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Hardy spaces associated with different homogeneities and boundedness of composition operators

Yongsheng Han, Chincheng Lin, Guozhen Lu, Zhuoping Ruan* and Eric.T.Sawyer

Abstract. It is well known that standard Calderón-Zygmund singular integral operators with the isotropic and non-isotropic homogeneities are bounded on the classical $H^p(\mathbb{R}^m)$ and non-isotropic $H^p_h(\mathbb{R}^m)$, respectively. In this paper, we develop a new Hardy space theory and prove that the composition of two Calderón-Zygmund singular integral operators with different homogeneities is bounded on this new Hardy space. It is interesting that such a Hardy space has surprisingly a multiparameter structure associated with the underlying mixed homogeneities arising from two singular integral operators under consideration. The Calderón-Zygmund decomposition and an interpolation theorem hold on such new Hardy spaces.

1. Introduction and statement of results

The purpose of this paper is to develop a new Hardy space theory and prove that the composition of two Calderón-Zygmund singular integrals associated with different homogeneities, respectively, is bounded on these new Hardy spaces. Indeed, the composition of operators was considered by Calderón and Zygmund when introducing the first generation of Calderón-Zygmund convolution operators. Calderón and Zygmund discovered that to compose two convolution operators, T_1 and T_2, it is enough to employ the product of the corresponding multipliers $m_1(\xi)$ and $m_2(\xi)$. However, the symbol $m_3(\xi) = m_1(\xi)m_2(\xi)$ does not necessarily have zero integral on the unit sphere, so they considered the algebra of operators $cI + T$, where c is a constant, I is the identity operator and T is the operator introduced by them. In 1965, Calderón considered again the problem of the symbolic calculus of the second generation of Calderón-Zygmund singular integral operators with the minimal regularity with respect to x on kernels $L_1(x, y)$ and $L_2(x, y)$, corresponding to

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*Corresponding author.

operators T_1 and T_2. This problem reduced to the study of the commutator which was the first non-convolution operator raised in harmonic analysis.

In the present paper, we consider the composition of two operators associated with different homogeneities. To be more precise, let $e(\xi)$ be a function on $\mathbb{R}^m$ homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, suppose that $h(\xi)$ is a function on $\mathbb{R}^m$ homogeneous of degree 0 in the non-isotropic sense related to the heat equation, and also smooth away from the origin. Then it is well-known that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\widehat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type (1, 1). It was well known that T_1 and T_2 are bounded on the classical isotropic and non-isotropic Hardy spaces, respectively. Rivieré in [24] asked the question: Is the composition $T_1 \circ T_2$ still of weak-type (1,1)? Phong and Stein in [22] answered this question and gave a necessary and sufficient condition for which $T_1 \circ T_2$ is of weak-type (1,1). The operators Phong and Stein studied are in fact compositions with different kind of homogeneities which arise naturally in the $\bar{\partial}$-Neumann problem. This motivates the present work in this paper.

In order to describe more precisely questions and results studied in this paper, we begin with considering all functions and operators defined on $\mathbb{R}^m$. We write $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$. We consider two kinds of homogeneities

$$\delta : (x', x_m) \to (\delta x', \delta x_m), \delta > 0$$

and

$$\delta : (x', x_m) \to (\delta x', \delta^2 x_m), \delta > 0.$$

The first are the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second are non-isotropic and related to the heat equations (also Heisenberg groups).

For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$ we denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$. The singular integrals considered in this paper are defined by

Definition 1.1. *A locally integrable function $\mathcal{K}_1$ on $\mathbb{R}^m/\{0\}$ is said to be a Calderón-Zygmund kernel associated with the isotropic homogeneity if*

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{K}_1(x)\right| \le A|x|_e^{-m-|\alpha|} \qquad \text{for all } |\alpha| \ge 0, \tag{1.1}$$

$$\int_{r_1<|x|_e<r_2} \mathcal{K}_1(x)\, dx = 0 \tag{1.2}$$

for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_1 is a Calderón-Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p.v.(\mathcal{K}_1 * f)(x)$, where $\mathcal{K}_1$ satisfies conditions in (1.1) and (1.2).

Definition 1.2. *Suppose $\mathcal{K}_2 \in L^1_{loc}(\mathbb{R}^m \setminus \{0\})$. $\mathcal{K}_2$ is said to be a Calderón-Zygmund kernel associated with the non-isotropic homogeneity if*

$$\left|\frac{\partial^\alpha}{\partial(x')^\alpha}\frac{\partial^\beta}{\partial(x_m)^\beta}\mathcal{K}_2(x',x_m)\right| \leq B|x|_h^{-m-1-|\alpha|-2\beta} \qquad \text{for all } |\alpha| \geq 0,\ \beta \geq 0, \tag{1.3}$$

$$\int_{r_1<|x|_h<r_2} \mathcal{K}_2(x)\, dx = 0 \tag{1.4}$$

for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_2 is a Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity if $T_2(f)(x) = p.v.(\mathcal{K}_2 * f)(x)$, where $\mathcal{K}_2$ satisfies the conditions in (1.3) and (1.4).

It is well-known that any Calderón-Zygmund singular integral operator associated with the isotropic homogeneity is bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$ and is also bounded on the classical Hardy space $H^p(\mathbb{R}^m)$ with $0 < p \leq 1$. Here the classical Hardy space $H^p(\mathbb{R}^m)$ is introduced by Fefferman and Stein in [FS]. This space is associated with the isotropic homogeneity. To see this, let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\text{supp } \widehat{\psi^{(1)}} \subseteq \{(\xi',\xi_m) \in \mathbb{R}^{m-1}\times\mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2\}, \tag{1.5}$$

and

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)|^2 = 1 \text{ for all } (\xi',\xi_m) \in \mathbb{R}^{m-1}\times\mathbb{R}/\{(0,0)\}. \tag{1.6}$$

The Littlewood-Paley-Stein square function of $f \in \mathcal{S}'(\mathbb{R}^m)$ then is defined by

$$g(f)(x) = \Big\{\sum_{j\in\mathbb{Z}} |\psi_j^{(1)} * f(x)|^2\Big\}^{\frac{1}{2}},$$

where $\psi_j^{(1)}(x',x_m) = 2^{jm}\psi^{(1)}(2^j x', 2^j x_m)$. Note that the isotropic homogeneity is involved in $g(f)$. The classical Hardy space $H^p(\mathbb{R}^m)$ then can be characterized by

$$H^p(\mathbb{R}^m) = \{f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^m) : g(f) \in L^p(\mathbb{R}^m)\},$$

where $\mathcal{S}'/\mathcal{P}$ denotes the space of distributions modulo polynomials. If $f \in H^p(\mathbb{R}^m)$, the H^p norm of f is defined by $\|f\|_{H^p} = \|g(f)\|_{L^p}$.

As we mentioned above, a Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity is bounded on $L^p, 1 < p < \infty$. It is not bounded on the classical Hardy space but bounded on the non-isotropic Hardy space. The non-isotropic Hardy space can also be characterized by the non-isotropic Littlewood-Paley-Stein square function. To be more precise, let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\text{supp } \widehat{\psi^{(2)}} \subseteq \{(\xi',\xi_m) \in \mathbb{R}^{m-1}\times\mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq 2\}, \tag{1.7}$$

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \text{ for all } (\xi',\xi_m) \in \mathbb{R}^{m-1}\times\mathbb{R}\setminus\{(0,0)\}. \tag{1.8}$$

We then define $g_h(f)$, the non-isotropic Littlewood-Paley-Stein square function of $f \in \mathcal{S}'(\mathbb{R}^m)$, by

$$g_h(f)(x) = \Big\{ \sum_{k\in\mathbb{Z}} |\psi_k^{(2)} * f(x)|^2 \Big\}^{\frac{1}{2}},$$

where $\psi_k^{(2)}(x', x_m) = 2^{k(m+1)}\psi(2^k x', 2^{2k}x_m)$. Note again that the non-isotropic homogeneity is involved in $g_h(f)$. The non-isotropic Hardy space $H_h^p(\mathbb{R}^m)$ then can be characterized by

$$H_h^p(\mathbb{R}^m) = \{f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^m) : g_h(f) \in L^p(\mathbb{R}^m)\}$$

and if $f \in H_h^p(\mathbb{R}^m)$, the H_h^p norm of f is defined by $\|f\|_{H_h^p} = \|g_h(f)\|_{L^p}$.

If T_1 and T_2 are Calderón-Zygmund singular integrals with isotropic and non-isotropic homogeneities, respectively, then the composition $T_1 \circ T_2$ is always bounded on $L^p, 1 < p < \infty$, however, in general, bounded neither on the classical Hardy space $H^p(\mathbb{R}^m)$ nor on the non-isotropic Hardy space $H_h^p(\mathbb{R}^m)$. Our goal of this paper is to develop a new Hardy space theory associated with different homogeneities such that the composition $T_1 \circ T_2$ is bounded on this new Hardy space. A new idea to achieve this is to establish the Littlewood-Paley-Stein theory associated with different homogeneities. More precisely, suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (1.5) - (1.6) and (1.7) - (1.8), respectively. Let $\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x)$. Define a new Littlewood-Paley-Stein square function by

$$\mathcal{G}_{com}(f)(x) = \Big\{ \sum_{j,k\in\mathbb{Z}} |\psi_{j,k} * f(x)|^2 \Big\}^{\frac{1}{2}}.$$

We remark that a significant feature is that the multiparameter structure is involved in the above Littlewood-Paley-Stein square function. As in the classical case, it is not difficult to check that for $1 < p < \infty$,

$$\|\mathcal{G}_{com}(f)\|_{L^p} \approx \|f\|_{L^p}. \tag{1.9}$$

The estimates above suggest us to define the H^p norm of f in terms of the L^p norm of $\mathcal{G}_{com}(f)$ when $0 < p \le 1$. However, this continuous version of the Littlewood-Paley-Stein square function $\mathcal{G}_{com}(f)$ is convenient to deal with the case for $1 < p < \infty$ but not for the case when $0 < p \le 1$. See further remark about this below. The crucial idea is to replace the continuous version $\mathcal{G}_{com}(f)$ by the discrete version $\mathcal{G}_\psi^d(f)$ as follows.

To define the discrete version $\mathcal{G}_\psi^d(f)$, the key tool is discrete Calderón's identity. To be more precise, we first recall classical continuous Calderón's identity on $L^2(\mathbb{R}^m)$. Let $\psi^{(1)}$ be a function satisfying the conditions of (1.5) and (1.6). By taking the Fourier transform, we have the following classical continuous Calderón's identity:

$$f(x) = \sum_{j\in\mathbb{Z}} \psi_j^{(1)} * \psi_j^{(1)} * f(x),$$

where the series converges in $L^2(\mathbb{R}^m)$ and in $\mathcal{S}_0(\mathbb{R}^m) := \{f \in \mathcal{S}(\mathbb{R}^m) : \int_{\mathbb{R}^m} f(x)x^\alpha dx = 0$ for any$|\alpha| \geq 0\}$.

Note that the Fourier transforms of both $\psi_j^{(1)}$ and $\psi_j^{(1)} * f$ are compactly supported. Using a similar idea as in the Shannon sampling theorem, one can decompose $\psi_j^{(1)} * \psi_j^{(1)} * f(x)$ by

$$\sum_{\ell \in \mathbb{Z}^m} \psi_j^{(1)}(x - 2^{-j}\ell)(\psi_j^{(1)} * f)(2^{-j}\ell).$$

Then classical discrete Calderón's identity is given by

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}^m} \psi_j^{(1)}(x - 2^{-j}\ell)(\psi_j^{(1)} * f)(2^{-j}\ell), \tag{1.10}$$

where the series converges in $L^2(\mathbb{R}^m)$, $\mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}_0'(\mathbb{R}^m)$. See [9] and [10] for more details.

Now by considering $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$ and taking the Fourier transform, we obtain the following continuous Calderón's identity:

$$f(x) = \sum_{j,k \in \mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x), \tag{1.11}$$

where the series converges in $L^2(\mathbb{R}^m)$, $\mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}_0'(\mathbb{R}^m)$. Furthermore, we will prove the following discrete Calderón's identity.

Theorem 1.3. *Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (1.5) - (1.6) and (1.7) - (1.8), respectively. Let $\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x)$. Then*

$$\begin{aligned} f(x', x_m) = \sum_{j,k \in \mathbb{Z}} \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} & 2^{-(m-1)(j \wedge k)}\ 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)}\ell', 2^{-(j \wedge 2k)}\ell_m) \\ & \times \psi_{j,k}(x' - 2^{-(j \wedge k)}\ell', x_m - 2^{-(j \wedge 2k)}\ell_m) \qquad (1.12), \end{aligned}$$

where the series converges in $L^2(\mathbb{R}^m), \mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}_0'(\mathbb{R}^m)$.

This discrete Calderón's identity leads to the following discrete Littlewood-Paley-Stein square function.

