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$H^p \to H^p$ boundedness implies $H^p \to L^p$ boundedness

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Abstract. In this paper, we explore a general method to derive $H^p \to L^p$ boundedness from $H^p \to H^p$ boundedness of linear operators, an idea originated in the work of Han and Lu in dealing with the multiparameter flag singular integrals ([19]). These linear operators include many singular integral operators in one parameter and multiparameter settings. In this paper, we will illustrate further that this method will enable us to prove the $H^p \to L^p$ boundedness on product spaces of homogeneous type in the sense of Coifman and Weiss ([5]) where maximal function characterization of Hardy spaces is not available. Moreover, we also provide a particularly easy argument in those settings such as one parameter or multiparameter Hardy spaces $H^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ where the maximal function characterization exists. The key idea is to prove $\|f\|_{L^p} \leq C\|f\|_{H^p}$ for $f \in L^q \cap H^p$ $(1 < q < \infty, 0 < p \leq 1)$. It is surprising that this simple result even in this classical setting has been absent in the literature.

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1 Introduction

The purpose of this paper is to develop a general method to derive $H^p \to L^p$ boundedness from $H^p \to H^p$ boundedness of linear operators for the product spaces of homogeneous type in the sense of Coifman and Weiss ([5]). The original idea was first used in the recent work of Han and Lu [19] where the multiparameter Hardy space H^p theory associated with the flag singular integrals and boundedness of flag singular integrals on H^p spaces and from H^p to L^p spaces were established. The crucial idea is to prove the following inequality:

$$\|f\|_{L^p} \leq C\|f\|_{H^p} \quad \text{for } f \in L^q \cap H^p,\ 1 < q < \infty,\ 0 < p \leq 1. \qquad (1.1)$$

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Using this inequality, the H^p to L^p boundedness follows immediately from the boundedness on H^p spaces. We should point out that, in the product spaces of homogeneous type, the maximal function characterizations of Hardy spaces seem to be impossible at the present time. Thus, establishing (1.1) is not a trivial task in this setting.

In this paper, we will illustrate further that this inequality also holds on product spaces of homogeneous type in the sense of Coifman and Weiss ([5]) and thus enables us to prove the $H^p \to L^p$ boundedness. Moreover, we also provide a particularly easy argument in those settings such as the classical one parameter or multiparameter Hardy spaces $H^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ where the maximal function characterizations do exist. The key idea proving $\|f\|_{L^p} \le C\|f\|_{H^p}$ for $f \in L^q \cap H^p$ $(1 < q < \infty, 0 < p \le 1)$ in these classical settings using the maximal function characterizations is rather simple. However, it is surprising that the $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ boundedness for singular integral operators has been missing in the literature even in these classical cases. It is the main goal of this paper to demonstrate this crucial idea in proving the H^p to L^p boundedness.

As we pointed out earlier, an application of an inequality of type (1.1) can easily lead to the $H^p \to L^p$ boundedness from the $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ boundedness of singular integral operators. We state these results as

Theorem 1.1. *Let $0 < p \le 1$. If T is a linear operator which is bounded on $L^q(\mathbb{R}^n)$ for some q, $1 < q < \infty$, and on $H^p(\mathbb{R}^n)$, then T can be extended to be a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Theorem 1.2. *Let $0 < p \le 1$. If T is a linear operator which is bounded on $L^q(\mathbb{R}^n \times \mathbb{R}^m)$ for some q, $1 < q < \infty$, and on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, then T can be extended to be a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.*

It turns out that the proofs of these two theorems are surprisingly simple, though these theorems have been absent in the literature. Indeed, in the books of Garcia-Cuerva and Rubio De Francia [11, Theorem 7.7, page 320], Stein [30, Theorem 4, page 115] and Grafakos [12, Theorem 6.7.3, page 474], it was proved that convolution operators satisfying certain conditions are bounded on $H^p(\mathbb{R}^n)$. Therefore, these results, by the above Theorems 1.1 and 1.2, directly imply the $H^p \to L^p$ boundedness for all these convolution operators as corollaries of our theorems. We state this result as

Corollary 1.3. *Let $0 < p \le 1$ and $N = [\frac{n}{p} - n]$. Let K be a C^N function on $\mathbb{R}^n \backslash \{0\}$ that satisfies*

$$|\partial^\beta K(x)| \le A|x|^{-n-|\beta|}$$

for all multiindices $|\beta| \le N$ and all $x \ne 0$. Let W be a tempered distribution that coincides with K on $\mathbb{R}^n \backslash \{0\}$ whose Fourier transform is a bounded function satisfying $|\widehat{W}(\xi)| \le B$. Then the operator $T(f) = f * W$ initially defined for f in the Schwartz class whose support vanishes in a neighborhood of the origin admits an extension which is both bounded on $H^p(\mathbb{R}^n)$ and bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Namely,

$$\|T(f)\|_{H^p} \le C_{n,p}(A+B)\|f\|_{H^p}$$

and

$$\|T(f)\|_{L^p} \le C_{n,p}(A+B)\|f\|_{H^p}$$

for some constant $C_{n,p}$.

We will also provide some examples on the $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness in the multiparameter settings in Section 4.

Corollary 1.4. *The singular integral operators of convolution type or Journé's type in product spaces defined in Section 4 are $H^p \to L^p$ bounded for all $0 < p \le 1$.*

In particular, these linear operators include certain classes of singular integrals studied by R. Fefferman and E. M. Stein [9], J. L. Journé [24] and J. Pipher [27], etc.

We mention in passing that the product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ was first introduced by Gundy and Stein [13]. Chang–Fefferman [3, 4] developed the theory of atomic decomposition. Atomic decomposition of the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is more complicated than the classical $H^p(\mathbb{R}^n)$. Indeed it was conjectured that the product atomic Hardy space on $\mathbb{R}^m \times \mathbb{R}^n$ could be defined by rectangle atoms. However, this conjecture was disproved by a counter-example constructed by Carleson [1]. This leads that the role of cubes in the classical atomic decomposition of $H^p(\mathbb{R}^n)$ was replaced by arbitrary open sets of finite measures in the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. This was carried out in [3], [4].

Since it is more complicated to state our applications in multiparameter setting, we will state these results and give some brief proofs in Section 4 with more details. The proof of Corollary 1.4 could also be derived using the deep atomic decomposition of Chang–R. Fefferman and the R. Fefferman method by considering its action on rectangle atoms and combining it with Journé's covering lemma. This was done in [6]. Our approach of proving $H^p \to L^p$ boundedness from $H^p \to H^p$ boundedness without using Journé's covering lemma thus provides an alternative way different from that of R. Fefferman [6]

Proof of Theorem 1.1. Suppose $f \in L^q(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ and $1 < q < \infty$. Let ϕ be a Schwartz function in $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. We recall that $f \in H^p(\mathbb{R}^n)$ can be characterized as $\sup_{t>0} |\phi_t * f| \in L^p$. Note $\sup_{t>0} |\phi_t * f(x)| \le CM(f)(x)$. Thus, $\sup_{t>0} |\phi_t * f(x)| \in L^q(\mathbb{R}^n)$. Since

$$\lim_{t\to 0} \|\phi_t * f - f\|_{L^q(\mathbb{R}^n)} = 0,$$

there is a sequence of $t_j \to 0$ such that $\lim_{t_j\to 0} \phi_{t_j} * f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$. Then we have for all $0 < p \le 1$

$$\|f\|_{L^p(\mathbb{R}^n)} \le \lim_{t_j\to 0} \|\phi_{t_j} * f\|_{L^p(\mathbb{R}^n)}$$

and thus

$$\|f\|_{L^p(\mathbb{R}^n)} \le \|f\|_{H^p(\mathbb{R}^n)}.$$ □

Proof of Theorem 1.2. Suppose $f \in L^q(\mathbb{R}^n \times \mathbb{R}^m) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Let $\phi^{(1)}$ be a Schwartz function in $\mathbb{R}^n$ and $\phi^{(2)}$ be a Schwartz function $\mathbb{R}^m$ such that $\int_{\mathbb{R}^n} \phi^{(1)}(x)\,dx = 1$, and $\int_{\mathbb{R}^m} \phi^{(1)}(y)\,dy = 1$. Set

$$\phi_{ts}(x, y) = t^{-n}s^{-m}\phi^{(1)}(\frac{x}{t})\phi^{(2)}(\frac{y}{s}).$$

