Some recent works on multi-parameter Hardy space theory and discrete Littlewood-Paley Analysis

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Dedicated to Professor Guangchang Dong on the occasion of his 80th birthday

Abstract

The main purpose of this paper is to briefly review the earlier works of multiparameter Hardy space theory and boundedness of singular integral operators on such spaces defined on product of Euclidean spaces, and to describe some recent developments in this direction. These recent works include discrete multiparameter Calderón reproducing formulas and Littlewood-Paley theory in the framework of product of two homogeneous spaces, product of Carnot-Caretheodory spaces, multiparameter structures associated with flag singular integrals and the Zygmund dilation. Using these discrete multiparameter analysis, we are able to establish the theory of multiparameter Hardy spaces associated to the aforementioned multiparameter structures and prove the boundedness of singular integral operators on such Hardy H^p spaces and from H^p to L^p for all $0 < p \leq 1$, and derive the dual spaces of the Hardy spaces. These Hardy spaces are canonical and intrinsic to the underlying structures since they satisfy Calderón-Zygmund decomposition for functions in such spaces and interpolation properties between them. Proving boundedness of singular integral operators on product Hardy spaces was an extremely difficult task two decades ago. Our method avoids the use of very difficult Journe's geometric lemma and is a unified approach to the multiparameter theory of Hardy spaces in all aforementioned settings.

1 Introduction

The Hardy space theory has a long history. It was first introduced by Hardy for complex analytic functions on the complex plane. Following Hardy, an H^p function is a complex analytic function $F(z)$ in the upper half-plane R^2_+ such that the L^p norms

$$\Big(\int_{-\infty}^{\infty} |F(x+iy)|^p dx\Big)^{1/p}$$

are bounded independent of $y > 0$. It is clear that the H^p space is very similar to the classical L^p space when $p > 1$. One of the main results of H^p spaces is that H^p-functions, $p > 0$, have

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boundary values, i.e., when $F(z) \in H^p(R^2_+)$, then $\lim_{y\to 0} F(x+iy)$ exists for a.e. $x \in R^1$. To extend the theory of Hardy spaces to R^{n+1}_+, the upper half-space in R^{n+1}, that is, $\{(x,y) : x \in R^n, y > 0\}$, Stein and Weiss [SW] considered $H^p(R^{n+1}_+)$ functions as systems of $n+1$ harmonic functions, $F(x,y) = \{u_i(x,y)\}, i = 0,1,...,n$, defined on R^{n+1}_+, which are conjugate in the sense that they satisfy the generalized Cauchy-Riemann equations

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \ \sum_{i=0}^{n} \frac{\partial u_i}{\partial x_i} = 0 \text{ where } y = x_0$$

and such that

$$\sup_{y>0} \big(\int_{R^n} |F(x+iy)|^p dx \big)^{1/p} < \infty.$$

Here

$$|F(x,y)| = \big(\sum_{i=0}^{n} |u_i(x,y)|^2 \big)^{1/2}.$$

Stein and Weiss then proved the following theorem

Theorem 1.1. *If $F(x,y) \in H^p(R^{n+1}_+)$ and $p \geq (n-1)/n$, then $\lim_{y\to 0} F(x,y)$ exists for a.e. $x \in R^n$.*

Using non-tangential maximal function, Burkholder, Gundy and Silverstein ([BGS]) proved

Theorem 1.2. *If $u(z)$ is real valued and harmonic in upper half-plane and $u^*(x) \in L^1(\mathbb{R})$ where $u^*(x) = \sup_{|x-t|\leq y} |u(z)|, z = t + iy$, then $u = \mathcal{R}F$, where $F(z)$ is an analytic function in upper half-plane and $F \in H^1$ and $\mathcal{R}F$ is the real part of u.*

Instead of using analytic functions and system of conjugate harmonic functions, C. Fefferman and Stein [FeS2] characterized the real Hardy spaces using maximal function and the Littlewood-Paley square function. To define the maximal function on R^n, one starts with Schwartz functions $\phi(x)$ and $\psi(x)$ satisfying $\int_{R^n} \phi(x)dx = 1$ and $\int \psi(x) x^\alpha dx = 0$ for all multi-indexes α. Then letting $\phi_t(x) = t^{-n}\phi(x/t)$ for $t > 0$, and similarly to $\psi_t(x)$, the maximal function of f, for a tempered distribution f on R^n, is defined by

$$f^+(x) = \sup_{t>0} |\phi_t * f(x)|$$

and

$$f^*(x) = \sup_{|x-y|\leq t, t>0} |\phi_t * f(y)|.$$

The Littlewood-Paley square function of f is defined by

$$g(f)(x) = \big(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \big)^{1/2}.$$

C. Fefferman and Stein proved

Theorem 1.3. *All L^p-norms of f^+, f^* and $g(f)$ are equivalent, i.e.,*

$$\|f^+\|_p \approx \|f^*\|_p \approx \|g(f)\|_p$$

for all $p > 0$.

Therefore, C. Fefferman and Stein introduced the real Hardy spaces H^p as collection of all tempered distributions f such that $\|f^*\|_p < \infty$. Using these characterizations of the real H^p spaces, they also showed that Calderón-Zygmund singular integrals preserve these H^p spaces. Thus, we can regard H^p spaces as the appropriate substitute for $L^p, p > 1$. In particular, the space H^1 can be used as a replacement of space L^1 on which the singular integral operators are only weak $(1,1)$ bounded, but not bounded. It is also known that the Calderón-Zygmund operators are not bounded on L^∞. Thus, a good substitute for this function space is $BMO(R^n)$, namely, the space, introduced by John and Nirenberg, of functions satisfying

$$\frac{1}{|Q|}\int\limits_Q |f(x) - f_Q| dx \le C,$$

where f_Q denotes the mean value of f over the cube Q, and C is independent of Q.

There have been several characterizations of BMO space and we now consider a basic result of Carleson. A positive measure μ on R^{n+1}_+ is called a Carleson measure provided that $\mu(S(Q)) \le C|Q|$ for all cubes Q in R^n, where the Carleson region is defined by

$$S(Q) = \{(x,t) : x \in Q, 0 < t < \ell(Q)\},$$

where $\ell(Q)$ is the side-length of Q. Carleson then proved in [Car1]

Theorem 1.4. *μ is a Carleson measure if and only if for each $f \in L^p, p > 1$*

$$\int\limits_{R^{n+1}_+}\int |\phi_t * f(x)|^p d\mu \le C \int\limits_{R^n} |f(x)|^p dx,$$

where the function ϕ is the same as above.

Subsequently, C. Fefferman established that $f \in BMO(R^n)$ if and only if $|\psi_t * f(x)|^2 dx\frac{dt}{t}$ is a Carleson measure in R^{n+1}_+. Using this characterization, C. Fefferman ([Fe]) showed

Theorem 1.5. *The dual space of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$.*

Soon after Fefferman's celebrated duality result, R. R. Coifman ([Co]) found an atomic decomposition characterization of H^p on $\mathbb{R}^1$ and gave a powerful proof of this duality theorem. Coifman's work was extended to high dimensional case by Latter [La]. Then we have

Theorem 1.6. *$f \in H^p(\mathbb{R}^n)$ if and only if $f(x) = \sum\limits_k \lambda_k a_k(x)$ where the a_k are H^p-atoms, i.e., a_k is supported in a cube $Q_k, \int a_k(x)x^\alpha dx = 0, |\alpha| \le [n(1/p-1]$ and satisfies $\|a_k\|_2 \le |Q_k|^{1/2-1/p}, \sum |\lambda_k|^p < \infty$, and the series converges in the sense of distributions.*

We would like to point out that all aforementioned results of the Hardy spaces $H^p(R^n)$ share one common feature. Namely, they all deal with operators indexed by one parameter or are invariant with respect to a one-parameter family of dilations on R^n. On the other hand, if we consider the group of product dilations,

$$\delta(x_1, x_2, \cdots, x_n) = (\delta_1 x_1, \cdots, \delta_n x_n),\ \delta_i > 0,\ i = 1, \cdots, n,$$

then the study of these operators is quite different and becomes more complicated. Operators which are invariant under this group, such as the strong maximal function, Marcinkiewicz multipliers, generalizations of multiple Hilbert transform and multiparameter Hardy space theory, have been studied extensively in the past several decades and by now a fairly satisfactory theory has been established (see [CF1-3], [Ch], [GS], [Car1-2], [F1-4], [FS], [J1-2], [P] and in particular the beautiful survey articles of Chang and R. Fefferman [CF3] and R. Fefferman [F3] for developments in this area).

This multi-parameter dilation is also one of the objectives associated with problems in the theory of differentiation of integrals. A theorem of Jensen-Marcinkiewcz-Zygmund [JMZ] says that the strong maximal function in $\mathbb{R}^n$ defined by

$$M_n(f)(x) = \sup_{x \in R} \frac{1}{|R|} \int\limits_R |f(y)| dy,$$

where R are rectangles with sides parallel to the axes, is bounded from the Orlicz space $L(1 + (\log^+ L)^{n-1})$ to weak L^1. The basic idea of their proof is to dominate the operator M_n by the composition of $M_{x_1} M_{x_2} \cdots M_{x_n}$ where each M_{x_i} is the one dimensional Hardy-Littlewood maximal operator in the direction of the ith coordinate axis. A geometric proof of this result has been given by Cordoba-R. Fefferman using the deep understanding of the geometry of rectangles which illustrates its intimate connection to the strong maximal function [CoF]. On the other hand, Zygmund conjectured that if the rectangles in R^n had n side lengths which involve only k independent variables, then the resulting maximal operator should behave like M_k, the k-parameter strong maximal operator. More precisely, for $1 \le k \le n$, and for positive functions $\phi_1, \cdots, \phi_n$ as the side-lengths of the given collection of rectangles where the maximal function is defined, each one depending on parameters $t_1 > 0, t_2 > 0, \cdots, t_k > 0$, assuming arbitrarily small values and increasing in each variable separately, then the resulting maximal function would be bounded from $L(1 + (\log^+ L)^{k-1})$ to weak L^1 according to Zygmund's conjecture. For $k = n$, this is just the result of [JMZ].

It is well-known that there is a basic obstacle to the pure product Hardy and BMO space theory associated with multiparameter product dilations. Indeed it was conjectured that the product atomic Hardy space on $R \times R$ could be defined by rectangle atoms. Here a rectangle atom is a function $a(x, y)$ supported on a rectangle $R = I \times J$ have the property that

$$\|a\|_2 \le |R|^{1/2}, \int\limits_I a(x, y) dx = \int\limits_J a(x, y) dy = 0$$

for every $(x, y) \in R$. Then $H^1_{rect}(R \times R)$ is the space of functions $\sum\limits_k \lambda_k a_k$ with each a_k a rectangle atom and $\sum\limits_k |\lambda_k| < \infty$. However, this conjecture was disproved by Carleson by constructing a counter-example of a measure satisfying the product form of the Carleson measure, that is, the measure μ satisfies

$$\int\limits_{S(I)\times S(J)} d\mu \leq C|I \times J|$$

for all intervals I, J in R and $S(I)$ is the Carleson region associated with I. Carleson([Car2]) showed that the measure he constructed is not bounded on the product Hardy space $H^1(R\times R)$.

Let us consider a little bit more details about the product Hardy space theory. We will follow Chang-R. Fefferman [CF3] for the description of Hardy space theory on polydisks. More precisely, let $\mathbb{D}$ denote the unit disc $\{z : |z| < 1\}$ and T be its boundary. For each $z_0 = re^{i\theta_0}$, let I_{z_0} denote the arc $\{e^{i\theta} : |\theta - \theta_0| < 1 - r\}$ and for each arc $I \subset T$, let $S(I)$ denote the region $\{z : I_z \subset T\}$. For any $f \in L^p(T), p > 1$, let $u(re^{i\theta}) = (P_r * f)(\theta)$, be the Poisson integral of f. Now let $\mathbb{D}^2$ be the bi-disc, $f \in L^p(T^2)$ with bi-Poisson integral $u(r_1e^{\theta_1}, r_2e^{\theta_2}) = (P_{r_2} * (P_{r_1} * f))(\theta_1, \theta_2)$. Then we can define, similar to the situation in R^{n+1}_+, the nontangential maximal function of u as

$$u^*(z_1, z_2) = \sup_{(w_1,w_2)\in I_{(z_1,z_2)}} |u(w_1, w_2)|,$$

where $I_{(z_1,z_2)} = I_{z_1} \times I_{z_2}$. The original strategy of C. Fefferman and Stein of proving Theorem 1.2 is based on the following observation: For $\lambda > 0$, consider the set $G = \{e^{i\theta} : u^*(\theta) \leq \lambda\}$, and the region $G^+ = \cup_{\theta\in G} I_\theta$. The key fact used by C. Fefferman and Stein is that the boundary of G^+ consists of a sawtooth type region which can be approximated by Lipschitz region and on which Green's theorem can be applied. However, on the bidisc the corresponding region G^+ has a quite complicated boundary, and it is not clear how to apply Green's Theorem in this domain. In 1977, M.P. and P. Malliavin [MM] overcame this geometric difficulty and developed the product Hardy space theory by use of some delicate and complicated algebraic arguments. Essentially what they did is that instead of applying Green's theorem in G^+, they considered some function $u^2\widetilde{\chi}_{G^+}$, where $\widetilde{\chi}_{G^+}$ is a smooth version of the characteristic function χ_{G^+}, and applied Stokes' Theorem to $u^2\widetilde{\chi}_{G^+}$ on the entire domain D^2. These techniques were later generalized and simplified by Gundy and Stein to establish the product Hardy spaces. Indeed, Gundy and Stein([GS]) proved

Theorem 1.7. *For all $0 < p < \infty, \|u^*\|_p \approx \|S(u)\|_p$ where $S(u)$ is the product version of area integral.*

At almost the same time, S. Y. A. Chang found that the classical Carleson region should be replaced by any open set in R^2 with finite measure. Chang showed in [Ch]

Theorem 1.8. *A positive measure μ on D^2 is bounded in $L^p(T^2), 1 < p < \infty$, i.e.,*

$$\Big(\int\int\limits_{D^2} |u(z_1, z_2)|^p d\mu(z_1, z_2)\Big)^{1/p} \leq C\Big(\int\limits_{T^2} |f(e^{\theta_1}, e^{\theta_2})|^p d\theta_1 d\theta_2\Big)^{1/p}$$

for every $f \in L^p(T^2)$, holds if and only if

$$\mu(S(U)) \leq C|U|$$

for all connected, open sets $U \subset T$, where the region $S(U)$ is defined by $\{(z_1, z_2) : I_{z_1} \times I_{z_2} \subset T\}$.

This leads that the role of cubes in the classical atomic decomposition of $H^p(R^n)$ was replaced by arbitrary open sets of finite measures in the product $H^p(R^n \times R^m)$ and the Hardy space $H^p(R^n \times R^m)$ theory was finally developed by Chang and R. Fefferman ([Ch],[CF1-3]). Chang and Fefferman([CF2]) proved

Theorem 1.9. *$f \in H^p(R^n \times R^m)$ if and only if $f(x,y) = \sum\limits_k \lambda_k a_k(x,y)$ where $\sum\limits_k |\lambda_k|^p < \infty$ and $a_k(x,y)$ are $(2,p)-$atoms, that is, each $a_k(x,y)$ is supported in an open set Ω with finite measure satisfying the following properties:*

$$\|a_k\|_2 \leq |\Omega|^{1/2-1/p};$$

each $a_k(x,y)$ can be further decomposed by

$$a_k(x,y) = \sum_{R \subset \Omega} a_R(x,y)$$

where $R = I \times J \subset \Omega$, and I, J are dyadic rectangles in R^n and R^m, respectively, and $a_R(x,y)$ satisfy

$$\int\limits_I a_k(x,y) x^\alpha dx = \int\limits_J a_k(x,y) y^\beta dy = 0$$

for $0 \leq |\alpha|, |\beta| \leq N_p$, where N_p is a large integer depending on p, and

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} a_R(x,y)\right| \leq d_R |I|^{-n-|\alpha|} |J|^{-m-|\beta|}$$

with

$$\sum_{R \subset \Omega} |R| d_R^2 \leq |\Omega|^{1-2/p}.$$

Chang and R. Fefferman ([CF1]) also proved

Theorem 1.10. *The dual of $H^1(R^n \times R^m)$ is the product BMO space where $f \in BMO(R^n \times R^m)$ if $|\psi_{t,s} * f(x,y)|^2 dxdy \frac{dt}{t} \frac{ds}{s}$ is a product Carleon measure defined in Theorem 1.8.*

Because of the complicated nature of atoms in product space, it was an extremely difficult task to prove boundedness of singular integral operators on multi-parameter Hardy spaces. This was first overcame by Journé. Indeed, Journé([J1]) proved the following covering lemma.

Lemma 1.11. *Let $M(\Omega)$ denote the family of all maximal dyadic subrectangles of Ω. Then*

$$\sum_{R \in M(\Omega)} |R| \gamma(R)^{-\delta} \leq C_\delta |\Omega|,$$

for any $\delta > 0$, where $\gamma(R)$ is a factor which reflects how much R can be stretched and still remain inside the expansion of $\Omega, \widetilde{\Omega} = \{M_S(\chi_\Omega) > 1/2\}$.

Using this geometric covering lemma and the atomic decomposition provided by Chang and R. Fefferman ([CF2]), R. Fefferman ([F4-5]) discovered the boundedness criterion of singular integral operators. To describe this result, we introduce the following definition.

Definition 1.12. *A function $a(x,y)$ supported in a rectangle $R = I \times J \subset R^2$ is called an H^p rectangle atom provided*

$$\int_I a(x,y)x^\alpha dx = 0, \int_J a(x,y)y^\alpha dy = 0,$$

for all $\alpha = 0,1,2,...,[1/p-1]$, and

$$\|a\|_{L^2(R)} \leq |R|^{1/2-1/p}.$$

As mentioned above, according to Carleson's counterexample, the H^p rectangle atoms do not span the product Hardy space $H^1(R \times R)$ as was expected prior to his work. However, R. Fefferman ([F4-5]) proved

Theorem 1.13. *Fix $0 < p \leq 1$. Let T be a linear operator which is bounded in $L^2(R^2)$ and which satisfies*

$$\int_{\widetilde{R}_\gamma} |T(a)|^p dxdy \leq C\gamma^{-\delta}$$

for all $\gamma \geq 2$ and for some $\delta > 0$ and for each H^p rectangle atom a supported in R. Then T is bounded from $H^p(R \times R)$ to $L^p(R^2)$.