Definition 1.4. *For $f \in \mathcal{S}_0'(\mathbb{R}^m)$, $\mathcal{G}_\psi^d(f)$, the discrete Littlewood-Paley-Stein square function of f, is defined by*

$$\begin{aligned} \mathcal{G}_\psi^d(f)(x', x_m) = \Big\{ \sum_{j,k \in \mathbb{Z}} \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} & |(\psi_{j,k} * f) \quad (2^{-(j \wedge k)}\ell', 2^{-(j \wedge 2k)}\ell_m)|^2 \\ & \times \chi_I(x')\chi_J(x_m) \Big\}^{\frac{1}{2}}, \end{aligned}$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $\ell(I) = 2^{-(j \wedge k)}$ and $\ell(J) = 2^{-(j \wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j \wedge k)}\ell'$ and $2^{-(j \wedge 2k)}\ell_m$, respectively.

Now we formally define the Hardy spaces associated with two different homogeneities by the following

Definition 1.5. *Let* $0 < p \leq 1$. $H^p_{com}(\mathbb{R}^m) = \{f \in \mathcal{S}'_0(\mathbb{R}^m) : \mathcal{G}^d_\psi(f) \in L^p(\mathbb{R}^m)\}$. *If* $f \in H^p_{com}(\mathbb{R}^m)$, *the norm of* f *is defined by* $\|f\|_{H^p_{com}(\mathbb{R}^m)} = \|\mathcal{G}^d_\psi(f)\|_{L^p(\mathbb{R}^m)}$.

Note that, as mentioned above for the Littlewood-Paley-Stein square function, the multiparameter structures are involved again in the discrete Calderón's identity and the Hardy spaces $H^p_{com}(\mathbb{R}^m)$. To see that these Hardy spaces are well defined, we need to show that $H^p_{com}(\mathbb{R}^m)$ is independent of the choice of the functions $\psi^{(1)}$ and $\psi^{(2)}$. This will directly follow from the following theorem

Theorem 1.6. *If* φ *satisfies the same conditions as* ψ, *then for* $0 < p \leq 1$ *and* $f \in \mathcal{S}'_0(\mathbb{R}^m)$,

$$\|\mathcal{G}^d_\psi(f)\|_{L^p(\mathbb{R}^m)} \approx \|\mathcal{G}^d_\varphi(f)\|_{L^p(\mathbb{R}^m)}.$$

We would like to point out that one can define the Hardy space $H^p_{com}(\mathbb{R}^m)$ in terms of $\mathcal{G}_{com}(f)$, the Littlewood-Paley-Stein square function. Then one has to show the following sup-inf principle for all $0 < p < \infty$:

$$\|\Big\{\sum_{j,k\in\mathbb{Z},I,J} \sup_{u\in I,v\in J} |\psi_{j,k} * f(u,v)|^2 \chi_I(x')\chi_J(x_m)\Big\}^{\frac{1}{2}}\|_{L^p}$$

$$\approx \|\Big\{\sum_{j,k\in\mathbb{Z},I,J} \inf_{u\in I,v\in J} |\phi_{j,k} * f(u,v)|^2 \chi_I(x')\chi_J(x_m)\Big\}^{\frac{1}{2}}\|_{L^p}$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $\ell(I) = 2^{-(j\wedge k)}$ and $\ell(J) = 2^{-(j\wedge 2k)}$, $\psi_{j,k}(x) = \psi^{(1)}_j * \psi^{(2)}_k(x)$, $\phi_{j,k}(x) = \phi^{(1)}_j * \phi^{(2)}_k(x)$ and $\psi^{(1)}$, $\phi^{(1)}$ and $\psi^{(2)}$, $\phi^{(2)}$ are functions satisfying conditions in (1.5) - (1.6) and (1.7) - (1.8), respectively. This will actually imply the equivalence of the L^p norms of the two square functions $\mathcal{G}_{com}(f)$ and $\mathcal{G}^d_\psi(f)$ and allow us to use the discrete Littlewood-Paley-Stein square function to define the Hardy space. In the case of the multiparameter structure associated with the flag singular integrals, it was done in [15] (see Theorem 1.9 there). However, such a proof in our case is more complicated than using the discrete Littlewood-Paley-Stein square function directly as we are doing here. This is why, instead of using $\mathcal{G}_{com}(f)$, we decide to choose $\mathcal{G}^d_\psi(f)$ to define the Hardy space $H^p_{com}(\mathbb{R}^m)$. Indeed, by applying a similar argument as in [9], one can also show that for all $0 < p < \infty$,

$$\|\mathcal{G}_{com}(f)\|_p \approx \|\mathcal{G}^d_\psi(f)\|_p.$$

We omit the details of the proof and refer reader to [9] for further details.

We now state the main results of this paper.

Theorem 1.7. *Let* T_1 *and* T_2 *be Calderón-Zygmund singular integral operators with the isotropic and non-isotropic homogeneity, respectively. Then for* $0 < p \leq 1$, *the composition operator* $T = T_1 \circ T_2$ *is bounded on* $H^p_{com}(\mathbb{R}^m)$.

It is well known that the atomic decomposition of the classical Hardy spaces is the main tool to study the $H^p - L^p$ boundedness for classical Calderón-Zygmund operators. See [4], [6], [12] and [13]. However, to get an atomic decomposition for the Hardy space $H^p_{com}(\mathbb{R}^m)$ with miltiparameter structures, as in the classical case, one needs first to establish Journé's covering lemma in this setting. See [1], [2], [3], [19], [7], [8] and [21] for more details. Our approach is quite different from this scheme. Indeed, we will prove the following theorem

Theorem 1.8. *Let $0 < p \leq 1$. If $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, then there is a constant $C = C(p)$ such that*

$$\|f\|_{L^p(\mathbb{R}^m)} \leq C\|f\|_{H^p_{com}(\mathbb{R}^m)},$$

where the constant C is independent of f.

We remark that the proof of the above theorem does not use atomic decomposition and hence Journé's covering lemma is not required. As a consequence, we obtain

Theorem 1.9. *Let $0 < p \leq 1$. Suppose that T is a composition of T_1 and T_2 as given in Theorem 1.7. Then T extends to a bounded operator from $H^p_{com}(\mathbb{R}^m)$ to $L^p(\mathbb{R}^m)$.*

Next we provide the Calderón-Zygmund decomposition and prove an interpolation theorem on $H^p_{com}(\mathbb{R}^m)$. We note that $H^p_{com}(\mathbb{R}^m) = L^p(\mathbb{R}^m)$ for $1 < p < \infty$.

Theorem 1.10. *(Calderón-Zygmund decomposition for H^p_{com}) Let $0 < p_2 \leq 1, p_2 < p < p_1 < \infty$ and let $\alpha > 0$ be given and $f \in H^p_{com}$. Then we may write $f = g + b$ where $g \in H^{p_1}_{com}$ and $b \in H^{p_2}_{com}$ such that $\|g\|^{p_1}_{H^{p_1}_{com}} \leq C\alpha^{p_1-p}\|f\|^p_{H^p_{com}}$ and $\|b\|^{p_2}_{H^{p_2}_{com}} \leq C\alpha^{p_2-p}\|f\|^p_{H^p_{com}}$, where C is an absolute constant.*

Theorem 1.11. *(Interpolation theorem on H^p_{com}) Let $0 < p_2 < p_1 < \infty$ and T be a linear operator which is bounded from $H^{p_2}_{com}$ to L^{p_2} and bounded from $H^{p_1}_{com}$ to L^{p_1}, then T is bounded from H^p_{com} to L^p for all $p_2 < p < p_1$. Similarly, if T is bounded on $H^{p_2}_{com}$ and $H^{p_1}_{com}$, then T is bounded on H^p_{com} for all $p_2 < p < p_1$.*

Before we end this section, several remarks must be in order. First, as mentioned before, the continuous version of the Littlewood-Paley-Stein square function $\mathcal{G}_{com}(f)$ is convenient to deal with the case for $1 < p < \infty$ but not for the case when $0 < p \leq 1$. However, we can still use this continuous version $\mathcal{G}_{com}(f)$ to define the Hardy spaces $H^p_{com}(\mathbb{R}^m)$ for $0 < p \leq 1$. More precisely, suppose that $\psi^{(1)} \in \mathcal{S}_0$ satisfies

$$\int_0^\infty |\widehat{\psi^{(1)}}(t\xi', t\xi_m)|^2 \frac{dt}{t} = 1 \text{ for all } (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}/\{(0,0)\}$$

and $\psi^{(2)} \in \mathcal{S}_0$ satisfies

$$\int_0^\infty |\widehat{\psi^{(2)}}(s\xi', s^2\xi_m)|^2 \frac{ds}{s} = 1 \text{ for all } (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R}/\{(0,0)\}.$$

Set $\psi_{t,s} = \psi_t^{(1)} * \psi_s^{(2)}$, where $\psi_t^{(1)}(x', x_m) = t^{-m}\psi^{(1)}(\frac{x'}{t}, \frac{x_m}{t})$ and $\psi_s^{(2)}(x', x_m) = s^{-m-1}\psi^{(2)}(\frac{x'}{s}, \frac{x_m}{s^2})$. Then one can argue that the $H^p_{com}(\mathbb{R}^m)$ norm of f defined in Definition 1.5 is equivalent to

$$\|\{\int_0^\infty \int_0^\infty |\psi_{t,s} * f(x', x_m)|^2 \frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_{L^p}.$$

The same ideas in this paper can be carried out to the proof of the above equivalent norms of such defined two Hardy spaces.

Secondly, in this paper, we restrict our attention to the above two very specific dilations. However, all results in this paper can be carried out to the composition with more singular integral operators associated with more general non-isotropic homogeneities. To see this, let $T_i(f)(x) = p.v.K_i * f(x), 1 \le i \le n$, be singular integral operators associated with non-isotropic dilations given by

$$\delta_i : (x_1, x_2, \cdot, \cdot, x_m) \to (\delta_i^{\lambda_{i,1}} x_1, \delta_i^{\lambda_{i,2}} x_2, \cdot, \cdot, \delta_i^{\lambda_{i,m}} x_m)$$

for $\delta_i > 0, \lambda_{i,\ell} > 0, 1 \le i \le n$ and $1 \le \ell \le m$.

For $x \in \mathbb{R}^m$ we denote $|x|_i = \sqrt{|x_1|^{\frac{2}{\lambda_{i,1}}} + |x_2|^{\frac{2}{\lambda_{i,2}}} + \cdot\cdot + |x_m|^{\frac{2}{\lambda_{i,m}}}}$. Let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\text{supp } \widehat{\psi^{(i)}} \subseteq \{(\xi_1, \xi_2, \cdot\cdot \xi_m) \in \mathbb{R}^m : \frac{1}{2} \le |\xi|_i \le 2\},$$

and

$$\sum_{j_i \in \mathbb{Z}} |\widehat{\psi^{(i)}}(2^{-j_i\lambda_{i,1}}\xi_1, 2^{-j_i\lambda_{i,2}}\xi_2, \cdot\cdot, 2^{-j_i\lambda_{i,m}}\xi_m)|^2 = 1 \text{ for all } (\xi_1, \xi_2, \cdot, \xi_m) \in \mathbb{R}^m/\{0\}.$$

Set $\psi_{j_1,j_2,\cdot\cdot,j_n}(x) = \psi_{j_1}^{(1)} * \psi_{j_2}^{(2)} * \cdot\cdot\cdot * \psi_{j_n}^{(n)}(x)$, where

$$\psi_{j_i}^{(i)}(x) = 2^{j_i(\lambda_{i,1}+\lambda_{i,2}+\cdot\cdot+\lambda_{i,m})}\psi^{(i)}(2^{j_i\lambda_{i,1}}x_1, 2^{j_i\lambda_{i,2}}x_2, \cdot\cdot, 2^{j_i\lambda_{i,m}}x_m).$$

Define a Littlewood-Paley-Stein square function by

$$\widetilde{\mathcal{G}}_{com}(f)(x) = \Big\{ \sum_{j_1,j_2,\cdot\cdot,j_n \in \mathbb{Z}} |\psi_{j_1,j_2,\cdot\cdot,j_n} * f(x)|^2 \Big\}^{\frac{1}{2}}.$$

Applying the same line as in this paper, one can develop the Hardy space theory associated with these more general non-isotropic dilations. The details of the proofs seem to be rather lengthy to be written out. Therefore, we shall not discuss these in more details in this paper.

Thirdly, the regularity conditions on kernels can be weakened if one considers only the $H^p_{com}(\mathbb{R}^m)$ boundedness for the certain range of p.

Finally, we would like to remark that the method of discrete Littlewood-Paley-Stein analysis in the multiparameter settings used in this paper has been used in a number of other cases earlier. This method allows us to avoid using the Journé covering lemma to prove the boundedness of multiparameter singular integrals from the Hardy spaces. It first appeared in [15] where the theory of the multiparameter Hardy spaces associated with the flag singular integrals was developed and in [16] where the discrete Littlewood-Paley-Stein theory was established in the multiparameter structure associated with the Zygmund dilation (see also the expository article [17]). A recent development for the implicit multiparameter Hardy space and the Marcinkiewicz multiplier theory on the Heisenberg group has been successfully carried out in [18]. We also refer to [5], [20], [23], [14] for this discrete Littlewood-Paley-Stein analysis in other settings such as weighted multiparameter Hardy spaces in Euclidean spaces or multiparameter theory in homogeneous spaces.