By a result of Merryfield [26], $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$ can be characterized as $\sup_{t,s>0} |\phi_{ts} * f| \in L^p$. Note $\sup_{t,s>0} |\phi_{ts} * f(x, y)| \le CM_s(f)(x, y)$, where

$$M_s(f)(x, y) = \sup_R \frac{1}{|R|}\int_R |f(x, y)|\,dxdy$$

is the strong maximal function and the above supremum is taken among all rectangles R in $\mathbb{R}^n \times \mathbb{R}^m$. Thus, $\sup_{t,s>0} |\phi_{ts} * f(x, y)| \in L^q(\mathbb{R}^n \times \mathbb{R}^m)$. Since

$$\lim_{t\to 0,\, s\to 0} \|\phi_{ts} * f - f\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m)} = 0,$$

the same argument as in the proof of Theorem 1.1 shows that we have for all $0 < p \le 1$

$$\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le \|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}$$

for $f \in L^q(\mathbb{R}^n \times \mathbb{R}^m) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$. □

The second main purpose of this paper is to prove such a general result when the maximal characterization of the Hardy space is not available, for instance, $H^p(\mathcal{X} \times \mathcal{X})$, where only the Littlewood–Paley characterization exists. The main theorem is as follows:

Theorem 1.5. *Let $\frac{1}{1+\theta} < p \le 1$. If T is a linear operator which is bounded on $L^q(\mathcal{X}\times\mathcal{X})$ for some q, $1 < q < \infty$, and on $H^p(\mathcal{X}\times\mathcal{X})$, then T can be extended to be a bounded operator from $H^p(\mathcal{X}\times\mathcal{X})$ to $L^p(\mathcal{X}\times\mathcal{X})$.*

Here the Hardy space $H^p(\mathcal{X}\times\mathcal{X})$ for $\frac{1}{1+\theta} < p \le 1$ was established in [21] on the product $\mathcal{X}\times\mathcal{X}$ of two spaces of homogeneous type in the sense of Coifman and Weiss ([5], see more details in the next section).

The crucial ideas to prove Theorem 1.5 can be summarized as follows:

Step 1. Establish the density result of $L^q(\mathcal{X}\times\mathcal{X}) \cap H^p(\mathcal{X}\times\mathcal{X})$ in $H^p(\mathcal{X}\times\mathcal{X})$ for $1 < q < \infty$ and $0 < p \le 1$ close to 1.

Step 2. Establish $\|f\|_{L^p(\mathcal{X}\times\mathcal{X})} \le C\|f\|_{H^p(\mathcal{X}\times\mathcal{X})}$ for $f \in L^q(\mathcal{X}\times\mathcal{X}) \cap H^p(\mathcal{X}\times\mathcal{X})$.

As we mentioned earlier, there is no maximal characterization for the product Hardy space $H^p(\mathcal{X}\times\mathcal{X})$. Therefore, establishing the above two steps is not completely trivial. After we have derived the above two steps, we will conclude that for each $f \in L^q \cap H^p$, $1 < q < \infty$, the L^p norm of Tf is dominated by the H^p norm of Tf and hence, the proof of Theorem 1.5 will follow.

The organization of this paper is as follows: In Section 2, we recall some preliminaries on multiparameter Hardy spaces in spaces of homogeneous type. Section 3 proves a density result which states that for $1 < q < \infty$, $L^q(\mathcal{X}\times\mathcal{X}) \cap H^p(\mathcal{X}\times\mathcal{X})$ is dense in $H^p(\mathcal{X}\times\mathcal{X})$. In Section 3, we also show that if $f \in L^q(\mathcal{X}\times\mathcal{X}) \cap H^p(\mathcal{X}\times\mathcal{X})$, then $f \in L^p(\mathcal{X}\times\mathcal{X})$ and there is a constant $C_p > 0$ which is independent of the L^q norm of f such that $\|f\|_p \le C_p\|f\|_{H^p}$. These two results lead us to prove the $H^p \to L^p$ boundedness for singular integrals in multiparameter Hardy spaces of homogeneous type where the maximal function characterization is not available. Section 4 provides some examples of $H^p \to L^p$ boundedness of singular integral operators in multiparameter setting which includes those studied by R. Fefferman and Stein [9] and Journé [24]. These boundedness results can be obtained by our general principle demonstrated in this paper and avoid Journé's covering lemma [24].

2 Preliminaries

We begin by recalling some necessary definitions and notation on spaces of homogeneous type.

A quasi-metric ρ on a set $\mathcal{X}$ is a function $\rho : \mathcal{X}\times\mathcal{X} \longrightarrow [0, \infty)$ satisfying

(1) $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{X}$;

(3) there exists a constant $A \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$,

$$\rho(x, y) \le A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in \mathcal{X} : \rho(x, y) < r\}$ form a base. However, the balls themselves need not be open when $A > 1$.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [5].

Definition 2.1. Let $\theta \in (0, 1]$. A *space of homogeneous type*, $(\mathcal{X}, \rho, \mu)_\theta$, is a set $\mathcal{X}$ together with a quasi-metric ρ and a nonnegative Borel regular measure μ on $\mathcal{X}$ and there exist constants $C_0 > 0$ such that for all $0 < r < \operatorname{diam} \mathcal{X}$ and all $x, x^{'}, y \in \mathcal{X}$,

$$\mu(B(x, r)) \sim r, \tag{2.1}$$

$$|\rho(x, y) - \rho(x^{'}, y)| \le C_0 \rho(x, x^{'})^\theta [\rho(x, y) + \rho(x^{'}, y)]^{1-\theta}. \tag{2.2}$$

Through out the paper, we assume that $\mu(\mathcal{X}) = \infty$.

We first recall the following construction given independently by Christ in [2] and by Sawyer–Wheeden in [28], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. We will follow the statement given in [2].

Lemma 2.2. *Let $(\mathcal{X}, \rho, \mu)$ be a space of homogeneous type. Then there exists a collection $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constant $\delta = 1/2$, and $C_1, C_2 > 0$, such that*

(i) $\mu(\mathcal{X} \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \Phi$ if $\alpha \ne \beta$;*

(ii) *for any α, β, k, l with $l \ge k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \Phi$;*

(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*

(iv) $\operatorname{diam}(Q_\alpha^k) \le C_1 (\frac{1}{2})^k$*;*

(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_2(\frac{1}{2})^k)$, where $z_\alpha^k \in \mathcal{X}$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter rough $(\frac{1}{2})^k$ centered at z_α^k. As a result, we consider CQ_α^k to be the dyadic cube with the same center as Q_α^k and diameter $C \operatorname{diam}(Q_\alpha^k)$. In the following, for $k \in \mathbb{Z}$ and $\tau \in I_k$, we will denote by $Q_\tau^{k,v}$, $v = 1, 2, \ldots, N(k, \tau)$, the set of all cubes $Q_\tau^{k+J} \subset Q_\tau^k$, where J is a fixed large positive integer, and denote by $y_\tau^{k,v}$ a point in $Q_\tau^{k,v}$.

Now we introduce the approximation to identity on $\mathcal{X}$.

Definition 2.3 ([23]). A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of operators is said to be *an approximation to identity of order* $\varepsilon \in (0,\theta]$, if there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in \mathcal{X}$, $S_k(x,y)$, the kernel of S_k, is a function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

$$|S_k(x,y)| \le C\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}}; \tag{2.3}$$

$$\begin{aligned} |S_k(x,y) - S_k(x',y)| &\le C\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}} \\ &\quad \text{for } \rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y)); \end{aligned} \tag{2.4}$$

$$\begin{aligned} |S_k(x,y) - S_k(x,y')| &\le C\Big(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}} \\ &\quad \text{for } \rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y)); \end{aligned} \tag{2.5}$$

$$\begin{aligned} &|S_k(x,y) - S_k(x,y') - S_k(x',y) + S_k(x',y')| \\ &\qquad \le C\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\Big)^{\varepsilon}\Big(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\Big)^{\varepsilon} \\ &\qquad\quad \times \frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{1+\varepsilon}} \\ &\qquad\quad \text{for } \rho(x,x'),\ \rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y)); \end{aligned} \tag{2.6}$$

$$\int_{\mathcal{X}} S_k(x,y)d\mu(y) = \int_{\mathcal{X}} S_k(x,y)\,d\mu(x) = 1. \tag{2.7}$$

We remark that by a construction of Coifman, in what follows, we will use an approximation to the identity of order with $\varepsilon = \theta$.

