1}| \leqslant \frac{1}{2}|x_1 - x_{I_1}|$ *and* $|y_2 - x_{I_2}| \leqslant \frac{1}{2}|x_2 - x_{I_2}|$, *then*

$$\Bigg\| \bigg\{ \int_{\mathbb{R}^{n_3}} \big[T_{t_1,t_2,t_3}(x_1, x_2, \cdot, y_1, y_2, y_3) - T_{t_1,t_2,t_3}(x_1, x_2, \cdot, x_{I_1}, y_2, y_3) - T_{t_1,t_2,t_3}(x_1, x_2, \cdot, y_1, x_{I_2}, y_3) + T_{t_1,t_2,t_3}(x_1, x_2, \cdot, x_{I_1}, x_{I_2}, y_3)\big] f(y_3)\, dy_3 \bigg\} \Bigg\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_3})} \leqslant C \frac{|y_1 - x_{I_1}|^{\varepsilon'}}{|x_1 - x_{I_1}|^{n_1+\varepsilon'}} \frac{|y_2 - x_{I_2}|^{\varepsilon'}}{|x_2 - x_{I_2}|^{n_2+\varepsilon'}} \|f\|_{L^2(\mathbb{R}^{n_3})};$$

(v) *Suppose* $f \in L^2(\mathbb{R}^{n_2})$. *If* $|y_1 - x_{I_1}| \leqslant \frac{1}{2}|x_1 - x_{I_1}|$ *and* $|y_3 - x_{I_3}| \leqslant \frac{1}{2}|x_3 - x_{I_3}|$, *then*

$$\Bigg\| \bigg\{ \int_{\mathbb{R}^{n_2}} \big[T_{t_1,t_2,t_3}(x_1, \cdot, x_3, y_1, y_2, y_3) - T_{t_1,t_2,t_3}(x_1, \cdot, x_3, x_{I_1}, y_2, y_3) - T_{t_1,t_2,t_3}(x_1, \cdot, x_3, y_1, y_2, x_{I_3}) + T_{t_1,t_2,t_3}(x_1, \cdot, x_3, x_{I_1}, y_2, x_{I_3})\big] f(y_2)\, dy_2 \bigg\} \Bigg\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_2})} \leqslant C \frac{|y_1 - x_{I_1}|^{\varepsilon'}}{|x_1 - x_{I_1}|^{n_1+\varepsilon'}} \frac{|y_3 - x_{I_3}|^{\varepsilon'}}{|x_3 - x_{I_3}|^{n_3+\varepsilon'}} \|f\|_{L^2(\mathbb{R}^{n_2})};$$

(vi) *Suppose* $f \in L^2(\mathbb{R}^{n_1})$. *If* $|y_2 - x_{I_2}| \leqslant \frac{1}{2}|x_2 - x_{I_2}|$ *and* $|y_3 - x_{I_3}| \leqslant \frac{1}{2}|x_3 - x_{I_3}|$, *then*

$$\Bigg\| \bigg\{ \int_{\mathbb{R}^{n_1}} \big[T_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, y_3) - T_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, x_{I_2}, y_3) - T_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, x_{I_3}) + T_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, x_{I_2}, x_{I_3})\big] f(y_1)\, dy_1 \bigg\} \Bigg\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_1})} \leqslant C \frac{|y_2 - x_{I_2}|^{\varepsilon'}}{|x_2 - x_{I_2}|^{n_2+\varepsilon'}} \frac{|y_3 - x_{I_3}|^{\varepsilon'}}{|x_3 - x_{I_3}|^{n_3+\varepsilon'}} \|f\|_{L^2(\mathbb{R}^{n_1})}.$$

Indeed, it is easy to see that (B_1) and (B_2) of Theorem 3.1 yield (i), (ii) and (iii) in Corollary 3.2 and (B_2) and (B_3) of Theorem 3.1 imply (iv)–(vi) of Corollary 3.2. We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. To obtain the estimates (B_1)–(B_3), as we mentioned in Section 1, we decompose T_{t_1,t_2,t_3} based on the following principle: if only some one $s_i > t_i$, $i = 1, 2, 3$, then we take the difference for $\psi^i_{s_i}$. For instance, if $s_3 > t_3$, we write

$$\psi^1_{t_1}(x_1 - u_1)\psi^2_{t_2}(x_2 - u_2)\psi^3_{t_3}(x_3 - u_3)\mathcal{K}(u_1, u_2, u_3, v_1, v_2, v_3)\psi^1_{s_1} * \psi^1_{s_1}(v_1 - y_1) \times \psi^2_{s_2} * \psi^2_{s_2}(v_2 - y_2)\big[\psi^3_{s_3}(v_3 - z_3) - \psi^3_{s_3}(x_3 - z_3)\big]\psi^3_{s_3}(z_3 - y_3).$$

Similarly, if two of s_1, s_2, s_3 are larger than the corresponding two of t_1, t_2, t_3, for example, $s_2 > t_2$ and $s_3 > t_3$, then we write

$$\psi^1_{t_1}(x_1-u_1)\psi^2_{t_2}(x_2-u_2)\psi^3_{t_3}(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3)\psi^1_{s_1}*\psi^1_{s_1}(v_1-y_1) \times\big[\psi^2_{s_2}(v_2-z_2)-\psi^2_{s_2}(x_2-z_2)\big]\big[\psi^3_{s_3}(v_3-z_3)-\psi^3_{s_3}(x_3-z_3)\big].$$

If $s_1>t_1, s_2>t_2$ and $s_3>t_3$, we take all difference for $\psi^1_{s_1}$, $\psi^2_{s_2}$ and $\psi^3_{s_3}$. Of course, we have to add all remainder terms. Therefore, there are totally twenty seven terms, but each term belongs to one and only one of the following forms.