Section 1 deals with Theorem 1.3. The proof of Theorem 1.6 is given in section 3. The method of the proof will be applied to the proof of Theorem 1.8 and Theorem 1.10. To show Theorem 1.7, we provide a discrete Calderón-type identity, Theorem 4.1 which has its own interest. These will be given in Section 4. Theorem 1.8 and Theorem 1.9 are proved in Section 5. In the last section, we prove the Calderón-Zygmund decomposition and interpolation theorems.

2. Proof of Theorem 1.3

As mentioned in the previous section, by taking the Fourier transform, we obtain the following continuous Calderón's identity:

$$f(x) = \sum_{j,k\in\mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x), \tag{2.1}$$

where the convergence of series in $L^2(\mathbb{R}^m)$, $\mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}_0'(\mathbb{R}^m)$ follows from the results in the classical case. See [9] and [10] for more details.

To get a discrete version of Calderon's identity, we need to decompose $\psi_{j,k} * \psi_{j,k} * f$ in (2.1). Similar to a method as in [10], set $g = \psi_{j,k} * f$ and $h = \psi_{j,k}$. The Fourier transforms of g and h are given by

$$\widehat{g}(\xi', \xi_m) = \widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)\widehat{f}(\xi', \xi_m)$$

and

$$\widehat{h}(\xi', \xi_m) = \widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m).$$

Note that the Fourier transforms of g and h are both compactly supported. More precisely,

$$\text{supp } \widehat{g}, \text{ supp } \widehat{h} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : |\xi'| \le 2^{j\wedge k}\pi, |\xi_m| \le 2^{j\wedge 2k}\pi\}.$$

Thus, we first expand $\widehat{g}$ in a Fourier series on the rectangle $R_{j,k} = \{\xi' \in \mathbb{R}^{m-1}, \xi_m \in \mathbb{R} : |\xi'| \le 2^{j\wedge k}\pi, |\xi_m| \le 2^{j\wedge 2k}\pi\}$:

$$\begin{aligned}\widehat{g}(\xi',\xi_m) = \textstyle\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)}(2\pi)^{-m}\\ & \times \textstyle\int_{R_{j,k}} \widehat{g}(\eta',\eta_m) e^{i(2^{-(j\wedge k)}\ell'\cdot\eta' + 2^{-(j\wedge 2k)}\ell_m\eta_m)} d\eta' d\eta_m\\ & \times e^{-i(2^{-(j\wedge k)}\ell'\cdot\xi' + 2^{-(j\wedge 2k)}\ell_m\xi_m)}\end{aligned}$$

and then replace $R_{j,k}$ by $\mathbb{R}^m$ since $\widehat{g}$ is supported in $R_{j,k}$. Finally, we obtain

$$\begin{aligned}\widehat{g}(\xi',\xi_m) = \textstyle\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)} g(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\\ & \times\ e^{-i(2^{-(j\wedge k)}\ell'\cdot\xi' + 2^{-(j\wedge 2k)}\ell_m\xi_m)}.\end{aligned}$$

Multiplying $\widehat{h}(\xi',\xi_m)$ from both sides yields

$$\begin{aligned}\widehat{g}(\xi',\xi_m)\widehat{h}(\xi',\xi_m) = \quad \textstyle\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)}\ g(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\\ & \times\widehat{h}(\xi',\xi_m)\ e^{-i(2^{-(j\wedge k)}\ell'\cdot\xi' + 2^{-(j\wedge 2k)}\ell_m\xi_m)}.\end{aligned}$$

Note that $\widehat{h}(\xi',\xi_m)\ e^{-i(2^{-(j\wedge k)}\ell'\cdot\xi' + 2^{-(j\wedge 2k)}\ell_m\xi_m)} = \widehat{h}(\cdot - 2^{-(j\wedge k)}\ell', \cdot - 2^{-(j\wedge 2k)}\ell_m)(\xi',\xi_m)$. Therefore, applying the identity $g * h = (\widehat{g}\ \widehat{h})^\vee$ implies that

$$\begin{aligned}(g*h)(x',x_m) = \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)} g(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\\ & \times h(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m). \qquad (2.2)\end{aligned}$$

Substituting g by $\psi_{j,k} * f$ and h by $\psi_{j,k}$ into Calderón's identity in (2.1) gives the discrete Calderón's identity in (1.12) and the convergence of the series in the $L^2(\mathbb{R}^m)$.

It remains to prove that the series in (1.12) converges in $\mathcal{S}_0(\mathbb{R}^m)$. To do this, it suffices to show that

$$\begin{aligned}\sum_{|j|\ge N_1 \text{or} |k|\ge N_2}\ \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)}(\psi_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)\\ & \times\psi_{j,k}(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m) \qquad (2.3)\end{aligned}$$

tend to zero in $\mathcal{S}_0(\mathbb{R}^m)$ as N_1 and N_2 tend to infinity.

For the sake of convenience, we denote $x_I = 2^{-(j\wedge k)}\ell'$ and $x_J = 2^{-(j\wedge 2k)}\ell_m$. Let I be dyadic cubes in $\mathbb{R}^{m-1}$ and J be dyadic intervals in $\mathbb{R}$ with side-length $\ell(I) = 2^{-(j\wedge k)}$ and $\ell(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are x_I and x_J, respectively. Then the above limit will follow from the following estimates: for any fixed j, k and any given integer $M > 0, |\alpha| \ge 0$, there exists a constant $C = C(M,\alpha) > 0$ which is independent of j and k such that

$$\Big|\sum_{I\times J} |I|\ |J|\ (\psi_{j,k} * f)(x_I, x_J)(D^\alpha\psi_{j,k})(x' - x_I, x_m - x_J)\Big| \le C 2^{-|j|} 2^{-|k|}(1 + |x'| + |x_m|)^{-M}. \qquad (2.4)$$

To show (2.4), we apply the classical almost orthogonality argument. To be more precise, for any given positive integers L_1 and L_2, there exists a constant $C = C(L_1, L_2) > 0$ such that

$$|\psi_j^{(1)} * \psi_{j'}^{(1)}(x', x_m)| \leq C \frac{2^{-|j-j'|L_1}\ 2^{(j\wedge j')m}}{(1 + 2^{(j\wedge j')}|x'| + 2^{(j\wedge j')}|x_m|)^{L_2}} \tag{2.5}$$

and

$$|\psi_k^{(2)} * \psi_{k'}^{(2)}(x', x_m)| \leq C \frac{2^{-|k-k'|L_1}\ 2^{(k\wedge k')(m+1)}}{(1 + 2^{(k\wedge k')}|x'| + 2^{2(k\wedge k')}|x_m|)^{L_2}}. \tag{2.6}$$

Applying (2.6) with $\psi_0^{(2)} = f, L_1 = L + 2M + m + 1$ and $L_2 = M$, where L and M are any fixed positive integers, we obtain

$$\begin{aligned} |(\psi_k^{(2)} * f)(x', x_m)| \leq&\ C\ 2^{-|k|(L+2M+m+1)} \tfrac{2^{-(k\wedge 0)(m+1)}}{(1+2^{(k\wedge 0)}|x'|+2^{(2k\wedge 0)}|x_m|)^M} \\ \leq&\ C\ 2^{-|k|L} \tfrac{1}{(1+|x'|+|x_m|)^M}, \end{aligned}$$

where the last inequality is obvious if $k \geq 0$, and when $k \leq 0$,

$$\frac{2^{-k(m+1)}}{(1 + 2^k|x'| + 2^{2k}|x_m|)^M} \leq 2^{|k|(2M+m+1)} \frac{1}{(1 + |x'| + |x_m|)^M}.$$

Note that $\psi_k^{(2)} * f \in \mathcal{S}_0(\mathbb{R}^m)$. Similarly, we have that

$$|(\psi_j^{(1)} * (\psi_k^{(2)} * f))(u', u_m)| \leq C\ 2^{-|k|L} 2^{-|j|L} \frac{1}{(1 + |u'| + |u_m|)^M}. \tag{2.7}$$

From the size conditions of the functions $\psi^{(1)}$ and $\psi^{(2)}$, we have that for any fixed large M,

$$\begin{aligned} |D^\alpha \psi_{j,k}(u', u_m)| &= |D^\alpha(\psi_j^{(1)} * \psi_k^{(2)})(u', u_m)| \\ &\leq C 2^{|j||\alpha|+2|k||\alpha|} \int \frac{2^{j(m-1)}}{(1 + 2^j|u' - v'| + 2^j|u_m - v_m|)^M} \frac{2^{k(m+1)}}{(1 + 2^k|v'| + 2^{2k}|v_m|)^M}\ dv' dv_m \\ &\leq C 2^{|j||\alpha|+2|k||\alpha|} \frac{2^{(j\wedge k)(m-1)} 2^{(j\wedge 2k)}}{(1 + 2^{j\wedge k}|u'| + 2^{j\wedge 2k}|u_m|)^M} \\ &\leq C 2^{|j|(M+m+|\alpha|)} 2^{|k|(2M+2+2|\alpha|)} \frac{1}{(1 + |u'| + |u_m|)^M}. \end{aligned} \tag{2.8}$$

Estimates in (2.7) and (2.8) yield

$$\begin{aligned} &\Big|\sum_{I\times J} |I|\ |J|\ (D^\alpha \psi_{j,k})(x' - x_I, x_m - x_J)(\psi_{j,k} * f)(x_I, x_J)\Big| \\ &\leq C 2^{-|k|(L-2M-2|\alpha|-2)} 2^{-|j|(L-M-m-|\alpha|)} \\ &\quad \times \sum_{I\times J} |I|\ |J|\ \frac{1}{(1 + |x_I| + |x_J|)^M\ (1 + |x' - x_I| + |x_m - x_J|)^M} \\ &= C 2^{-|k|(L-2M-2|\alpha|-2)} 2^{-|j|(L-M-m-|\alpha|)} \\ &\quad \times \sum_{I\times J} \int_{I\times J} \frac{dy' dy_m}{(1 + |x_I| + |x_J|)^M\ (1 + |x' - x_I| + |x_m - x_J|)^M}. \end{aligned} \tag{2.9}$$

Note that if $y' \in I$ and $y_m \in J$, then $\ell(I)+|x'-x_I| \sim \ell(I)+|x'-y'|, \ell(I)+|x_I| \sim \ell(I)+|y'|$, $\ell(J)+|x_m-x_J| \sim \ell(J)+|x_m-y_m|$, and $\ell(J)+|x_J| \sim \ell(J)+|y_m|$. The simple calculation gives

$$\begin{aligned}\frac{1}{(1+|x'-x_I|+|x_m-x_J|)^M} &\leq \frac{2^{|j|2M}2^{|k|3M}}{(\ell(I)+\ell(J)+|x'-x_I|+|x_m-x_J|)^M}\\ &\leq \frac{2^{|j|4M}2^{|k|6M}}{(1+|x'-y'|+|x_m-y_m|)^M}.\end{aligned}$$

Similarly,

$$\frac{1}{(1+|x_I|+|x_J|)^M} \leq \frac{2^{|j|4M}2^{|k|6M}}{(1+|y'|+|y_m|)^M}.$$

This implies that the last term in (2.9) is dominated by

$$C2^{-|k|(L-20M-2|\alpha|-2)}2^{-|j|(L-20M-m-|\alpha|)}\frac{1}{(1+|x'|+|x_m|)^M}.$$

Choosing $L = 20M+2|\alpha|+m+3$, we derive the estimates in (2.4) and hence the series in (2.3) converges to zero as N_1 and N_2 tend to infinity. Therefore, the series in (1.12) converges in $\mathcal{S}_0(\mathbb{R}^m)$. By the duality argument, we obtain the series in (1.12) converges in $\mathcal{S}_0'(\mathbb{R}^m)$. The proof of Theorem 1.3 is concluded.

3. Proof of Theorem 1.6

In this section, we first derive almost orthogonality estimates in Lemma 3.1 and discrete version of maximal estimate in Lemma 3.2. Lemmas 3.1 and 3.2 together with Theorem 1.3 yield Theorem 1.6.