To recall the definition of $H^p(\mathcal{X}\times\mathcal{X})$, we need to introduce the space of test functions on $\mathcal{X}\times\mathcal{X}$.

Definition 2.4 ([21]). For $i = 1, 2$, fix $\gamma_i > 0$ and $\beta_i > 0$. A function f defined on $\mathcal{X}\times\mathcal{X}$ is said to be *a test function of type* $(\beta_1,\beta_2;\gamma_1,\gamma_2)$ *centered at* $(x_0,y_0) \in \mathcal{X}\times\mathcal{X}$ *with width* $r_1, r_2 > 0$ if f satisfies the following conditions:

(1) $\displaystyle |f(x,y)| \le C\frac{r_1^{\gamma_1}}{(r_1+\rho(x,x_0))^{1+\gamma_1}}\frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}};$

(2) $|f(x,y)-f(x',y)|$

$$\le C\Big(\frac{\rho(x,x')}{r_1+\rho(x,x_0)}\Big)^{\beta_1}\frac{r_1^{\gamma_1}}{(r_1+\rho(x,x_0))^{1+\gamma_1}}\frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}}$$

for $\rho(x,x')\le \frac{1}{2A}[r_1+\rho(x,x_0)]$;

(3) $|f(x,y)-f(x,y')|$

$$\le C\Big(\frac{\rho(y,y')}{r_2+\rho(y,y_0)}\Big)^{\beta_2}\frac{r_1^{\gamma_1}}{(r_1+\rho(x,x_0))^{1+\gamma_1}}\frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}}$$

for $\rho(y,y')\le \frac{1}{2A}[r_2+\rho(y,y_0)]$;

(4) $|[f(x,y)-f(x',y)]-[f(x,y')-f(x',y')]|$

$$\le C\Big(\frac{\rho(x,x')}{r_1+\rho(x,x_0)}\Big)^{\beta_1}\Big(\frac{\rho(y,y')}{r_2+\rho(y,y_0)}\Big)^{\beta_2}\frac{r_1^{\gamma_1}}{(r_1+\rho(x,x_0))^{1+\gamma_1}}$$
$$\times\frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}}$$

for $\rho(x,x')\le \frac{1}{2A}[r_1+\rho(x,x_0)]$ and $\rho(y,y')\le \frac{1}{2A}[r_2+\rho(y,y_0)]$;

(5) $\int_{\mathcal{X}} f(x,y)d\mu(x)=0$ for all $y\in\mathcal{X}$;

(6) $\int_{\mathcal{X}} f(x,y)d\mu(y)=0$ for all $x\in\mathcal{X}$.

If f is a test function of type $(\beta_1,\beta_2;\gamma_1,\gamma_2)$ centered at $(x_0,y_0)\in\mathcal{X}\times\mathcal{X}$ with width $r_1,r_2>0$, then we write $f\in\mathcal{G}(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)$ and define the norm of f by $\|f\|_{\mathcal{G}(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)}=\inf\{C:$ (1), (2), (3) and (4) hold$\}$.

We denote by $\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ the class of $\mathcal{G}(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)$ with $r_1=r_2=1$ for fixed $(x_0,y_0)\in\mathcal{X}\times\mathcal{X}$. It is easy to see that

$$\mathcal{G}(x_1,y_1;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)=\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$$

with an equivalent norm for all $(x_1,y_1)\in\mathcal{X}\times\mathcal{X}$. We can easily check that $\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$.

For any $0<\beta_1,\beta_2,\gamma_1,\gamma_2<\theta$, the space $\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ is defined to be the completion of $\mathcal{G}(\theta,\theta;\theta,\theta)$ in $\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$. We define $\|f\|_{\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)}=\|f\|_{\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)}$. Then, obviously, $\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ is a Banach space. Hence we can define the dual space $\big(\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$ to be the set of all linear functionals $\mathcal{L}$ from $\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists a $C \geq 0$ such that for all $f \in \mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

$$|\mathcal{L}(f)| \leq C \|f\|_{\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)}.$$

In [21], to define the product Hardy space $H^p(\mathcal{X} \times \mathcal{X})$, they first introduced the Littlewood–Paley–Stein square function on $\mathcal{X} \times \mathcal{X}$ by

$$g(f)(x_1, x_2) = \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |D_{k_1} D_{k_2}(f)(x_1, x_2)|^2 \right\}^{1/2},$$

where $D_{k_i} = S_{k_i} - S_{k_{i-1}}$ with S_{k_i} being an approximation to the identity for $i = 1, 2$, and proved that $\|g(f)\|_p \approx \|f\|_p$ for $1 < p < \infty$. Then $H^p(\mathcal{X} \times \mathcal{X})$ is defined as follows.

Definition 2.5. Let $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order θ, $i = 1, 2$. Set $D_{k_i} = S_{k_i} - S_{k_{i-1}}$ for all $k_i \in \mathbb{Z}$. For $\frac{1}{1+\theta} < p \leq 1$ and $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$, the Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ is defined to be the set of all $f \in \big(\mathring{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\big)'$ such that $\|g(f)\|_{L^p(\mathcal{X} \times \mathcal{X})} < \infty$, and we define

$$\|f\|_{H^p(\mathcal{X} \times \mathcal{X})} = \|g(f)\|_{L^p(\mathcal{X} \times \mathcal{X})}.$$

In order to verify that the definition of $H^p(\mathcal{X} \times \mathcal{X})$ is independent of the choice of approximations to the identity, in [21] the following min-max type inequality for $H^p(\mathcal{X} \times \mathcal{X})$ is proved.

Lemma 2.6. *Let all the notation be the same as in Definition 2.5. Moreover, for $i = 1, 2$, let $\{P_{k_i}\}_{k_i \in \mathbb{Z}}$ be another approximation to the identity of order θ and $E_{k_i} = P_{k_i} - P_{k_i - 1}$ for all $k_i \in \mathbb{Z}$. And let $\{Q_{\tau_i}^{k_i, v_i} : k_i \in \mathbb{Z}, \tau_i \in I_{k_i}, v_i = 1, \ldots, N(k_i, \tau_i)\}$ and $\{Q_{\tau_i'}^{k_i', v_i'} : k_i' \in \mathbb{Z}, \tau_i' \in I_{k_i'}, v_i' = 1, \ldots, N(k_i', \tau_i')\}$ be sets of dyadic cubes of $\mathcal{X}$ as mentioned in Lemma 2.2. Then, for $\frac{1}{1+\theta} < p < \infty$, there is a constant $C > 0$ such that for all $f \in \big(\mathring{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\big)'$ with $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$,*

$$\Bigg\| \Bigg\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)}$$

$$\sup_{z_1 \in Q_{\tau_1}^{k_1, v_1}, z_2 \in Q_{\tau_2}^{k_2, v_2}} |D_{k_1} D_{k_2}(f)(z_1, z_2)|^2 \chi_{Q_{\tau_1}^{k_1, v_1}}(\cdot) \chi_{Q_{\tau_2}^{k_2, v_2}}(\cdot) \Bigg\}^{1/2} \Bigg\|_{L^p(\mathcal{X} \times \mathcal{X})}$$

$$\leq C\bigg\|\bigg\{\sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{v_1'=1}^{N(k_1',\tau_1')}\sum_{v_2'=1}^{N(k_2',\tau_2')}$$

$$\inf_{z_1\in Q_{\tau_1'}^{k_1',v_1'},z_2\in Q_{\tau_2'}^{k_2',v_2'}}|E_{k_1'}E_{k_2'}(f)(z_1,z_2)|^2\chi_{Q_{\tau_1'}^{k_1',v_1'}}(\cdot)\chi_{Q_{\tau_2'}^{k_2',v_2'}}(\cdot)\bigg\}^{1/2}\bigg\|_{L^p(\mathcal{X}\times\mathcal{X})}.$$

As a consequence of Lemma 2.6, we define the discrete Littlewood–Paley–Stein square function by

$$\begin{aligned}g_d(f)&(x_1,x_2)\\&=\bigg\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}|D_{k_1}D_{k_2}(f)(y_1,y_2)|^2\\&\qquad\times\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\bigg\}^{1/2},\end{aligned}$$

where y_1 and y_2 are any fixed points in $Q_{\tau_1}^{k_1,v_1}$ and $Q_{\tau_2}^{k_2,v_2}$, respectively, and we have

$$\|f\|_{H^p(\mathcal{X}\times\mathcal{X})}=\|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})}\approx\|g_d(f)\|_{L^p(\mathcal{X}\times\mathcal{X})}.$$

To prove Lemma 2.6, in [21] they established the discrete Calderón reproducing formula on $\mathcal{X}\times\mathcal{X}$.