Form (1):

$$\int_0^{t_3}\int_0^{t_2}\int_0^{t_1}\int\int \psi^1_{t_1}(x_1-u_1)\psi^2_{t_2}(x_2-u_2)\psi^3_{t_3}(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3) \times\psi^1_{s_1}*\psi^1_{s_1}(v_1-y_1)\psi^2_{s_2}*\psi^2_{s_2}(v_2-y_2) \times\psi^3_{s_3}*\psi^3_{s_3}(v_3-y_3)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\frac{ds_3}{s_3}.$$

Form (2):

$$\int_{t_3}^{\infty}\int_0^{t_2}\int_0^{t_1}\int_{\mathbb{R}^{n_3}}\int\int \psi^1_{t_1}(x_1-u_1)\psi^2_{t_2}(x_2-u_2)\psi^3_{t_3}(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3) \times\psi^1_{s_1}*\psi^1_{s_1}(v_1-y_1)\psi^2_{s_2}*\psi^2_{s_2}(v_2-y_2)\big[\psi^3_{s_3}(v_3-z_3)-\psi^3_{s_3}(x_3-z_3)\big] \times\psi^3_{s_3}(z_3-y_3)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,dz_3\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\frac{ds_3}{s_3}.$$

Form (3):

$$\int_0^{t_2}\int_0^{t_1}\int\int \psi^1_{t_1}(x_1-u_1)\psi^2_{t_2}(x_2-u_2)\psi^3_{t_3}(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3) \times\psi^1_{s_1}*\psi^1_{s_1}(v_1-y_1)\psi^2_{s_2}*\psi^2_{s_2}(v_2-y_2)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\phi^3_{t_3}(x_3-y_3).$$

Form (4):

$$\int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_0^{t_1}\int_{\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}}\int\int \psi^1_{t_1}(x_1-u_1)\psi^2_{t_2}(x_2-u_2)\psi^3_{t_3}(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3) \times\psi^1_{s_1}*\psi^1_{s_1}(v_1-y_1)\big[\psi^2_{s_2}(v_2-z_2)-\psi^2_{s_2}(x_2-z_2)\big]\big[\psi^3_{s_3}(v_3-z_3)-\psi^3_{s_3}(x_3-z_3)\big] \times\psi^2_{s_2}(z_2-y_2)\psi^3_{s_3}(z_3-y_3)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,dz_2\,dz_3\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\frac{ds_3}{s_3}.$$

Form (5):

$$\int_{t_2}^{\infty}\int_0^{t_1}\int_{\mathbb{R}^{n_2}}\int\int \psi_{t_1}^1(x_1-u_1)\psi_{t_2}^2(x_2-u_2)\psi_{t_3}^3(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3)$$
$$\times \psi_{s_1}^1*\psi_{s_1}^1(v_1-y_1)\big[\psi_{s_2}^2(v_2-z_2)-\psi_{s_2}^2(x_2-z_2)\big]$$
$$\times \psi_{s_2}^2(z_2-y_2)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,dz_2\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\phi_{t_3}^3(x_3-y_3).$$

Form (6):

$$\int_0^{t_1}\int\int \psi_{t_1}^1(x_1-u_1)\psi_{t_2}^2(x_2-u_2)\psi_{t_3}^3(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3)$$
$$\times \psi_{s_1}^1*\psi_{s_1}^1(v_1-y_1)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,\phi_{t_2}^2(x_2-y_2)\,\frac{ds_1}{s_1}\,\phi_{t_3}^3(x_3-y_3).$$

Form (7):

$$\int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_1}^{\infty}\int\int\int \psi_{t_1}^1(x_1-u_1)\psi_{t_2}^2(x_2-u_2)\psi_{t_3}^3(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3)$$
$$\times \big[\psi_{s_1}^1(v_1-z_1)-\psi_{s_1}^1(x_1-z_1)\big]\big[\psi_{s_2}^2(v_2-z_2)-\psi_{s_2}^2(x_2-z_2)\big]$$
$$\times \big[\psi_{s_3}^3(v_3-z_3)-\psi_{s_3}^3(x_3-z_3)\big]\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\frac{ds_3}{s_3}.$$

Form (8):

$$\int_{t_2}^{\infty}\int_{t_1}^{\infty}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int\int \psi_{t_1}^1(x_1-u_1)\psi_{t_2}^2(x_2-u_2)\psi_{t_3}^3(x_3-u_3)\mathcal{K}(u_1,u_2,u_3,v_1,v_2,v_3)$$
$$\times \big[\psi_{s_1}^1(v_1-z_1)-\psi_{s_1}^1(x_1-z_1)\big]\big[\psi_{s_2}^2(v_2-z_2)-\psi_{s_2}^2(x_2-z_2)\big]$$
$$\times \psi_{s_1}^1(z_1-y_1)\psi_{s_2}^2(z_2-y_2)\,du_1\,du_2\,du_3\,dv_1\,dv_2\,dv_3\,dz_1\,dz_2\,\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\,\phi_{t_3}^3(x_3-y_3).$$

Form (9):

$$\psi_{t_1,t_2,t_3}*T(1)(x_1,x_2,x_3)\phi_{t_1}^1(x_1-y_1)\phi_{t_2}^2(x_2-y_2)\phi_{t_3}^3(x_3-y_3)$$

where $\phi_{t_i}^i(x_i)=\int_{t_i}^{\infty}\psi_{s_i}^i*\psi_{s_i}^i\,\frac{ds_i}{s_i}$, $i=1,2,3$.