Lemma 3.1. *(Almost orthogonality estimates)*
Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in (1.5)-(1.8). Then for any given integers L and M, there exists a constant $C = C(L,M) > 0$ such that

$$\begin{aligned}|\psi_{j,k}*\varphi_{j',k'}(x',x_m)| &\leq C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}}\\ &\qquad\times\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)^{(M+1)}}.\end{aligned}$$

Proof: We first write

$$(\psi_{j,k}*\varphi_{j',k'})(x',x_m) = \int_{\mathbb{R}^{m-1}\times\mathbb{R}}(\psi_j^{(1)}*\varphi_{j'}^{(1)})(x'-y',x_m-y_m)(\psi_k^{(2)}*\varphi_{k'}^{(2)})(y',y_m)dy'dy_m.$$

By the almost orthogonal estimates as in (2.4) and (2.5), we have

$$(3.1)\quad |\psi_j^{(1)}*\varphi_{j'}^{(1)}(u',u_m)| \leq C\frac{2^{(j\wedge j')m}\ 2^{-|j-j'|L}}{(1+2^{(j\wedge j')}|u'|)^{(M+m-1)}\ (1+2^{(j\wedge j')}|u_m|)^{(M+1)}}.$$

and

$$(3.2)\quad |\psi_k^{(2)} * \varphi_{k'}^{(2)}(y', y_m)| \le C \frac{2^{(k\wedge k')(m+1)}\ 2^{-|k-k'|L}}{(1+2^{(k\wedge k')}|y'|)^{(M+m-1)}\ (1+2^{2(k\wedge k')}|y_m|)^{(M+1)}}.$$

The estimates in (3.1) and (3.2) imply that

$$(3.3)\qquad |(\psi_{j,k} * \varphi_{j',k'})(x', x_m)| \le C2^{-|j-j'|L}\ 2^{-|k-k'|L}\ AB,$$

where

$$\begin{aligned} A &= \int_{\mathbb{R}} \frac{2^{(j\wedge j')}}{(1+2^{(j\wedge j')}|y_m|)^{(M+1)}}\ \frac{2^{2(k\wedge k')}}{(1+2^{2(k\wedge k')}|x_m-y_m|)^{(M+1)}} dy_m \\ &\le C\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}} \end{aligned}$$

and

$$\begin{aligned} B &= \int_{\mathbb{R}^{m-1}} \frac{2^{(j\wedge j')(m-1)}}{(1+2^{(j\wedge j')}|y'|)^{(M+m-1)}}\ \frac{2^{(k\wedge k')(m-1)}}{(1+2^{(k\wedge k')}|x'-y'|)^{(M+m-1)}} dy' \\ &\le C\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)^{(M+1)}}. \end{aligned}$$

This implies the conclusion of Lemma 3.1. □

Now we prove the following estimate of the discrete version of the maximal function.

Lemma 3.2. *Let I, I' be dyadic cubes in $\mathbb{R}^{m-1}$ and J, J' be dyadic intervals in $\mathbb{R}$ with the side lengths $\ell(I) = 2^{-(j\wedge k)}, \ell(I') = 2^{-(j'\wedge k')}$ and $\ell(J) = 2^{-(j\wedge 2k)}$, $\ell(J') = 2^{-(j'\wedge 2k')}$, and the left lower corners of I, I' and the left end points of J, J' are $2^{-(j\wedge k)}\ell', 2^{-(j'\wedge k')}\ell'', 2^{-(j\wedge 2k)}\ell_m$ and $2^{-(j'\wedge 2k')}\ell'_m$, respectively. Then for any $u', v' \in I$, $u_m, v_m \in J$, and any $\frac{m-1}{M+m-1} < \delta \le 1$,*

$$\begin{aligned} &\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}\ 2^{-(m-1)(j'\wedge k')}\ 2^{-(j'\wedge 2k')}}{(1+2^{j\wedge j'\wedge k\wedge k'}|u'-2^{-(j'\wedge k')}\ell''|)^{(M+m-1)}} \\ &\qquad\times\frac{|(\varphi_{j',k'} * f)(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)|}{(1+2^{j\wedge j'\wedge 2k\wedge 2k'}|u_m-2^{-(j'\wedge 2k')}\ell'_m|)^{(M+1)}} \\ &\le C_1\Big\{\mathcal{M}_s\Big[\Big(\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)|^2\chi_{I'}\chi_{J'}\Big)^{\delta/2}\Big]\Big\}^{1/\delta}(v', v_m) \end{aligned}$$

where $C_1 = C2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k'-j\wedge k)_+}\ 2^{(\frac{1}{\delta}-1)(j'\wedge 2k'-j\wedge 2k)_+}$, here $(a-b)_+ = \max\{a-b, 0\}$, and $\mathcal{M}_s$ is the strong maximal function.

Before proving Lemma 3.2, we would like to point out that this lemma is the key tool to show Theorem 1.6 and 1.7. The discrete version plays a crucial role for this maximal function estimate. And this is why we choose the discrete Littlewood-Paley-Stein square function and use it to define the Hardy space.

Proof of Lemma 3.2: For the sake of convenience, we denote by $x_I = 2^{-(j\wedge k)}\ell', x_{I'} = 2^{-(j'\wedge k')}\ell''$ the left lower corners of I, I' and by $x_J = 2^{-(j\wedge 2k)}\ell_m, x_{J'} = 2^{-(j'\wedge 2k')}\ell'_m$ the left end of points of J, J', respectively. Set

$$A_0 = \Big\{I' : \frac{|u' - x_{I'}|}{2^{-(j\wedge j'\wedge k\wedge k')}} \leq 1\Big\}, \quad B_0 = \Big\{J' : \frac{|u_m - x_{J'}|}{2^{-(j\wedge j'\wedge 2k\wedge 2k')}} \leq 1\Big\},$$

and for $r \geq 1$ and $s \geq 1$,

$$A_r = \Big\{I' : 2^{r-1} < \frac{|u' - x_{I'}|}{2^{-(j\wedge j'\wedge k\wedge k')}} \leq 2^r\Big\}, \quad B_s = \Big\{J' : 2^{s-1} < \frac{|u_m - x_{J'}|}{2^{-(j\wedge j'\wedge 2k\wedge 2k')}} \leq 2^s\Big\}.$$

For any fixed $r, s \geq 0$, denote

$$E = \{(w', w_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \quad |w' - u'| \leq 2^{r-(j\wedge j'\wedge k\wedge k')} + 2^{-(j\wedge k)}, \quad |w_m - u_m| \leq 2^{r-(j\wedge j'\wedge 2k\wedge 2k')} + 2^{-(j\wedge 2k)}\}.$$

Then $A_r \times B_s \subset E$ and for any $(v', v_m) \in I \times J$, $(v', v_m) \in E$. Obviously,

$$|E| \leq C 2^{(m-1)[r-(j\wedge j'\wedge k\wedge k')]}\ 2^{[s-(j\wedge j'\wedge 2k\wedge 2k')]}$$

Thus for $\frac{m-1}{M+m-1} < \delta \leq 1$,

$$
\begin{aligned}
&\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}\ 2^{-(m-1)(j'\wedge k')}\ 2^{-(j'\wedge 2k')}}{(1+2^{j\wedge j'\wedge k\wedge k'}|u'-2^{-(j'\wedge k')}\ell''|)^{(M+m-1)}} \\
&\qquad\qquad \times \frac{|(\varphi_{j',k'}*f)(2^{-(j'\wedge k')}\ell'',2^{-(j'\wedge 2k')}\ell'_m)|}{(1+2^{j\wedge j'\wedge 2k\wedge 2k'}|u_m-2^{-(j'\wedge 2k')}\ell'_m|)^{(M+1)}} \\
\leq\ & C\sum_{r,s\geq 0} 2^{-r(M+m-1)}2^{-s(M+1)}2^{(m-1)(j\wedge j'\wedge k\wedge k')}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}\ 2^{-(m-1)(j'\wedge k')}\ 2^{-(j'\wedge 2k')} \\
&\times\Bigg(\sum_{I'\times J'\in A_r\times B_s}|(\varphi_{j',k'}*f)(x_{I'},x_{J'})|^\delta\Bigg)^{1/\delta} \\
=\ & C\sum_{r,s\geq 0} 2^{-r(M+m-1)}2^{-s(M+1)}2^{(m-1)(j\wedge j'\wedge k\wedge k')}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}\ |I'|\ |J'|\ |E|^{1/\delta} \\
&\times\Bigg\{\frac{1}{|E|}\int_E\sum_{I'\times J'\in A_r\times B_s}|I'|^{-1}\ |J'|^{-1}\ |(\varphi_{j',k'}*f)(x_{I'},x_{J'})|^\delta\chi_{I'}\chi_{J'}dx\Bigg\}^{1/\delta} \\
\leq\ & C\sum_{r,s\geq 0}\ 2^{-r(M+m-1)}2^{-s(M+1)}2^{(m-1)(j\wedge j'\wedge k\wedge k')}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}\ |I'|^{1-\frac{1}{\delta}}\ |J'|^{1-\frac{1}{\delta}}\ |E|^{1/\delta} \\
&\times\Bigg\{\mathcal{M}_s\Big(\sum_{I'\times J'\in A_r\times B_s}|(\varphi_{j',k'}*f)(x_{I'},x_{J'})|^\delta\chi_{I'}\chi_{J'}\Big)(v',v_m)\Bigg\}^{1/\delta} \\
\leq\ & C_1\Bigg\{\mathcal{M}_s\Big(\sum_{I'\times J'}|(\varphi_{j',k'}*f)(x_{I'},x_{J'})|^\delta\chi_{I'}\chi_{J'}\Big)(v',v_m)\Bigg\}^{1/\delta} \\
=\ & C_1\Bigg\{\mathcal{M}_s\Big[\Big(\sum_{I'\times J'}|(\varphi_{j',k'}*f)(x_{I'},x_{J'})|^2\chi_{I'}\chi_{J'}\Big)^{\delta/2}\Big](v',v_m)\Bigg\}^{1/\delta} \\
=\ & C_1\Bigg\{\mathcal{M}_s\Big[\Big(\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|(\varphi_{j',k'}*f)(2^{-(j'\wedge k')}\ell'',2^{-(j'\wedge 2k')}\ell'_m)|^2\chi_{I'}\chi_{J'}\Big)^{\delta/2}\Big](v',v_m)\Bigg\}^{1/\delta}.
\end{aligned}
$$

Now we return to

Proof of Theorem 1.6: Let $f\in\mathcal{S}'_0(\mathbb{R}^m)$. We denote $x_I = 2^{-(j\wedge k)}\ell', x_J = 2^{-(j\wedge 2k)}\ell_m, x_{I'} = 2^{-(j'\wedge k')}\ell''$ and $x_{J'} = 2^{-(j'\wedge 2k')}\ell'_m$. Discrete Calderón's identity on $\mathcal{S}'/\mathcal{P}(\mathbb{R}^m)$ and the almost orthogonality estimates yield that for $\frac{m-1}{M+m-1} < \delta <$

$p \le 1$ and any $v' \in I, v_m \in J$,

$$
\begin{aligned}
& |(\psi_{j,k} * f)(x_I, x_J)| \\
= \;& |\sum_{j',k'} \sum_{(\ell'',\ell'_m)} 2^{-(m-1)(j'\wedge k')} 2^{-(j'\wedge 2k')} (\psi_{j,k} * \varphi_{j',k'})(x_I - x_{I'}, x_J - x_{J'})(\varphi_{j',k'} * f)(x_{I'}, x_{J'})| \\
\le \;& C \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} \sum_{(\ell'',\ell'_m)} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}\, 2^{j\wedge j'\wedge 2k\wedge 2k'}\, 2^{-(m-1)(j'\wedge k')}\, 2^{-(j'\wedge 2k')}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|x_I - x_{I'}|)^{(M+m-1)}} \\
& \times \frac{|(\varphi_{j',k'} * f)(x_{I'}, x_{J'})|}{(1 + 2^{j\wedge j'\wedge 2k\wedge 2k'}|x_J - x_{J'}|)^{(M+1)}} \\
\le \;& C \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \Big\{ \mathcal{M}_s \Big[\Big(\sum_{(\ell'',\ell'_m)} |(\varphi_{j',k'} * f)(x_{I'}, x_{J'})|^2 \chi_{I'} \chi_{J'} \Big)^{\delta/2} \Big] (v', v_m) \Big\}^{1/\delta}
\end{aligned}
$$

where the last inequality follows from Lemma 3.2. Squaring both sides, then multiplying χ_I, χ_J, summing over all $j, k \in \mathbb{Z}$ and $(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}$, and finally applying Hölder's inequality we obtain that for any $x' \in I, x_m \in J$, and $\max\{\frac{m+1}{L+m+1}, \frac{m-1}{M+m-1}\} < \delta < p \le 1$,

$$
\begin{aligned}
& |\mathcal{G}^d_\psi(f)(x', x_m)|^2 \\
\le \;& C \sum_{j,k} \Big\{ \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k' - j\wedge k)_+} 2^{(\frac{1}{\delta}-1)(j'\wedge 2k' - j\wedge 2k)_+} \Big\} \\
& \times \Big\{ \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k' - j\wedge k)_+} 2^{(\frac{1}{\delta}-1)(j'\wedge 2k' - j\wedge 2k)_+} \\
& \times \Big\{ \mathcal{M}_s \Big[\Big(\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)|^2 \chi_{I'} \chi_{J'} \Big)^{\frac{\delta}{2}} \Big] (x', x_m) \Big\}^{2/\delta} \Big\} \\
\le \;& C \Big\{ \sum_{j',k'} \Big\{ \mathcal{M}_s \Big[\Big(\sum_{(\ell'',\ell'_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)|^2 \chi_{I'} \chi_{J'} \Big)^{\frac{\delta}{2}} \Big] (x', x_m) \Big\}^{2/\delta} \Big\},
\end{aligned}
$$

where in the last inequality we use the facts that $(j' \wedge k' - j \wedge k)_+ \le |j - j'| + |k - k'|$, $(j' \wedge 2k' - j \wedge 2k)_+ \le |j - j'| + 2|k - k'|$ and if choose $L > (m+1)(\frac{1}{\delta} - 1)$ then

$$\sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k' - j\wedge k)_+} 2^{(\frac{1}{\delta}-1)(j'\wedge 2k' - j\wedge 2k)_+} \le C$$

and

$$\sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(m-1)(\frac{1}{\delta}-1)(j'\wedge k' - j\wedge k)_+} 2^{(\frac{1}{\delta}-1)(j'\wedge 2k' - j\wedge 2k)_+} \le C.$$

Applying Fefferman-Stein's vector-valued strong maximal inequality on $L^{p/\delta}(\ell^{2/\delta})$ yields

$$\|\mathcal{G}^d_\psi(f)\|_{L^p(\mathbb{R}^m)} \le C \|\mathcal{G}^d_\varphi(f)\|_{L^p(\mathbb{R}^m)}.$$

The conclusion of Theorem 1.6 follows. □

As a consequence of Theorem 1.6, $L^2(\mathbb{R}^m)\cap H^p_{com}(\mathbb{R}^m)$ is dense in $H^p_{com}(\mathbb{R}^m)$. Indeed we have the following result

Corollary 3.3. *$\mathcal{S}_0(\mathbb{R}^m)$ is dense in $H^p_{com}(\mathbb{R}^m)$.*

Proof: Let $f\in H^p_{com}(\mathbb{R}^m)$. For any fixed $N>0$, denote

$$E=\{(j,k,\ell',\ell_m):|j|\le N,|k|\le N,|\ell'|\le N,|\ell_m|\le N\},$$

and

$$\begin{aligned}f_N(x',x_m):=\sum_{(j,k,\ell',\ell_m)\in E}\quad & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)}(\psi_{j,k}*f)(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)\\ &\times\psi_{j,k}(x'-2^{-(j\wedge k)}\ell',x_m-2^{-(j\wedge 2k)}\ell_m)\end{aligned}$$

where $\psi_{j,k}$ is the same as in Theorem 1.3.