Lemma 2.7. *Let all the notation be the same as in Definition 2.5. Then there are families of linear operators $\{\tilde{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ and $\{\bar{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ such that for all $f\in\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ with $\beta_i,\gamma_i\in(0,\theta)$,*

$$\begin{aligned}f(x_1,x_2)&=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\\&\qquad\times\widetilde{D}_{k_1}\widetilde{D}_{k_2}(x_1,x_2,y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})\\&=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\\&\qquad\times D_{k_1}D_{k_2}(x_1,x_2,y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2}),\end{aligned}\tag{2.8}$$

where $y_{\tau_i}^{k_i,v_i}$ is any point in $Q_{\tau_i}^{k_i,v_i}$ for $i = 1, 2$ and the series converges in both the norm of $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and the norm of $L^p(\mathcal{X} \times \mathcal{X})$ with $1 < p < \infty$. Moreover, $\tilde{D}_{k_i}(x, y)$, the kernel of $\tilde{D}_{k_i}$ satisfies the conditions (2.3) *and* (2.4) *of Definition 2.3 with θ replaced by any $\varepsilon < \theta$ and*

$$\int_{\mathcal{X}} \tilde{D}_{k_i}(x, y) d\mu(y) = \int_{\mathcal{X}} \tilde{D}_{k_i}(x, y) d\mu(x) = 0; \tag{2.9}$$

similarly, $\bar{D}_{k_i}(x, y)$, the kernel of $\bar{D}_{k_i}$ satisfies the conditions (2.3) *and* (2.5) *of Definition 2.3 with θ replaced by any $\varepsilon < \theta$ and* (2.9)*, for all $k_i \in \mathbb{Z}$ with $i = 1, 2$.*

For any $f \in \big(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$, (2.8) *also holds in $\big(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$.*

In this paper, we use the notation $a \sim b$ and $b \lesssim c$ for $a, b, c \geq 0$ to mean that there exists a $C > 0$, so that $a/C \leq b \leq C \cdot a$ and $b \leq C \cdot c$, respectively. The value of C varies from one usage to the next, but it depends only on constants quantified in the relevant preceding hypotheses. We use $a \vee b$ and $a \wedge b$ to mean $\max(a, b)$ and $\min(a, b)$ for any $a, b \in \mathbb{R}$, respectively.

3 A density result and bounding the L^p norm by H^p norm

In this section, we prove Theorem 1.5. To do this, we need the following two results.

Proposition 3.1. *For $\frac{1}{1+\theta} < p \leq 1$ and $1 < q < \infty$, $L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$ is dense in $H^p(\mathcal{X} \times \mathcal{X})$.*

Proposition 3.2. *For $\frac{1}{1+\theta} < p \leq 1 < q < \infty$, if $f \in L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$, then $f \in L^p(\mathcal{X} \times \mathcal{X})$ and there is a constant $C_p > 0$ which is independent of the L^q norm of f such that*

$$\|f\|_p \leq C_p \|f\|_{H^p}.$$

Proof of Theorem 1.5. Let us assume the two propositions first. Then for $f \in L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$, we have $T(f) \in L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$. From Proposition 3.2, we get that $\|T(f)\|_p \leq C_p \|T(f)\|_{H^p}$ and C_p is independent of the L^q norm of $T(f)$. Since T is bounded on $H^p(\mathcal{X} \times \mathcal{X})$, we have $\|T(f)\|_{H^p} \leq C\|f\|_{H^p}$. Moreover, by Proposition 3.1, we obtain that T can be extended to a bounded operator from $H^p(\mathcal{X} \times \mathcal{X})$ to $L^p(\mathcal{X} \times \mathcal{X})$. This completes the proof of Theorem 1.5. □

Now we begin to prove the above two propositions.

Proof of Proposition 3.1. Suppose $f \in H^p(\mathcal{X} \times \mathcal{X})$ and set

$$W = \{(k_1, k_2; \tau_1, \tau_2; v_1, v_2) : |k_1| \le L_1,\ |k_2| \le L_2,\ Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset B(x_0, r) \times B(x_0, r)\},$$

where k_1, k_2 are positive integers and $B(x_0, r)$ is a ball in $\mathcal{X}$ centered at $x_0 \in \mathcal{X}$ with radius $r > 0$. It is easy to see that

$$\sum_{(k_1,k_2;\tau_1,\tau_2;v_1,v_2)\in W} \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\widetilde{D}_{k_1}\widetilde{D}_{k_2}(x_1, x_2, y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2}) \times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})$$

is an L^q function, $1 < q < \infty$, for any fixed L_1, L_2 and r. To show this proposition, it suffices to show that

$$\sum_{(k_1,k_2;\tau_1,\tau_2;v_1,v_2)\in W^c} \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\widetilde{D}_{k_1}\widetilde{D}_{k_2}(x_1, x_2, y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2}) \times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})$$

tends to zero in the H^p norm as L_1, L_2 and r tend to infinity. In fact, from the min-max type inequality in Lemma 2.6, we can see that

$$\|f\|_{H^p} \approx \Bigg\|\Bigg\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)} |D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})|^2 \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\Bigg\}^{\frac{1}{2}}\Bigg\|_p. \tag{3.1}$$

And repeating the same proof of Lemma 2.6 yields that

$$\begin{aligned}
&\Bigg\|\sum_{(k_1,k_2;\tau_1,\tau_2;v_1,v_2)\in W^c} \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\widetilde{D}_{k_1}\widetilde{D}_{k_2}(x_1, x_2, y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2}) \\
&\qquad \times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\Bigg\|_{H^p} \\
&\le \Bigg\|\Bigg\{\sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\Bigg|\sum_{(k_1,k_2;\tau_1,\tau_2;v_1,v_2)\in W^c} \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2}) \\
&\qquad \times D_{k_1'}D_{k_2'}\widetilde{D}_{k_1}\widetilde{D}_{k_2}(x_1, x_2, y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2}) \\
&\qquad \times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\Bigg|^2\Bigg\}^{\frac{1}{2}}\Bigg\|_p.
\end{aligned}$$

$$\leq C \left\| \left\{ \sum_{(k_1,k_2;\tau_1,\tau_2;v_1,v_2)\in W^c} |D_{k_1} D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})|^2 \right.\right.$$
$$\left.\left. \times \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2) \right\}^{\frac{1}{2}} \right\|_p.$$

Since $f \in H^p(\mathcal{X} \times \mathcal{X})$, from (3.1) we can get that the last term tends to zero as L_1, L_2 and r tend to infinity. This completes the proof of the proposition. □

Proof of Proposition 3.2. To show this result, we first use Coifman's idea to construct an approximation to the identity $\{S_k\}_k$ on $\mathcal{X}$ which satisfies the following conditions: There exists a constant $\overline{C} > 0$ such that for all $k \in \mathbb{Z}$ and $x, x', y \in \mathcal{X}$,

(i) $S_k(x, y) = 0$ if $\rho(x, y) > \overline{C}2^{-k}$ and $\|S_k\|_\infty \leq \overline{C}2^k$;

(ii) $|S_k(x, y) - S_k(x', y)| \leq \overline{C}2^{k(1+\epsilon)}\rho(x, x')^\epsilon$;

(iii) $\int_{\mathcal{X}} S_k(x, y)\, d\mu(y) = 1$, a.e. $x \in \mathcal{X}$;

(iv) $S_k(x, y) = S_k(y, x)$.

We can check that such a $\{S_k\}_k$ satisfies all the conditions in Definition 2.3. Moreover, we can see that for each fixed y, when considering $S_k(x, y)$ as a function of variable x, it supports on $\{x \in \mathcal{X} : \rho(x, y) \leq \overline{C}2^{-k}\}$. Set $D_k = S_k - S_{k-1}$, then we can see that similar results hold for D_k with only (iii) replaced by

(iii)′ $\int_{\mathcal{X}} D_k(x, y)\, d\mu(y) = 0$, a.e. $x \in \mathcal{X}$.