Here $\widetilde{R}_\gamma$ denotes the $\gamma-$fold concentric enlargement of R. (see [J1], [J2] and [P]). Such a geometric lemma also played an important role in the study of the boundedness of product singular integrals on $BMO(R^n \times R^m)$ (see [J1], [J2] and [P]). Recently, using a new version of Journé covering lemma, Ferguson and Lacey in [FL] (see also [FSa]) gave a new characterization of the product $BMO(R \times R)$ by bicommutator. They prove

Theorem 1.14. *These exist two constants C_1 and C_2 such that*

$$C_1\|b\|_{BMO(R\times R)} \leq \|[[M_b, H_1], H_2]\|_{L^2 \to L^2} \leq C_2\|b\|_{BMO(R\times R)},$$

where M_b is the multiplication operator by $b(x,y)$ defined on $R \times R$, and $[T,S]$ is commutator and H_1 and H_2 are the Hilbert transform with respect to the variables x and y, respectively.

Since the original Journé covering lemma holds only for two parameters, namely, $R \times R$, R. Fefferman's result only holds for the domain with only two parameters. To generalize Journé covering lemma and Fefferman's result to a domain with any number of factors, Pipher([P]) extended Journé's lemma and proved

Lemma 1.15. *Let $\Omega \subset R^3$ be open and bounded. Given $S = I \times J \times R \in M_3(\Omega)$, there exists $I \subset \hat{I}$ and $J \subset \hat{J}$ such that*

$$\| \cup_{S \in M_3(\Omega)} \hat{I} \times \hat{J} \times R \| \leq C|\Omega|$$

and

$$\sum_{S \in M_3(\Omega)} |S| w\big(\frac{|I|}{|\hat{I}|}\big) w\big(\frac{|J|}{|\hat{J}|}\big) \leq C|\Omega|,$$

where $M_3(\Omega)$ is the collection of dyadic rectangles $S \subset \Omega$ that are maximal in the x_3-direction and the function $w(x)$ is increasing and satisfies $\sum_{k>0} kw(2^{-k}) < \infty$.

While great progress has been made in the case of pure product structure of two Euclidean spaces for both L^p and H^p theory, multi-parameter analysis has only been developed in recent years for the L^p theory when the underlying multi -parameter structure is not explicit, but implicit, such as the flag multi-parameter structure studied by Muller-Ricci-Stein and Nagel-Ricci-Stein in [MRS1,2] and [NRS]. One of the main goals of this article is to exhibit a theory of Hardy space in this setting. Another is to develop the corresponding theory in the setting of nonclassical Zygmund dilation. Moreover, we also carry out the multiparameter Hardy space theory in the framework of the product of two homogeneous spaces in the sense of Coifman and Weiss and also in the setting of product Carnot-Carathedory spaces where L^p theory has been recently established by Nagel-Stein [NS3]. One of the main ideas of our program is to develop a discrete version of Calderón reproducing formula associated with the underlying multiparameter structure, and thus prove a Min-Max type inequality in this setting. This discrete scheme of Littlewood-Paley-Stein analysis is particularly useful in dealing with the Hardy spaces H^p for $0 < p \leq 1$. Using this method of discretizing, we will be able to show that the singular integral operators are bounded on H^p for all $0 < p \leq 1$, from H^p to L^p for all $0 < p \leq 1$. This method offers an alternate approach of R. Fefferman's idea of restricting singular integral operator's action on the rectangle atoms. Thus, we bypass the use of Journe's covering lemma in proving the H^p to L^p boundedness for all $0 < p \leq 1$.

In this survey, we will then describe some recent developments of the multiparameter Hardy space theory. This includes (1) the product Hardy spaces on spaces of homogeneous type, which includes the multiparameter Hardy space theory on the product of two stratified groups such as the Heisenberg group; (2) the product Hardy spaces on spaces of Carnot-Carathéodory spaces where the L^p theory has been developed recently by Nagel and Stein [NS3] ; (3) the multiparameter Hardy spaces with Zygmund dilations on R^3, where the L^p theory was studied by Ricci-Stein([RS]); and (4) theory of multiparameter Hardy spaces associated with the implicit flag singular integrals as recently studied by Muller-Ricci-Stein [MRS1,2] and Nagel Ricci-Stein [NRS].

The purpose of this program is to give a uniform approach to deal with all these multiparameter Hardy space theory using the discrete Littlewood-Paley-Stein analysis. This approach goes as follows:

(1) Calderón's identity

Let $\psi \in L^1(R^n)$ be a function whose integral is zero and whose Fourier transform $\hat{\psi}(\xi)$ satisfies $\int\limits_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1$, for each $\xi \neq 0$. We put $\tilde{\psi}(x) = \bar{\psi}(-x), \psi_t(x) = t^{-n}\psi(x/t)$ and similarly for $\tilde{\psi}_t$. Then, for every function $f \in L^2(R^n)$, Calderón's identity is given by

$$f = \int\limits_0^\infty f * \tilde{\psi}_t * \psi_t \frac{dt}{t}.$$

Thus the above formula provides the one-parameter Calderón's identity. It is known nowadays that atomic decomposition of Hardy space $H^p(R^n)$ and continuous version of wavelets were obtained by making a discrete version of the above Calderón's identity. Moreover, it is also clear that one parameter dilation on R^n is involved in this identity. On the other hand, if one lets $\psi^1, \psi^2 \in L^1(R^n)$ be functions with the same properties as ψ given above, then we have the product version of Calderón's identity: For every function $f \in L^2(R^n \times R^n)$,

$$f(x,y) = \int\limits_0^\infty f * \tilde{\psi}_{t,s} * \psi_{t,s} \frac{dtds}{ts},$$

where $\psi_{t,s}(x,y) = \psi^1_t(x)\psi^2_s(y)$ and similarly for $\tilde{\psi}_{t,s}$.

This product version of Calderón's identity played a crucial role in the product $H^p(R^n \times R^n)$ theory. The starting point of our approach is to establish all kind of Calderón's identities in all different settings. Moreover, we will discretize such formulas to adapt to the H^p theory for $0 < p \leq 1$. To be more precise, we will construct a sequence of operators $D_{j,k}$ such that there exists a family of operators $\widetilde{D}_{j,k}$ so that for each $f \in L^2$,

$$f = \sum_{j,k} \widetilde{D}_{j,k} D_{j,k}(f),$$

where the series converges in the L^2 norms.

The construction of the sequence $\{D_{j,k}\}$ varies from case to case in the aforementioned multiparameter structures. Such a sequence of the operators $D_{j,k}$ on product spaces of homogeneous type follows from the construction of approximations to the identity given by Coifman. We would like to point out that the condition (2.2) played a crucial role in the construction of Coifman. On product Carnot-Carathéodory spaces, the condition (2.2) is not satisfied. However these operators $D_{j,k}$ were constructed by the spectrum theory from the heat kernel given by Nagel and Stein. In the case of Zygmund dilation, we will use the Fourier transform to construct the operators $D_{j,k}$ which will be used for the product Hardy spaces associated with Zygmund dilations on R^3. The flag structure is used in the construction of such a sequence of the operators $D_{j,k}$ which will be used to establish the flag Hardy spaces.

For our purpose to study the Hardy space theory, the above Calderón's identity on L^2 is not powerful enough because we will have to work on spaces of distributions in order to establish the Hardy space theory. To this end, we will need such an identity to hold on some appropriate spaces of distributions. To achieve this, we need to introduce suitable test function spaces. Roughly speaking, these test functions satisfy the size, smoothness and cancellation conditions. Indeed, these conditions will follow from all conditions which are satisfied by $D_{j,k}$. See part 2-4 for more details. Then we have to show that the above Calderón's identity converges in test function spaces. By the duality argument, we have Calderón's identity on the spaces of distributions.

However, to deal with the Hardy spaces theory for $0 < p \leq 1$, the above continuous version of Calderón's identity is not convenient. More precisely, we need a discrete version of Calderón's identity. By use of the Calderón-Zygmund operator theory, discretizing the continuous version of Calderón's identity provides the following discret version of Calderón's identity:

$$f(x,y) = \sum_{j,k} \sum_{R} \widetilde{D}_{j,k,R}(x,y) D_{j,k}(f)(x_R, y_R),$$

where R are dyadic rectangles or (metric balls in appropriate sense) whose sides lengths (or radii) are associated with j,k and (x_R, y_R) are arbitrary points in R.

This discrete Calderón's identity provides the following discrete Littlewood-Paley analysis.

(2) Discrete Littlewood-Paley analysis

We now define the discrete Littlewood-Paley square function by

$$g_d(f)(x,y) = \{\sum_{j,k} \sum_{R} |D_{j,k}(f)(x_R, y_R)|^2 \chi_R(x,y)\}^{\frac{1}{2}},$$

where χ_R are characteristic functions of R.

By the almost orthogonality argument on product spaces of homogeneous type, the spectrum theory on product Carnot-Carathéodory spaces and the Fourier transform on multiparameter structures associated with Zygmund dilations and flag singular integrals, one can easily obtain

$$\|g_d(f)\|_2 \approx \|f\|_2.$$

Then apply the Calderón-Zygmund L^p theory, we have

$$\|g_d(f)\|_p \approx \|f\|_p$$

for all $1 < p < \infty$. This together with the characterizations of $H^p(R^n)$ given by Fefferman and Stein leads to define the multiparameter Hardy spaces by the collection of suitable distributions f such that

$$\|g_d(f)\|_p < \infty$$

where $0 < p \leq 1$.

Of course, this definition makes sense if and only if the L^p norms of $\|g_d(f)\|_p$ are independent of the choice of the operators $D_{j,k}$. Therefore, we prove the following so-called the Min-Max inequalities. Roughly speaking, we have sup and inf inequalities.

$$\|\{\sum_{j,k} \sum_{R} \sup_{(u,v)\in R} |D_{j,k}(f)(u,v)|^2 \chi_R(x,y)\}^{\frac{1}{2}}\|_p \approx \|\{\sum_{j,k} \sum_{R} \sup_{(u,v)\in R} |E_{j,k}(f)(u,v)|^2 \chi_R(x,y)\}^{\frac{1}{2}}\|_p,$$

where $D_{j,k}$ and $E_{j,k}$ satisfy the same properties and χ_R are characteristic functions of R.

Obviously, the above Min-Max inequalities show that the definition of the multiparameter Hardy spaces are well defined. Using discrete Littlewood-paley analysis together with the almost orthogonality arguments, we can prove the boundedness of certain Calderón-Zygmund operator on the multiparameter Hardy spaces. We would like to point out that using this discrete Littlewood-Paley analysis we also prove a fairly general result. Namely, if an oprator T is bounded on L^2 and on the multiparameter Hardy sapces H^p, then T extends to a bounded operator from the multiparameter Hardy spaces H^p to $L^p, 0 < p \leq 1$. This principle is broad enough to prove the H^p to L^p boundedness in most settings under consideration of this paper.

(3) Generalized Carleson measure spaces

To simplify our description here, we only discuss the case of pure product $R^n \times R^m$. For other cases, the reader can find details in the subsequent sections. As we mentioned above, the product BMO spaces on $R^n \times R^m$ was defined by Carleson measure condition. This suggest us to introduce a generalized Carleson measure spaces CMO^p for $0 < p \leq 1$ defined by

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \int_{\Omega} \sum_{I,J: I\times J \subseteq \Omega} |D_{j,k} * f(x,y)|^2 \chi_I(x)\chi_J(y) dxdy \right\}^{\frac{1}{2}}$$

where f is a suitable distribution, Ω in $R^n \times R^m$ are open sets with finite measures, I, J are dyadic cubes, and $D_{j,k}$ are operators as mentioned in **(2)**.

As in the case for the multiparameter Hardy spaces, the generalized Carleson measure spaces CMO^p is well defined if and only if the norm given above is independent of the choice of operators $D_{j,k}$. Again, this can be proved by the following Min-Max inequality.

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j} \sum_{k} \sum_{I\times J \subseteq \Omega} \sup_{u\in I, v\in J} |D_{j,k} * f(u,v)|^2 |I||J| \right\}^{\frac{1}{2}} \approx$$

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j} \sum_{k} \sum_{I\times J \subseteq \Omega} \inf_{u\in I, v\in J} |E_{j,k} * f(u,v)|^2 |I||J| \right\}^{\frac{1}{2}}.$$

We will prove that CMO^p is dual of multiparameter Hardy spaces H^p for all $0 < p \leq 1$. The proof follows from a very general line. We first introduce the sequence spaces. The sequence space s^p is the collection of all sequences $s = \{s_{I\times J}\}$ such that

$$\|s\|_{s^p} = \left\| \left\{ \sum_{j,k} \sum_{I,J} |s_{I\times J}|^2 |I|^{-1} |J|^{-1} \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}} \right\|_{L^p} < \infty,$$

where the sum runs over all dyadic cubes I, J and χ_I, and χ_J are indicator functions of I and

J respectively. The sequence space c^p is the collection of all sequences $s = \{s_{I\times J}\}$ such that

$$\|s\|_{c^p} = \sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I,J: I\times J\subseteq \Omega} |s_{I\times J}|^2 \right\}^{\frac{1}{2}} < \infty,$$

where Ω are all open sets with finite measures and the sum runs over all dyadic cubes I and J.

We should mention that the sequence spaces s^p and c^p on one parameter space R^n were introduced by Frazier and Jawerth([FJ]). We then prove that the dual of s^p is c^p. Finally, using discrete Calderón's identity, we define lefting operator S and projection operator T, and prove that S is bounded from H^p to s^p and from CMO^p to c^p, and T is bounded from s^p to H^p and from c^p to CMO^p. Moreover, TS is identity on H^p and on CMO^p. This clearly implies that the dual of H^p is CMO^p.

This article is organized as follows. In section 2, we focus on the multiparameter Hardy space theory in the product of two homogeneous spaces. We discuss the results on discrete Littlewood-Paley theory and Calderón's identities in this general setting of pure product. The Hardy space theory in this setting developed includes the atomic decomposition, Journe's covering lemma and boundedness of singular integral operators on Hardy spaces H^p and from H^p to L^p following R. Fefferman's ideas by restricting the action of the operator to rectangle atoms. Finally, we establish the duality theory of Hardy spaces H^p for all $0 < p \leq 1$. The theory developed in this part includes many examples of multiparameter Hardy space theory for product of the Euclidean spaces $R^n \times R^m$, product of stratified groups such as the Heisenberg group, and many others. The duality theory extends to all $0 < p \leq 1$ that of Chang and R. Fefferman where they proved the dual space of product $H^1(R^n \times R^m)$ is the product $BMO(R^n \times R^m)$ space. Section 3 is devoted to the theory of product Hardy spaces on Carnot-Carathéodory spaces. As we pointed it out earlier, condition (2.2) in section 2 for general spaces of homogeneous type is not satisfied for the Carnot-Caratheodory spaces. Therefore, the multiparameter theory for product Carnot-Caratheodory spaces does not follow from our results in section 2. The L^p theory in Carnot-Caratheodory spaces for $1 < p < \infty$ was developed by Nagel-Stein. Thus, our results on boundedness on Hardy spaces and BMO spaces can be viewed as the endpoint results of Nagel-Stein. In section 4, we develop a satisfactory theory of multiparameter Hardy spaces associated with the well-known Zygmund dilation on R^3. This is perhaps the simplest example beyond the pure product dilation. The singular integral operators introduced by Ricci-Stein [RS] are invariant under this Zygmund dilation and the L^p boundedness of such operators was established by Ricci-Stein for all L^p $(1 < p < \infty)$ and weighted L^p boundedness was proved by R. Fefferman and Pipher [FP]. In the last section of this article, we build up the the multiparameter Hardy space theory associated with the flag singular integral operators on $R^n \times R^m$. Such L^p theory was developed by Muller-Ricci-Stein [MRS1,2] and Nagel-Ricci-Stein [NRS]. This last section also includes the Calderón-Zygmund decomposition on Hardy spaces and interpolation theorem on such spaces. Such interpolation theorem on pure product spaces was first established by Chang and R. Fefferman([CF2]). We finally remark that section 5 also contains some ideas and outline of proofs of various theorems. These proofs provide some insights to those in different settings considered in sections 2, 3, 4 as well. We have chosen to include these outlines of proofs in the last section so that a reader who is only interested in the results rather than their proofs do not have to go through these in the earlier sections.

Some final words on the introduction of this paper. We have purposely written this paper in such a way that each section can be virtually read independently without relying on the rest of the paper. If a reader is only interested in the multiparameter Hardy spaces associated to Zygmund dilation, he/she can simply go to Section 4. If a reader is particularly interested in the flag Hardy spaces, then Section 5 will be the only section needed. Likewise, if a reader is only eager to know the multiparameter theory in homogeneous spaces, Sections 2 and 3 are sufficient. Nevertheless, as we have pointed out earlier, Section 5 also provides some details which can be extended to other sections.

2 Product H^p Theory on Spaces of Homogeneous Type

The main purpose of this part is to develop a satisfactory product theory for $0 < p \leq 1$ on product of two spaces of homogeneous type, namely, the theory of Hardy spaces (including atomic decomposition) and boundeness of singular operators on such Hardy spaces H^p and from H^p to L^p and duality of such Hardy spaces. Results in this part include the product H^p theory, developed in [HL1] and [HL2], of two stratified groups such as the Heisenberg group as a special case. Our methods are quite different from those given in the classical product theory of Euclidean spaces in [CF1, CF2, CF3, F1, F4, F6] because we mainly establish the Hardy space theory using the Calderón reproducing formula and Littlewood-Paley analysis which hold in test function spaces in the product of homogeneous spaces, which are particularly suitable for the H^p theory when $0 < p \leq 1$.

Part of the work described here is taken from [HLY], the duality result is from [HLL1].

2.1 Hardy spaces, atomic decomposition and boundedness of singular integrals

To develop the product H^p theory on spaces of homogeneous type,we begin with recalling some necessary definitions and notation on spaces of homogeneous type.