Since $\psi_{j,k}\in\mathcal{S}_0(\mathbb{R}^m)$, we obviously have $f_N\in\mathcal{S}_0(\mathbb{R}^m)$. Repeating the proof of Theorem 1.6, we can conclude that $\|f_N\|_{H^p_{com}(\mathbb{R}^m)}\le C\|f\|_{H^p_{com}(\mathbb{R}^m)}$. To see that f_N tends to f in H^p_{com}, by the discrete Calderón's identity in $\mathcal{S}_0'(\mathbb{R}^m)$ in Theorem 1.3,

$$\begin{aligned}(f-f_N)(x',x_m)=\sum_{(j,k,\ell',\ell_m)\in E^c}\quad & 2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)}(\psi_{j,k}*f)(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)\\ &\times\psi_{j,k}(x'-2^{-(j\wedge k)}\ell',x_m-2^{-(j\wedge 2k)}\ell_m),\end{aligned}$$

where the series converges in $\mathcal{S}_0'(\mathbb{R}^m)$.

Therefore,

$$\begin{aligned}\mathcal{G}_\psi(f-f_N):\ &=\ \Big\{\sum_{j',k'}\sum_{(\ell'',\ell_m)}|\psi_{j',k'}*(f-f_N)(2^{-(j'\wedge k')}\ell'',2^{-(j'\wedge 2k')}\ell'_m)|^2\ \chi_{I'}\chi_{J'}\Big\}^{1/2}\\ &=\ \Big\{\sum_{j',k'}\sum_{(\ell'',\ell_m)}|\psi_{j',k'}*\sum_{(j,k,\ell',\ell_m)\in E^c}2^{-(m-1)(j\wedge k)}\ 2^{-(j\wedge 2k)}\\ &\qquad\times(\psi_{j,k}*f)(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)\\ &\qquad\times\psi_{j',k'}*\psi_{j,k}(2^{-(j'\wedge k')}\ell''-2^{-(j\wedge k)}\ell',2^{-(j'\wedge 2k')}\ell'_m-2^{-(j\wedge 2k)}\ell_m)|^2\ \chi_{I'}\chi_{J'}\Big\}^{1/2}\end{aligned}$$

Repeating the proof of Theorem 1.6,

$$\|\mathcal{G}_\varphi(f-f_N)\|_{L^p(\mathbb{R}^m)}\le C\|\sum_{(j,k,\ell',\ell_m)\in E^c}|\psi_{j,k}*f(2^{-(j\wedge k)}\ell',2^{-(j\wedge 2k)}\ell_m)|^2\chi_I\chi_J\}^{\frac12}\|_{L^p(\mathbb{R}^m)},$$

where the last term tends to 0 as N tends to infinity. This implies that f_N tends to f in the $H^p_{com}(\mathbb{R}^m)$ norm as N tend to infinity. □

4. Proof of Theorem 1.7

To show Theorem 1.7, we need a discrete Calderón-type identity on $L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, which has its own interests. To do this, let $\phi^{(1)} \in \mathcal{S}(\mathbb{R}^m)$ with supp $\phi^{(1)} \subseteq B(0,1)$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\phi^{(1)}}(2^{-j}\xi)|^2 = 1 \qquad \text{for all } \ \xi \in \mathbb{R}^m \setminus \{0\}, \tag{4.1}$$

and

$$\int_{\mathbb{R}^m} \phi^{(1)}(x)x^\alpha dx = 0 \qquad \text{for all } \ |\alpha| \le 10M, \tag{4.2}$$

where M is a fixed large positive integer depending on p. We also let $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ with supp $\phi^{(2)} \subseteq B(0,1)$,

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \qquad \text{for all } \ (\xi',\xi_m) \in \mathbb{R}^{m-1}\times\mathbb{R}\setminus\{(0,0)\}, \tag{4.3}$$

and

$$\int_{\mathbb{R}^m} \phi^{(2)}(x)x^\beta dx = 0 \qquad \text{for all } \ |\beta| \le 10M. \tag{4.4}$$

Set $\phi_{j,k} = \phi^{(1)}_j * \phi^{(2)}_k$, where $\phi^{(1)}_j(x) = 2^{jm}\phi^{(1)}(2^j x)$ and $\phi^{(2)}_k(x', x_m) = 2^{k(m+1)}\phi^{(2)}(2^k x', 2^{2k}x_m)$.

The discrete Calderón-type identity is given by the following

Theorem 4.1 Let $\phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions from (4.1) to (4.4). Then for any $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, there exists $h \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$ such that for a sufficiently large $N \in \mathbb{N}$,

$$\begin{aligned} f(x', x_m) = \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} &|I|\ |J|\ \phi_{j,k}(x' - 2^{-(j\wedge k)-N}\ell', x_m - 2^{-(j\wedge 2k)-N}\ell_m) \\ &\times(\phi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m), \end{aligned} \tag{4.5}$$

where the series converges in L^2, I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with side-length $\ell(I) = 2^{-(j\wedge k)-N}$ and $\ell(J) = 2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively. Moreover,

$$\|f\|_{L^2(\mathbb{R}^m)} \approx \|h\|_{L^2(\mathbb{R}^m)}, \tag{4.6}$$

and

$$\|f\|_{H^p_{com}(\mathbb{R}^m)} \approx \|h\|_{H^p_{com}(\mathbb{R}^m)}. \tag{4.7}$$

We point out that the main difference between the discrete Calderón-type identity above and the discrete Calderón's identity given in Theorem 1.3 is that for any fixed $j,k\in\mathbb{Z}$, $\phi_{j,k}(x',x_m)$ in (4.5) have compact supports but $\psi_{j,k}(x',x_m)$ in (1.17) don't. Being of compact support allows to use the orthogonality argument in the proof of Theorem 1.7.

Proof of Theorem 4.1: By taking the Fourier transform, we have that for any $f\in L^2(\mathbb{R}^m)$,

$$f(x',x_m)=\sum_{j,k}\phi_{j,k}*\phi_{j,k}*f(x',x_m).$$

Applying Coifman's decomposition of the identity operator, we obtain

$$\begin{aligned} f(x',x_m) &= \sum_{j,k}\sum_{(\ell',\ell_m)}|I|\;|J|\;\phi_{j,k}(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m)\\ &\qquad\times(\phi_{j,k}*f)(2^{-(j\wedge k)-N}\ell',2^{-(j\wedge 2k)-N}\ell_m)+\mathcal{R}_N(f)(x',x_m)\\ &:= T_N(f)(x',x_m)+\mathcal{R}_N(f)(x',x_m), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{R}_N(f)(x',x_m)\\ =&\sum_{j,k}\sum_{(\ell',\ell_m)}\int_{I\times J}[\phi_{j,k}(x'-y',x_m-y_m)(\phi_{j,k}*f)(y',y_m)\\ &\qquad-\phi_{j,k}(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m)(\phi_{j,k}*f)(2^{-(j\wedge k)-N}\ell',2^{-(j\wedge 2k)-N}\ell_m)]dy'dy_m\\ =&\sum_{j,k}\sum_{(\ell',\ell_m)}\int_{I\times J}[\phi_{j,k}(x'-y',x_m-y_m)-\phi_{j,k}(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m)]\\ &\qquad\times(\phi_{j,k}*f)(y',y_m)dy'dy_m\\ &+\sum_{j,k}\sum_{(\ell',\ell_m)}\int_{I\times J}\phi_{j,k}(x'-2^{-(j\wedge k)-N}\ell',x_m-2^{-(j\wedge 2k)-N}\ell_m)\\ &\qquad\times[\phi_{j,k}*f(y',y_m)-\phi_{j,k}*f(2^{-(j\wedge k)-N}\ell',2^{-(j\wedge 2k)-N}\ell_m)]dy'dy_m\\ :=&\mathcal{R}_N^1(f)(x',x_m)+\mathcal{R}_N^2(x',x_m), \end{aligned}$$

here I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with side-length $\ell(I)=2^{-(j\wedge k)-N}$ and $\ell(J)=2^{-(j\wedge 2k)-N}$ and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively. We claim that for $i=1,2$,

$$\|\mathcal{R}_N^i(f)\|_{L^2(\mathbb{R}^m)}\le C2^{-N}\|f\|_{L^2(\mathbb{R}^m)},\tag{4.8}$$

and

$$\|\mathcal{R}_N^i(f)\|_{H^p_{com}(\mathbb{R}^m)}\le C2^{-N}\|f\|_{H^p_{com}(\mathbb{R}^m)},\tag{4.9}$$

where C is the constant independent of f and N.

Assume the claim for the moment, then, by choosing sufficiently large N, $T_N^{-1}=\sum\limits_{n=0}^{\infty}(\mathcal{R}_N)^n$ is bounded in both L^2 and H^p_{com}, which implies that

$$\|T_N^{-1}(f)\|_{L^2(\mathbb{R}^m)}\approx\|f\|_{L^2(\mathbb{R}^m)}$$

and

$$\|T_N^{-1}(f)\|_{H^p_{com}(\mathbb{R}^m)} \approx \|f\|_{H^p_{com}(\mathbb{R}^m)}.$$

Moreover, for any $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, set $h = T_N^{-1}(f)$, then

$$\begin{aligned} f(x', x_m) &= T_N(T_N^{-1}(f))(x', x_m) \\ &= \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |I||J|\phi_{j,k}(x' - 2^{-(j\wedge k)-N}\ell', x_m - 2^{-(j\wedge 2k)-N}\ell_m) \\ &\qquad \times(\phi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m), \end{aligned}$$

where the series converges in L^2.

Now we show the claim. Since the proofs for $\mathcal{R}_N^1$ and $\mathcal{R}_N^2$ are similar, we only give the proof for $\mathcal{R}_N^1$. Roughly speaking, the proof is similar to Theorem 1.6. To see this, let $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$. Applying discrete Calderón's identity in $L^2(\mathbb{R}^m)$ in Theorem 1.3 yields

$$\begin{aligned} &\psi_{j',k'} * \mathcal{R}_N^1(f)(x', x_m) \\ = &\sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \int_{I\times J} \psi_{j',k'} * [\phi_{j,k}(\cdot - y', \cdot - y_m) \\ &\quad -\phi_{j,k}(\cdot - 2^{-(j\wedge k)-N}\ell', \cdot - 2^{-(j\wedge 2k)-N}\ell_m)](x', x_m)(\phi_{j,k} * f)(y', y_m)dy'dy_m \\ = &\sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} \int_{I\times J} \psi_{j',k'} * [\phi_{j,k}(\cdot - y', \cdot - y_m) \\ &\quad -\phi_{j,k}(\cdot - 2^{-(j\wedge k)-N}\ell', \cdot - 2^{-(j\wedge 2k)-N}\ell_m)](x', x_m) \\ &\quad \times\phi_{j,k} * \Bigg\{ \sum_{j'',k''\in\mathbb{Z}} \sum_{(\ell''',\ell''_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |I''||J''|\psi_{j'',k''}(\cdot - 2^{-(j''\wedge k'')}\ell''', \cdot - 2^{-(j''\wedge 2k'')}\ell''_m) \\ &\qquad (\psi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell''_m)\Bigg\}(y', y_m)dy'dy_m, \end{aligned} \tag{4.10}$$

where I'' are dyadic cubes in $\mathbb{R}^{m-1}$ and J'' are dyadic intervals in $\mathbb{R}$ with the side length $\ell(I'') = 2^{-(j''\wedge k'')}$ and J'' are dyadic intervals in $\mathbb{R}$ with the side length $\ell(J'') = 2^{-(j''\wedge 2k'')}$, and the left lower corners of I'' and the left end points of J'' are $2^{-(j''\wedge k'')}\ell'''$ and $2^{-(j''\wedge 2k'')}\ell''_m$, respectively.