Let S_{k_1} and S_{k_2} be two approximations to the identity on $\mathcal{X}$ that satisfy all the above conditions, $D_{k_1} = S_{k_1} - S_{k_1-1}$, $D_{k_2} = S_{k_2} - S_{k_2-1}$. And then we substitute such $D_{k_1}D_{k_2}$ into Definition 2.5, Lemma 2.6 and 2.7. And then we define a square function as follows:

$$\mathcal{G}(f)(x_1, x_2) = \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1\in I_{k_1}} \sum_{\tau_2\in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \right.$$
$$\left. |\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})|^2 \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2) \right\}^{\frac{1}{2}}, \tag{3.2}$$

where $\overline{D}_{k_1}\overline{D}_{k_2}$ is the same as that in (2.8), Lemma 2.6. By Lemma 2.6, for $f \in L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$, we have

$$\|\mathcal{G}(f)\|_p \leq C\|f\|_{H^p}.$$

Now let $f \in L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$, and set

$$\begin{aligned}
\Omega_i &= \{(x_1, x_2) \in \mathcal{X} \times \mathcal{X} : \mathcal{G}(f)(x_1, x_2) > 2^i\};\\
\widetilde{\Omega}_i &= \Big\{(x_1, x_2) \in \mathcal{X} \times \mathcal{X} : M_s(\chi_{\Omega_i})(x_1, x_2) > \frac{\overline{\overline{C}}}{100}\Big\};\\
B_i &= \Big\{R \subset \mathcal{X} \times \mathcal{X} : \mu(R \cap \Omega_i) > \frac{1}{2}\mu(R),\ \mu(R \cap \Omega_{i+1}) \le \frac{1}{2}\mu(R)\Big\},
\end{aligned}$$

where R ranges over all the dyadic rectangles in $\mathcal{X} \times \mathcal{X}$ and $\overline{\overline{C}}$ is a constant to be chosen later. It is easy to see that each dyadic rectangle R belongs to only one B_i.

Since $f \in L^q(\mathcal{X} \times \mathcal{X})$, then by the discrete Calderón reproducing formula (2.8) in Lemma 2.7,

$$\begin{aligned}
f(x_1, x_2) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\\
&\quad \times D_{k_1} D_{k_2}(x_1, x_2, y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\\
&\quad \times \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\\
&= \sum_i \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)}\\
&\quad \chi_{\{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \in B_i\}}(k_1, k_2; \tau_1, \tau_2; v_1, v_2)\\
&\qquad \times \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\\
&\qquad \times D_{k_1} D_{k_2}(x_1, x_2, y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\\
&\qquad \times \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\\
&\triangleq \sum_i f_i(x_1, x_2),
\end{aligned}$$

where the series converges in L^q norm hence it also converges almost everywhere.

We *claim* that

$$\|f_i\|_p^p \le C2^{ip}\mu(\Omega_i),$$

which together with the fact $\frac{1}{1+\theta} < p \le 1$ yields that

$$\|f\|_p^p \le \sum_i \|f_i\|_p^p \le \sum_i C2^{ip}\mu(\Omega_i) \le C\|\mathcal{G}(f)\|_p^p \le C\|f\|_{H^p}^p.$$

To show the claim, note that Coifman's construction yields that $D_{k_i}(x_i, y_{\tau_i}^{k_i,v_i})$ is a function of x_i with compact support $\{x_i \in \mathcal{X} : \rho(x_i, y_{\tau_i}^{k_i,v_i}) \le 2\overline{C}2^{-k_i}\}$, for $i = 1, 2$. Thus, by choosing a constant $\overline{\overline{C}}$ small enough, when $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \in B_i$, we can obtain that $D_{k_1}(x_1, y_{\tau_1}^{k_1,v_1})D_{k_2}(x_2, y_{\tau_2}^{k_2,v_2})$ is supported in $\widetilde{\Omega}_i$. Here $\overline{\overline{C}}$ only depends on $\overline{C}$, A and J. This yields that for each i, $f_i(x_1, x_2)$ is supported in $\widetilde{\Omega}_i$. Thus, by using the Hölder inequality, we have

$$\|f_i(x_1, x_2)\|_p^p \le \mu(\widetilde{\Omega}_i)^{1-\frac{p}{q}}\|f_i\|_q^p.$$

We now estimate the L^q norm of f_i. By the duality argument, for all $h \in L^{q'}(\mathcal{X} \times \mathcal{X})$ with $\|h\|_{q'} \le 1$,

$$\begin{aligned}
|\langle f_i, h\rangle| = \Bigg|\int_{\mathcal{X}\times\mathcal{X}} &\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)} \\
&\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1, k_2; \tau_1, \tau_2; v_1, v_2)\mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2}) \\
&\times D_{k_1}D_{k_2}(h)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2}) \\
&\times \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\, d\mu(x_1)d\mu(x_2)\Bigg| \\
\le \Bigg\|\Bigg(&\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)} \\
&\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1, k_2; \tau_1, \tau_2; v_1, v_2) \\
&\times |D_{k_1}D_{k_2}(h)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})|^2 \\
&\times \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\, d\mu(x_1)d\mu(x_2)\Bigg)^{\frac{1}{2}}\Bigg\|_q \\
\times \Bigg\|\Bigg(&\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)} \\
&\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1, k_2; \tau_1, \tau_2; v_1, v_2) \\
&\times |\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})|^2 \\
&\times \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)d\mu(x_1)d\mu(x_2)\Bigg)^{\frac{1}{2}}\Bigg\|_{q'}.
\end{aligned}$$

Since

$$
\begin{aligned}
&\Big\|\Big(\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\\
&\qquad\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1,k_2;\tau_1,\tau_2;v_1,v_2)\\
&\qquad\quad\times|D_{k_1}D_{k_2}(h)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})|^2\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\Big)^{\frac{1}{2}}\Big\|_{q'}\\
&\quad\le\Big\|\Big(\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}|D_{k_1}D_{k_2}(h)(x_1,x_2)|^2\\
&\qquad\qquad\times\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)d\mu(x_1)d\mu(x_2)\Big)^{\frac{1}{2}}\Big\|_{q'}\\
&\quad\le C\|h\|_{q'}\\
&\quad\le C,
\end{aligned}
$$

the claim follows from the following estimate:

$$
\begin{aligned}
&C2^{qi}\mu(\Omega_i)\\
&\quad\ge\int_{\widetilde{\Omega}_i\setminus\Omega_i}\mathcal{G}(f)^q(x_1,x_2)d\mu(x_1)d\mu(x_2)\\
&\quad\ge\int_{\widetilde{\Omega}_i\setminus\Omega_i}\Big(\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\\
&\qquad\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1,k_2;\tau_1,\tau_2;v_1,v_2)|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})|^2\\
&\qquad\quad\times\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\Big)^{\frac{q}{2}}d\mu(x_1)d\mu(x_2)\\
&\quad=\int_{\mathcal{X}\times\mathcal{X}}\Big(\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\\
&\qquad\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1,k_2;\tau_1,\tau_2;v_1,v_2)|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})|^2\\
&\qquad\quad\times\chi_{\{(Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2})\cap\widetilde{\Omega}_i\setminus\Omega_i\}}(x_1,x_2)\Big)^{\frac{q}{2}}d\mu(x_1)d\mu(x_2)
\end{aligned}
$$

$$\geq C \int_{\mathcal{X}\times\mathcal{X}} \Bigg(\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)}$$

$$\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1,k_2;\tau_1,\tau_2;v_1,v_2)|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})|^2$$

$$\times M_s\Big(\chi_{\{(Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2})\cap\widetilde{\Omega}_i\backslash\Omega_i\}}\Big)^2(x_1,x_2)\Bigg)^{\frac{q}{2}} d\mu(x_1)d\mu(x_2)$$

$$\geq C \int_{\mathcal{X}\times\mathcal{X}} \Bigg\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)}$$

$$\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i\}}(k_1,k_2;\tau_1,\tau_2;v_1,v_2)|\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2})|^2$$

$$\times \chi_{Q_{\tau_1}^{k_1,v_1}(x_1)}\chi_{Q_{\tau_2}^{k_2,v_2}(x_2)}\Bigg\}^{\frac{q}{2}} d\mu(x_1)d\mu(x_2),$$

where in the last inequality we have used the fact that

$$\mu\Big(Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\cap\widetilde{\Omega}_i\backslash\Omega_i\Big) > \frac{1}{2}\mu(Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2})$$

when $Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\in B_i$, and thus

$$\chi_{(Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2})}(x_1,x_2) \leq \frac{1}{2}M_s\Big(\chi_{\ Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\cap(\widetilde{\Omega}_i\ \backslash\Omega_i)}\Big)(x_1,x_2),$$

and in the second to the last inequality we have used the vector-valued Fefferman–Stein inequality for strong maximal functions. This finishes the proof of the proposition. □

4 Examples of multiparameter singular integrals bounded on $H^p(\mathbb{R}^n\times\mathbb{R}^m)$

We end this paper with some examples on how our general results of boundedness on Hardy spaces $H^p(\mathcal{X}\times\mathcal{X})$ make sense and imply in the simplest case of product spaces of two Euclidean spaces. In particular, our Theorem 1.2 will imply the $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ boundedness of a certain class of product singular integrals. We first remark that our results hold on $\mathcal{X}\times\mathcal{Y}$ with two different homogeneous spaces $\mathcal{X}$ and $\mathcal{Y}$. Second, all the theorems proved in this paper on $\mathcal{X}\times\mathcal{Y}$ can be made very precise on $\mathbb{R}^n\times\mathbb{R}^m$ by using Calderón reproducing formulas with explicitly constructed approximation of identity via Fourier transform. In particular, the definitions of Hardy spaces $H^p(\mathcal{X} \times \mathcal{Y})$ can be made for all $0 < p \leq 1$ when $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^m$.