Now we write $T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)=\sum_{j=1}^{27}T_{t_1,t_2,t_3}^j(x_1,x_2,x_3,y_1,y_2,y_3)$, where for each j, $1\leqslant j\leqslant 27$, $T_{t_1,t_2,t_3}^j(x_1,x_2,x_3,y_1,y_2,y_3)$ belongs to one of the forms (1)–(9).

Thus, the estimates (B_1)–(B_3) in Theorem 3.1 will follow from the following lemma.

Lemma 3.3. *For $1 \leqslant j \leqslant 27$ and $t_1, t_2, t_3 > 0$, there exists a constant C such that*:

(D_1) *For $0 < \varepsilon'' < \varepsilon' < \varepsilon$, if $|y_1 - y_1'| \leqslant \frac{1}{2} t_1$, then*

$$\begin{aligned}&\left|T^j_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big)\right| \\ &\quad \leqslant C\left(\frac{|y_1-y_1'|}{t_1}\right)^{\varepsilon''} \frac{(t_1)^{\varepsilon'}}{(t_1+|x_1-y_1|)^{n_1+\varepsilon'}} \frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}} \frac{(t_3)^{\varepsilon'}}{(t_3+|x_3-y_3|)^{n_3+\varepsilon'}}.\end{aligned}$$

The same estimates hold if the difference is taken over y_2, and y_3, respectively.
(D_2) *For $0 < \varepsilon'' < \varepsilon' < \varepsilon$, if $|y_1 - y_1'| \leqslant \frac{1}{2} t_1$ and $|y_2 - y_2'| \leqslant \frac{1}{2} t_2$, then*

$$\begin{aligned}&\big|T^j_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big) - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3\big) \\ &\qquad + T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3\big)\big| \\ &\quad \leqslant C\left(\frac{|y_1-y_1'|}{t_1}\right)^{\varepsilon''}\left(\frac{|y_2-y_2'|}{t_2}\right)^{\varepsilon''} \frac{(t_1)^{\varepsilon'}}{(t_1+|x_1-y_1|)^{n_1+\varepsilon'}} \frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}} \\ &\qquad \times \frac{(t_3)^{\varepsilon'}}{(t_3+|x_3-y_3|)^{n_3+\varepsilon'}}.\end{aligned}$$

The same estimates hold if the double difference is taken over y_1, y_3, and y_2, y_3, respectively.
(D_3) *For $0 < \varepsilon'' < \varepsilon' < \varepsilon$, if $|y_1 - y_1'| \leqslant \frac{1}{2} t_1$, $|y_2 - y_2'| \leqslant \frac{1}{2} t_2$ and $|y_3 - y_3'| \leqslant \frac{1}{2} t_3$, then*

$$\begin{aligned}&\big|T^j_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big) - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3\big) \\ &\qquad + T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3\big) - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2,y_3'\big) \\ &\qquad + T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3'\big) + T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3'\big) \\ &\qquad - T^j_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3'\big)\big| \\ &\quad \leqslant C\left(\frac{|y_3-y_3'|}{t_3}\right)^{\varepsilon''}\left(\frac{|y_2-y_2'|}{t_2}\right)^{\varepsilon''}\left(\frac{|y_3-y_3'|}{t_3}\right)^{\varepsilon''} \frac{(t_1)^{\varepsilon'}}{(t_1+|x_1-y_1|)^{n_1+\varepsilon'}} \frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}} \\ &\qquad \times \frac{(t_3)^{\varepsilon'}}{(t_3+|x_3-y_3|)^{n_3+\varepsilon'}}.\end{aligned}$$

To prove Lemma 3.3, we need the following classical almost orthogonal estimates on Euclidean space. See [14] for the proof.

Proposition 3.4. *Suppose $\mathcal{S}$ is a Calderón–Zygmund operator with regularity exponent ε associated with a kernel $\mathcal{S}(u,v)$ and $\mathcal{S}^*(1) = 0$. Then for all $\psi \in C^\infty_{0,0}(\mathbb{R}^n)$ and $0 < \varepsilon'' < \varepsilon' < \varepsilon$,*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-u)\mathcal{S}(u,v)\psi_s(v-y)\,du\,dv\right| \leqslant C\|\mathcal{S}\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''} \frac{t^{\varepsilon'}}{(t+|x-y|)^{n+\varepsilon'}} \quad \text{if } s \leqslant t, \tag{3.1}$$

and

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-u)\mathcal{S}(u,v)\big[\psi_s(v-y)-\psi_s(x-y)\big]\,du\,dv\right| \leqslant C\|\mathcal{S}\|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon''}\frac{s^{\varepsilon'}}{(s+|x-y|)^{n+\varepsilon'}} \quad \textit{if } t<s. \tag{3.2}$$