Set $\widetilde{\phi}_{j,k} = \phi_{j,k}(z' - y', z_m - y_m) - \phi_{j,k}(z' - 2^{-(j\wedge k)-N}\ell', z_m - 2^{-(j\wedge 2k)-N}\ell_m)$. Then by the almost orthogonality arguments in Lemma 3.1, we obtain

$$\begin{aligned} |\psi_{j',k'} * \widetilde{\phi}_{j,k}(x', x_m)| &\lesssim 2^{-N}2^{-10M|j-j'|}2^{-10M|k-k'|}\frac{2^{(j'\wedge k')(m-1)}}{(1 + 2^{j'\wedge k'}|x' - y'|)^{(M+m-1)}} \\ &\quad \times\frac{2^{j'\wedge 2k'}}{(1 + 2^{j'\wedge 2k'}|x_m - y_m|)^{(M+1)}} \end{aligned}$$

and similarly, for $y' \in I, y_m \in J$,

$$\begin{aligned}
&|\phi_{j,k} * \psi_{j'',k''}(y' - 2^{-(j''\wedge k'')}\ell'', y_m - 2^{-(j''\wedge 2k'')}\ell'_m)| \\
\lesssim\ & 2^{-10M|j-j''|}2^{-10M|k-k''|}\frac{2^{(j''\wedge k'')(m-1)}}{(1+2^{j''\wedge k''}|y' - 2^{-(j''\wedge k'')}\ell''|)^{(M+m-1)}} \\
&\times\frac{2^{j''\wedge 2k''}}{(1+2^{j''\wedge 2k''}|y_m - 2^{-(j''\wedge 2k'')}\ell'_m)|)^{(M+1)}}.
\end{aligned}$$

Substituting these estimates into the last term in (4.10) yields

$$\begin{aligned}
&|\psi_{j',k'} * \mathbb{R}^1_N(f)(x', x_m)| \\
\lesssim\ & \sum_{j'',k''\in\mathbb{Z}}\sum_{(\ell''',\ell''_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|I''||J''||(\psi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell''_m)| \\
&\times\sum_{j,k\in\mathbb{Z}}\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}\int_{I\times J}2^{-N}2^{-|j-j'|3M}2^{-|k-k'|3M} \\
&\times\frac{2^{(j'\wedge k')(m-1)}}{(1+2^{j'\wedge k'}|x'-y'|)^{(M+m-1)}}\frac{2^{j'\wedge 2k'}}{(1+2^{j'\wedge 2k'}|x_m - y_m|)^{(M+1)}}2^{-|j-j''|3M}2^{-|k-k''|3M} \\
&\times\frac{2^{(j''\wedge k'')(m-1)}}{(1+2^{j''\wedge k''}|y' - 2^{-(j''\wedge k'')}\ell''|)^{(M+m-1)}}\frac{2^{j''\wedge 2k''}}{(1+2^{j''\wedge 2k''}|y_m - 2^{-(j''\wedge 2k'')}\ell'_m)|)^{(M+1)}}dy'dy_m \\
\\
\lesssim\ & 2^{-N}\sum_{j'',k''}\sum_{(\ell''',\ell''_m)}2^{-|j'-j''|3M}2^{-|k'-k''|3M}|I''||J''|\frac{2^{(j'\wedge j''\wedge k'\wedge k'')(m-1)}}{(1+2^{j'\wedge j''\wedge k'\wedge k''}|x' - 2^{-(j''\wedge k'')}\ell'''|)^{(M+m-1)}} \\
&\times\frac{2^{(j'\wedge j'')\wedge 2(k\wedge k')}}{(1+2^{(j'\wedge j'')\wedge 2(k\wedge k')}|x_m - 2^{-(j''\wedge 2k'')}\ell''_m|)^{(M+1)}}|(\psi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell''_m)|.
\end{aligned}$$

By the L^2 boundedness of the discrete Littlewood-Paley-Stein square function $\mathcal{G}^d_\psi(f)$, we have

$$\begin{aligned}
\|\mathcal{R}^1_N(f)\|_{L^2} \lesssim\ & \|\mathcal{G}^d_\psi(\mathcal{R}^1_N f)(x', x_m)\|_{L^2} \\
\lesssim\ & 2^{-N}\|\{\sum_{j'',k''\in\mathbb{Z}}\sum_{(\ell''',\ell''_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|(\psi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell''_m)|^2\chi''_I\chi''_J\}^{\frac{1}{2}}\|_{L^2} \\
\lesssim\ & 2^{-N}\|f\|_{L^2}.
\end{aligned}$$

Repeating the same proof as in Theorem 1.6 implies

$$\begin{aligned}
&\|\mathbb{R}^1_N(f)\|_{H^p_{com}} \lesssim \|\mathcal{G}^d_\psi(\mathcal{R}^1_N f)(x', x_m)\|_{L^p} \\
\lesssim\ & 2^{-N}\|\{\sum_{j'',k''\in\mathbb{Z}}\sum_{(\ell''',\ell''_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}}|(\psi_{j'',k''} * f)(2^{-(j''\wedge k'')}\ell''', 2^{-(j''\wedge 2k'')}\ell''_m)|^2\chi''_I\chi''_J\}^{\frac{1}{2}}\|_{L^p} \\
\lesssim\ & 2^{-N}\|f\|_{H^p_{com}}.
\end{aligned}$$

The claim is concluded and hence Theorem 4.1 follows. □

Repeating the same proof of Theorem 1.6, we have

Corollary 4.1. *Let $0 < p \le 1$. Suppose $\phi_{j,k}$ satisfies the same conditions as in Theorem 4.1 with a large M depending on p. Then for a large N as in Theorem 4.1 and $f \in L^2 \cap H^p_{com}$,*

$$\|f\|_{H^p_{com}} \approx \|\bigg(\sum_{j,k\in\mathbb{Z}}\sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\phi_{j,k}*f)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)|^2\chi_I\chi_J\bigg)^{1/2}\|_{L^p}$$

We now prove Theorem 1.7.

Proof of Theorem 1.7: We may assume that $\mathcal{K}_i$ is the kernel of the convolution operator T_i, $i = 1, 2$, and $\mathcal{K}$ is the kernel of the composition operator $T = T_1 \circ T_2$. Then $T(f) = \mathcal{K} * f$ and $\mathcal{K} = \mathcal{K}_1 * \mathcal{K}_2$. For $f \in L^2 \cap H^p_{com}, 0 < p \le 1$, by the L^2 boundedness of T and applying discrete Calderon's identity of f on $L^2 \cap H^p_{com}$ in Theorem 4.1, we conclude

$$\begin{aligned}
& \|T(f)\|_{H^p_{com}} \\
\le\ & C\|\{\sum_{j,k}\sum_{(\ell',\ell_m)} |(\phi_{j,k}*\mathcal{K}*f)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)|^2\chi_I\chi_J\}^{\frac{1}{2}}\|_{L^p} \\
=\ & C\|\{\sum_{j,k}\sum_{(\ell',\ell_m)} |\sum_{j',k'}\sum_{(\ell'',\ell'_m)} 2^{-(m-1)(j'\wedge k')}\ 2^{-(j'\wedge 2k')}(\phi_{j',k'}*h)(2^{-(j'\wedge k')-N}\ell'', 2^{-(j'\wedge 2k')-N}\ell'_m) \\
& \times(\mathcal{K}*\phi_{j,k}*\phi_{j',k'})(2^{-(j\wedge k)-N}\ell' - 2^{-(j'\wedge k')-N}\ell'', 2^{-(j\wedge 2k)-N}\ell_m - 2^{-(j'\wedge 2k')-N}\ell'_m)|^2\chi_I\chi_J\}^{\frac{1}{2}}\|_{L^p},
\end{aligned}$$

where $\phi_{j,k}, \phi_{j',k'}, h$ and N are the same as in Theorem 4.1.
We claim that for any given $M > 0$,

$$|\mathcal{K}_1 * \phi^{(1)}_k(x', x_m)| \le C\frac{2^{km}}{(1+2^k|x'|)^{M+m-1}(1+2^k|x_m|)^{M+1}}, \tag{4.11}$$

and

$$|\mathcal{K}_2 * \phi^{(2)}_k(x', x_m)| \le C\frac{2^{k(m+1)}}{(1+2^k|x'|)^{M+m+1}\ (1+2^{2k}|x_m|)^{M+1}}. \tag{4.12}$$

We only show (4.12) here since the proof of (4.11) is similar. We consider the following two cases:
Case 1. $|x|_h \le 2\ 2^{-k}$:
In this case, $2^k|x'| \le 2$ and $2^{2k}|x_m| \le 4$, which imply that

$$1 + 2^k|x'| \sim 1 \ \text{ and } \ 1 + 2^{2k}|x_m| \sim 1.$$

By the fact supp $\phi^{(2)}_k \subset \{x : |x|_h \le 2^{-k}\}$ and the cancellation condition in (4.4),

$\mathcal{K}_2 * \phi_k^{(2)}(x)$ is bounded by

$$\begin{aligned}
|\mathcal{K}_2 * \phi_k^{(2)}(x)| &= |\lim_{\varepsilon\to 0}\int_{\varepsilon\le|x-y|_h\le 10\ 2^{-k}} \mathcal{K}_2(x-y)\phi_k^{(2)}(y)dy| \\
&= |\lim_{\varepsilon\to 0}\int_{\varepsilon\le|x-y|_h\le 3\ 2^{-k}} \mathcal{K}_2(x-y)\ [\phi_k^{(2)}(y)-\phi_k^{(2)}(x)]\ dy| \\
&\le C2^{k(m+1)}\int_{|x'-y'|\le 3}(|x'-y'|)^{-(m-1)+1}dy'\int_{|x_m-y_m|\le 9}|x_m-y_m|^{-2+2}dy_m \\
&\le C2^{k(m+1)} \le C\frac{2^{k(m+1)}}{(1+2^k|x'|)^{M+m-1}(1+2^{2k}|x_m|)^{M+1}}.
\end{aligned}$$

Case 2. $|x|_h > 2\ 2^{-k}$:
In this case, $2^k|x'|>2$ or $2^{2k}|x_m|>4$, which imply that

$$1+2^k|x'|\sim 2^k|x'| \ \text{ or }\ 1+2^{2k}|x_m|\sim 2^{2k}|x_m|.$$

By the cancellation condition of $\phi^{(2)}$ with order $4M$ in (4.4) and the size condition of $\mathcal{K}_2$ in (1.3),

$$\begin{aligned}
|\mathcal{K}_2 * \phi_k^{(2)}(x)| &= |\int_{|y|_h\le 2^{-k}}[\mathcal{K}_2(x-y)-\sum_{|\alpha|=|\alpha_1|+|2\alpha_2|\le 4M}\frac{1}{\alpha!}D_{x'}^{\alpha_1}D_{x_m}^{\alpha_2}\mathcal{K}_2(x',x_m)y^\alpha]\ \phi_k^{(2)}(y)dy| \\
&\le C\int_{|y|_h\le 2^{-k}}\frac{(|y|_h)^{4M+1}}{(|x|_h)^{m+1+4M+1}}|\phi_k^{(2)}(y)|\ dy \\
&\le C\frac{2^{k(m+1)}}{(1+2^k|x'|)^{M+m-1}(1+2^{2k}|x_m|)^{M+1}}.
\end{aligned}$$

Thus the claim follows. By the classical orthogonality argument, for any fixed L amd M,

$$(4.13)\quad |\phi_j^{(1)} * \phi_{j'}^{(1)}(x',x_m)| \le C\frac{2^{-|j-j'|L}\ 2^{m(j\wedge j')}}{(1+2^{(j\wedge j')}|x'|)^{(M+m-1)}\ (1+2^{(j\wedge j')}|x_m|)^{(M+1)}},$$

and

$$(4.14)\quad |\phi_k^{(2)} * \phi_{k'}^{(2)}(x',x_m)| \le C\frac{2^{-|k-k'|L}\ 2^{(k\wedge k')(m+1)}}{(1+2^{(k\wedge k')}|x'|)^{(M+m-1)}\ (1+2^{2(k\wedge k')}|x_m|)^{(M+1)}}.$$

Estimates from (4.11) to (4.14) yield that

$$\begin{aligned}
|\mathcal{K} * \phi_{j,k} * \phi_{j',k'}(x',x_m)| &= |[\mathcal{K}_1 * \phi_j^{(1)} * \phi_{j'}^{(1)}] * [\mathcal{K}_2 * \psi_k^{(2)} * \psi_{k'}^{(2)}](x',x_m)| \\
&\le C\frac{2^{-|j-j'|L}2^{-|k-k'|L}2^{(j\wedge j'\wedge k\wedge k')(m-1)}\ 2^{j\wedge j'\wedge 2k\wedge 2k'}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}\ (1+2^{j\wedge j'\wedge 2k\wedge 2k'}|x_m|)^{(M+1)}}.
\end{aligned}\tag{4.15}$$

Using the estimates in (4.15) and applying the same proof as in Theorem 1.6 yield that for $f\in L^2\cap H^p_{com}$ and $0<\delta<p\le 1$,

$$\begin{aligned}
\|T(f)\|_{H^p_{com}} &\le C\Big\|\Big\{\sum_{j',k'}\{\mathcal{M}_s[(\sum_{(\ell'',\ell'_m)}|(\phi_{j',k'} * h)(2^{-(j'\wedge k')-N}\ell'',2^{-(j'\wedge 2k')-N}\ell'_m)|\chi_{I'}\chi_{J'})^\delta]\}^{\frac{2}{\delta}}\Big\}^{\frac12}\Big\|_{L^p} \\
&\le C\|h\|_{H^p_{com}} \le C\|f\|_{H^p_{com}}.
\end{aligned}$$

Since $L^2 \cap H^p_{com}$ is dense in H^p_{com}, we conclude the proof of Theorem 1.7. $\square$

5. Proofs of Theorems 1.8 and 1.9

In this section using Theorem 4.1, we prove Theorem 1.8. Theorem 1.9 then follows directly from Theorem 1.8.