To state the realization of our main results on $\mathbb{R}^n \times \mathbb{R}^m$, we need to start with some preliminaries. Let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all Schwartz functions in $\mathbb{R}^n$. Then the test functions defined on $\mathbb{R}^n \times \mathbb{R}^m$ can be given by

$$\psi(x, y) = \psi^{(1)}(x)\psi^{(2)}(y)$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, and satisfy $\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi_1)|^2 = 1$ for all $\xi_1 \in \mathbb{R}^n \backslash \{(0)\}$, and $\sum_{k\in\mathbb{Z}} |\widehat{\psi}^{(2)}(2^{-k}\xi_2)|^2 = 1$ for all $\xi_2 \in \mathbb{R}^m \backslash \{0\}$, and the moment conditions

$$\int_{\mathbb{R}^n} x^\alpha \psi^{(1)}(x)\, dx = \int_{\mathbb{R}^m} \psi^{(2)}(y) y^\beta\, dy = 0$$

for all nonnegative integers α and β.

Let $f \in L^p$, $1 < p < \infty$. Thus $g(f)$, the Littlewood–Paley–Stein square function of f, is defined by

$$g(f)(x, y) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(x, y)|^2 \right\}^{\frac{1}{2}},$$

where

$$\psi_{j,k}(x, y) = 2^{jn+km} \psi^{(1)}(2^j x)\psi^{(2)}(2^k y). \tag{4.1}$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón's identity holds on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$,

$$f(x, y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x, y). \tag{4.2}$$

Using the orthogonal estimates and together with Calderón's identity on L^2 allows us to obtain the L^p estimates of g for $1 < p < \infty$. Namely, there exist constants C_1 and C_2 such that for $1 < p < \infty$,

$$C_1\|f\|_p \leq \|g(f)\|_p \leq C_2\|f\|_p.$$

In order to use the Littlewood–Paley–Stein square function g to define the Hardy space, one needs to extend the Littlewood–Paley–Stein square function to be defined on a suitable distribution space. For this purpose, we introduce the product test function space on $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 4.1. A Schwartz test function $f(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to be a *product test function on* $\mathbb{R}^n \times \mathbb{R}^m$ if $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and

$$\int_{\mathbb{R}^n} f(x, y)x^\alpha\,dx = \int_{\mathbb{R}^m} f(x, y)y^\beta\,dy = 0$$

for all indices α, β of nonnegative integers.

If f is a product test function on $\mathbb{R}^n \times \mathbb{R}^m$ we denote $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and the norm of f is defined by the norm of Schwartz test functions.

We denote by $(\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m))'$ the dual of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

Now we need to establish the discrete Calderón reproducing formula as follows.

Theorem 4.2. *Suppose that* $\psi_{j,k}$ *are the same as in* (4.1). *Then*

$$f(x, y) = \sum_{j,k} \sum_{I,J} |I||J|\widetilde{\psi}_{j,k}(x, x_I, y, y_J)\psi_{j,k} * f(x_I, y_J), \tag{4.3}$$

where $\widetilde{\psi}_{j,k}(x, x_I, y, y_J) \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$*,* $I \subset \mathbb{R}^n$*,* $J \subset \mathbb{R}^m$ *are dyadic cubes with side-length* $\ell(I) = 2^{-j-N}$ *and* $\ell(J) = 2^{-k-N}$ *for a large fixed integer* N*, and* x_I, y_J *are any fixed points in* I, J*, respectively. Moreover, the series in* (4.3) *converges in the norm of* $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ *and in the dual space* $(\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m))'$.

The proof of this theorem is similar to that of Lemma 2.7 and we shall omit it here.

Since the functions $\psi_{j,k}$ constructed above belong to the space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, the Littlewood–Paley–Stein square function g can be defined for all distributions in $(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m))'$. Formally, we can define the multiparameter Hardy space as follows.

Definition 4.3. Let $0 < p < \infty$. The *multiparameter Hardy space* is defined as $H^p(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S})' : g(f) \in L^p(\mathbb{R}^n \times \mathbb{R}^m)\}$. If $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$, the norm of f is defined by $\|f\|_{H^p} = \|g(f)\|_p$.

To establish the Hardy space theory on $\mathbb{R}^n \times \mathbb{R}^m$, we need the following discrete Calderón-type identity.

Theorem 4.4. *Let* $0 < p \le 1$ *and* M *be a large fixed integer such that* $M > \max\big(n(\frac{1}{p} - 1), m(\frac{1}{p} - 1)\big)$. *Suppose that* $\psi_{j,k}$ *are the same as in* (4.1) *and that*

ψ^1, ψ^2 are Schwartz functions supported in the unit ball in R^n and R^m, respectively, and satisfy the moment conditions

$$\int_{\mathbb{R}^n} x^\alpha \psi^{(1)}(x)\, dx = \int_{\mathbb{R}^m} \psi^{(2)}(y) y^\beta\, dy = 0$$

for $0 \le |\alpha|, |\beta| \le M$. Then there exists an operator T_N^{-1} such that for $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$,

$$f(x,y) = \sum_{j,k} \sum_{I,J} |I||J| \widetilde{\psi}_{j,k}(x - x_I, y - y_J) \big(\psi_{j,k} * (T_N^{-1} f)\big)(x_I, y_J), \quad (4.4)$$

where $I \subset \mathbb{R}^n$, $J \subset \mathbb{R}^m$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$ for a fixed large integer N depending on M, and x_I, y_J are any fixed points in I, J respectively. For each j, k, I, J and x_I, y_J as above, $\widetilde{\psi}_{j,k}(x - x_I, y - y_J)$ is also a Schwartz function with compact support and satisfies the same moment conditions as $\psi_{j,k}$. Moreover, the series in (4.4) *converges in the norm of $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.*

Remark 4.5. This theorem can be proved in the same way as in [19]. The difference between Theorem 4.2 (see also the second equality in Lemma 2.7) and Theorem 4.4 are that our Calderón reproducing formula in Theorem 4.4 has the operator T_N^{-1} acting on f. The purpose to preserve T_N^{-1} in the Calderón reproducing formula is that we would like to make $\widetilde{\psi}_{j,k}(x - x_I, y - y_J)$ have compact support, which plays an important role in the discrete Littlewood–Paley characterization of the Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof. For any $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$, by using (4.2), we have

$$\begin{aligned}
f(x,y) &= \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x,y) \\
&= \sum_{j,k} \sum_{I,J} \int_I \int_J \psi_{j,k}(x-u, y-v) \psi_{j,k} * f(u,v)\, du dv \\
&= \sum_{j,k} \sum_{I,J} \Big[\int_I \int_J \psi_{j,k}(x-u, y-v)\, du dv \Big] \psi_{j,k} * f(x_I, y_J) \\
&\qquad + \mathcal{R}(f)(x,y),
\end{aligned}$$

where for each j, k, $I \subset \mathbb{R}^n$, $J \subset \mathbb{R}^m$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$ for a fixed large integer N depending on M and x_I, y_J are any fixed points in I, J, respectively.