For such $\mathcal{S}$ in Proposition 3.4, we obtain the following almost orthogonal estimates from (3.1) and (3.2)

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-z)\mathcal{S}(z,w)\psi_s*\psi_s(w-y)\,dz\,dw\right| \leqslant C\|\mathcal{S}\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x-y|)^{n+\varepsilon'}} \quad \text{for } s\leqslant t, \tag{3.3}$$

and

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-z)\mathcal{S}(z,w)\big[\psi_s(w-u)-\psi_s(x-u)\big]\psi_s(u-y)\,du\,dw\,dz\right| \leqslant C\|\mathcal{S}\|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon''}\frac{s^{\varepsilon'}}{(s+|x-y|)^{n+\varepsilon'}} \quad \text{for } t<s. \tag{3.4}$$

Moreover, by the almost orthogonal estimate in (3.1) and the smoothness condition on ψ_s,

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-z)\mathcal{S}(z,w)\psi_s(w-u)\big[\psi_s(u-y)-\psi_s\big(u-y'\big)\big]\,du\,dw\,dz\right| \leqslant C\|\mathcal{S}\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''-\varepsilon'''}\left(\frac{|y-y'|}{t}\right)^{\varepsilon'''}\frac{t^{\varepsilon'}}{(t+|x-y|)^{n+\varepsilon'}} \quad \text{for } \big|y-y'\big|\leqslant\frac{t}{2} \text{ and } s\leqslant t, \tag{3.5}$$

and by the estimate (3.2) and the smoothness condition on ψ_s,

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-z)\mathcal{S}(z,w)\big[\psi_s(w-u)-\psi_s(x-u)\big]\big[\psi_s(u-y)-\psi_s\big(u-y'\big)\big]\,du\,dw\,dz\right| \leqslant C\|\mathcal{S}\|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon''}\left(\frac{|y-y'|}{s}\right)^{\varepsilon'''}\frac{s^{\varepsilon'}}{(s+|x-y|)^{n+\varepsilon'}} \quad \text{for } \big|y-y'\big|\leqslant\frac{t}{2} \text{ and } t<s. \tag{3.6}$$

Proof of Lemma 3.3. We first show that $T^1_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)$, which belongs to the form (1), satisfies (D_1)–(D_3). To show $T^1_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)$ satisfies (D_1), we only show the case when its difference is taken over y_1 since the other one is similar. Set

$$\mathcal{K}_{3,1}(u_3,v_3)(u_1,v_1) = \int_0^{t_2}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} \psi_{t_2}^2(x_2-u_2)\psi_{s_2}^2 * \psi_{s_2}^2(v_2-y_2)\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)(u_2,v_2)\,du_2\,dv_2\,\frac{ds_2}{s_2},$$

where $\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)$ is the same as in Definition 1.3.

Since

$$\big\|\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)\big\|_{CZ} \leqslant C|u_1-v_1|^{-n_1}|u_3-v_3|^{-n_3},$$

we have from $T_2^*(1)=0$ and the almost orthogonality estimate in (3.3) that

$$\begin{aligned}\big|\mathcal{K}_{3,1}(u_3,v_3)(u_1,v_1)\big| &\leqslant C\int_0^{t_2}\big\|\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)\big\|_{CZ}\left(\frac{s_2}{t_2}\right)^{\varepsilon''}\frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}}\frac{ds_2}{s_2} \\ &\leqslant C\frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}}|u_1-v_1|^{-n_1}|u_3-v_3|^{-n_3}.\end{aligned}$$

Similarly, when u_3, v_3, u_1, u_1' and v_1 are fixed, we have

$$\begin{aligned}&\mathcal{K}_{3,1}(u_3,v_3)(u_1,v_1)-\mathcal{K}_{3,1}(u_3,v_3)\big(u_1',v_1\big) \\ &\quad= \int_0^{t_2}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} \psi_{t_2}^2(x_2-u_2)\big[\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)(u_2,v_2)-\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)\big(u_1',v_1\big)(u_2,v_2)\big] \\ &\qquad\times \psi_{s_2}^2 * \psi_{s_2}^2(v_2-y_2)\,du_2\,dv_2\,\frac{ds_2}{s_2}.\end{aligned}$$

Note again that $\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)(u_2,v_2) - \widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1',v_1)(u_2,v_2)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_2}\times\mathbb{R}^{n_2}$ with the norm

$$\begin{aligned}&\big\|\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)-\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)\big(u_1',v_1\big)\big\|_{CZ} \\ &\quad\leqslant C|u_3-v_3|^{-n_3}\big|u_1-u_1'\big|^{\varepsilon}|u_1-v_1|^{-n_1-\varepsilon}, \quad \text{if } \big|u_1-u_1'\big|\leqslant\frac{1}{2}|u_1-v_1|.\end{aligned}$$