Proof of Theorem 1.8:
For any $f \in L^2(\mathbb{R}^m) \cap H^p_{com}(\mathbb{R}^m)$, set

$$\Omega_i = \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \widetilde{\mathcal{G}}^d_\phi(f)(x', x_m) > 2^i\},$$

where

$$\widetilde{\mathcal{G}}^d_\phi(f) = \{\sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |(\phi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)|^2 \chi_I\chi_J\}^{\frac{1}{2}},$$

here $\phi_{j,k}$ and h are given by Theorem 4.1. Denote

$$\mathcal{B}_i = \{(j,k,I,J) : \ |(I\times J)\cap\Omega_i| > \frac{1}{2}|I\times J|, \ \ |(I\times J)\cap\Omega_{i+1}| \le \frac{1}{2}|I\times J|\},$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side lengths $\ell(I) = 2^{-(j\wedge k)-N}$ and $\ell(J) = 2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively.

By Theorem 4.1, we write

$$f(x', x_m) = \sum_i \sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ \phi_{j,k}(x' - 2^{-(j\wedge k)-N}\ell', x_m - 2^{-(j\wedge 2k)-N}\ell_m)$$
$$\times(\phi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m),$$

where the series converges in the L^2 norm. We claim that

$$\Big\| \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\phi_{j,k}(\cdot - 2^{-(j\wedge k)-N}\ell', \cdot - 2^{-(j\wedge 2k)-N}\ell_m)$$
$$\times(\phi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)\Big\|^p_{L^p}$$
$$\le C 2^{ip}|\Omega_i|,$$

which together with the fact $0 < p \le 1$ yields

$$\|f\|^p_{L^p} \le C\sum_i 2^{ip}|\Omega_i| \le C\|\widetilde{\mathcal{G}}^d_\phi(f)\|^p_{L^p} \le C\|h\|^p_{H^p_{com}} \le C\|f\|^p_{H^p_{com}}.$$

Now we show the claim. Note that functions $\phi^{(1)}$ and $\phi^{(2)}$ are supported in unit balls. Hence if $(j,k,I,J) \in \mathcal{B}_i$, then $\phi_{j,k}$ are supported in

$$\widetilde{\Omega}_i = \{(x', x_m) : M_s(\chi_{\Omega_i})(x', x_m) > \frac{1}{100^m}\}.$$

For the sake of convenience, we denote $x_I = 2^{-(j\wedge k)-N}\ell'$ and $x_J = 2^{-(j\wedge 2k)-N}\ell_m$. Since $|\widetilde{\Omega}_i| \leq C|\Omega_i|$, by Hölder's inequality we obtain

$$\begin{aligned}
& \| \sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ \phi_{j,k}(\cdot - x_I, \cdot - x_J)(\phi_{j,k} * h)(x_I, x_J)\|_{L^p}^p \\
\leq\ & |\Omega_i|^{1-\frac{p}{2}}\ \| \sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ \phi_{j,k}(\cdot - x_I, \cdot - x_J)(\phi_{j,k} * h)(x_I, x_J)\|_2^p.
\end{aligned}$$

By the duality argument, we estimate the L^2 norm of

$$\sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ \phi_{j,k}(\cdot - x_I, \cdot - x_J)(\phi_{j,k} * h)(x_I, x_J)$$

as follows: For all $g \in L^2$ with $\|g\|_2 \leq 1$,

$$\begin{aligned}
& | < \sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ \phi_{j,k}(\cdot - x_I, \cdot - x_J)(\phi_{j,k} * h)(x_I, x_J), g > | \\
\leq\ & C \left(\sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ |(\phi_{j,k} * h)(x_I, x_J)|^2 \right)^{\frac{1}{2}} \left(\sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ |(\phi_{j,k} * g)(x_I, x_J)|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

While,

$$\begin{aligned}
& \sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ |(\phi_{j,k} * g)(x_I, x_J)|^2 \\
=\ & \int_{\mathbb{R}^{m-1}\times\mathbb{R}} \left\{ \sum_{(j,k,I,J)\in\mathcal{B}_i} |(\phi_{j,k} * g)(x_I, x_J))|^2 \chi_I(x')\chi_J(x_m) \right\}^{\frac{1}{2}\cdot 2} dx'dx_m \\
\leq\ & \int_{\mathbb{R}^{m-1}\times\mathbb{R}} \widetilde{\mathcal{G}}_\phi^d(g)(x', x_m)^2\ dx'dx_m \\
\leq\ & \|g\|_{L^2}^2.
\end{aligned}$$

In addition,

$$\begin{aligned}
C2^{2i}|\Omega_i| \ &\geq\ \int_{\widetilde{\Omega}_i\backslash\Omega_{i+1}} [\widetilde{\mathcal{G}}_\phi^d(f)(x,y)]^2\ dx'dx_m \\
&\geq\ \sum_{(j,k,I,J)\in\mathcal{B}_i} |(\phi_{j,k} * h)(x_I, x_J)|^2\ |(I\times J)\cap\widetilde{\Omega}_i\backslash\Omega_{i+1}| \\
&\geq\ \frac{1}{2} \sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\ |J|\ |(\phi_{j,k} * h)(x_I, x_J)|^2,
\end{aligned}$$

where in the last inequality we use the fact that $|(I\times J)\cap\widetilde{\Omega_i}\backslash\Omega_{i+1}| > \frac{1}{2}|I\times J|$ when $(j,k,I,J)\in\mathcal{B}_i$. This completes the proof of Theorem 1.8. □

Proof of Theorem 1.9: Suppose $f \in H^p_{com} \cap L^2$. By Theorem 1.7, T is bounded on H^p_{com}, which together with the fact that T is also bounded on L^2 yields that $T(f) \in H^p_{com} \cap L^2$, so applying first Theorem 1.8 and then Theorem 1.7 we obtain

$$\|T(f)\|_{L^p} \leq C\|T(f)\|_{H^p_{com}} \leq C\|f\|_{H^p_{com}} \quad \text{for any} \quad f \in L^2 \cap H^p_{com}.$$

Since $H^p_{com} \cap L^2$ is dense in H^p_{com}, the composition operator T extends to a bounded operator from H^p_{com} to L^p. □

6. Proofs of Theorems 1.10 and 1.11

We now prove the Calderón-Zygmund decomposition and the interpolation theorem on $H^p_{com}(\mathbb{R}^m)$.

Proof of Theorem 1.10: We first assume $f \in L^2 \cap H^p_{com}$. Let $\alpha > 0$ and $\Omega_\ell = \{x \in \mathbb{R}^m : \widetilde{\mathcal{G}}^d_\phi(f)(x) > \alpha 2^\ell\}$, where $\widetilde{\mathcal{G}}^d_\phi(f)$ is defined in the the proof of Theorem 1.8.

Let

$$\mathcal{R}_0 = \left\{ I \times J : \ |(I \times J) \cap \Omega_0| < \frac{1}{2}|I \times J| \right\}$$

and for $\ell \geq 1$

$$\mathcal{R}_\ell = \left\{ I \times J : \ |(I \times J) \cap \Omega_{\ell-1}| \geq \frac{1}{2}|I \times J|, \ |(I \times J) \cap \Omega_\ell| < \frac{1}{2}|I \times J| \right\},$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side lengths $\ell(I) = 2^{-(j\wedge k)-N}$ and $\ell(J) = 2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively.

By the discrete Calderón-type identity in Theorem 4.1,

$$\begin{aligned}
f(x', x_m) &= \sum_{j,k}\sum_{I,J} |I||J|\phi_{j,k}(x' - x_I, x_m - y_J)\phi_{j,k} * h(x_I, y_J) \\
&= \sum_{j,k}\sum_{\ell \geq 1}\sum_{I \times J \in \mathcal{R}_\ell} |I||J|\phi_{j,k}(x' - x_I, x_m - y_J)\phi_{j,k} * h(x_I, y_J) \\
&\quad + \sum_{j,k}\sum_{I \times J \in \mathcal{R}_0} |I||J|\phi_{j,k}(x' - x_I, x_m - y_J)\phi_{j,k} * h(x_I, y_J) \\
&= b(x', x_m) + g(x', x_m),
\end{aligned}$$

where $x_I = 2^{-(j\wedge k)-N}\ell'$ and $y_J = 2^{-(j\wedge 2k)-N}\ell_m$.

When $p_1 > 1$, using duality argument as in the proof of Theorem 1.8, it is easy to show

$$\|g\|_{p_1} \leq C\| \left\{ \sum_{j,k}\sum_{I \times J \in \mathcal{R}_0} |\phi_{j,k} * h(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \|_{p_1}.$$

Next, we estimate $\|g\|_{H^{p_1}_{com}}$ when $0 < p_1 \le 1$. Clearly, the duality argument will not work here. Nevertheless, we can estimate the $H^{p_1}_{com}$ norm directly by using discrete Calderón's identity in Theorem 1.3. To this end, we note that

$$\|g\|_{H^{p_1}_{com}} \le \| \left\{ \sum_{j',k'} \sum_{I',J'} |\left(\psi_{j'k'} * g\right)(x_{I'}, y_{J'})|^2 \chi_{I'}(x)\chi_{J'}(y) \right\}^{\frac{1}{2}} \|_{L^{p_1}},$$

where I' are dyadic cubes in $\mathbb{R}^{m-1}$ and J' are dyadic intervals in $\mathbb{R}$ with the side lengths $\ell(I') = 2^{-(j' \wedge k')}$ and $\ell(J') = 2^{-(j' \wedge 2k')}$, and the left lower corners of I' and the left end points of J' are $2^{-(j' \wedge k')}\ell''$ and $2^{-(j' \wedge 2k')}\ell'_m$, respectively.

Since

$$\left(\psi_{j',k'} * g\right)(x_{I'}, y_{J'}) = \sum_{j,k} \sum_{I \times J \in \mathcal{R}_0} |I||J| \left(\psi_{j'k'} * \phi_{j,k}\right)(x_{I'} - x_I, y_{J'} - y_J)\phi_{j,k} * h(x_I, y_J)$$

Repeating the same proof of Theorem 1.6, we have

$$\begin{aligned} &\| \left\{ \sum_{j',k'} \sum_{I',J'} |\left(\psi_{j'k'} * g\right)(x_{I'}, y_{J'})|^2 \chi_{I'}(x)\chi_{J'}(y) \right\}^{\frac{1}{2}} \|_{p_1} \\ &\le C\| \left\{ \sum_{j,k} \sum_{I \times J \in \mathcal{R}_0} |\phi_{j,k} * h(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \|_{p_1}. \end{aligned}$$

This shows that for all $0 < p_1 < \infty$

$$\|g\|_{H^{p_1}_{com}} \le C\| \left\{ \sum_{j,k} \sum_{I \times J \in \mathcal{R}_0} |\phi_{j,k} * h(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \|_{p_1}.$$

Claim 1:

$$\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m) \le \alpha} (\widetilde{\mathcal{G}}^d(f))^{p_1}(x', x_m)dx'dx_m \ge C\| \left\{ \sum_{j,k} \sum_{I \times J \in \mathcal{R}_0} |\phi_{j,k} * h(x_I, y_J)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \|^{p_1}_{p_1}.$$

This claim implies

$$\begin{aligned} \|g\|^{p_1}_{H^{p_1}_{com}} &\le C \int_{\widetilde{\mathcal{G}}^d(f)(x',x_m) \le \alpha} (\widetilde{\mathcal{G}}^d(f))^{p_1}(x', x_m)dx'dx_m \\ &\le C\alpha^{p_1 - p} \int_{\widetilde{\mathcal{G}}^d(f)(x',x_m) \le \alpha} (\widetilde{\mathcal{G}}^d(f))^{p}(x', x_m)dx'dx_m \\ &\le C\alpha^{p_1 - p}\|f\|^p_{H^p_{com}}. \end{aligned}$$