A quasi-metric ρ on a set X is a function $\rho:\ X \times X \to [0, \infty)$ satisfying that

(i) $\rho(x, y) = 0$ if and only if $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ for all $x,\ y \in X$;

(iii) There exists a constant $A \in [1, \infty)$ such that for all $x,\ y$ and $z \in X$,

$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)]. \tag{2.1}$$

Any quasi-metric defines a topology, for which the balls

$$B(x, r) = \{y \in X :\ \rho(y, x) < r\}$$

for all $x \in X$ and all $r > 0$ form a basis.

Definition 2.1. *Let $d > 0$ and $\theta \in (0, 1]$. A space of homogeneous type, $(X, \rho, \mu)_{d,\theta}$, is a set X together with a quasi-metric ρ and a nonnegative Borel regular measure μ on X, and there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam} X$ and all $x,\ x',\ y \in X$,*

$$\mu(B(x, r)) \sim r^d \tag{2.2}$$

and

$$|\rho(x, y) - \rho(x', y)| \leq C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta}. \tag{2.3}$$

The space of homogeneous type was first introduced by Coifman and Weiss [CW1] and its theory has developed significantly in the past three decades. For a variant of the space of homogeneous type as given in the above definition, we refer to ([MS]). In [MS], Macias and Segovia have proved that one can replace the quasi-metric ρ of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar{\rho}$ which yields the same topology on X as ρ such that $(X, \bar{\rho}, \mu)$ is the space defined by Definition 2.1 with $d = 1$.

We emphasize that conditions (2.2) and (2.3) are crucial for our product H^p spaces on spaces of homogeneous type. Throughout this part, we will always assume that $\mu(X) = \infty$.

Let us now recall the definition of the space of test functions on spaces of homogeneous type.

Definition 2.2. *([H1]) Let X be a space of homogeneous type as in Definition 2.1. Fix $\gamma > 0$ and $\beta > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:*

(i) $|f(x)| \leq C \dfrac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$;

(ii) $|f(x) - f(y)| \leq C \left(\dfrac{\rho(x, y)}{r + \rho(x, x_0)} \right)^\beta \dfrac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$

$$for\ \rho(x, y) \leq \frac{1}{2A}[r + \rho(x, x_0)];$$

(iii) $\int_X f(x)\, d\mu(x) = 0$.

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C :\ (i)\ and\ (ii)\ hold\}.$$

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with an equivalent norm for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta,\gamma)$. Also, let the dual space $(\mathcal{G}(\beta,\gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \mathcal{G}(\beta,\gamma)$,

$$|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta,\gamma))'$ and $f \in \mathcal{G}(\beta,\gamma)$. Clearly, for all $h \in (\mathcal{G}(\beta,\gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well-known that even when $X = \mathbb{R}^n$, $\mathcal{G}(\beta_1,\gamma)$ is not dense in $\mathcal{G}(\beta_2,\gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows, for a given $\epsilon \in (0,\theta]$, we let $\mathring{\mathcal{G}}(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$ when $0 < \beta,\ \gamma < \epsilon$.

Definition 2.3. *([H1]) Let X be a space of homogeneous type as in Definition 2.1. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0,\theta]$ if there exists $C_1 > 0$ such that for all $k \in \mathbb{Z}$ and all $x,\ x',\ y$ and $y' \in X$, $S_k(x,y)$, the kernel of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying*

(1) $|S_k(x,y)| \leq C_1 \dfrac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$;

(2) $$|S_k(x,y) - S_k(x',y)| \leq C_1 \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$$ $$\text{for} \quad \rho(x,x') \leq \frac{1}{2A}(2^{-k}+\rho(x,y));$$

(3) $$|S_k(x,y) - S_k(x,y')| \leq C_1 \left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$$ $$\text{for} \quad \rho(y,y') \leq \frac{1}{2A}(2^{-k}+\rho(x,y));$$

(4) $$|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \leq C_1 \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon} \times \left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$$ $$\text{for } \rho(x,x') \leq \frac{1}{2A}(2^{-k}+\rho(x,y)) \text{ and } \rho(y,y') \leq \frac{1}{2A}(2^{-k}+\rho(x,y));$$

(5) $\int_X S_k(x,y)\, d\mu(y) = 1$;

(6) $\int_X S_k(x,y)\, d\mu(x) = 1$.

Moreover, A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0,\theta]$ having compact support if there exist constants $C_2,\ C_3 > 0$ such that for all $k \in \mathbb{Z}$ and all $x,\ x',\ y$ and $y' \in X$, $S_k(x,y)$, the kernel of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying (5), (6) and

(7) $S_k(x,y)=0$ *if* $\rho(x,y)\ge C_2 2^{-k}$ *and* $\|S_k\|_{L^\infty(X\times X)}\le C_3 2^{kd}$;

(8) $|S_k(x,y)-S_k(x',y)|\le C_3 2^{k(d+\epsilon)}\rho(x,x')^\epsilon$;

(9) $|S_k(x,y)-S_k(x,y')|\le C_3 2^{k(d+\epsilon)}\rho(y,y')^\epsilon$;

(10) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\le C_3 2^{k(d+2\epsilon)}\rho(x,x')^\epsilon\rho(y,y')^\epsilon$.

Remark 2.4. *By Coifman's construction in [DJS], one can construct an approximation to the identity of order θ having compact support satisfying the above Definition 2.3.*

We now recall the continuous Calderón reproducing formulae on spaces of homogeneous type in [HS, H1].

Lemma 2.5. *Let X be a space of homogeneous type as in Definition 2.1, $\epsilon\in(0,\theta]$, $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ and $D_k=S_k-S_{k-1}$ for $k\in\mathbb{Z}$. Then there are families of linear operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ and $\{\overline{D}_k\}_{k\in\mathbb{Z}}$ such that for all $f\in\mathcal{G}(\beta,\gamma)$ with $\beta,\ \gamma\in(0,\epsilon)$,*

$$f=\sum_{k=-\infty}^{\infty}\widetilde{D}_kD_k(f)=\sum_{k=-\infty}^{\infty}D_k\overline{D}_k(f), \tag{2.4}$$

where the series converge in the norm of both the space $\mathcal{G}(\beta',\gamma')$ with $0<\beta'<\beta$ and $0<\gamma'<\gamma$ and the space $L^p(X)$ with $p\in(1,\infty)$. Moreover, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$ for all $k\in\mathbb{Z}$ satisfies the conditions (i) and (ii) of Definition 2.3 with ϵ replaced by any $\epsilon'\in(0,\epsilon)$, and

$$\int_X\widetilde{D}_k(x,y)\,d\mu(y)=0=\int_X\widetilde{D}_k(x,y)\,d\mu(x); \tag{2.5}$$

$\overline{D}_k(x,y)$, the kernel of $\overline{D}_k$ satisfies the conditions (i) and (iii) of Definition 2.3 with ϵ replaced by any $\epsilon'\in(0,\epsilon)$ and (2.5).

By an argument of duality, the above continuous Calderón reproducing formulae on spaces of distributions, $\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$ with $\beta,\ \gamma\in(0,\epsilon)$ is also established.

Lemma 2.6. *With all the notation as in Lemma 2.1, then for all $f\in\left(\mathring{\mathcal{G}}(\beta,\gamma)\right)'$ with $\beta,\ \gamma\in(0,\epsilon)$, (2.4) holds in $\left(\mathring{\mathcal{G}}(\beta',\gamma')\right)'$ with $\beta<\beta'<\epsilon$ and $\gamma<\gamma'<\epsilon$.*

Let now $(X_i,\rho_i,\mu_i)_{d_i,\theta_i}$ for $i=1,\ 2$ be two spaces of homogeneous type as in Definition 2.1 and ρ_i satisfies (2.3) with A replaced by A_i for $i=1,\ 2$. We now introduce the space of test functions on the product space $X_1\times X_2$ of spaces of homogeneous type.

Definition 2.7. *For $i=1,\ 2$, fix $\gamma_i>0$ and $\beta_i>0$. A function f defined on $X_1\times X_2$ is said to be a test function of type $(\beta_1,\beta_2,\gamma_1,\gamma_2)$ centered at $(x_0,y_0)\in X_1\times X_2$ with width $r_1,\ r_2>0$ if f satisfies the following conditions:*

(i) $|f(x,y)| \le C \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$;

(ii) $|f(x,y) - f(x',y)| \le C \left(\frac{\rho_1(x,x')}{r_1+\rho_1(x,x_0)}\right)^{\beta_1} \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$

$$\text{for } \rho_1(x,x') \le \frac{1}{2A_1}[r_1+\rho_1(x,x_0)];$$

(iii) $|f(x,y) - f(x,y')| \le C \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \left(\frac{\rho_2(y,y')}{r_2+\rho_2(y,y_0)}\right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}}$

$$\text{for } \rho_2(y,y') \le \frac{1}{2A_2}[r_2+\rho_2(y,y_0)];$$

(iv)

$$\begin{aligned} &|[f(x,y) - f(x',y)] - [f(x,y') - f(x',y')]| \\ &\le C \left(\frac{\rho_1(x,x')}{r_1+\rho_1(x,x_0)}\right)^{\beta_1} \frac{r_1^{\gamma_1}}{(r_1+\rho_1(x,x_0))^{d_1+\gamma_1}} \\ &\quad \times \left(\frac{\rho_2(y,y')}{r_2+\rho_2(y,y_0)}\right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2+\rho_2(y,y_0))^{d_2+\gamma_2}} \end{aligned}$$

$$\text{for } \rho_1(x,x') \le \frac{1}{2A_1}[r_1+\rho_1(x,x_0)] \text{ and } \rho_2(y,y') \le \frac{1}{2A_2}[r_2+\rho_2(y,y_0)];$$

(v) $\int_{X_1} f(x,y)\, d\mu_1(x) = 0$ *for all* $y \in X_2$*;*

(vi) $\int_{X_2} f(x,y)\, d\mu_2(y) = 0$ *for all* $x \in X_1$*.*

If f is a test function of type $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ centered at $(x_0, y_0) \in X_1 \times X_2$ with width $r_1,\ r_2 > 0$, we write $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and we define the norm of f by

$$\|f\|_{\mathcal{G}(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C : \ (i),\ (ii),\ (iii) \text{ and } (iv) \text{ hold}\}.$$

Remark 2.8. *In the sequel, if $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$, we will then simply write*

$$f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta; \gamma).$$

We now denote by $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $r_1 = r_2 = 1$ for fixed $(x_0, y_0) \in X_1 \times X_2$. It is easy to see that

$$\mathcal{G}(x_1, y_1; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$$

with an equivalent norm for all $(x_1, y_1) \in X_1 \times X_2$. We can easily check that the space $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space. Also, we denote by $(\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ its dual space which is the set of all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

$$|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ and $f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$. Clearly, for all $h \in (\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $(x_0, y_0) \in X_1 \times X_2$, $r_1 > 0$ and $r_2 > 0$. By the same reason as the case of non product spaces, we denote by $\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the completion of the space $\mathcal{G}(\epsilon_1, \epsilon_2)$ in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ when $0 < \beta_1,\ \gamma_1 < \epsilon_1$ and $0 < \beta_2,\ \gamma_2 < \epsilon_2$.

We then have the following

Lemma 2.9. *Let $(x_1, x_2) \in X_1 \times X_2$, $r_i > 0$, $\epsilon_i \in (0, \theta_i]$ and $0 < \beta_i,\ \gamma_i < \epsilon_i$ for $i = 1,\ 2$. If the linear operators T_1 and T_2 are respectively bounded on the spaces $\mathcal{G}(x_1, r_1, \beta_1, \gamma_1)$ and $\mathcal{G}(x_2, r_2, \beta_2, \gamma_2)$ with operator norms C_1 and C_2, then the operator T_1T_2 is bounded on $\mathcal{G}(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with an operator norm C_1C_2.*

To establish the continuous Calderón reproducing formulae on the product spaces $X_1 \times X_2$, we first need to recall some details of the proof of the same formulae for the one-parameter case in [H1], namely Lemma 2.5. One of the keys for establishing these formulae is Coifman's idea in [DJS]. Let X be a space of homogeneous type as in Definition 2.1, $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity of order $\epsilon \in (0, \theta]$ on X as in Definition 2.3 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then, it is easy to see that

$$I = \sum_{k=-\infty}^{\infty} D_k \quad \text{in} \quad L^2(X) \tag{2.6}$$

Let $N \in \mathbb{N}$. Coifman's idea is to rewrite (2.6) into

$$\begin{aligned} I &= \left(\sum_{k=-\infty}^{\infty} D_k\right)\left(\sum_{j=-\infty}^{\infty} D_j\right) \\ &= \sum_{|j|>N} \sum_{k=-\infty}^{\infty} D_{k+j} D_k + \sum_{k=-\infty}^{\infty} \sum_{|j|\leq N} D_{k+j} D_k \\ &= R_N + T_N, \end{aligned} \tag{2.7}$$

where

$$R_N = \sum_{|j|>N} \sum_{k=-\infty}^{\infty} D_{k+j} D_k \tag{2.8}$$

and

$$T_N = \sum_{k=-\infty}^{\infty} D_k^N D_k \tag{2.9}$$

with

$$D_k^N = \sum_{|j|\le N} D_{k+j}.$$

It was proved in [H1] that there are constants $C > 0$ and $\delta > 0$ independent of $N \in \mathbb{N}$ such that for all $f \in \mathcal{G}(x_1, r, \beta, \gamma)$ with $x_1 \in X$, $r > 0$ and $0 < \beta,\ \gamma < \epsilon$,

$$\|R_N f\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \le C 2^{-N\delta} \|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)}. \tag{2.10}$$

Thus, if we choose $N \in \mathbb{N}$ such that

$$C 2^{-N\delta} < 1, \tag{2.11}$$

then T_N in (2.9) is invertible in the space $\mathcal{G}(x_1, r, \beta, \gamma)$, namely, T_N^{-1} exists in the space $\mathcal{G}(x_1, r, \beta, \gamma)$ and there is a constant $C > 0$ such that for all $f \in \mathcal{G}(x_1, r, \beta, \gamma)$,

$$\|T_N^{-1} f\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \le C \|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)}.$$

For such chosen $N \in \mathbb{N}$, letting

$$\widetilde{D}_k = T_N^{-1} D_k^N, \tag{2.12}$$

we then obtain the first formula in (2.4). The proof of the second formula in (2.4) is similar.

Using this idea, we can obtain the following continuous Calderón reproducing formula of separable variable type on product spaces of homogeneous-type spaces, which is also the main theorem of this part 1.

By a procedure similar to the proof of Lemma 2.5, we can establish another continuous Calderón reproducing formulae. We leave the details to the reader.

Theorem 2.10. *Let $i = 1,\ 2$ and $\{D_{k_i}\}_{k_i\in\mathbb{Z}}$ be the same as in Lemma 2.5. Then there are families of linear operators $\{\overline{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ on X_i such that for all $f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $\beta_i,\ \gamma_i \in (0, \epsilon_i)$ for $i = 1,\ 2$,*

$$f = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} D_{k_1} D_{k_2} \overline{D}_{k_1} \overline{D}_{k_2}(f),$$

where the series converge in the norm of both the space $\mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2')$ with $\beta_i' \in (0, \beta_i)$ and $\gamma_i' \in (0, \gamma_i)$ for $i = 1,\ 2$, and $L^p(X_1 \times X_2)$ with $p \in (1, \infty)$. Moreover, $\overline{D}_{k_i}(x_i, y_i)$, the kernel of $\overline{D}_{k_i}$ for $x_i,\ y_i \in X_i$ and all $k_i \in \mathbb{Z}$ satisfies the conditions (1) and (3) of Definition 2.3 with ϵ_i replaced by any $\epsilon_i' \in (0, \epsilon_i)$, and

$$\int_{X_i} \overline{D}_{k_i}(x_i, y_i)\, d\mu_i(y_i) = 0 = \int_{X_i} \overline{D}_{k_i}(x_i, y_i)\, d\mu_i(x_i),$$

where $i = 1,\ 2$.

To establish the following continuous Calderón reproducing formulae in spaces of distributions, we need to use the theory of Calderón-Zygmund operators on these spaces developed in [H1]. We first recall some definitions.

Let X be a space of homogeneous type as in Definition 2.1. For $\eta \in (0, \theta]$, we define $C_0^\eta(X)$ to be the set of all functions having compact support such that

$$\|f\|_{C_0^\eta(X)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology and let $(C_0^\eta(X))'$ be its dual space.

Definition 2.11. *Let $\epsilon \in (0, \theta]$ and X be a space of homogeneous type as in Definition 2.1. A continuous complex-valued function $K(x,y)$ on*

$$\Omega = \{(x,y) \in X \times X : x \neq y\}$$

is called a Calderón-Zygmund kernel of type ϵ if there exist a constant $C > 0$ such that

(i) $|K(x,y)| \leq C\rho(x,y)^{-d}$,

(ii) $|K(x,y) - K(x',y)| \leq C\rho(x,x')^\epsilon \rho(x,y)^{-d-\epsilon}$ *for* $\rho(x,x') \leq \dfrac{\rho(x,y)}{2A}$,

(iii) $|K(x,y) - K(x,y')| \leq C\rho(y,y')^\epsilon \rho(x,y)^{-d-\epsilon}$ *for* $\rho(y,y') \leq \dfrac{\rho(x,y)}{2A}$.

A continuous linear operator $T : C_0^\eta(X) \to (C_0^\eta(X))'$ for all $\eta \in (0,\theta]$ is a Calderón-Zygmund singular integral operator of type ϵ if there is a Calderón-Zygmund kernel $K(x,y)$ of the type ϵ as above such that

$$\langle Tf, g\rangle = \int_X \int_X K(x,y) f(y) g(x)\, d\mu(x)\, d\mu(y)$$

for all $f,\ g \in C_0^\eta(X)$ with disjoint supports. In this case, we write $T \in CZO(\epsilon)$.

We also need the following notion of the strong weak boundedness property in [HS].

Definition 2.12. *Let X be a space of homogeneous type as in Definition 2.1. A Calderón-Zygmund singular integral operator T of the kernel K is said to have the strong weak boundedness property, if there exist $\eta \in (0,\theta]$ and constant $C > 0$ such that*

$$|\langle K, f\rangle| \leq Cr^d$$

for all $r > 0$ and all continuous f on $X \times X$ with $\operatorname{supp} f \subseteq B(x_1, r) \times B(y_1, r)$, where x_1 and $y_1 \in X$, $\|f\|_{L^\infty(X\times X)} \leq 1$, $\|f(\cdot,y)\|_{C_0^\eta(X)} \leq r^{-\eta}$ for all $y \in X$ and $\|f(x,\cdot)\|_{C_0^\eta(X)} \leq r^{-\eta}$ for all $x \in X$. We denote this by $T \in SWBP$.