Thus,

$$\begin{aligned}&\big|\mathcal{K}_{3,1}(u_3,v_3)(u_1,v_1)-\mathcal{K}_{3,1}(u_3,v_3)\big(u_1',v_1\big)\big| \\ &\quad\leqslant C\int_0^{t_2}\big\|\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)-\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)\big(u_1',v_1\big)\big\|_{CZ}\left(\frac{s_2}{t_2}\right)^{\varepsilon''}\frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}}\frac{ds_2}{s_2} \\ &\quad\leqslant C|u_3-v_3|^{-n_3}\big|u_1-u_1'\big|^{\varepsilon}|u_1-v_1|^{-n_1-\varepsilon}\frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}} \quad \text{if } \big|u_1-u_1'\big|\leqslant\frac{1}{2}|u_1-v_1|.\end{aligned}$$

A similar argument shows if $|v_1 - v_1'| \leqslant \frac{1}{2}|u_1 - v_1|$, then

$$\begin{aligned}
&\big|\mathcal{K}_{3,1}(u_3, v_3)(u_1, v_1) - \mathcal{K}_{3,1}(u_3, v_3)\big(u_1, v_1'\big)\big| \\
&\quad \leqslant C|u_3 - v_3|^{-n_3} \frac{|v_1 - v_1'|^{\varepsilon}}{|u_1 - v_1|^{n_1+\varepsilon}} \frac{(t_2)^{\varepsilon'}}{(t_2 + |x_2 - y_2|)^{n_2+\varepsilon'}}.
\end{aligned}$$

These imply that for fixed u_3 and v_3 on $\mathbb{R}^{n_3}$, $\mathcal{K}_{3,1}(u_3, v_3)(u_1, v_1)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ with the norm

$$\big\|\mathcal{K}_{3,1}(u_3, v_3)\big\|_{CZ} \leqslant C \frac{(t_2)^{\varepsilon'}}{(t_2 + |x_2 - y_2|)^{n_2+\varepsilon'}} |u_3 - v_3|^{-n_3}. \tag{3.7}$$

Set

$$\begin{aligned}
\widehat{\mathcal{K}}_3(u_3, v_3) = &\int_0^{t_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \psi_{t_1}^1(x_1 - u_1)\mathcal{K}_{3,1}(u_3, v_3)(u_1, v_1)\psi_{s_1}^1(v_1 - z_1) \\
&\times \big[\psi_{s_1}^1(z_1 - y_1) - \psi_{s_1}^1\big(z_1 - y_1'\big)\big]\, dz_1\, du_1\, dv_1 \frac{ds_1}{s_1}.
\end{aligned} \tag{3.8}$$

Then by estimates in (3.5) and (3.7), we could derive that $\widehat{\mathcal{K}}_3(u_3, v_3)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_3} \times \mathbb{R}^{n_3}$ and for $|y_1 - y_1'| < \frac{t_1}{2}$,

$$\|\widehat{\mathcal{K}}_3\|_{CZ} \leqslant C \frac{(t_1)^{\varepsilon'}}{(t_1 + |x_1 - y_1|)^{n_1+\varepsilon'}} \frac{(t_2)^{\varepsilon'}}{(t_2 + |x_2 - y_2|)^{n_2+\varepsilon'}} \left(\frac{|y_1 - y_1'|}{t_1}\right)^{\varepsilon''}. \tag{3.9}$$

From (3.8) we can write

$$\begin{aligned}
&T_{t_1,t_2,t_3}^1(x_1, x_2, x_3, y_1, y_2, y_3) - T_{t_1,t_2,t_3}^1\big(x_1, x_2, x_3, y_1', y_2, y_3\big) \\
&\quad = \int_0^{t_3} \int_{\mathbb{R}^{n_3}} \int_{\mathbb{R}^{n_3}} \psi_{t_3}^3(x_3 - u_3)\psi_{s_3}^3 * \psi_{s_3}^3(v_3 - y_3)\widehat{\mathcal{K}}_3(u_3, v_3)\, du_3\, dv_3 \frac{ds_3}{s_3}.
\end{aligned}$$

For $|y_1 - y_1'| < \frac{t_1}{2}$, $T_3^*(1) = 0$ and the almost orthogonality estimate in (3.3) together with (3.9) yield that

$$\begin{aligned}
&\big|T_{t_1,t_2,t_3}^1(x_1, x_2, x_3, y_1, y_2, y_3) - T_{t_1,t_2,t_3}^1\big(x_1, x_2, x_3, y_1', y_2, y_3\big)\big| \\
&\quad \leqslant C\left(\frac{|y_1 - y_1'|}{t_1}\right)^{\varepsilon''} \frac{(t_1)^{\varepsilon'}}{(t_1 + |x_1 - y_1|)^{n_1+\varepsilon'}} \frac{(t_2)^{\varepsilon'}}{(t_2 + |x_2 - y_2|)^{n_2+\varepsilon'}} \frac{(t_3)^{\varepsilon'}}{(t_3 + |x_3 - y_3|)^{n_3+\varepsilon'}}.
\end{aligned}$$

Now we prove that $T_{t_1,t_2,t_3}^1(x_1, x_2, x_3, y_1, y_2, y_3)$ satisfies (D_2) for its double difference taken over y_1 and y_2.