Note that

$$
\begin{aligned}
&\mathcal{R}(f)(x,y)\\
&= \sum_{j,k}\sum_{I,J}\int_I\int_J \psi_{j,k}(x-u,y-v)\big[\psi_{j,k}*f(u,v)-\psi_{j,k}*f(x_I,y_J)\big]\,dudv\\
&= \sum_{j,k}\sum_{I,J}\int_I\int_J \psi_{j,k}(x-u,y-v)\int_{\mathbb{R}^n\times\mathbb{R}^m}\big[\psi_{j,k}(u-u',v-v')\\
&\qquad\qquad - \psi_{j,k}(x_I-u',y_J-v')\big]f(u',v')\,du'dv'dudv\\
&= \int_{\mathbb{R}^n\times\mathbb{R}^m}\mathcal{R}(x,y,u',v')f(u',v')\,du'dv',
\end{aligned}
$$

where $\mathcal{R}(x,y,u',v')$ is the kernel of $\mathcal{R}$.

Next, we need the following: there exists a constant $C>0$ such that for any $f\in L^2(\mathbb{R}^n\times\mathbb{R}^m)\cap H^p(\mathbb{R}^n\times\mathbb{R}^m)$,

(a) $\|\mathcal{R}(f)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\le C2^{-N}\|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}$;

(b) $\|\mathcal{R}(f)\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}\le C2^{-N}\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}$.

In fact, for any $0<p\le 1$, using the discrete Calderón reproducing formula for $f\in L^2(\mathbb{R}^n\times\mathbb{R}^m)\cap H^p(\mathbb{R}^n\times\mathbb{R}^m)$, we have

$$
\begin{aligned}
\|g(\mathcal{R}(f))\|_p &\le \Big\|\Big\{\sum_{j,k}\sum_{I,J}\big|\psi_{j,k}*\mathcal{R}(f)(\cdot,\cdot)\big|^2\chi_I(\cdot)\chi_J(\cdot)\Big\}^{\frac12}\Big\|_p\\
&\le \Big\|\Big\{\sum_{j,k}\sum_{I,J}\sum_{j',k'}\sum_{I',J'}\big|\psi_{j,k}*\mathcal{R}\big(|I'||J'|\widetilde{\psi}_{j',k'}(\cdot,x_{I'},\cdot,y_{J'})\cdot\psi_{j',k'}\\
&\qquad *f(x_{I'},y_{J'})\big)\big|^2\chi_I(\cdot)\chi_J(\cdot)\Big\}^{\frac12}\Big\|_p,
\end{aligned}
$$

where j,k,I,J and $j',k',I',J',x_{I'},y_{J'}$ are the same as in Theorem 4.2.

Now, from the definition of $\mathcal{R}(x,y,u',v')$ and using the cancellation and smoothness conditions, we can obtain the following almost orthogonality estimate:

$$
\begin{aligned}
&\Big|\Big(\psi_{j,k}*\mathcal{R}\big(\widetilde{\psi}_{j',k'}(\cdot,x_{I'},\cdot,y_{J'})\big)\Big)(x,y)\Big|\\
&\qquad\le C2^{-N}2^{-|j-j'|K}2^{-|k-k'|K}\frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')}+|x-x_{I'}|)^{(n+K)}}\\
&\qquad\quad\times\frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')}+|y-y_{J'}|)^{(m+K)}},
\end{aligned}
$$

where $K < M$ and $\max\left(\frac{n}{n+K}, \frac{m}{m+K}\right) < p$. Then, using the above almost orthogonality estimate, we can obtain that for any $x \in I$ and $y \in J$,

$$
\begin{aligned}
&\left|\psi_{j,k} * \mathcal{R}(f)(x,y)\right| \\
&\qquad \le C2^{-N}2^{-|j-j'|K}2^{-|k-k'|K} \\
&\qquad\qquad \times \Big\{M_s\Big(\sum_{I',J'} |\psi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{I'}(\cdot)\chi_{J'}(\cdot)\Big)^r\Big\}^{\frac{1}{r}}(x,y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|g(\mathcal{R}(f))\|_p \\
&\quad \le C2^{-N}\Big\|\Big\{\sum_{j',k'}\Big\{M_s\Big(\sum_{I',J'} |\psi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{I'}\chi_{J'}\Big)^r\Big\}^{\frac{2}{r}}(\cdot,\cdot)\Big\}^{\frac{1}{2}}\Big\|_p \\
&\quad \le C2^{-N}\Big\|\Big\{\sum_{j',k'}\sum_{I',J'} |\psi_{j',k'} * f(x_{I'}, y_{J'})|^2\chi_{I'}\chi_{J'}\Big\}^{\frac{1}{2}}\Big\|_p \\
&\quad \le C2^{-N}\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}.
\end{aligned}
$$

This implies that (b) holds. It is clear that the above estimates still hold when p is replaced by 2, which implies that (a) holds.

We now denote $T_N^{-1} = \sum_{i=0}^{\infty}\mathcal{R}^i$, where

$$
T_N(f)(x,y) = \sum_{j,k}\sum_{I,J}\Big[\int_I\int_J \psi_{j,k}(x-u, y-v)\,du dv\Big]\psi_{j,k} * f(x_I, y_J).
$$

Then, (a) and (b) together show that if N is large enough, then both T_N and T_N^{-1} are bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Hence, we can get the following reproducing formula:

$$
f(x,y) = \sum_{j,k}\sum_{I,J} |I||J|\widetilde{\psi}_{j,k}(x-x_I, y-y_J)\psi_{j,k} * (T_N^{-1}f)(x_I, y_J),
$$

where $\widetilde{\psi}_{j,k}(x-x_I, y-y_J) = \frac{1}{|I|}\frac{1}{|J|}\int_I\int_J \psi_{j,k}(x-u, y-v)\,du dv$ satisfies the properties mentioned in Theorem 4.4.

This completes the proof of Theorem 4.4. □

We now recall some basic definitions about the product Hardy space on $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 4.6. Suppose that $f(x_1, x_2) \in L^2((\mathbb{R}^n \times \mathbb{R}^m))$. Let $\psi^{(1)}(x_1)$, $\psi^{(2)}(x_2)$ be functions as above. The *discrete Littlewood–Paley function of* f, $g_N(f)$, is defined by

$$g_N(f)(x_1,x_2) = \left\{ \sum_{j,k} \sum_{I,J} |\psi_{j,k} * (T_N^{-1} f)(x_I, y_J)|^2 \chi_I(x_1)\chi_J(x_2) \right\}^{\frac{1}{2}}, \quad (4.5)$$

where T_N^{-1} is the same as above, $I \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^m$ are cubes with $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, N is a large fixed integer, and χ_I and χ_J are the characteristic function of I and J, x_I, y_J are any fixed points in I and J, respectively.

Similar to [19] and [20], the above discrete Littlewood–Paley function in (4.5) can characterize the product Hardy spaces. More precisely, we have

Theorem 4.7. *For $0 < p \le 1$, let N be the same as in Theorem 4.4. Then for $f \in L^2((\mathbb{R}^n \times \mathbb{R}^m)) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$, we have*

$$\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} \approx \|g_N(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}.$$

This theorem is a direct consequence of the following min-max type inequality.

Lemma 4.8. *Let $0 < p \le 1$ and let M, $\psi_{j,k}$, $\Psi_{j,k}$, $\phi_{j,k}$ and $\Phi_{j,k}$ be the same as in Theorem 4.4 and N, T_N^{-1} be the same as in Theorem 4.4. Then for any $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$, we have*

$$\begin{aligned}
&\left\| \left\{ \sum_{j,k} \sum_{I,J} \sup_{u\in I, v\in J} |\psi_{j,k} * f(u,v)|^2 \chi_I(\cdot)\chi_J(\cdot) \right\}^{\frac{1}{2}} \right\|_p \\
&\quad \approx \left\| \left\{ \sum_{j,k} \sum_{I,J} \inf_{u\in I, v\in J} |\Psi_{j,k} * f(u,v)|^2 \chi_I(\cdot)\chi_J(\cdot) \right\}^{\frac{1}{2}} \right\|_p \\
&\quad \approx \left\| \left\{ \sum_{j,k} \sum_{I,J} \sup_{u\in I, v\in J} |\phi_{j,k} * (T_N^{-1} f)(u,v)|^2 \chi_I(\cdot)\chi_J(\cdot) \right\}^{\frac{1}{2}} \right\|_p \\
&\quad \approx \left\| \left\{ \sum_{j,k} \sum_{I,J} \sup_{u\in I, v\in J} |\Phi_{j,k} * (T_N^{-1} f)(u,v)|^2 \chi_I(\cdot)\chi_J(\cdot) \right\}^{\frac{1}{2}} \right\|_p,
\end{aligned}$$

where I, J are the same as in Theorem 4.2.

The proof of the above lemma can be obtained by applying the Calderón reproducing formulae in Theorem 4.2 and 4.4 and then repeating the same proof of the min-max type inequality as in Lemma 2.6. For the detail, we omit it here.