The following theorem is the variant on space of homogeneous type of Theorem 1.19 in [H1].

Lemma 2.13. *Let $\epsilon \in (0, \theta]$ and X be a space of homogeneous type as in Definition 2.1. Let $T \in CZO(\epsilon)$, $T(1) = T^*(1) = 0$, and $T \in SWBP$. Furthermore, $K(x, y)$, the kernel of T, satisfies the following smoothness condition*

$$
\begin{aligned}
\tag{2.13} &|[K(x, y) - K(x', y)] - [K(x, y') - K(x', y')]| \\
&\leq C\rho(x, x')^{\epsilon}\rho(y, y')^{\epsilon}\rho(x, y)^{-d-2\epsilon}
\end{aligned}
$$

for all $x, x', y, y' \in X$ such that $\rho(x, x'), \rho(y, y') \leq \frac{\rho(x,y)}{3A^2}$. Then for any $x_0 \in X$, $r > 0$ and $0 < \beta, \gamma < \epsilon$, T maps $\mathcal{G}(x_0, r, \beta, \gamma)$ into itself. Moreover, if we let $\|T\|$ be the norm of T on L^2, then there exists a constant $C > 0$ such that

$$\|Tf\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \leq C\|T\|\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)}.$$

We also need the following construction given by Christ in [Chr], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. A similar construction was independently given by Sawyer and Wheeden [SaW].

Lemma 2.14. *Let X be a space of homogeneous type as in Definition 2.1. Then there exist a collection*

$$\{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$$

of open subsets, where I_k is some index set, and constants $\delta \in (0, 1)$ and $C > 0$ such that

(i) $\mu(X \setminus \cup_\alpha Q^k_\alpha) = 0$ *for each fixed k and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$;*

(ii) *for any α, β, k, l with $l \geq k$, either $Q^l_\beta \subset Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$;*

(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q^k_\alpha \subset Q^l_\beta$;*

(iv) $\operatorname{diam}(Q^k_\alpha) \leq C\delta^k$;

(v) *each Q^k_α contains some ball $B(z^k_\alpha, C\delta^k)$, where $z^k_\alpha \in X$.*

In fact, we can think of Q^k_α as being a dyadic cube with diameter roughly δ^k and centered at z^k_α. In what follows, we always suppose $\delta = 1/2$. See [HS] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}_+$ and $\tau \in I_k$, we will denote by $Q^{k,\nu}_\tau$, $\nu = 1, 2, \cdots, N(k, \tau)$, the set of all cubes $Q^{k+j}_{\tau'} \subset Q^k_\tau$, where j is a fixed large positive integer. Denote by $y^{k,\nu}_\tau$ a point in $Q^{k,\nu}_\tau$. For any dyadic cube Q and any $f \in L^1_{\mathrm{loc}}(X)$, we set

$$m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x).$$

Using Theorem 2.10, we can establish the following continuous Calderón reproducing formulae in spaces of distributions.

Theorem 2.15. *Let all the notation be the same as in Theorem 2.10. Then for all* $f \in \Big(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\Big)'$,

$$f = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} D^*_{k_1}D^*_{k_2}\widetilde{D}^*_{k_1}\widetilde{D}^*_{k_2}(f)$$

holds in $\Big(\mathring{\mathcal{G}}(\beta'_1,\beta'_2;\gamma'_1,\gamma'_2)\Big)'$ *with* $\beta'_i \in (\beta_i,\epsilon_i)$ *and* $\gamma'_i \in (\gamma_i,\epsilon_i)$ *for* $i = 1,\ 2$, *where* $D^*_{k_i}(x,y) = D_{k_i}(y,x)$ *and* $\widetilde{D}^*_{k_i}(x,y) = \widetilde{D}_{k_i}(y,x)$.

Similarly, from Theorem 2.15, we can deduce the following continuous Calderón reproducing formulae in spaces of distributions.

Theorem 2.16. *Let all the notation be the same as in Theorem 2.15. Then for all* $f \in \Big(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\Big)'$,

$$f = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \overline{D}^*_{k_1}\overline{D}^*_{k_2}D^*_{k_1}D^*_{k_2}(f)$$

holds in $\Big(\mathring{\mathcal{G}}(\beta'_1,\beta'_2;\gamma'_1,\gamma'_2)\Big)'$ *with* $\beta'_i \in (\beta_i,\epsilon_i)$ *and* $\gamma'_i \in (\gamma_i,\epsilon_i)$ *for* $i = 1,\ 2$, *where* $D^*_{k_i}(x,y) = D_{k_i}(y,x)$ *and* $\overline{D}^*_{k_i}(x,y) = \overline{D}_{k_i}(y,x)$.

Let $i = 1,\ 2$. Note that $D^*_{k_i}$, $\widetilde{D}^*_{k_i}$ and $\overline{D}^*_{k_i}$ respectively have the same properties as D_{k_i}, $\overline{D}_{k_i}$ and $\widetilde{D}_{k_i}$. From this, it is easy to see that we can re-state Theorem 2.10 as the following theorem, which will simplify the notation in the following applications of these formulae.

Theorem 2.17. *Let all the notation be the same as in Theorem 2.10. Then for all* $f \in \Big(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\Big)'$,

$$f = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \widetilde{D}_{k_1}\widetilde{D}_{k_2}D_{k_1}D_{k_2}(f) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} D_{k_1}D_{k_2}\overline{D}_{k_1}\overline{D}_{k_2}(f)$$

holds in $\Big(\mathring{\mathcal{G}}(\beta'_1,\beta'_2;\gamma'_1,\gamma'_2)\Big)'$ *with* $\beta'_i \in (\beta_i,\epsilon_i)$ *and* $\gamma'_i \in (\gamma_i,\epsilon_i)$ *for* $i = 1,\ 2$.

We now recall the discrete Calderón reproducing formulae on spaces of homogeneous type in [H3].

Lemma 2.18. *With all the notation as in Lemma 2.14, then for all* $f \in \mathcal{G}(\beta,\gamma)$ *with* $\beta,\ \gamma \in (0,\epsilon)$ *and any* $y^{k,\nu}_\tau \in Q^{k,\nu}_\tau$,

$$\begin{aligned}
f(x) &= \sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau)D_k(f)(y^{k,\nu}_\tau)\\
&= \sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)D_k(x,y^{k,\nu}_\tau)\overline{D}_k(f)(y^{k,\nu}_\tau),
\end{aligned} \tag{2.14}$$

where the series converge in the norm of both the space $\mathcal{G}(\beta',\gamma')$ *with* $0 < \beta' < \beta$ *and* $0 < \gamma' < \gamma$ *and the space* $L^p(X)$ *with* $p \in (1,\infty)$.

By an argument of duality, the following discrete Calderón reproducing formulae on spaces of distributions, $\Big(\mathring{\mathcal{G}}(\beta,\gamma)\Big)'$ with $\beta,\ \gamma\in(0,\epsilon)$ is also established in [H3].

Lemma 2.19. *With all the notation as in Lemma 2.6, then for all $f\in\Big(\mathring{\mathcal{G}}(\beta,\gamma)\Big)'$ with $\beta,\ \gamma\in(0,\epsilon)$, (2.14) holds in $\Big(\mathring{\mathcal{G}}(\beta',\gamma')\Big)'$ with $\beta<\beta'<\epsilon$ and $\gamma<\gamma'<\epsilon$.*

By a procedure similar to the proofs of Lemma 2.18, we can also establish the following discrete Calderón reproducing formulae on product spaces of homogeneous-type spaces. We only state the results and leave the details to the reader.

Theorem 2.20. *Let all the notation as in Theorems 2.17 and Lemma 2.18, and*

$$\{Q_{\tau_1}^{k_1,\nu_1}:\ k_1\in\mathbb{Z},\ \tau_1\in I_{k_1},\ \nu_1=1,\cdots,N(k_1,\tau_1)\}$$

and $\{Q_{\tau_2}^{k_2,\nu_2}:\ k_2\in\mathbb{Z},\ \tau_2\in I_{k_2},\ \nu_2=1,\cdots,N(k_2,\tau_2)\}$ respectively be the dyadic cubes of X_1 and X_2 defined above with $j_1,\ j_2\in\mathbb{N}$ large enough. Then for all $f\in\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ with $\beta_i,\ \gamma_i\in(0,\epsilon_i)$ for $i=1,\ 2$ and any $y_{\tau_1}^{k_1,\nu_1}\in Q_{\tau_1}^{k_1,\nu_1}$ and $y_{\tau_2}^{k_2,\nu_2}\in Q_{\tau_2}^{k_2,\nu_2}$,

$$\begin{aligned}
f(x_1,x_2)&=\sum_{k_1=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_1(Q_{\tau_1}^{k_1,\nu_1})\mu_2(Q_{\tau_2}^{k_2,\nu_2})\\
&\quad\times\widetilde{D}_{k_1}(x_1,y_{\tau_1}^{k_1,\nu_1})\widetilde{D}_{k_2}(x_2,y_{\tau_2}^{k_2,\nu_2})D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2})\\
&=\sum_{k_1=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\nu_1=1}^{N(k_1,\tau_1)}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_2\in I_{k_2}}\sum_{\nu_2=1}^{N(k_2,\tau_2)}\mu_1(Q_{\tau_1}^{k_1,\nu_1})\mu_2(Q_{\tau_2}^{k_2,\nu_2})\\
&\quad\times D_{k_1}(x_1,y_{\tau_1}^{k_1,\nu_1})D_{k_2}(x_2,y_{\tau_2}^{k_2,\nu_2})\overline{D}_{k_1}\overline{D}_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1},y_{\tau_2}^{k_2,\nu_2}),
\end{aligned}\tag{2.15}$$

where the series converge in the norm of both the space $\mathcal{G}(\beta_1',\beta_2';\gamma_1',\gamma_2')$ with $\beta_i'\in(0,\beta_i)$ and $\gamma_i'\in(0,\gamma_i)$ for $i=1,\ 2$, and $L^p(X_1\times X_2)$ with $p\in(1,\infty)$.

Theorem 2.21. *Let all the notation be the same as in Theorem 2.20. Then for all $f\in\Big(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\Big)'$, (2.15) holds in $\Big(\mathring{\mathcal{G}}(\beta_1',\beta_2';\gamma_1',\gamma_2')\Big)'$ with $\beta_i'\in(\beta_i,\epsilon_i)$ and $\gamma_i'\in(\gamma_i,\epsilon_i)$ for $i=1,\ 2$.*

Using the Calderón reproducing formulas, we now establish the Littlewood-Paley theorem on product spaces of spaces of homogeneous type. To this end, we recall the Littlewood-Paley theorem on spaces of homogeneous type in [DJS].

Lemma 2.22. *Let X be a space of homogeneous type as in Definition 2.1, $\epsilon\in(0,\theta)$, $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ as in Definition 2.3 and $D_k=S_k-S_{k-1}$ for $k\in\mathbb{Z}$. If $1<p<\infty$, then there is a constant $C_p>0$ such that for all $f\in L^p(X)$,*

$$C_p^{-1}\|f\|_{L^p(X)}\le\left\|\left\{\sum_{k=-\infty}^{\infty}|D_k(f)|^2\right\}^{1/2}\right\|_{L^p(X)}\le C_p\|f\|_{L^p(X)}.\tag{2.16}$$

The Littlewood-Paley theorem on product spaces of homogeneous-type spaces can be stated as follows, whose proof can be deduced from the well-known discrete vector-valued Littlewood-Paley theorem on spaces of homogenous type, see also the proof of Theorem 2 in [FS].

Theorem 2.23. *Let $i = 1,\ 2$, X_i be a space of homogeneous type as in Definition 2.1, $\epsilon_i \in (0, \theta_i]$, $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order ϵ_i on space of homogeneous type, X_i, and $D_{k_i} = S_{k_i} - S_{k_i - 1}$ for all $k_i \in \mathbb{Z}$. If $1 < p < \infty$, then there is a constant $C_p > 0$ such that for all $f \in L^p(X_1 \times X_2)$,*

$$C_p^{-1} \|f\|_{L^p(X_1 \times X_2)} \le \|g_2(f)\|_{L^p(X_1 \times X_2)} \le C_p \|f\|_{L^p(X_1 \times X_2)}, \tag{2.17}$$

where $g_q(f)$ for $q \in (0, \infty)$ is called the discrete Littlewood-Paley g-function defined by

$$g_q(f)(x_1, x_2) = \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |D_{k_1} D_{k_2}(f)(x_1, x_2)|^q \right\}^{1/q}$$

for $x_1 \in X_1$ and $x_2 \in X_2$.

We now define the Littlewood-Paley S-function S_q on the product space $X_1 \times X_2$ by

$$\begin{aligned} S_q(f)(x_1, x_2) &= \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \int_{\rho_1(x_1,y_1) \le C_{11,1} 2^{-k_1}} \int_{\rho_2(x_2,y_2) \le C_{11,2} 2^{-k_2}} 2^{k_1 d_1 + k_2 d_2} \right. \\ &\quad \left. \times |D_{k_1} D_{k_2}(f)(y_1, y_2)|^q \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/q} \end{aligned} \tag{2.18}$$

for $x_1 \in X_1$ and $x_2 \in X_2$.

Using the Calderón reproducing formula we have the following

Lemma 2.24. *Let $1 < p,\ q < \infty$. Then there exists a constant $C_{p,q} > 0$ such that for all $f \in L^p(X_1 \times X_2)$,*

$$\|S_q(f)\|_{L^p(X_1 \times X_2)} \le C_{p,q} \|g_q(f)\|_{L^p(X_1 \times X_2)}.$$

and

Lemma 2.25. *Let $1 < p,\ q < \infty$. Then there exists a constant $C_p > 0$ such that for all $f \in L^p(X_1 \times X_2)$,*

$$\|f\|_{L^p(X_1 \times X_2)} \le C_p \|S_2(f)\|_{L^p(X_1 \times X_2)}.$$

Lemma 2.24, Lemma 2.25 and Theorem 2.23 imply the following equivalence of the Littlewood-Paley S-function and g-function in $L^p(X_1 \times X_2)$-norm.

Theorem 2.26. *Let all the notation be the same as in Theorem 3.1, g_2 and S_2 be defined respectively as in Theorem 2.23 and (2.18). If $1 < p < \infty$, then there is a constant $C_p > 0$ such that for all $f \in L^p(X_1 \times X_2)$,*

$$C_p^{-1} \|S_2(f)\|_{L^p(X_1\times X_2)} \le \|g_2(f)\|_{L^p(X_1\times X_2)} \le C_p \|S_2(f)\|_{L^p(X_1\times X_2)}.$$

We are now ready to introduce the product H^p spaces on spaces of homogeneous type. We first apply the discrete Calderón reproducing formulae to establish the equivalence between the Littlewood-Paley S-function and g-function in $L^p(X_1 \times X_2)$-norm with $p \le 1$. Such a result for one-parameter spaces was already obtained in [H3] via a Min-Max inequality. We use the same ideas as in [H3] here. Thus, we first establish a product-type Min-Max inequality. To this end, we need the following lemma which can be found in [FJ, pp. 147-148] for $\mathbb{R}^n$ and [HS, p. 93] for spaces of homogeneous type.

Lemma 2.27.. *Let X be a space of homogeneous type as in Definition 2.1, $0 < r \le 1$, k, $\eta \in \mathbb{Z}_+$ with $\eta \le k$ and for any dyadic cube $Q_\tau^{k,\nu}$,*

$$|f_{Q_\tau^{k,\nu}}(x)| \le (1 + 2^\eta \rho(x, y_\tau^{k,\nu}))^{-d-\gamma},$$

where $x \in X$, $y_\tau^{k,\nu}$ is any point in $Q_\tau^{k,\nu}$ and $\gamma > d(1/r - 1)$. Then

$$\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}||f_{Q_\tau^{k,\nu}}(x)| \le C2^{(k-\eta)d/r} \left[M \left(\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}|^r \chi_{Q_\tau^{k,\nu}} \right)(x) \right]^{1/r},$$

where C is independent of x, k and η, and M is the Hardy-Littlewood maximal operator on X.

The product-type Min-Max inequalities are the following

Theorem 2.28. *Let the notation be the same as in Theorem 2.20. Moreover, let*

$$\{Q_{\tau_1'}^{k_1',\nu_1'} : \ k_1' \in \mathbb{Z},\ \tau_1' \in I_{k_1'},\ \nu_1' = 1, \cdots, N(k_1', \tau_1')\}$$

and $\{Q_{\tau_2'}^{k_2',\nu_2'} : \ k_2' \in \mathbb{Z},\ \tau_2' \in I_{k_2'},\ \nu_2' = 1, \cdots, N(k_2', \tau_2')\}$ respectively be another set of dyadic cubes of X_1 and X_2 defined above with j_1', $j_2' \in \mathbb{N}$ large enough, let $\{P_{k_i}\}_{k_i\in\mathbb{Z}}$ be another approximation to the identity of order ϵ_i on homogeneous-type space X_i and $E_{k_i} = P_{k_1} - P_{k_i-1}$ for $k_i \in \mathbb{Z}$ and $i = 1,\ 2$. If $\max\left\{\frac{d_1}{d_1+\epsilon_1}, \frac{d_2}{d_2+\epsilon_2}\right\} < p,\ q \le \infty$, then there is a constant $C > 0$

such that for all $f \in \left(\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ *with* $\beta_i,\ \gamma_i \in (0, \epsilon_i)$ *for* $i = 1,\ 2,$

$$\begin{aligned}(2.19)\quad &\Bigg\| \Bigg\{ \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \sup_{z_1 \in Q_{\tau_1}^{k_1,\nu_1}, z_2 \in Q_{\tau_2}^{k_2,\nu_2}} |D_{k_1} D_{k_2}(f)(z_1, z_2)|^q \\ &\quad \times \chi_{Q_{\tau_1}^{k_1,\nu_1}}(\cdot) \chi_{Q_{\tau_2}^{k_2,\nu_2}}(\cdot) \Bigg\}^{1/q} \Bigg\|_{L^p(X_1 \times X_2)} \\ &\le C \Bigg\| \Bigg\{ \sum_{k_1'=-\infty}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\nu_1'=1}^{N(k_1',\tau_1')} \sum_{k_2'=-\infty}^{\infty} \sum_{\tau_2' \in I_{k_2'}} \sum_{\nu_2'=1}^{N(k_2',\tau_2')} \\ &\quad \times \inf_{z_1 \in Q_{\tau_1'}^{k_1',\nu_1'}, z_2 \in Q_{\tau_2'}^{k_2',\nu_2'}} \left| E_{k_1'} E_{k_2'}(f)(z_1, z_2) \right|^q \chi_{Q_{\tau_1'}^{k_1',\nu_1'}}(\cdot) \chi_{Q_{\tau_2'}^{k_2',\nu_2'}}(\cdot) \Bigg\}^{1/q} \Bigg\|_{L^p(X_1 \times X_2)}.\end{aligned}$$

The basic tool to prove the above theorem is the discrete Calderón reproducing formula.