Set

$$\overline{\overline{\mathcal{K}}}_{3,1}(u_3,v_3)(u_1,v_1)=\int_0^{t_2}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\psi_{t_2}^2(x_2-u_2)\widetilde{\mathcal{K}}^{3,1}(u_3,v_3)(u_1,v_1)(u_2,v_2)\psi_{s_2}^2 \times\left[\psi_{s_2}^2(z_2-y_2)-\psi_{s_2}^2\left(z_2-y_2'\right)\right]du_2\,dv_2\,dz_2\,\frac{ds_2}{s_2},$$

and

$$\overline{\overline{\mathcal{K}}}_3(u_3,v_3)=\int_0^{t_1}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_1}}\psi_{t_1}^1(x_1-u_1)\overline{\overline{\mathcal{K}}}_{3,1}(u_3,v_3)(u_1,v_1)\psi_{s_1}^1(v_1-z_1) \times\left[\psi_{s_1}^1(z_1-y_1)-\psi_{s_1}^1\left(z_1-y_1'\right)\right]du_1\,dv_1\,dz_1\,\frac{ds_1}{s_1}.$$

Following a similar argument to that in (D_1), we can deduce that for fixed u_3 and v_3 on $\mathbb{R}^{n_3}$, $\overline{\overline{\mathcal{K}}}_{3,1}(u_3,v_3)(u_1,v_1)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}$ with the norm

$$\left\|\overline{\overline{\mathcal{K}}}_{3,1}(u_3,v_3)\right\|_{CZ}\leqslant C\frac{t_2^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}}|u_3-v_3|^{-n_3}\left(\frac{|y_2-y_2'|}{t_2}\right)^{\varepsilon''}\quad \text{for } \left|y_2-y_2'\right|\leqslant\frac{t_2}{2}.$$

We can also conclude that $\overline{\overline{\mathcal{K}}}_3(u_3,v_3)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_3}\times\mathbb{R}^{n_3}$ with the norm

$$\|\overline{\overline{\mathcal{K}}}_3\|_{CZ}\leqslant C\frac{t_1^{\varepsilon'}}{(t_1+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{t_2^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}}\left(\frac{|y_1-y_1'|}{t_1}\right)^{\varepsilon''}\left(\frac{|y_2-y_2'|}{t_2}\right)^{\varepsilon''}\tag{3.10}$$

when $|y_1-y_1'|\leqslant\frac{t_1}{2}$ and $|y_2-y_2'|\leqslant\frac{t_2}{2}$. Then from $T_3^*(1)=0$ and estimates in (3.3) and (3.10), we have

$$\begin{aligned}
&\left|T^1_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)-T^1_{t_1,t_2,t_3}\left(x_1,x_2,x_3,y_1',y_2,y_3\right)\right.\\
&\quad\left.-T^1_{t_1,t_2,t_3}\left(x_1,x_2,x_3,y_1,y_2',y_3\right)+T^1_{t_1,t_2,t_3}\left(x_1,x_2,x_3,y_1',y_2',y_3\right)\right|\\
&=\left|\int_0^{t_3}\int_{\mathbb{R}^{n_3}}\int_{\mathbb{R}^{n_3}}\psi_{t_3}^3(x_3-u_3)\overline{\overline{\mathcal{K}}}_3(u_3,v_3)\psi_{s_3}^3*\psi_{s_3}^3(v_3-y_3)\,du_3\,dv_3\,\frac{ds_3}{s_3}\right|\\
&\leqslant C\left(\frac{|y_1-y_1'|}{t_1}\right)^{\varepsilon''}\left(\frac{|y_2-y_2'|}{t_2}\right)^{\varepsilon''}\frac{(t_1)^{\varepsilon'}}{(t_1+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{(t_2)^{\varepsilon'}}{(t_2+|x_2-y_2|)^{n_2+\varepsilon'}}\\
&\quad\times\frac{(t_3)^{\varepsilon'}}{(t_3+|x_3-y_3|)^{n_3+\varepsilon'}}.
\end{aligned}$$

Finally, we prove that $T^1_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)$ satisfies (D_3). To do this, we write

$$\begin{aligned}
&T_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3\big)\\
&\quad + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3\big) - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3\big)\\
&\quad + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2,y_3\big) + T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1,y_2',y_3'\big)\\
&\quad - T_{t_1,t_2,t_3}\big(x_1,x_2,x_3,y_1',y_2',y_3'\big)\\
&= \int_0^{t_3}\int_{\mathbb{R}^{n_3}}\int_{\mathbb{R}^{n_3}}\int_{\mathbb{R}^{n_3}} \psi^3_{t_3}(x_3-u_3)\widehat{K}_3(u_3,v_3)\psi^3_{s_3}(v_3-z_3)\\
&\quad \times \big[\psi^3_{s_3}(z_3-y_3) - \psi^3_{s_3}\big(z_3-y_3'\big)\big]\,du_3\,dv_3\,dz_3\,\frac{ds_3}{s_3}.
\end{aligned}$$

Estimates in (3.5) and (3.9) together with $T_3^*(1)=0$ derive directly that $T^1_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)$ satisfies (D_3).