The Calderón–Zygmund convolution operators on the product space $\mathbb{R}^n \times \mathbb{R}^m$ studied by Fefferman and Stein [8] generalizes the double Hilbert transform in R^2, $H(f) = \text{p.v.} f * \frac{1}{xy}$. These operators are defined by $T(f) = f * K$ acting on functions on $\mathbb{R}^n \times \mathbb{R}^m$ with the kernel $K = K(x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. The kernel satisfies the following basic assumptions:

(1) Size properties:

$$|\partial^{\alpha+\beta} K(x, y)| \leq C_{\alpha,\beta} |x|^{-n-|\alpha|} |y|^{-m-|\beta|}$$

for all multiindices α and β;

(2) Cancellation properties:

$$\int_{a<|x|<b} K(x, y)\, dx = 0$$

for all $0 < a < b < \infty$ and $y \in \mathbb{R}^m$, and

$$\int_{a<|x|<b} K(x, y)\, dy = 0$$

for all $0 < a < b < \infty$ and $x \in \mathbb{R}^n$.

To see that such convolution operators are bounded on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $\max(\frac{n}{n+1}, \frac{m}{m+1}) < p \leq 1$ for each $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^n \times \mathbb{R}^m)$, using Theorem (4.7) and the discrete Calderón identity we have

$$\begin{aligned}
\|Tf\|_{H^p} &= \|g_N(Tf)\|_p \\
&= \Big\| \Big\{ \sum_{j,k} \sum_{I,J} |\psi_{j,k} * K * (T_N^{-1} f)(x_I, y_J)|^2 \chi_I(x_1) \chi_J(x_2) \Big\}^{\frac{1}{2}} \Big\|_p \\
&= \Big\| \Big\{ \sum_{j,k} \sum_{I,J} |\psi_{j,k} * K * \sum_{j',k'} \sum_{I',J'} \psi_{j',k'}(x_I - x_{I'}, y_J - y_{J'}) \\
&\qquad \times |I'||J'| |\psi_{j',k'} * (T_N^{-1} f)(x_{I'}, y_{J'})|^2 \chi_I(x_1) \chi_J(x_2) \Big\}^{\frac{1}{2}} \Big\|_p,
\end{aligned} \tag{4.6}$$

where in the above we have used the dyadic cubes I, I' in $\mathbb{R}^n$ and J, J' in $\mathbb{R}^m$ with $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-k-N}$, $\ell(I') = 2^{-j'-N}$ and $\ell(J') = 2^{-k'-N}$.

By an orthogonal estimate, that is

$$\begin{aligned}
\Big|\psi_{j,k} * \sum_{j',k'} \psi_{j',k'}(x_1, x_2)\Big| &\leq C_M 2^{-|j-j'|M} 2^{-|k-k'|M} \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_1|)^{n+M+1}} \\
&\quad \times \frac{2^{-(k\wedge k')M}}{(2^{-(k\wedge k')} + |x_2|)^{m+M+1}}
\end{aligned}$$

for any fixed large integer M.

Applying the similar estimate as in [9, page 125], we have

$$\begin{aligned}\Big|\psi_{j,k} * K * \sum_{j',k'} \psi_{j',k'}(x_1,x_2)\Big| \\ \leq C_M 2^{-|j-j'|M} 2^{-|k-k'|M} \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_1|)^{n+M+1}} \\ \times \frac{2^{-(k\wedge k')M}}{(2^{-(k\wedge k')} + |x_2|)^{m+M+1}}.\end{aligned}$$

Substituting this estimate back into (4.6) and applying Fefferman–Stein's vector valued maximal function inequality, we can obtain the H^p boundedness of T. By our main theorem, such an operator can be extended to be a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Similarly, we can consider a class of the product Calderón–Zygmund operators T studied by Journé, which is defined by

$$Tf(x_1,x_2) = \int_{\mathbb{R}^n\times\mathbb{R}^m} K(x_1,x_2,y_1,y_2) f(y_1,y_2)\, dy_1 dy_2, \tag{2.4}$$

where the kernel $K(x_1,x_2,y_1,y_2)$ is defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m$ and there exist constants $C > 0$ and $\varepsilon > 0$ such that

(i) $\displaystyle |K(x_1,x_2,y_1,y_2)| \leq \frac{C}{|x_1-y_1|^n |x_2-y_2|^m}$;

(ii)
$$|K(x_1,x_2,y_1,y_2) - K(x_1',x_2,y_1,y_2)| \leq C \frac{|x_1-x_1'|^\varepsilon}{|x_1-y_1|^{n+\varepsilon}|x_2-y_2|^m},$$
$$|K(x_1,x_2,y_1,y_2) - K(x_1,x_2,y_1',y_2)| \leq C \frac{|y_1-y_1'|^\varepsilon}{|x_1-y_1|^{n+\varepsilon}|x_2-y_2|^m},$$
$$|K(x_1,x_2,y_1,y_2) - K(x_1,x_2',y_1,y_2)| \leq C \frac{|x_2-x_2'|^\varepsilon}{|x_1-y_1|^{n}|x_2-y_2|^{m+\varepsilon}},$$
$$|K(x_1,x_2,y_1,y_2) - K(x_1,x_2,y_1,y_2')| \leq C \frac{|y_2-y_2'|^\varepsilon}{|x_1-y_1|^{n}|x_2-y_2|^{m+\varepsilon}}$$

for $2|x_1-x_1'| \leq |x_1-y_1|$, $2|y_1-y_1'| \leq |x_1-y_1|$, $2|x_2-x_2'| \leq |x_2-y_2|$, $2|y_2-y_2'| \leq |x_2-y_2|$, respectively;

(iii)
$$\begin{aligned}\big|[K(x_1,x_2,y_1,y_2) - K(x_1',x_2,y_1,y_2)] \\ - [K(x_1,x_2',y_1,y_2) - K(x_1',x_2',y_1,y_2)]\big| \\ \leq C \frac{|x_1-x_1'|^\varepsilon}{|x_1-y_1|^{n+\varepsilon}} \frac{|x_2-x_2'|^\varepsilon}{|x_2-y_2|^{m+\varepsilon}},\end{aligned}$$

$$\begin{aligned}&\big|[K(x_1,x_2,y_1,y_2)-K(x_1,x_2,y_1',y_2)]\\&\qquad\qquad\qquad-[K(x_1,x_2,y_1,y_2')-K(x_1,x_2,y_1',y_2')]\big|\\&\quad\le C\frac{|y_1-y_1'|^{\varepsilon}}{|x_1-y_1|^{n+\varepsilon}}\frac{|y_2-y_2'|^{\varepsilon}}{|x_2-y_2|^{m+\varepsilon}},\end{aligned}$$

for $2|x_1-x_1'|\le|x_1-y_1|$, $2|x_2-x_2'|\le|x_2-y_2|$ and $2|y_1-y_1'|\le|x_1-y_1|$, $2|y_2-y_2'|\le|x_2-y_2|$, respectively.

For the Calderón–Zygmund operator T as above, the following result has been proved in the recent paper of Han, Lee, Lin and Lin (see [16]).

Theorem 4.9 ([16]). *Let T be a Calderón–Zygmund operator associated to the kernel K that satisfies* (i)–(iii) *with $T_1^*(1)=T_2^*(1)=0$. Then T is bounded on $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ for* $\max\{\frac{n}{n+\varepsilon},\frac{m}{m+\varepsilon}\}<p\le 1$.

Here, $T_1^*(1)=0$ and $T_2^*(1)=0$ mean that, for all $\varphi\in C_{0,0}^{\infty}(\mathbb{R}^n\times\mathbb{R}^m)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\times\mathbb{R}^m}K(x_1,x_2,y_1,y_2)\varphi(y_1,y_2)\,dy_1dy_2dx_1=0$$

and

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^n\times\mathbb{R}^m}K(x_1,x_2,y_1,y_2)\varphi(y_1,y_2)\,dy_1dy_2dx_2=0,$$

where $C_{0,0}^{\infty}$ is the class of all functions in C^{∞} with support on the unit ball and $\int\varphi=0$.

Therefore, by Theorem 1.2, such operators can be extended to be bounded operators from $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^p(\mathbb{R}^n\times\mathbb{R}^m)$. This implies Corollary 1.4.

This type of theorem in the more difficult and complicated setting of three or more parameters has been established by Y. Han, G. Lu and Z. Ruan [22].

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