To show Claim 1, we choose $0 < q < p_1$ and note that

$$
\begin{aligned}
&\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)\le\alpha} (\widetilde{\mathcal{G}}^d(f))^{p_1}(x',x_m)dx'dx_m \\
= &\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)\le\alpha} \left\{\sum_{j,k}\sum_{I,J} |\phi_{j,k} * h(x_I,y_J)|^2\chi_I(x')\chi_J(x_m)\right\}^{\frac{p_1}{2}} dx'dx_m \\
\ge &\; C\int_{\Omega_0^c} \left\{\sum_{j,k}\sum_{R=I\times J\in\mathcal{R}_0} |\phi_{j,k} * h(x_I,y_J)|^2\chi_I\chi_J\right\}^{\frac{p_1}{2}} dx'dx_m \\
= &\; C\int_{\mathbb{R}^{m-1}\times\mathbb{R}} \left\{\sum_{j,k}\sum_{R\in\mathcal{R}_0} |\phi_{j,k} * h(x_I,y_J)|^2\chi_{R\cap\Omega_0^c}(x',x_m)\right\}^{\frac{p_1}{2}} dx'dx_m \\
\ge &\; C\int_{\mathbb{R}^{m-1}\times\mathbb{R}} \left\{\left\{\sum_{j,k}\sum_{R\in\mathcal{R}_0} \left(M_s\left(|\phi_{j,k} * h(x_I,y_J)|^q\chi_{R\cap\Omega_0^c}\right)(x',x_m)\right)^{\frac{2}{q}}\right\}^{\frac{q}{2}}\right\}^{\frac{p_1}{q}} dx'dx_m \\
\ge &\; C\int_{\mathbb{R}^{m-1}\times\mathbb{R}} \left\{\sum_{j,k}\sum_{R\in\mathcal{R}_0} |\phi_{j,k} * h(x_I,y_J)|^2\chi_R(x',x_m)\right\}^{\frac{p_1}{2}} dx'dx_m
\end{aligned}
$$

where in the last inequality we have used the fact that $|\Omega_0^c \cap R| \ge \frac{1}{2}|R|$ for $R = I\times J \in \mathcal{R}_0$, and thus

$$\chi_I(x')\chi_J(x_m) \le 2^{\frac{1}{q}} M_s(\chi_{R\cap\Omega_0^c})^{\frac{1}{q}}(x',x_m)$$

and in the second to the last inequality we have used the vector-valued Fefferman-Stein inequality for strong maximal functions

$$\|\left(\sum_{k=1}^{\infty}(M_s(f_k))^r\right)^{\frac{1}{r}}\|_p \le C\|\left(\sum_{k=1}^{\infty}|f_k|^r\right)^{\frac{1}{r}}\|_p$$

with the exponents $r = 2/q > 1$ and $p = p_1/q > 1$. Thus the claim follows.

We now recall $\widetilde{\Omega}_\ell = \{(x',x_m) \in \mathbb{R}^{m-1}\times\mathbb{R} : M_s(\chi_{\Omega_\ell}) > \frac{1}{2}\}$.

Claim 2: For any $0 < p_2 \le 1$ and $\ell \ge 1$,

$$\|\sum_{j,k}\sum_{I\times J\in\mathcal{R}_\ell} |I||J|\widetilde{\phi}_{j,k}(x'-x_I, x_m-y_J)\phi_{j,k} * h(x_I,y_J)\|^{p_2}_{H^{p_2}_{com}} \le C(2^\ell\alpha)^{p_2}|\widetilde{\Omega_{\ell-1}}|.$$

Claim 2 implies

$$\begin{aligned}
\|b\|_{H^{p_2}_{com}}^{p_2} &\leq \sum_{\ell\geq 1}(2^\ell\alpha)^{p_2}|\widetilde{\Omega_{\ell-1}}| \\
&\leq C\sum_{\ell\geq 1}(2^\ell\alpha)^{p_2}|\Omega_{\ell-1}| \leq C\int_{\widetilde{\mathcal{G}}^d(f)(x,y)>\alpha}(\widetilde{\mathcal{G}}^d)^{p_2}(f)(x',x_m)dx'dx_m \\
&\leq C\alpha^{p_2-p}\int_{\widetilde{\mathcal{G}}^d(f)(x,y)>\alpha}(\widetilde{\mathcal{G}}^d)^{p}(f)(x',x_m)dx'dx_m \leq C\alpha^{p_2-p}\|f\|^p_{H^p_{com}}.
\end{aligned}$$

To show Claim 2, again we have

$$\begin{aligned}
&\|\sum_{j,k}\sum_{I\times J\in\mathcal{R}_\ell}|I|\ |J|\ \phi_{j,k}(x'-x_I,x_m-y_J)\phi_{j,k}*h(x_I,y_J)\|_{H^{p_2}_{com}} \\
&\leq C\|\left\{\sum_{j'k'}\sum_{I',J'}\left|\sum_{j,k}\sum_{I\times J\in\mathcal{R}_\ell}|I||J|\left(\psi_{j'k'}*\phi_{j,k}\right)(x_{I'}-x_I,y_{J'}-y_J)\phi_{j,k}*h(x_I,y_J)\right|^2\right\}^{\frac12}\|_{L^{p_2}} \\
&\leq C\|\left\{\sum_{j,k}\sum_{I\times J\in\mathcal{R}_\ell}|\phi_{j,k}*h(x_I,y_J)|^2\chi_I\chi_J\right\}^{\frac12}\|_{L^{p_2}}
\end{aligned}$$

where we can use a similar argument in the proof of Theorem 1.8 to prove the last inequality.

However, as in the proof of the claim 1, choosing $0<q<2$ and $q<p_2$ implies that

$$\begin{aligned}
&(2^\ell\alpha)^{p_2}|\widetilde{\Omega}_{\ell-1}| \\
&\geq\int_{\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell}\widetilde{\mathcal{G}}^d(f)^{p_2}(x',x_m)dx'dx_m \\
&=\int_{\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell}\left\{\sum_{j,k}\sum_{I,J}|\phi_{j,k}*h(x_I,y_J)|^2\chi_I(x')\chi_J(x_m)\right\}^{\frac{p_2}{2}}dx'dx_m \\
&=\int_{\mathbb{R}^{m-1}\times\mathbb{R}}\left\{\sum_{j,k}\sum_{I,J}|\phi_{j,k}*h(x_I,y_J)|^2\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell)}(x',x_m)\right\}^{\frac{p_2}{2}}dx'dx_m \\
&\geq C\int_{\mathbb{R}^{m-1}\times\mathbb{R}}\left\{\left\{\sum_{j,k}\sum_{I,J}\left(M_s\left(|\phi_{j,k}*h(x_I,y_J)|^q\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell)}(x',x_m)\right)\right)^{\frac2q}\right\}^{\frac q2}\right\}^{\frac{p_2}{q}}dx'dx_m \\
&\geq C\int_{\mathbb{R}^{m-1}\times\mathbb{R}}\left\{\sum_{I\times J\in\mathcal{R}_\ell}|\phi_{j,k}*h(x_I,y_J)|^2\chi_I(x')\chi_J(x_m)\right\}^{\frac{p_2}{2}}dx'dx_m.
\end{aligned}$$

In the above string of inequalities, we have used the fact that for $I \times J \in \mathcal{R}_\ell$ we have

$$|(I \times J) \cap \Omega_{\ell-1}| > \frac{1}{2}|I \times J| \text{ and } |(I \times J) \cap \Omega_\ell| \leq \frac{1}{2}|I \times J|$$

and consequently $I \times J \subset \widetilde{\Omega}_{\ell-1}$. Therefore $|(I \times J) \cap (\widetilde{\Omega}_{\ell-1} \backslash \Omega_\ell)| > \frac{1}{2}|I \times J|$ for $I \times J \in \mathcal{R}_\ell$. Thus

$$\chi_I(x')\chi_J(x_m) \leq 2^{\frac{1}{q}} M_s(\chi_{(I\times J)\cap\widetilde{\Omega}_{\ell-1}\backslash\Omega_\ell)})^{\frac{1}{q}}(x', x_m).$$

This gives the proof of the claim 2. Since $L^2(\mathbb{R}^m) \cap H^p_{com}$ is dense in H^p_{com}. □

We are now ready to prove the interpolation theorem on Hardy spaces H^p_{com} for all $0 < p < \infty$.

Proof of Theorem 1.11: Suppose that T is bounded from $H^{p_2}_{com}$ to L^{p_2} and from $H^{p_1}_{com}$ to L^{p_1}. For any given $\lambda > 0$ and $f \in H^p_{com}$, by the Calderón-Zygmund decomposition,

$$f(x) = g(x) + b(x)$$

with

$$\|g\|^{p_1}_{H^{p_1}_{com}} \leq C\lambda^{p_1-p}\|f\|^p_{H^p_{com}} \text{ and } \|b\|^{p_2}_{H^{p_2}_{com}} \leq C\lambda^{p_2-p}\|f\|^p_{H^p_{com}}.$$

Moreover, we have proved the estimates

$$\|g\|^{p_1}_{H^{p_1}_{com}} \leq C \int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)\leq\alpha} \widetilde{\mathcal{G}}^d(f)^{p_1}(x', x_m)dx'dx_m$$

and

$$\|b\|^{p_2}_{H^{p_2}_{com}} \leq C \int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)>\alpha} \widetilde{\mathcal{G}}^d(f)^{p_2}(x', x_m)dx'dx_m$$

which implies that

$$\begin{aligned}
\|Tf\|^p_p &= p\int_0^\infty \alpha^{p-1}|\{(x', x_m) : |Tf(x', x_m)| > \lambda\}|d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\left|\left\{(x', x_m) : |Tg(x', x_m)| > \frac{\lambda}{2}\right\}\right|d\alpha \\
&\quad + p\int_0^\infty \alpha^{p-1}\left|\left\{(x', x_m) : |Tb(x', x_m)| > \frac{\lambda}{2}\right\}\right|d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)\leq\alpha} \widetilde{\mathcal{G}}^d(f)^{p_1}(x', x_m)dx'dx_m d\alpha \\
&\quad + p\int_0^\infty \alpha^{p-1}\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)>\alpha} \widetilde{\mathcal{G}}^d(f)^{p_2}(x', x_m)dx'dx_m d\alpha \\
&\leq C\|f\|^p_{H^p_{com}}
\end{aligned}$$

Thus,

$$\|Tf\|_p \leq C\|f\|_{H^p_{com}}$$

for any $p_2 < p < p_1$. Hence, T is bounded from H^p_{com} to L^p.

To prove the second assertion that T is bounded on H^p_{com} for $p_2 < p < p_1$, for any given $\lambda > 0$ and $f \in H^p_{com}$, by the Calderón-Zygmund decomposition again

$$\begin{aligned}
&\left|\{(x', x_m) : |g(Tf)(x', x_m)| > \alpha\}\right| \\
&\le \left|\left\{(x', x_m) : |g(Tg)(x', x_m)| > \tfrac{\alpha}{2}\right\}\right| + \left|\left\{(x', x_m) : |g(Tb)(x', x_m)| > \tfrac{\alpha}{2}\right\}\right| \\
&\le C\alpha^{-p_1}\|Tg\|^{p_1}_{H^{p_1}_{com}} + C\alpha^{-p_2}\|Tb\|^{p_2}_{H^{p_2}_{com}} \\
&\le C\alpha^{-p_1}\|g\|^{p_1}_{H^{p_1}_{com}} + C\alpha^{-p_2}\|b\|^{p_2}_{H^{p_2}_{com}} \\
&\le C\alpha^{-p_1}\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)\le\alpha} \widetilde{\mathcal{G}}^d(f)^{p_1}(x', x_m)dx'dx_m \\
&\quad + C\alpha^{-p_2}\int_{\widetilde{\mathcal{G}}^d(f)(x',x_m)>\alpha} \widetilde{\mathcal{G}}^d(f)^{p_2}(x', x_m)dx'dx_m
\end{aligned}$$

which, as above, shows that $\|Tf\|_{H^p_{com}} \le C\|g(Tf)\|_p \le C\|f\|_{H^p_{com}}$ for any $0 < p_2 < p < p_1 < \infty$.

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Received ??

YONGSHEN HAN: Department of Mathematics
Auburn University
Auburn, AL 36849, USA
E-mail: `hanyong@mail.auburn.edu`

CHINCHENG LIN: Department of Mathematics
National Central University
Taiwan, 32054, Taiwan
E-mail: `clin@math.ncu.edu.tw`

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Guozhen Lu: Department of Mathematics
Wayne State University
Detroit, MI 48202, USA
E-mail: gzlu@math.wayne.edu

Zhuoping Ruan: Department of Mathematics & IMS
Nanjing University
Nanjing, 210093, P.R. China
E-mail: zhuopingruan@nju.edu.cn

Eric T. Sawyer: Department of Mathematics & Statistics
McMaster University
Hamilton, Ontario, L8S 4K1, Canada
E-mail: sawyer@mcmaster.ca