We now estimate $T^3_{t_1,t_2,t_3}(x_1,x_2,x_3,y_1,y_2,y_3)$ which belongs to the form (3). For fixed t_1,t_2, set

$$\mathcal{K}_{1,2}(u_1,v_1)(u_2,v_2) = \int_{\mathbb{R}^{n_3}}\int_{\mathbb{R}^{n_3}} \psi^3_{t_3}(x_3-u_3)\widetilde{\mathcal{K}}^{1,2}(u_1,v_1)(u_2,v_2)(u_3,v_3)\,du_3\,dv_3.$$

Note that for fixed (u_1,v_1) and (u_2,v_2), $\int_{\mathbb{R}^{n_3}}\widetilde{\mathcal{K}}^{1,2}(u_1,v_1)(u_2,v_2)(u_3,v_3)\,dv_3$, as a function of the variable u_3, is a *BMO* function with the norm

$$\Big\|\int_{\mathbb{R}^{n_3}}\widetilde{\mathcal{K}}^{1,2}(u_1,v_1)(u_2,v_2)(\cdot,v_3)\,dv_3\Big\|_{BMO(\mathbb{R}^{n_3})} \leqslant C|u_1-v_1|^{-n_1}|u_2-v_2|^{-n_2},$$

and $\psi^3_{t_3}(x_3)$ is a function in $H^1(\mathbb{R}^{n_3})$ with the $H^1(\mathbb{R}^{n_3})$ norm uniformly bounded for all x_3 and t_3, thus, $|\mathcal{K}_{1,2}(u_1,v_1)(u_2,v_2)| \leqslant C|u_1-v_1|^{-n_1}|u_2-v_2|^{-n_2}$.

Similarly, for $|u_2-u_2'| \leqslant \frac{1}{2}|u_2-v_2|$, we have

$$\begin{aligned}
&\big\|\big[\widetilde{\mathcal{K}}^{1,2}(u_1,v_1)(u_2,v_2)(\cdot,v_3) - \widetilde{\mathcal{K}}^{1,2}(u_1,v_1)\big(u_2',v_2\big)(\cdot,v_3)\big]\,dv_3\big\|_{BMO(\mathbb{R}^{n_3})}\\
&\quad \leqslant C\big\|\widetilde{\mathcal{K}}^{1,2}(u_1,v_1)(u_2,v_2) - \widetilde{\mathcal{K}}^{1,2}(u_1,v_1)\big(u_2',v_2\big)\big\|_{CZ}\\
&\quad \leqslant C|u_1-v_1|^{-n_1}\frac{|u_2-u_2'|^{\varepsilon}}{|u_2-v_2|^{n_2+\varepsilon}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\big|\mathcal{K}_{1,2}(u_1,v_1)(u_2,v_2) - \mathcal{K}_{1,2}(u_1,v_1)\big(u_2',v_2\big)\big|\\
&\quad \leqslant C\big\|\psi^3_{t_3}\big\|_{H^p(\mathbb{R}^{n_3})}\Big\|\int_{\mathbb{R}^{n_3}}\big[\widetilde{\mathcal{K}}^{1,2}(u_1,v_1)(u_2,v_2)(\cdot,v_3)
\end{aligned}$$

$$- \widetilde{\mathcal{K}}^{1,2}(u_1, v_1)(u_2', v_2)(\cdot, v_3)\big] \, dv_3 \Big\|_{BMO(\mathbb{R}^{n_3})}$$

$$\leqslant C|u_1 - v_1|^{-n_1} \frac{|u_2 - u_2'|^{\varepsilon}}{|u_2 - v_2|^{n_2+\varepsilon}} \quad \text{if } |u_2 - u_2'| \leqslant \frac{1}{2}|u_2 - v_2|.$$

Thus, for fixed $u_1, v_1 \in \mathbb{R}^{n_1}$, $\mathcal{K}_{1,2}(u_1, v_1)(u_2, v_2)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ with the norm

$$\big\|\mathcal{K}_{1,2}(u_1, v_1)\big\|_{CZ} \leqslant C|u_1 - v_1|^{-n_1}. \tag{3.11}$$

Denote that

$$\mathcal{K}_1(u_1, v_1) = \int_0^{t_2} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} \psi_{t_2}^2(x_2 - u_2)\mathcal{K}_{1,2}(u_1, v_1)(u_2, v_2)\psi_{s_2}^2 * \psi_{s_2}^2(v_2 - y_2) \, du_2 \, dv_2 \, \frac{ds_2}{s_2}.$$

By the similar argument as in $\mathcal{K}_2$, we can conclude that $\mathcal{K}_1(u_1, v_1)$ is a Calderón–Zygmund kernel on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ with the norm

$$\|\mathcal{K}_1\|_{CZ} \leqslant C \frac{(t_2)^{\varepsilon'}}{(t_2 + |x_2 - y_2|)^{n_2+\varepsilon'}}. \tag{3.12}$$

We write

$$\begin{aligned}
&T^3_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, y_3) - T^3_{t_1,t_2,t_3}\big(x_1, x_2, x_3, y_1', y_2, y_3\big) \\
&\quad = \int_0^{t_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \psi_{t_1}^1(x_1 - u_1)\mathcal{K}_1(u_1, v_1) \\
&\qquad \times \big[\psi_{s_1}^1 * \psi_{s_1}^1(v_1 - y_1) - \psi_{s_1}^1 * \psi_{s_1}^1\big(v_1 - y_1'\big)\big] \, du_1 \, dv_1 \frac{ds_1}{s_1} \phi_{t_3}^3(x_3 - y_3).
\end{aligned}$$