We now can use the Min-Max inequalities to generalize Theorem 2.26 to the case $p,\ q \le 1$.

Theorem 2.29. *Let all the notation be the same as in Theorem 2.8. If*

$$\max\left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p,\ q \le \infty,$$

then there is a constant $C_{p,q} > 0$ *such that for all* $f \in \left(\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ *with* $\beta_i,\ \gamma_i \in (0, \epsilon_i)$ *for* $i = 1,\ 2,$

$$(2.20)\qquad C_{p,q}^{-1} \|S_q(f)\|_{L^p(X_1 \times X_2)} \le \|g_q(f)\|_{L^p(X_1 \times X_2)} \le C_{p,q} \|S_q(f)\|_{L^p(X_1 \times X_2)}.$$

Now we are in the position to introduce the Hardy spaces $H^p(X_1 \times X_2)$ for some $p \le 1$ and establish their atomic decomposition characterization.

Definition 2.30. *Let* X_i *be a homogeneous-type space as in Definition 2.1,* $\epsilon_i \in (0, \theta_i]$ *and* $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ *be the same as in Theorem 3.1 for* $i = 1,\ 2.$ *Let*

$$\max\left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p < \infty$$

and for $i = 1,\ 2,$

$$(2.21)\qquad d_i(1/p - 1)_+ < \beta_i,\ \gamma_i < \epsilon_i.$$

The Hardy space $H^p(X_1 \times X_2)$ *is defined to be the set of all* $f \in \left(\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ *such that* $\|g_2(f)\|_{L^p(X_1 \times X_2)} < \infty$*, and we define*

$$\|f\|_{H^p(X_1 \times X_2)} = \|g_2(f)\|_{L^p(X_1 \times X_2)},$$

where $g_2(f)$ *is defined as in Theorem 2.26.*

We first consider the reasonability of the definition of the Hardy space $H^p(X_1 \times X_2)$.

Proposition 2.31. *Let all the notation be the same as in Definition 2.30. Then the definition of the Hardy space $H^p(X_1 \times X_2)$ is independent of the choice of the approximations to the identity and the spaces of distributions with β_i and γ_i satisfying (2.21), where $i = 1, 2$.*

Thus, Definition 2.30 is reasonable by Proposition 2.31. We remark that in the proof of Proposition 2.31, we actually only require that $0 < \gamma_i < \epsilon_i$ for $i = 1, 2$. However, if γ_i and β_i for $i = 1, 2$ are as in (2.21), we then can verify that the space of test functions, $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, is contained in the Hardy space $H^p(X_1 \times X_2)$. To be precise, we have the following propositions.

Proposition 2.32. *Let p and the space $H^p(X_1 \times X_2)$ be the same as in Definition 2.30. If $0 < \beta_i < \epsilon_i$ and $d_i(1/p - 1)_+ < \gamma_i < \epsilon_i$ for $i = 1, 2$, then*

$$\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \subset H^p(X_1 \times X_2).$$

Proposition 2.33. *If $1 < p < \infty$, then the space $H^p(X_1 \times X_2)$ is the same space as the space $L^p(X_1 \times X_2)$ with an equivalent norm.*

and

Proposition 2.34. *Let p and the space $H^p(X_1 \times X_2)$ be the same as in Definition 2.30, and S_2 be defined as in (2.18) with $q = 2$. If β_i and γ_i with $i = 1, 2$ are as in (2.21), then $f \in H^p(X_1 \times X_2)$ if and only if $f \in \left(\mathring{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ and $S_2(f) \in L^p(X_1 \times X_2)$. Moreover,*

$$\|f\|_{H^p(X_1 \times X_2)} \sim \|S_2(f)\|_{L^p(X_1 \times X_2)}.$$

We now use Proposition 2.34 to obtain the atomic decomposition of the Hardy space $H^p(X_1 \times X_2)$. Before we do so, we establish Journé's covering lemma in the setting of homogeneous-type spaces.

We recall some notation. Let $\{Q^{k_i}_{\alpha_i} \subset X_i : \ k_i \in \mathbb{Z}, \ \alpha_i \in I_{k_i}\}$ for $i = 1, 2$ be the same as in Lemma 2.14. Then the open set $Q^{k_1}_{\alpha_1} \times Q^{k_2}_{\alpha_2}$ for $k_1, \ k_2 \in \mathbb{Z}, \ \alpha_1 \in I_{k_1}$ and $\alpha_2 \in I_{k_2}$ is called a dyadic rectangle of $X_1 \times X_2$. Let $\Omega \subset X_1 \times X_2$ be an open set of finite measure and $\mathcal{M}_i(\Omega)$ denote the family of dyadic rectangles $R \subset \Omega$ which are maximal in the x_i "direction", where $i = 1, 2$. In what follows, we denote by $R = B_1 \times B_2$ any dyadic rectangle of $X_1 \times X_2$. Given $R = B_1 \times B_2 \in \mathcal{M}_1(\Omega)$, let $\widehat{B}_2 = \widehat{B}_2(B_1)$ be the "longest" dyadic cube containing B_2 such that

$$(\mu_1 \times \mu_2)(B_1 \times \widehat{B}_2 \cap \Omega) > \frac{1}{2}(\mu_1 \times \mu_2)(B_1 \times \widehat{B}_2); \tag{2.22}$$

and given $R = B_1 \times B_2 \in \mathcal{M}_2(\Omega)$, let $\widehat{B}_1 = \widehat{B}_1(B_2)$ be the "longest" dyadic cube containing B_1 such that

$$(\mu_1 \times \mu_2)(\widehat{B}_1 \times B_2 \cap \Omega) > \frac{1}{2}(\mu_1 \times \mu_2)(\widehat{B}_1 \times B_2). \tag{2.23}$$

If $B_i = Q^{k_i}_{\alpha_i} \subset X_i$ for some $k_i \in \mathbb{Z}$ and some $\alpha_i \in I_{k_i}$, $(B_i)_k$ for $k \in \mathbb{N}$ is used to denote any dyadic cube $Q^{k_i - k}_{\alpha_i}$ containing $Q^{k_i}_{\alpha_i}$ and $(B_i)_0 = B_i$, where $i = 1, 2$. Also, let $w(x)$ be any increasing function such that $\sum_{j=0}^{\infty} jw(C2^{-j}) < \infty$, where $C > 0$ is any given constant. In particular, we may take $w(x) = x^{\delta}$ for any $\delta > 0$.

Then we have the following variant of Journé's covering lemma in the setting of homogeneous type whose idea of proof comes from the work of Pipher [P].

Lemma 2.35. *Assume that $\Omega \subset X_1 \times X_2$ is an open set with finite measure. Let all the notation be the same as above and $\mu = \mu_1 \times \mu_2$. Then*

$$\sum_{R=B_1\times B_2\in\mathcal{M}_1(\Omega)} \mu(R) w\left(\frac{\mu_2(B_2)}{\mu_2(\widehat{B}_2)}\right) \le C\mu(\Omega) \tag{2.24}$$

and

$$\sum_{R=B_1\times B_2\in\mathcal{M}_2(\Omega)} \mu(R) w\left(\frac{\mu_1(B_1)}{\mu_1(\widehat{B}_1)}\right) \le C\mu(\Omega). \tag{2.25}$$

To give an atomic characterization of the product H^p spaces of homogeneous type, we first introduce the $H^p(X_1 \times X_2)$-atom. In what follows, for any open set Ω, we denote by $\mathcal{M}(\Omega)$ the set of all maximal dyadic rectangles contained in Ω.

Definition 2.36. *Let all the notation be the same as in Definition 2.30 and $\mu = \mu_1 \times \mu_2$. A function $a(x_1, x_2)$ on $X_1 \times X_2$ is called a $(p, 2)$-atom of $H^p(X_1 \times X_2)$, if it satisfies*

(1) $\operatorname{supp} a \subset \Omega$, *where Ω is an open set of $X_1 \times X_2$ with finite measure;*

(2) a can be further decomposed into

$$a = \sum_{R\in\mathcal{M}(\Omega)} a_R,$$

where

(i) supposing $R = Q_1 \times Q_2$ with $\operatorname{diam} Q_1 \sim 2^{-k_1}$ and $\operatorname{diam} Q_2 \sim 2^{-k_2}$, then

$$\operatorname{supp} a_R \subset B_1(z_1, A_1C2^{-k_1}) \times B_2(z_2, A_2C2^{-k_2}),$$

where z_i is the center of Q_i for $i = 1,\ 2$, C is the constant in Lemma 2.5, for X_1 and X_2.

(ii) for all $x_1 \in X_1$,

$$\int_{X_2} a_R(x_1, x_2)\, d\mu_2(x_2) = 0$$

and for all $x_2 \in X_2$,

$$\int_{X_1} a_R(x_1, x_2)\, d\mu_1(x_1) = 0;$$

(iii) $\|a\|_{L^2(X_1\times X_2)} \le \mu(\Omega)^{1/2-1/p}$ *and*

$$\left\{\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2(X_1\times X_2)}\right\}^{1/2} \le \mu(\Omega)^{1/2-1/p}.$$

Moreover, a_R *is called an* $H^p(X_1\times X_2)$ $(p,2)$*-rectangle atom, if* a_R *satisfies (i), (ii) and*

(iv) $\|a_R\|_{L^2(X_1\times X_2)} \le \mu(R)^{1/2-1/p}$.

The atomic decomposition of the Hardy space $H^p(X_1\times X_2)$ is stated in the following theorem.

Theorem 2.37. *Let* $i=1,\ 2,$ X_i *be a homogeneous-type space as in Definition 2.1,* $\epsilon_i\in(0,\theta_i]$ *and*

$$\max\left\{\frac{d_1}{d_1+\epsilon_1}, \frac{d_2}{d_2+\epsilon_2}\right\} < p \le 1.$$

Then $f\in H^p(X_1\times X_2)$ *if and only if* $f\in\left(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\right)'$ *for some* $\beta_i,\ \gamma_i$ *satisfying (2.21), where* $i=1,\ 2,$ *and there is a sequence of numbers,* $\{\lambda_k\}_{k\in\mathbb{Z}}$*, and a sequence of* $(p,2)$*-atoms of* $H^p(X_1\times X_2)$*,* $\{a_k\}_{k\in\mathbb{Z}}$*, such that* $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$ *and*

$$f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$$

in $\left(\mathring{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)\right)'$. *Moreover, in this case,*

$$\|f\|_{H^p(X_1\times X_2)} \sim \inf\left\{\left[\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right]^{1/p}\right\},$$

where the infimum is taken over all the decompositions as above.

The proof of this theorem is similar to classical case on R^n.

As the main application, we consider the boundedness of singular integrals on the product H^p spaces of homogeneous type.We first recall some notation. Let Ω be an open set in $X_1\times X_2$. We define

$$\widetilde{\Omega}=\{(x_1,x_2)\in X_1\times X_2:\ M_s\chi_\Omega(x_1,x_2)>1/2\}$$

and

$$\overline{\Omega}=\left\{(x_1,x_2)\in X_1\times X_2:\ M_s\chi_{\widetilde{\Omega}}(x_1,x_2)>1/2\right\}.$$

For any $R=Q_1\times Q_2\in\mathcal{M}(\Omega)$, we define $\widetilde{R}=\widetilde{Q}_1\times Q_2\in\mathcal{M}_1\left(\widetilde{\Omega}\right)$ such that

$$\mu\left(\widetilde{R}\cap\Omega\right)>\frac{1}{2}\mu\left(\widetilde{R}\right) \tag{2.26}$$

and $\overline{R}=\widetilde{Q}_1\times\widetilde{Q}_2\in\mathcal{M}_2\left(\overline{\Omega}\right)$ such that

$$\mu\left(\overline{R}\cap\Omega\right)>\frac{1}{2}\mu\left(\overline{R}\right). \tag{2.27}$$

Let $C\geq 1$ and we set

$$\overrightarrow{C}\overline{R}=C\widetilde{Q}_1\times C\widetilde{Q}_2, \tag{2.28}$$

where $C\widetilde{Q}_i$ means the "cube" with the same center as $\widetilde{Q}_i$ but with diameter C times the diameter of $\widetilde{Q}_i$. We also denote by $\widetilde{z}_i$ the center of $\widetilde{Q}_i$ for $i=1,\ 2$.

We first have the following general theorem on the boundedness of linear operators from $H^p(X_1\times X_2)$ to $L^p(X_1\times X_2)$ with $p\in(p_0,1]$, when the linear operators are assumed to be bounded on $L^2(X_1\times X_2)$. This is a generalization of R. Fefferman's theorem in pure product setting in Euclidean spaces, see Theorem 1 in [F4]. Here p_0 is some positive number less than 1.

Theorem 2.38. *Suppose that T is a bounded linear operator on $L^2(X_1\times X_2)$. Let $\epsilon_i\in(0,\theta_i]$ and*

$$\max\left\{\frac{d_1}{d_1+\epsilon_1},\frac{d_2}{d_2+\epsilon_2}\right\}<p\leq 1.$$

Suppose further that if a_R is an $H^p(X_1\times X_2)$ $(p,2)$-rectangle atom as in Definition 2.36 and $R=Q_1\times Q_2$. Let $\widetilde{Q}_1$ and $\widetilde{Q}_2$ be the same as in (2.26) and (2.27). If there exist fixed constant $\delta>0$ and some fixed large enough constant $C\geq 1$ such that for all $R=Q_1\times Q_2$,

$$\int_{X_2}\int_{(C\widetilde{Q}_1)^{\complement}}|T(a_R)(x_1,x_2)|^p\,d\mu_1(x_1)\,d\mu_2(x_2)\leq C\left(\frac{\mu_1(Q_1)}{\mu_1(\widetilde{Q}_1)}\right)^{\delta} \tag{2.29}$$

and

$$\int_{(C\widetilde{Q}_2)^{\complement}}\int_{X_1}|T(a_R)(x_1,x_2)|^p\,d\mu_1(x_1)\,d\mu_2(x_2)\leq C\left(\frac{\mu_2(Q_2)}{\mu_2(\widetilde{Q}_2)}\right)^{\delta}, \tag{2.30}$$

then T is a bounded operator from $H^p(X_1\times X_2)$ to $L^p(X_1\times X_2)$, where

$$\left(C\widetilde{Q}_i\right)^{\complement}=X_i\setminus C\widetilde{Q}_i,\ i=1,\ 2.$$

We now consider the boundedness on H^p space for a certain range of $p\in(p_0,1]$ for a class of singular integrals similar to [NS3].

Let $\eta_i\in(0,\theta_i]$, $i=1,\ 2$. We define $C_0^{\eta_1,\eta_2}(X_1\times X_2)=C_0^{\eta_1}(X_1)\otimes C_0^{\eta_2}(X_2)$. Also, for $i=1,\ 2$, we say φ is a bump function on X_i associated to a ball $B(x_i,\delta_i)$, if it is supported in that ball, and satisfies $\|\varphi\|_{L^\infty(X_i)}\leq 1$ and $\|\varphi\|_{C_0^\eta(X_i)}\leq C\delta_i^\eta$ for all $\eta\in(0,\theta_i]$, where $C\geq 0$ is independent of δ_i and x_i. In what follows, for its convenience, if $f\in L^\infty(X_i)$, we write $f\in C^0(X_i)$ and define

$$\|f\|_{C^0(X_i)}=\|f\|_{L^\infty(X_i)},$$

and for $\eta_i \in (0, \theta_i]$,

$$\|f\|_{C^{\eta_i}(X_i)} = \sup_{x_i, y_i \in X_i} \frac{|f(x_i) - f(y_i)|}{\rho_i(x_i, y_i)^{\eta_i}}, \ i = 1, \ 2.$$

Definition 2.39. *Let $\eta_i \in (0, \theta_i]$, $i = 1, \ 2$. A linear operator T initially defined from $C_0^{\eta_1,\eta_2}(X_1 \times X_2) = C_0^{\eta_1}(X_1) \otimes C_0^{\eta_2}(X_2)$ to its dual is called a singular integral if T has an associated distribution kernel $K(x_1, x_2; y_1, y_2)$ which is locally integrable away from the "cross"*

$$\{(x_1, x_2; y_1, y_2) : \ x_1 = y_1, \ or \ x_2 = y_2\}$$

satisfying the following additional properties

(i)

$$\begin{aligned}
&\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle \\
&\quad = \int_{X_1 \times X_2 \times X_1 \times X_2} K(x_1, x_2; y_1, y_2) \varphi_1(y_1) \varphi_2(y_2) \\
&\qquad \times \psi_1(x_1) \psi_2(x_2) \, d\mu_1(y_1) \, d\mu_2(y_2) \, d\mu_1(x_1) \, d\mu_2(x_2)
\end{aligned}$$

whenever $\varphi_1, \ \psi_1 \in C_0^{\eta_1}(X_1)$ and have disjoint supports, and $\varphi_2, \ \psi_2 \in C_0^{\eta_2}(X_2)$ and have disjoint supports;

(ii) For each bump function φ_2 on X_2 and each $x_2 \in X_2$, there exists a singular integral T^{φ_2, x_2} (of the one-factor type) on X_1, so that $x_2 \to T^{\varphi_2, x_2}$ is smooth in the sense make precise below, and so that

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int_{X_2} \langle T^{\varphi_2, x_2} \varphi_1, \psi_1 \rangle \psi_2(x_2) \, d\mu_2(x_2).$$