Since $T_1^*(1) = 0$, we have from the estimates in (3.3) and (3.11)

$$\begin{aligned}
&\big|T^3_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, y_3) - T^3_{t_1,t_2,t_3}\big(x_1, x_2, x_3, y_1', y_2, y_3\big)\big| \\
&\quad \leqslant C\bigg(\frac{|y_1 - y_1'|}{t_1}\bigg)^{\varepsilon''} \frac{(t_1)^{\varepsilon'}}{(t_1 + |x_1 - y_1|)^{n_1+\varepsilon'}} \frac{(t_2)^{\varepsilon'}}{(t_2 + |x_1 - y_2|)^{n_2+\varepsilon'}} \frac{(t_3)^{\varepsilon'}}{(t_3 + |x_3 - y_3|)^{n_3+\varepsilon'}},
\end{aligned}$$

which leads to (D_1). Since the proofs of the estimates in (D_2) and (D_3) are similar, we leave the details to the reader. Therefore, $T^3_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, y_3)$ satisfies (D_1)–(D_3).

Finally we estimate $T^{27}_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, y_3)$ which belongs to the form (9). Note that

$$T(1)(u_1, u_2, u_3) = \int \mathcal{K}(u_1, u_2, u_3, v_1, v_2, v_3) \, dv_1 \, dv_2 \, dv_3,$$

as a function of variables of u_1, u_2 and u_3 belongs to $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$ and $\psi^1_{t_1}(x_1 - u_1)\psi^2_{t_2}(x_2 - u_2)\psi^3_{t_3}(x_3 - u_3)$, as a function of variables u_1, u_2, u_3 is in $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$ with bounded norm uniformly for all t_1, t_2, t_3 and x_1, x_2, x_3. Using the size conditions of $\phi^1_{t_1}, \phi^2_{t_2}$ and $\phi^3_{t_3}$ we yield that $T^{27}_{t_1,t_2,t_3}(x_1, x_2, x_3, y_1, y_2, y_3)$ satisfies the estimates of (D_1)–(D_3). We omit the details here. The proof of Lemma 3.3 is complete and hence Theorem 3.1 follows. □

References

[1] C. Cabrelli Carlos, M. Lacey, U. Molter, J. Pipher, Variations on the theme of Journé's lemma, Houston J. Math. 32 (3) (2006) 833–861.

[2] A.P. Calderón, A. Zygmund, On the existence of certain singular integrals, Acta Math. 88 (1962) 85–139.

[3] A. Carbery, A. Seeger, H^p and L^p variants of multiparameter Calderón–Zygmund theory, Trans. Amer. Math. Soc. 334 (1992) 719–747.

[4] L. Carleson, A counterexample for measures bounded on H^p spaces for the bidisk, Mittag-Leffler Report 7, 1974.

[5] S.-Y.A. Chang, R. Fefferman, A continuous version of duality of H1 with BMO on the bidisc, Ann. of Math. 112 (1980) 179–201.

[6] S.-Y.A. Chang, R. Fefferman, Some recent developments in Fourier analysis and H^p theory on product domains, Bull. Amer. Math. Soc. 12 (1985) 143.

[7] R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977) 569–645.

[8] G. David, J.L. Journé, A boundedness criterion for generalized Calderón–Zygmund operators, Ann. of Math. 120 (1984) 371–397.

[9] R. Fefferman, Harmonic analysis on product spaces, Ann. of Math. 126 (1987) 109–130.

[10] C. Fefferman, E.M. Stein, H^p spaces of several variables, Acta Math. 129 (1972) 137–193.

[11] R. Fefferman, E.M. Stein, Singular integrals on product spaces, Adv. Math. 45 (1982) 117–143.

[12] S. Ferguson, M. Lacey, A characterization of product BMO by commutators, Acta Math. 189 (2) (2002) 143–160.

[13] R. Gundy, E.M. Stein, H^p spaces on the polydisc, Proc. Natl. Acad. Sci. 76 (1979) 1026–1029.

[14] Y. Han, E. Sawyer, Para-accretive functions, the weak boundedness property and the Tb theorem, Rev. Mat. Iberoamericana 6 (1990) 17–41.

[15] Y. Han, M.Y. Lin, C. Lin, Y. Lin, Calderón–Zygmund operators on product Hardy spaces, J. Funct. Anal. 258 (2010) 2834–2861.

[16] Y. Han, G. Lu, K. Zhao, Discrete Calderón's identity, atomic decomposition and boundedness criterion of operators on multi-parameter Hardy spaces, J. Geom. Anal. 20 (2010) 670–689.

[17] Y. Han, G. Lu, Z. Ruan, Boundedness of singular integrals in Journé's class on weighted multiparameter Hardy spaces, preprint.

[18] J.L. Journé, Calderón–Zygmund operators on product spaces, Rev. Mat. Iberoamericana 1 (1985) 55–92.

[19] J.L. Journé, Two problems of Calderón–Zygmund theory of product spaces, Ann. Inst. Fourier (Grenoble) 38 (1998) 111–132.

[20] G. Lu, Z. Ruan, Duality theory of weighted Hardy spaces with arbitrary number of parameters, Forum Math. (2012), http://dx.doi.org/10.1515/forum-2012-0018.

[21] J. Pipher, Journé's covering lemma and its extension to higher dimensions, Duke Math. J. 53 (1986) 683–690.