Moreover, we require that T^{φ_2, x_2} uniformly satisfies the following conditions that T^{φ_2, x_2} has a distribution kernel $K^{\varphi_2, x_2}(x_1, y_1)$ having the following properties:

(ii)$_1$ If $\varphi_1, \ \psi_1 \in C_0^{\eta_1}(X_1)$ have disjoint supports, then

$$\langle T^{\varphi_2, x_2} \varphi_1, \psi_1 \rangle = \int_{X_1 \times X_1} K^{\varphi_2, x_2}(x_1, y_1) \varphi_1(x_1) \psi_1(y_1) \, d\mu_1(x_1) \, d\mu_1(y_1);$$

(ii)$_2$ If φ_1 is a bump function associated to the ball $B(\bar{x}_1, r_1)$, then

$$\|T^{\varphi_2, x_2} \varphi_1\|_{C^{a_1}(X_1)} \leq C r_1^{-a_1}$$

for all $a_1 \in [0, \theta_1]$, where $C \geq 0$ is independent of φ_2, x_2, and r_1. Precisely, this means that for each $a_1 \geq 0$, there is a $b_1 \geq 0$ and a constant C_{a_1, b_1}, independent of φ_2, x_2 and r_1, so that whenever $\varphi \in C_0^{\theta_1}(X_1)$ supported in a ball $B(\bar{x}_1, r_1)$, then

$$r_1^{a_1} \|T^{\varphi_2, x_2} \varphi_1\|_{C^{a_1}(X_1)} \leq C_{a_1, b_1} \sup_{c_1 \leq b_1} r_1^{c_1} \|T^{\varphi_2, x_2} \varphi_1\|_{C^{c_1}(X_1)};$$

(ii)$_3$ There is a constant $C > 0$ independent of φ_2, x_2, and r_1 such that

(ii)$_{31}$ $|K^{\varphi_2,x_2}(x_1,y_1)| \le C\rho_1(x_1,y_1)^{-d_1}$,

(ii)$_{32}$ $|K^{\varphi_2,x_2}(x_1,y_1) - K^{\varphi_2,x_2}(x_1',y_1)| \le C\rho_1(x_1,x_1')^{\eta_1}\rho_1(x_1,y_1)^{-d_1-\eta_1}$ *for*

$$\rho_1(x_1,x_1') \le \frac{\rho_1(x_1,y_1)}{2A_1},$$

(ii)$_{33}$ $|K^{\varphi_2,x_2}(x_1,y_1) - K^{\varphi_2,x_2}(x_1,y_1')| \le C\rho_1(y_1,y_1')^{\eta_1}\rho_1(x_1,y_1)^{-d_1-\eta_1}$ *for*

$$\rho_1(y_1,y_1') \le \frac{\rho_1(x_1,y_1)}{2A_1};$$

(ii)$_4$ If φ_2 is a bump function associated to $B(\bar{x}_2,r_2)$, then for $a_2 \in (0,\theta_2]$,

$$r_2^{a_2}\rho_2(x_2,u_2)^{-a_2}\left[T^{\varphi_2,x_2} - T^{\varphi_2,u_2}\right]$$

also uniformly satisfies properties (ii)$_1$ through (ii)$_3$;

(ii)$_5$ Properties (ii)$_1$ through (ii)$_4$ also hold with x_1 and y_1 interchanged. That is, there properties also hold for the adjoint operator $(T^{\varphi_2,x_2})^t$ defined by

$$\left\langle (T^{\varphi_2,x_2})^t\varphi,\psi\right\rangle = \langle T\psi,\varphi\rangle;$$

(iii) The property (ii) hold when the index 1 and 2 are interchanged, namely, if the roles of X_1 and X_2 are interchanged;

(iv) There is a constant $C > 0$ such that for all bump functions φ_1 and φ_2, respectively, associated to $B(\bar{x}_1,r_1)$ and $B(\bar{x}_2,r_2)$,

$$\begin{aligned}
&|[T(\varphi_1\otimes\varphi_2)(x_1,x_2) - T(\varphi_1\otimes\varphi_2)(u_1,x_2)]\\
&\quad - [T(\varphi_1\otimes\varphi_2)(x_1,u_2) - T(\varphi_1\otimes\varphi_2)(u_1,u_2)]|\\
&\quad \le Cr_1^{-a_1}r_2^{-a_2}\rho_1(x_1,u_1)^{a_1}\rho_2(x_2,u_2)^{a_2}
\end{aligned}$$

for all $a_1 \in (0,\theta_1]$ and all $a_2 \in (0,\theta_2]$;

(v) The kernel $K(x_1,x_2;y_1,y_2)$ satisfies the following conditions:

(v)$_1$ $|K(x_1,x_2;y_1,y_2)| \le C\rho_1(x_1,y_1)^{-d_1}\rho_2(x_2,y_2)^{-d_2}$,

(v)$_2$ $|K(x_1,x_2;y_1,y_2) - K(x_1,x_2';y_1,y_2)| \le C\dfrac{1}{\rho_1(x_1,y_1)^{d_1}}\dfrac{\rho_2(x_2,x_2')^{\eta_2}}{\rho_2(x_2,y_2)^{d_2+\eta_2}}$ *for*

$$\rho_2(x_2,x_2') \le \frac{\rho_2(x_2,y_2)}{2A_2},$$

(v)$_3$ $|K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1,y_2')| \le C\frac{1}{\rho_1(x_1,y_1)^{d_1}}\frac{\rho_2(y_2,y_2')^{\eta_2}}{\rho_2(x_2,y_2)^{d_2+\eta_2}}$ *for*

$$\rho_2(y_2,y_2') \le \frac{\rho_2(x_2,y_2)}{2A_2},$$

(v)$_4$

$$\begin{aligned} &|[K(x_1,x_2;y_1,y_2)-K(x_1',x_2;y_1,y_2)]\\ &\quad -[K(x_1,x_2';y_1,y_2)-K(x_1',x_2';y_1,y_2)]|\\ &\quad \le C\frac{\rho_1(x_1,x_1')^{\eta_1}}{\rho_1(x_1,y_1)^{d_1+\eta_1}}\frac{\rho_2(x_2,x_2')^{\eta_2}}{\rho_2(x_2,y_2)^{d_2+\eta_2}}\end{aligned}$$

for $\rho_1(x_1,x_1') \le \frac{\rho_1(x_1,y_1)}{2A_1}$ *and* $\rho_2(x_2,x_2') \le \frac{\rho_2(x_2,y_2)}{2A_2}$,

(v)$_5$

$$\begin{aligned} &|[K(x_1,x_2;y_1,y_2)-K(x_1',x_2;y_1,y_2)]\\ &\quad -[K(x_1,x_2;y_1,y_2')-K(x_1',x_2;y_1,y_2')]|\\ &\quad \le C\frac{\rho_1(x_1,x_1')^{\eta_1}}{\rho_1(x_1,y_1)^{d_1+\eta_1}}\frac{\rho_2(y_2,y_2')^{\eta_2}}{\rho_2(x_2,y_2)^{d_2+\eta_2}}\end{aligned}$$

for $\rho_1(x_1,x_1') \le \frac{\rho_1(x_1,y_1)}{2A_1}$ *and* $\rho_2(y_2,y_2') \le \frac{\rho_2(x_2,y_2)}{2A_2}$,

(v)$_6$

$$\begin{aligned} &|[K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1',y_2)]\\ &\quad -[K(x_1,x_2;y_1,y_2')-K(x_1,x_2;y_1',y_2')]|\\ &\quad \le C\frac{\rho_1(y_1,y_1')^{\eta_1}}{\rho_1(x_1,y_1)^{d_1+\eta_1}}\frac{\rho_2(y_2,y_2')^{\eta_2}}{\rho_2(x_2,y_2)^{d_2+\eta_2}}\end{aligned}$$

for $\rho_1(y_1,y_1') \le \frac{\rho_1(x_1,y_1)}{2A_1}$ *and* $\rho_2(y_2,y_2') \le \frac{\rho_2(x_2,y_2)}{2A_2}$,

(v)$_7$ *The properties (iii)*$_2$ *to (iii)*$_6$ *hold when the index 1 and 2 are interchanged, that is, if the roles of* X_1 *and* X_2 *are interchanged.*

(vi) The same properties are assumed to hold for the 3 "transposes" of T*, i.e. those operators which arise by interchanging* x_1 *and* y_1*, or interchanging* x_2 *and* y_2*, or doing both interchanges.*

We can now establish the H^p-boundedness of these singular operators as defined in Definition 2.39 as follows.

Theorem 2.40. *Let $0 < \epsilon_i,\ \eta_i \le \theta_i,\ i = 1,\ 2,$ and*

$$\max\left\{\frac{d_1}{d_1+\epsilon_1},\ \frac{d_2}{d_2+\epsilon_2},\ \frac{d_1}{d_1+\eta_1},\ \frac{d_2}{d_2+\eta_2}\right\} < p < \infty.$$

Each product singular integral as in Definition 5.1 extends to a bounded operator on $H^p(X_1 \times X_2)$ to itself.

2.2 Duality of product Hardy spaces in homogeneous spaces

We now study the duality of the product H^P spaces of homogeneous type(see [HLL1]). To characterize the dual space of $H^p(\mathcal{X} \times \mathcal{X})$, we introduce the Carleson measure space CMO^p on $\mathcal{X} \times \mathcal{X}$, which is motivated by ideas of Chang and R. Fefferman ([CF1]).

Definition 2.41. *Let $i = 1, 2, 0 < \beta_i,\ \gamma_i < \theta$, $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order θ. Set $D_{k_i} = S_{k_i} - S_{k_i-1}$ for all $k_i \in \mathbb{Z}$. The Carleson measure space $CMO^p(\mathcal{X} \times \mathcal{X})$ is defined to be the set of all $f \in \big(\overset{\circ}{G}_\vartheta(\beta_1, \beta_2, \gamma_1, \gamma_2)\big)'$ such that*

$$\|f\|_{CMO^p(\mathcal{X}\times\mathcal{X})} = \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_\Omega \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}$$
$$\times \chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2)\big|D_{k_1}D_{k_2}(f)(x_1,x_2)\big|^2$$
$$\times \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)d\mu(x_1)d\mu(x_2)\Bigg)^{1/2} < \infty\ ,$$

where the sup *is taken over all open sets Ω in $\mathcal{X} \times \mathcal{X}$ with finite measures.*

In order to verify that the definition of $CMO^p(\mathcal{X} \times \mathcal{X})$ is independent of the choice of the approximations to identity, we establish Plancherel-Pôlya type inequality involving the CMO^p norm. To this end and for the sake of simplicity, we first give some notation as follows.

We write $R = Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$, $R' = Q_{\tau_1'}^{k_1',v_1'} \times Q_{\tau_2'}^{k_2',v_2'}$;

$$\sum_{R\subseteq\Omega} = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2);$$

$$\sum_{R'\subseteq\Omega} = \sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{v_1'=1}^{N(k_1',\tau_1')}\sum_{v_2'=1}^{N(k_2',\tau_2')}\chi_{\{Q_{\tau_1'}^{k_1',v_1'}\times Q_{\tau_2'}^{k_2',v_2'}\subset\Omega\}}(k_1',k_2',\tau_1',\tau_2',v_1',v_2');$$

$$\sum_{R'} = \sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{v_1'=1}^{N(k_1',\tau_1')}\sum_{v_2'=1}^{N(k_2',\tau_2')};$$

$$\mu(R) = \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2}); \qquad \mu(R') = \mu(Q_{\tau_1'}^{k_1',v_1'})\mu(Q_{\tau_2'}^{k_2',v_2'});$$

$$r(R,R^{'})=\Big(\frac{\mu(Q^{k_1,v_1}_{\tau_1})}{\mu(Q^{k_1^{'},v_1^{'}}_{\tau_1^{'}})}\wedge\frac{\mu(Q^{k_1^{'},v_1^{'}}_{\tau_1^{'}})}{\mu(Q^{k_1,v_1}_{\tau_1})}\Big)^{1+\varepsilon^{'}}\Big(\frac{\mu(Q^{k_2,v_2}_{\tau_2})}{\mu(Q^{k_2^{'},v_2^{'}}_{\tau_2^{'}})}\wedge\frac{\mu(Q^{k_2^{'},v_2^{'}}_{\tau_2^{'}})}{\mu(Q^{k_2,v_2}_{\tau_2})}\Big)^{1+\varepsilon^{'}};$$

$$v(R,R^{'})=\big(\mu(Q^{k_1,v_1}_{\tau_1})\vee\mu(Q^{k_1^{'},v_1^{'}}_{\tau_1^{'}})\big)\big(\mu(Q^{k_2,v_2}_{\tau_2})\vee\mu(Q^{k_2^{'},v_2^{'}}_{\tau_2^{'}})\big);$$

$$P(R,R^{'})=\frac{1}{\Big(1+\frac{\mathrm{dist}(Q^{k_1,v_1}_{\tau_1},Q^{k_1^{'},v_1^{'}}_{\tau_1^{'}})}{\mu(Q^{k_1,v_1}_{\tau_1})\vee\mu(Q^{k_1^{'},v_1^{'}}_{\tau_1^{'}})}\Big)^{1+\varepsilon'}}\frac{1}{\Big(1+\frac{\mathrm{dist}(Q^{k_2,v_2}_{\tau_2},Q^{k_2^{'},v_2^{'}}_{\tau_2^{'}})}{\mu(Q^{k_2,v_2}_{\tau_2})\vee\mu(Q^{k_2^{'},v_2^{'}}_{\tau_2^{'}})}\Big)^{1+\varepsilon'}};$$

$$S_R=\sup_{x_1\in Q^{k_1,v_1}_{\tau_1},x_2\in Q^{k_2,v_2}_{\tau_2}}|D_{k_1}D_{k_2}(f)(x_1,x_2)|^2;\qquad T_{R'}=\inf_{y_1^{'}\in Q^{k_1^{'},v_1^{'}}_{\tau_1^{'}},y_2^{'}\in Q^{k_2^{'},v_2^{'}}_{\tau_2^{'}}}\big|D_{k_1^{'}}D_{k_2^{'}}(f)(y_1^{'},y_2^{'})\big|^2.$$

Now we have the Min-Max inequality for the norm of $CMO^p(\mathcal{X}\times\mathcal{X})$.

Theorem 2.42. *Let all the notation be the same as above. For $\frac{2}{2+\theta}<p\le 1$ all $f\in CMO^p(\mathcal{X}\times\mathcal{X})$,*

$$\sup_{\Omega}\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subseteq\Omega}\mu(R)S_R\Big)^{1/2}\lesssim\sup_{\Omega}\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R'\subseteq\Omega}\mu(R')T_{R'}\Big)^{1/2},$$

where Ω ranges over the open sets in $\mathcal{X}\times\mathcal{X}$ with finite measures.

The proof of this theorem uses a simple geometrical argument, which is a generalization of Chang and R. Fefferman's idea, see more details in ([CF1]).

Before proving the duality theorem of the product H^p spaces of homogeneous type, we introduce the product sequence spaces s^p and c^p as follows.

Definition 2.43. *Let $\tilde{\chi}_Q(x)=\mu(Q)^{-1/2}\chi_Q(x)$. The product sequence space s^p, $0<p\le 1$, is defined as the collection of all complex-value sequences*

$$\lambda=\big\{\lambda_{Q^{k_1,v_1}_{\tau_1}\times Q^{k_2,v_2}_{\tau_2}}\big\}_{k_1,k_2\in\mathbb{Z};\ \tau_1\in I_{k_1},\tau_2\in I_{k_2};\ v_1=1,..N(k_1,\tau_1),\ v_2=1,..N(k_2,\tau_2)}$$

such that $\|\lambda\|_{s^p}$

$$=\Big\|\Big\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\big(|\lambda_{Q^{k_1,v_1}_{\tau_1}\times Q^{k_2,v_2}_{\tau_2}}|\cdot\tilde{\chi}_{Q^{k_1,v_1}_{\tau_1}}(\cdot)\tilde{\chi}_{Q^{k_2,v_2}_{\tau_2}}(\cdot)\big)^2\Big\}^{1/2}\Big\|_{L^p}<\infty.$$

Similarly, c^p, $0<p\le 1$, is defined as the collection of all complex-value sequences

$$t=\big\{t_{Q^{k_1,v_1}_{\tau_1}\times Q^{k_2,v_2}_{\tau_2}}\big\}_{k_1,k_2\in\mathbb{Z};\ \tau_1\in I_{k_1},\tau_2\in I_{k_2};\ v_1=1,..N(k_1,\tau_1),\ v_2=1,..N(k_2,\tau_2)}$$

such that $\|t\|_{c^p}$

$$= \sup_{\Omega}\Big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2)$$
$$\times\big(|t_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}|\cdot\tilde{\chi}_{Q_{\tau_1}^{k_1,v_1}}(x_1)\tilde{\chi}_{Q_{\tau_2}^{k_2,v_2}}(x_2)\big)^2 d\mu(x_1)d\mu(x_2)\Big)^{1/2} < \infty.$$

For simplicity, $\forall s \in s^p$, we rewrite $s = \{s_R\}_R$, and

$$\|s\|_{s^p} = \big\|\big\{\sum_R (s_R\tilde{\chi}_R(x_1,x_2)|^2\big\}^{1/2}\big\|_{L^p},$$

similarly, $\forall t \in c^p$, rewrite $t = \{t_R\}_R$, and

$$\|t\|_{c^p} = \sup_{\Omega}\big(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subseteq\Omega}|t_R|^2\big)^{1/2},$$

where R run over all the dyadic rectangles in $\mathcal{X}\times\mathcal{X}$. The main result in this section is the following duality theorem.

Theorem 2.44. *For* $p_0 < p \le 1$, $\big(s^p\big)' = c^p$.

The proof of this theorem uses the stopping time argument which was used in [CF2], for the sequence spaces s^p.

Now we have the following duality theorem.

Theorem 2.45. *For* $p_0 < p \le 1$, $\big(H^p(\mathcal{X}\times\mathcal{X})\big)' = CMO^p(\mathcal{X}\times\mathcal{X})$. *Namely, the dual space for* $H^p(\mathcal{X}\times\mathcal{X})$ *is* $CMO^p(\mathcal{X}\times\mathcal{X})$.

To prove the above duality theorem, we first define the lifting and projection operators as follows.

Definition 2.46. *Let* $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ *be an approximation to the identity of order* θ, $D_{k_i} = S_{k_i} - S_{k_i-1}$ *for* $i = 1, 2$. *For any* $f \in (\overset{\circ}{G}_{\vartheta}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ *with* $0 < \beta_i, \gamma_i < \varepsilon$, *define the lifting operator* S_D *by*

$$S_D(f) = \Big\{\mu(Q_{\tau_1}^{k_1,v_1})^{1/2}\mu(Q_{\tau_2}^{k_2,v_2})^{1/2}D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,v_1}, y_{\tau_2}^{k_2,v_2})\Big\}_{Q_{\tau_1}^{k_1,v_1}\ \ Q_{\tau_2}^{k_2,v_2}},$$

where $y_{\tau_i}^{k_i,v_i}$ *is the center of* $Q_{\tau_i}^{k_i,v_i}$, $k_i \in \mathbb{Z}$, $\tau_i \in I_{k_i}$, $v = 1, \cdots, N(\tau_i, k_i)$ *for* $i = 1, 2$.

Definition 2.47. *Let all the notation be the same as above. For any sequence* s, *define the projection operator* $T_{\tilde{D}}$ *by*

$$T_{\tilde{D}}(s)(x_1,x_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)} s_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}$$

$$\times\mu(Q_{\tau_1}^{k_1,v_1})^{1/2}\mu(Q_{\tau_2}^{k_2,v_2})^{1/2}\tilde{D}_{k_1}\tilde{D}_{k_2}(x_1,x_2,y_{\tau_1}^{k_1,v_1},y_{\tau_2}^{k_2,v_2}),$$

where $y_{\tau_i}^{k_i,v_i}$ *is the center of* $Q_{\tau_i}^{k_i,v_i}$ *and* $\tilde{D}_{k_i}$ *is the same operator as in the Calderón reproducing formula (2.15) associated with* D_{k_i} *for* $i=1,2$.

To work at the level of product sequences spaces, we still need the following two propositions.

Proposition 2.48. *Let all the notation be the same as above. Then for any* $f\in H^p(\mathcal{X}\times\mathcal{X})$, $\frac{1}{1+\theta}<p\le 1$,

$$\|S_D(f)\|_{s^p}\lesssim \|f\|_{H^p(\mathcal{X}\times\mathcal{X})}.$$

Conversely, for any $s\in s^p$,

$$\|T_{\tilde{D}}(s)\|_{H^p(\mathcal{X}\times\mathcal{X})}\lesssim \|s\|_{s^p}.$$

Moreover, $T_{\tilde{D}}\circ S_D$ *equals the identity on* $H^p(\mathcal{X}\times\mathcal{X})$.

Proposition 2.49. *Let all the notation be the same as above. Then for any* $f\in CMO^p(\mathcal{X}\times\mathcal{X})$, $\frac{2}{2+\varepsilon}<p\le 1$,

$$\|S_D(f)\|_{c^p}\lesssim \|f\|_{CMO^p(\mathcal{X}\times\mathcal{X})}.$$

Conversely, for any $t\in c^p$,

$$\|T_{\tilde{D}}(t)\|_{CMO^p(\mathcal{X}\times\mathcal{X})}\lesssim \|t\|_{c^p}.$$

Moreover, $T_{\tilde{D}}\circ S_D$ *is the identity on* $CMO^p(\mathcal{X}\times\mathcal{X})$.

The above two propositions give the proof of Theorem 2.45 with $p_0=\frac{2}{2+\theta}$.

2.3 Dual spaces of product Hardy spaces $H^p(R^n\times R^m)$

In this subsection we give some remarks on how our general results of duality theory of Hardy spaces $H^p(\chi\times\chi)$ imply in the simplest case of product spaces of two Euclidean spaces. We first remark that our results hold on $\mathcal{X}\times\mathcal{Y}$ with two different homogeneous spaces $\mathcal{X}$ and $\mathcal{Y}$. Second, all the theorems proved in this paper can be made very precise on $\mathbb{R}^n\times\mathbb{R}^m$ by using Calderón reproducing formulas with explicitly constructed approximation of identity via Fourier transform. In particular, the definitions of Hardy spaces $H^p(\mathcal{X}\times\mathcal{X})$ and their dual spaces $CMO^p(\mathcal{X}\times\mathcal{Y})$ can be made for all $0<p\le 1$ when $\mathcal{X}=\mathbb{R}^n$ and $\mathbb{Y}=\mathbb{R}^m$. Thus our results in this paper include the duality theory of Hardy spaces $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ for all $0<p\le 1$ and thus extend the earlier work of Chang and R. Fefferman [CF1] on $H^1(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ and $BMO(\mathbb{R}^2_+\times\mathbb{R}^2_+)$.

To state the realization of our main results on $\mathbb{R}^n\times\mathbb{R}^m$, we need to start with some preliminaries. Let $\mathcal{S}(\mathbb{R}^n)$ denote Schwartz functions in $\mathbb{R}^n$. Then the test function defined on $\mathbb{R}^n\times\mathbb{R}^m$ can be given by

$$\psi(x,y)=\psi^{(1)}(x)\psi^{(2)}(y)$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, and satisfy $\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi_1)|^2 = 1$ for all $\xi_1 \in \mathbb{R}^n\backslash\{(0)\}$, and $\sum_{k\in\mathbb{Z}} |\widehat{\psi}^{(2)}(2^{-k}\xi_2)|^2 = 1$ for all $\xi_2 \in \mathbb{R}^m\backslash\{0\}$, and the moment conditions

$$\int_{\mathbb{R}^n} x^\alpha \psi^{(1)}(x)dx = \int_{\mathbb{R}^m} \psi^{(2)}(y)y^\beta dy = 0$$

for all nonnegative integers α and β.

Let $f \in L^p, 1 < p < \infty$. Thus $g(f)$, the Littlewood-Paley-Stein square function of f, is defined by

$$g(f)(x,y) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(x,y)|^2 \right\}^{\frac{1}{2}}$$

where functions

$$\psi_{j,k}(x,y) = 2^{jn+km}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y). \tag{2.31}$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón's identity holds on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$,

$$f(x,y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y).$$

Using the orthogonal estimates and together with Calderón's identity on L^2 allows us to obtain the L^p estimates of g for $1 < p < \infty$. Namely, there exist constants C_1 and C_2 such that for $1 < p < \infty$,

$$C_1\|f\|_p \leq \|g(f)\|_p \leq C_2\|f\|_p.$$

In order to use the Littlewood-Paley-Stein square function g to define the Hardy space, one needs to extend the Littlewood-Paley-Stein square function to be defined on a suitable distribution space. For this purpose, we introduce the product test function space on $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 2.50. *A Schwartz test function $f(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to be a product test function on $\mathbb{R}^n \times \mathbb{R}^m$ if $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and*

$$\int_{\mathbb{R}^n} f(x,y)x^\alpha dx = \int_{\mathbb{R}^m} f(x,y)y^\beta dy = 0$$

for all indices α, β of nonnegative integers.

If f is a product test function on $\mathbb{R}^n \times \mathbb{R}^m$ we denote $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and the norm of f is defined by the norm of Schwartz test function.

We denote by $(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m))'$ the dual of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$.

We also denote $(\mathcal{S}_M(\mathbb{R}^n \times \mathbb{R}^m))$ by the collection of Schwartz test functions $f(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ with

$$\|f\|_{\mathcal{S}_M} = \sup_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} (1+|x|+|y|)^M \sum_{|\alpha|\leq M, |\beta|\leq M} |\frac{\partial^\alpha}{\partial^\alpha x}\frac{\partial^\beta}{\partial^\beta y} f(x,y)| < \infty,$$

and

$$\int_{\mathbb{R}^n} f(x,y)x^\alpha dx = \int_{\mathbb{R}^m} f(x,y)y^\beta dy = 0$$

for all indices $\alpha, \beta \leq M$.

Similarly, we denote $(\mathcal{S}_M(\mathbb{R}^n \times \mathbb{R}^m))'$ the dual of $\mathcal{S}_M(\mathbb{R}^n \times \mathbb{R}^m)$. Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, so the Littlewood-Paley-Stein square function g can be defined for all distributions in $(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m))'$. Formally, we can define the multi-parameter Hardy space as follows.

Definition 2.51. *Let $0 < p < \infty$. The multi-parameter Hardy space is defined as $H^p(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S})' : g(f) \in L^p(\mathbb{R}^n \times \mathbb{R}^m)\}$. If $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$, the norm of f is defined by $\|f\|_{H^p} = \|g(f)\|_p$.*

To establish the Hardy space theory on $\mathbb{R}^n \times \mathbb{R}^m$, we need the following discrete Calderón's identity.

Theorem 2.52. *Suppose that $\psi_{j,k}$ are the same as in (2.31). Then*

$$f(x,y) = \sum_{j,k} \sum_{I,J} |I||J| \widetilde{\psi}_{j,k}(x,y,x_I,y_J) \left(\psi_{j,k} * f\right)(x_I, y_J)$$

where $\widetilde{\psi}_{j,k}(x,y,x_I,y_J) \in \mathcal{S}_M(\mathbb{R}^n \times \mathbb{R}^m), I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ for a fixed large integer N, x_I, y_J are any fixed points in I, J respectively, and the series above converges in the norm of $\mathcal{S}_M(\mathbb{R}^n \times \mathbb{R}^m)$ and in the dual space $(\mathcal{S}_M(\mathbb{R}^n \times \mathbb{R}^m))'$.

The dual space $CMO^p(\mathbb{R}^n \times \mathbb{R}^m)$ can be defined using the Carleson measure characterization.

Definition 2.53. *Let $0 < p \leq 1$ and $\psi_{j,k}$ be the same as in Theorem 2.52. We say that $f \in CMO^p(\mathbb{R}^m \times \mathbb{R}^m)$ if $f \in (\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m))'$ with finite norm $\|f\|_{CMO^p}$ defined by*

$$\sup_\Omega \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \int_\Omega \sum_{I \times J \subset \Omega} |\psi_{j,k} * f(x,y)|^2 \chi_I(x) \chi_J(y) dx dy \right\}^{\frac{1}{2}}$$

for all open sets Ω in $\mathbb{R}^m \times \mathbb{R}^m$ with finite measures, and $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ for a large fixed positive integer.

We note that $\psi^{(1)}$ and $\psi^{(2)}$ are smooth and have moment condition of infinite order. As a consequence, the value of p in Definition 2.28 can be any number greater than 0. Therefore, we have the following duality result: $(H^p(\mathbb{R}^n \times \mathbb{R}^m))' = CMO^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $0 < p \leq 1$. The space $CMO^p(\mathbb{R}^n \times \mathbb{R}^m)$ when $p = 1$ coincides with the $BMO(\mathbb{R}^n \times \mathbb{R}^m)$ introduced by Chang and R. Fefferman in [CF1], and therefore our duality theorem of Hardy spaces on $\mathbb{R}^n \times \mathbb{R}^m$ extends the result of Chang and R. Fefferman to all $0 < p < 1$.

2.4 Hardy spaces on product homogenous groups

To explain how our results include the product H^p theory on two stratified groups such as the Heisenberg group, we give some preliminary introduction here.

We begin with some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [FoS] and [VSCC] for analysis on stratified groups. Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that

$$\mathcal{G} = \oplus_{i=1}^{s} V_i \ ,$$

with $[V_i, V_j] \subset V_{i+j}$ for $i + j \le s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \cdots, X_l$ be a basis for V_1 and suppose that $X_1, \cdots, X_l$ generate $\mathcal{G}$ as a Lie algebra. Then for $2 \le j \le s$, we can choose a basis $\{X_{ij}\}$, $1 \le i \le k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i,\ i = 1, \cdots, l$ and $k_1 = l$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in R^N$ for $1 \le i \le k_j$, $1 \le j \le s$ and $N = \sum_{j=1}^{s} k_j$ such that

$$g = \exp(\sum x_{ij} X_{ij}) \ .$$

A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by

$$|g| = (\sum |x_{ij}|^{2s!/j})^{1/2s!} \ ,$$

and $Q = \sum_{j=1}^{s} j k_j$ is said to be the **homogeneous dimension** of $\mathbb{G}$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp(\sum r^j x_{ij} X_{ij}) \ \text{ if } \ g = \exp(\sum x_{ij} X_{ij}).$$

We call a curve $\gamma : [a, b] \to \mathbb{G}$ "a horizontal curve" connecting two points $x, y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma^{'}(t) \in V_1$ for all t. Then the Carnot-Caratheodory distance between x, y is defined as

$$d_{cc}(x, y) = \inf_{\gamma} \int_a^b < \gamma^{'}(t), \gamma^{'}(t) >^{\frac{1}{2}} dt,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then d_{cc} is a left invariant metric on $\mathbb{G}$. We can define the metric ball centered at x and with radius r associated with this metric by

$$B_{cc}(x, r) = \{y : d_{cc}(x, y) < r\}.$$

We must notice that this metric d_{cc} is equivalent to the pseudo-metric $\rho(x, y) = |x^{-1}y|$ defined by the homogeneous norm $|\cdot|$ in the following sense (see [FS])

$$C\rho(x, y) \le d_{cc}(x, y) \le C\rho(x, y).$$

We denote the metric ball associated with ρ as $D(x, r) = \{y \in \mathbb{G} : \rho(x, y) < r\}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left invariant, namely,

$$d_{cc}(zx, zy) = d(x, y), B_{cc}(x, r) = xB_{cc}(0, r)$$

and

$$\rho(zx, zy) = \rho(x, y), D(x, r) = xD(0, r).$$

For simplicity, we will use the left invariant metric d_{cc} to study the product theory of two stratified groups. An important property of the metric ball is that

$$\mu(B_{cc}(x, r)) = c_Q r^Q$$

for all $x \in \mathbb{G}$ and $r > 0$, where μ is the Lebesgue measure on $\mathbb{G}$ and Q is the homogeneous dimension. Therefore, the space $(\mathbb{G}, d_{cc}, \mu)$ is a space of homogenous type.

If we consider two stratified groups $(\mathbb{G}_1, d_{cc}^1, \mu)$ and $(\mathbb{G}_2, d_{cc}^2, \mu)$, the product H^p theory developed in this section includes the case of product theory on $G_1 \times G_2$ as a special case. Of particular interests are the case $H^p(G_1 \times G_2)$ when G_1 or G_2 is the renowned Heisenberg group. Such product H^p theory was developed earlier by the first two authors in ([HL1], [HL2]). It is this work which motivated the generalization to the H^p product theory of two homogeneous spaces in the current section. In this special case, the duality theory of Theorem 2.45 works well for all $0 < p \leq 1$, namely p_0 can be taken 0 in the product of two stratified groups. The construction of the dual spaces is similar to the Euclidean case by considering group convolutions. Further generalizations to the product of two Carnot-Carathedory spaces are given in next section.

3 Product H^p spaces on Carnot-Carathéodory spaces

In [NS1], Nagel and Stein studied the initial value problem and the regularity properties of the heat operator $\mathcal{H} = \partial_s + \Box_b$ for the Kohn-Laplacian $\Box_b$ on M, where M is the boundary of a weakly pseudoconvex domain Ω of finite type in $\mathbb{C}^2$. And in [NS4], they obtained the optimal estimates for solution of the Kohn-Laplacian on q-forms, $\Box_b = \Box_b^{(q)}$, which is defined on the boundary $\overline{M} = \partial\Omega$ of a decoupled domain $\Omega \subseteq \mathbb{C}^n$. The method they used is to deduce the results about regularity of $\Box_b$ on $\overline{M}$ from corresponding results on $\widetilde{M} \subset \mathbb{C}^{2n}$ via projection, where $\widetilde{M} = M_1 \times \cdots \times M_n$ is the Cartesian product of boundaries of domains in $\mathbb{C}^2$ mentioned above. Namely, $\widetilde{M}$ is the Shilov boundary of the product domain $\Omega_1 \times \cdots \times \Omega_n$.

In [NS3], they developed an L^p $(1 < p < \infty)$ theory of product singular integral operators on product space $\widetilde{M} = M_1 \times \cdots \times M_n$ in sufficient generality, which can be used in a number of different situations, particularly for estimates of fundamental solutions of $\Box_b$ mentioned above. They carried this out by first considering the initial value problem of the heat operator $\mathcal{H} = \partial_s + \mathcal{L}$ for each M_i, where $\mathcal{L}$ is the sub-Laplacian on M_i in self-adjoint form, then using the heat kernel to introduce a Littlewood-Paley theory for each M_i and finally passing to the corresponding product theory.

In this part, we will follow the lines of Nagel and Stein([NS3]) to consider the the product space $\widetilde{M} = M_1 \times \cdots \times M_n$ and establish the Hardy spaces $H^p(\widetilde{M})$ for p less than 1 and near 1 and prove that the product singular integral operators are bounded on $H^p(\widetilde{M})$ and bounded from $H^p(\widetilde{M})$ to $L^p(\widetilde{M})$. Next, by imposing some natural conditions on each M_i, we can also establish the Carleson measure spaces $CMO^p(\widetilde{M})$ for p less than 1 and near 1. Then, we show that the duality of $H^p(\widetilde{M})$ is $CMO^p(\widetilde{M})$. In particular, when $p = 1$ we have $\big(H^1(\widetilde{M})\big)' = BMO(\widetilde{M})$. As a consequence, we can obtain that the product singular integral operators are bounded on $CMO^p(\widetilde{M})$ and bounded from $L^\infty(\widetilde{M})$ to $BMO(\widetilde{M})$. Results described here are joint work of Han, Li and Lu [HLL2].

To be more precise, let M is a connected smooth manifold and $\{\mathbb{X}_1, \cdots, \mathbb{X}_k\}$ are k given smooth real vector fields on M satisfying Hörmander condition of order m, i.e., these vector fields together with their commutators of order $\le m$ span the tangent space to M at each point.

In [NS3], for the sake of simplicity and because of the applications described in [NS4], Nagel and Stein focused their attention on two specific settings:

(A) Here M is a compact connected C^∞-manifold. We suppose that we are given k smooth real vector fields on M which are of finite type m in the sense that these vector fields together with their commutators of order $\le m$ span the tangent space to M at each point.

(B) Here M arises as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$. Let $\Omega = \{(z, w) \in \mathbb{C}^2 :\ \mathrm{Im}(w) > P(z)\}$, where P is a real, subharmonic, non-harmonic polynomial of degree m. Then $M = \partial\Omega$ can be identified with $\mathbb{C} \times \mathbb{R} = \{(z, t) :\ z \in \mathbb{C}, t \in \mathbb{R}\}$. The basic $(0, 1)$ Levi vector field is then $\bar{Z} = \frac{\partial}{\partial \bar{z}} - i\frac{\partial P}{\partial \bar{z}}\frac{\partial}{\partial t}$, and we write $\bar{Z} = \mathbb{X}_1 + i\mathbb{X}_2$. The real vector fields $\{\mathbb{X}_1, \mathbb{X}_1\}$ and their commutators of order $\le m$ span the tangent space to M at each point.

Thus this M is a non-compact variant, with $k = 2$, of the manifolds consider in (A).

One variant of the control distance is defined as follows:

For each $x, y \in M$, let $AC(x, y, \delta)$ denote the collection of absolutely continuous mapping $\varphi :\ [0, 1] \to M$ with $\varphi(0) = x,\ \varphi(1) = y$, and for almost every $t \in [0, 1]$, $\varphi'(t) = \sum_{j=1}^{k} a_j \mathbb{X}_j(\varphi(t))$ with $|a_j| \leq \delta$. The control distance $\rho(x, y)$ from x to y is the infimum of the set of $\delta > 0$ such that $AC(x, y, \delta) \neq \emptyset$. For details, see [NS3] and [NSW]. The result they needed is that there is a pseudo-metric $d \approx \rho$ [1] equivalent to this control metric which has the optimal smoothness ; i.e. $d(x, y)$ is C^∞ on $\{M \times M - \text{diagonal}\}$, and for $x \neq y$

$$|\partial_X^K \partial_Y^L d(x, y)| \lesssim d(x, y)^{1-K-L}.$$

Here ∂_X^K is a product of K of the vector fields $\{X_1, \cdots X_k\}$ acting as derivatives on the x variable, and ∂_Y^L are a corresponding L vector fields acting on the y variable.

It is clear that (M, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss. However, the measure μ does not satisfy the basic assumption in (2.2). Therefore, the methods used in section 2 for product of two homogeneous spaces can not be applied to (M, d, μ). To construct Calderón's identity, they considered a volume measure on M as follows. In the situation (A), they took any fixed smooth measure on M with strictly positive density. In the situation (B), they took Lebesgue measure on $\mathbb{C} \times \mathbb{R}$. Denote by $\mu(E)$ the measure of E. Define ball $B(x, \delta) = \{y \in M :\ d(x, y) < \delta\}$ with $0 < \delta \leq 1$ in case (A) and $0 < \delta < \infty$ in case (B). Then the following formulae hold for the volume $\mu(B(x, \delta))$:

$$\mu(B(x,\delta)) \approx \sum_{|I| \leq r} |\lambda_I(x)| \delta^{|I|} \qquad \text{in case (A);}$$
$$\mu(B(x,\delta)) \approx \sum_{k=2}^{m} \big(|\Lambda_k(x)\delta^k|\big)\delta^2 \qquad \text{in case (B).}$$

Here λ_I and Λ_k are the appropriate Levi-invariants, and are continuous, non-negative functions of M(see Theorem 2.2.4 and section 4.1 in [NS2]). The balls have the required doubling property

$$\mu(B(x, 2\delta)) \leq C\mu(B(x, \delta)) \quad \text{ for all } \ \delta > 0.$$

The volume functions are introduced as follows:

$$V_\delta(x) = \mu(B(x, \delta));$$
$$V(x, y) = \mu(B(x, d(x, y))).$$

More precisely, follow the steps in [NS3], we first focus on the case on $\widetilde{M} = M_1 \times M_2$. By discretizing the time scale t of the heat kernel, we restate the reproducing identity and

[1] Here, and subsequently, $A \approx B$ means that the ratio A/B is bounded and bounded away from zero by constants that do not depend on the relevant variables in A and B. $A \lesssim B$ means that the ratio A/B is bounded by a constant independent of the relevant variables

Littlewood-Paley theory obtained in [NS3]. Next, introduce the test function space and provide the continuous and discrete reproducing identity on the test function space and its dual space, and finally define the Hardy spaces $H^p(M)$. Then we show that the singular integral operator is bounded on $H^p(M)$ and from $H^p(M)$ to $L^p(M)$. Moreover, for the manifold M with some restrictions, we can establish the Carleson measure space $CMO^p(M)$ and prove that $\big(H^p(\widetilde{M})\big)' = CMO^p(\widetilde{M})$.

Finally, following the ideas and skills in ([HLL2]), we pass all the results of duality to the general product space $\widetilde{M} = M_1 \times \cdots \times M_n$.

3.1 The Heat Equation

In this subsection, we focus on the manifold $M_i, i = 1, 2$, with $|M_i| = \infty$.

We will use again the construction given by Christ in [Chr], which provides an analogue of the grid of Euclidean dyadic cubes on Carnot-Caratheodory spaces.

To construct the Littlewood-Paley square function, Negal and Stein in [NS3], see also [NS1], considered the sub-Laplacian $\mathcal{L}$ on M(here $M = M_i$ and dropping the index i) in self-adjoint form, given by

$$\mathcal{L} = \sum_{j=1}^{k} \mathbb{X}_j^* \mathbb{X}_j.$$

Here $(\mathbb{X}_j^*\varphi, \psi) = (\varphi, \mathbb{X}_j\psi)$, where $(\varphi, \psi) = \int\limits_M \varphi(x)\bar{\psi}(x)d\mu(x)$, and $\varphi, \psi \in C_0^\infty(M)$, the space of C^∞ functions on M with compact support. In general, $\mathbb{X}_j^* = -\mathbb{X}_j + a_j$, where $a_j \in C^\infty(M)$. The solution of the following initial value problem for the heat equation,

$$\frac{\partial u}{\partial s}(x, s) + \mathcal{L}_x u(x, s) = 0$$

with $u(x, 0) = f(x)$, is given by $u(x, s) = H_s(f)(x)$, where H_s is the operator given via the spectral theorem by $H_s = e^{-s\mathcal{L}}$, and an appropriate self-adjoint extension of the non-negative operator $\mathcal{L}$ initially defined on $C_0^\infty(M)$. And they proved that for $f \in L^2(X)$,

$$H_s(f)(x) = \int_M H(s, x, y) f(y) d\mu(y).$$

Moreover $H(s, x, y)$ has some nice properties (see Proposition 2.3.1 in [NS3] and Theorem 2.3.1 in [NS1]). We restate them as follows:

(1) $H(s, x, y) \in C^\infty\big([0, \infty) \times M \times M \backslash \{s = 0 \text{ and } x = y\}\big)$.

(2) For very integer $N \geq 0$,

$$\begin{aligned} &|\partial_s^j \partial_X^L \partial_Y^K H(s, x, y)| \\ &\lesssim \frac{1}{(d(x, y) + \sqrt{s})^{2j+K+L}} \frac{1}{V(x, y) + V_{\sqrt{s}}(x) + V_{\sqrt{s}}(y)} \left(\frac{\sqrt{s}}{d(x, y) + \sqrt{s}}\right)^{\frac{N}{2}} \end{aligned}$$

(3) For each integer $L \geq 0$ there exists an integer N_L and a constant C_L so that if $\varphi \in C_0^\infty(B(x_0, \delta))$, then for all $s \in (0, \infty)$

$$|\partial_X^L H_s[\varphi](x_0)| \leq C_L \delta^{-L} \sup_x \sum_{|J| \leq N_L} \delta^{|J|} |\partial_X^J \varphi(x)|.$$

(4) For all $(s, x, y) \in (0, \infty) \times M \times M$,

$$H(s,x,y) = H(s,y,x);$$
$$H(s,x,y) \geq 0.$$

(5) For all $(s, x) \in (0, \infty) \times M$, $\int H(s, x, y) dy = 1$.

(6) For $1 \leq p \leq \infty$, $\|H_s[f]\|_{L^p(M)} \leq \|f\|_{L^p(M)}$.

(7) For every $\varphi \in C_0^\infty(M)$ and every $t \geq 0$, $\lim\limits_{s \to 0} \|H_s[\varphi] - \varphi\|_t = 0$, where $\|\cdot\|_t$ denotes the Sobolev norm.

In [NS3], Nagel and Stein defined a bounded operator $Q_s = 2s\dfrac{\partial H_s}{\partial s}$, $s > 0$ on $L^2(M)$, and denote by $q_s(x, y)$ the kernel of Q_s, which has the following properties:

(a) $q_s(x, y) \in C^\infty\big(M \times M \backslash \{x = y\}\big)$.

(b) For every integer $N \geq 0$,

$$|\partial_X^L \partial_Y^K q_s(x,y)| \lesssim \frac{1}{(d(x,y) + \sqrt{s})^{K+L}} \frac{1}{V(x,y) + V_{\sqrt{s}}(x) + V_{\sqrt{s}}(y)} \left(\frac{\sqrt{s}}{d(x,y) + \sqrt{s}}\right)^{\frac{N}{2}}.$$

(c) $\int q_s(x, y) dy = \int q_s(x, y) dx = 0$.

Then they obtained the Littlewood-Paley theory on M by using the operator Q_s.

From the spectral theorem, we can see that $H_s \to Id$ on $L^2(M)$ as $s \to 0$ and $H_s \to 0$ on $L^2(M)$ as $s \to \infty$. Hence for any $f \in L^2(M)$,

$$\int_\epsilon^{\frac{1}{\epsilon}} Q_s(f) \frac{ds}{s} = \int_\epsilon^{\frac{1}{\epsilon}} 2s \frac{\partial H_s}{\partial s}(f) \frac{ds}{s} = 2 \int_\epsilon^{\frac{1}{\epsilon}} \frac{\partial H_s}{\partial s}(f) ds = 2H_s(f)\Big|_\epsilon^{\frac{1}{\epsilon}} \longrightarrow -2f$$

as $\epsilon \to 0$. Thus one obtains the following Calderón's identity:

$$\lim_{\epsilon \to 0} -\frac{1}{2} \int_\epsilon^{\frac{1}{\epsilon}} Q_s \frac{ds}{s} = Id \;\text{ on }\; L^2(M).$$

Let

$$Q_j = -\frac{1}{2} \int_{2^{-2j}}^{2^{-2j+2}} Q_s \frac{ds}{s},$$

then we have

$$\sum_j Q_j = Id \;\text{ on }\; L^2(M).$$

Denote by $q_j(x, y)$ the kernel of Q_j. From the estimates of $q_s(x, y)$, for each j, $q_j(x, y)$ satisfies that

(a') $q_j(x, y) \in C^\infty\big(M \times M \backslash \{x = y\}\big)$.

(b') For every integer $N \geq 0$,

$$|\partial_X^L \partial_Y^K q_j(x,y)| \lesssim \frac{1}{(d(x,y)+2^{-j})^{K+L}} \frac{1}{V(x,y)+V_{2^{-j}}(x)+V_{2^{-j}}(y)} \left(\frac{2^{-j}}{d(x,y)+2^{-j}}\right)^{\frac{N}{2}}.$$

(c') $\int q_j(x,y)dy = \int q_j(x,y)dx = 0$.

Then we have that for any $f \in L^2(M)$, $f = \sum\limits_j Q_j(f)$. Now we can restate the Littlewood-Paley theory as follows.

For $f \in L^2(M)$ we define the square function $S(f)$ by

$$S[f](x) = \left(\sum_j \left|Q_j[f](x)\right|^2\right)^{\frac{1}{2}}.$$

Proposition 3.1.
1. For $f \in L^2(M)$,

$$\|S[f]\|_{L^2(M)} = \|f\|_{L^2(M)}.$$

2. For $1 < p < \infty$, if $f \in L^p(M)$ then

$$\|S[f]\|_{L^p(M)} \approx \|f\|_{L^p(M)}.$$

3.2 Singular integral operators on M

We first recall that a bump function φ associated to a ball $B(x_0, \delta)$ if it is supported in that ball and satisfies the differential inequalities $|\partial_X^a \varphi| \lesssim \delta^{-a}$ for all monomials ∂_X in $X_1, \cdots, X_k$ of degree a and all $a \geq 0$.

A class of singular integral operators T are initially given as mappings from $C_0^\infty(M)$ to $C^\infty(M)$ with a distribution kernel $K(x,y)$ which is C^∞ away from the diagonal of $M \times M$, and we suppose the following four properties hold:

(I-1) If $\varphi, \psi \in C_0^\infty(M)$ have disjoint supports, then

$$< T\varphi, \psi >= \int_{M\times M} K(x,y)\varphi(y)\psi(x)dydx.$$

(I-2) If φ is a normalized bump function associated to a ball of radius r, then $|\partial_X^a T\varphi| \lesssim r^{-a}$ for each integer $a \geq 0$.

(I-3) If $x \neq y$, then for every $a \geq 0$,

$$|\partial_{X,Y}^a K(x,y)| \lesssim d(x,y)^{-a} V(x,y)^{-1}.$$

(I-4) Properties (I-1) through (I-3) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator T^t defined by

$$< T^t\varphi, \psi >=< T\psi, \varphi > .$$

The main result about this singular integral operator is as follows:

Theorem 3.2. *[NS3] Each singular integral T satisfying (I-1) through (I-4) extends to a bounded operator on $L^p(M)$ whenever $1 < p < \infty$.*

3.3 The product case of two factors

We assume that $\widetilde{M} = M_1 \times M_2$. Consider linear mappings T, initially defined from $C_0^\infty(\widetilde{M})$ to $C^\infty(\widetilde{M})$ which have an associated distribution kernel $K(x, y)$, which are C^∞ away from the "cross"$= \{(x, y) : x_1 = y_1 \text{ or } x_2 = y_2;\ x = (x_1, x_2), y = (y_1, y_2)\}$ and which satisfy the following additional properties:

(II-1) $\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int K(x_1, y_1, x_2, y_2)\varphi_1(y_1)\varphi_2(y_2)\psi_1(x_1)\psi_2(x_2)dydx$

$$\text{whenever} \begin{cases} \varphi_1, \psi_1 \in C_0^\infty(M_1) \;\text{ and have disjoint support,} \\ \varphi_2, \psi_2 \in C_0^\infty(M_2) \;\text{ and have disjoint support.} \end{cases}$$

(II-2) For each bump function φ_2 on M_2 and each $x_2 \in M_2$, there exists a singular integral T^{φ_2, x_2} (of the one factor type) on M_1, so that

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int_{M_2} \langle T^{\varphi_2, x_2}\varphi_1, \psi_1 \rangle \psi_2(x_2)dx_2.$$

Moreover, $x_2 \mapsto T^{\varphi_2, x_2}$ is smooth and uniform in the sense that T^{φ_2, x_2}, as well as $\rho_2^L \partial_{X_2}^L (T^{\varphi_2, x_2})$ for each $L \geq 0$, satisfy the conditions (I-1) to (I-4) uniformly.

(II-3) If φ_j is a bump function on a ball $B^j(r_j)$ in M_j, then

$$\left|\partial_{X_1}^{a_1}\partial_{X_2}^{a_2} T(\varphi_1 \otimes \varphi_2)\right| \lesssim r_1^{-a_1} r_2^{-a_2}.$$

In (II-2) and (II-3) both inequalities are taken in the sense of (I-2) whenever φ_2 is a bump function for $B^2_{(r_2)}$ in M_2.

(II-4) $\left|\partial_{X_1, Y_1}^{a_1}\partial_{X_2, Y_2}^{a_2} K(x_1, y_1; x_2, y_2)\right| \lesssim \dfrac{d_1(x_1, y_1)^{-a_1} d_2(x_2, y_2)^{-a_2}}{V_1(x_1, y_1)V_2(x_2, y_2)}$.

(II-5) The same conditions hold when the index 1 and 2 are interchanged, that is if the roles of M_1 and M_2 are interchanged.

(II-6) The same properties are assumed to hold for the 3 "transposes" of T, i.e. those operators which arise by interchanging x_1 and y_1, or interchanging x_2 and y_2, or doing both interchanges.

Remark 3.3. *([NS3]) If T_j are singular integral operators on M_j (for the one-factor case), $j = 1, 2$, then $T = T_1 \otimes T_2$ satisfies the above assumptions. Here $T^{\varphi_2, x_2} = T_1$ multiplied by the factor $T_2(\varphi_2)(x_2)$.*

The main result of Nagel-Stein concerning this singular integral operator is as follows.

Theorem 3.4. *([NS3]) For $1 < p < \infty$, each product singular integral satisfying conditions (II-1) to (II-6) extends to a bounded operator on $L^p(\widetilde{M})$ to itself.*

Theorem 3.4 can be obtained from the reproducing identity, the square function and the almost orthogonality estimate of the product case. We recall these results as follows.

Since $\widetilde{M} = M_1 \times M_2$. For each i, we have a heat operator $H^i_{s_i}$, and a corresponding $Q^i_{s_i}$, together with the projection E^i_0. If f is a function on $\widetilde{M}$ we define $Q^1_{s_1} \cdot Q^2_{s_2} = Q^1_{s_1} \otimes Q^2_{s_2}$, with Q^1 acting on the M_1 variable and Q^2 acting on the M_2 variable. We now also recall the almost orthogonality estimate:

Proposition 3.5. *[NS3] Suppose T is a product singular integral satisfying (II-1) to (II-6). Then*

$$\left|Q^1_{t_1} \cdot Q^2_{t_2}\ T\ Q^1_{s_1} \cdot Q^2_{s_2}(f)\right| \lesssim \left(\frac{t_1}{s_1} \wedge \frac{s_1}{t_1}\right)^{\frac{1}{2}} \left(\frac{t_2}{s_2} \wedge \frac{s_2}{t_2}\right)^{\frac{1}{2}} \mathcal{M}_1\mathcal{M}_2(f),$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the maximal function on M_1 and M_2, respectively.

Before considering the product Hardy space, we first introduce the Carleson measure space, the dual spaces of the Hardy spaces, in next subsection. We point out that this dual space is new even for the one-parameter Hardy spaces associated to the Carnot-Caratheodory spaces.

3.4 Carleson measure space and duality on one-parameter Carnot-Caratheodory spaces

To introduce the Carleson measure space, we need to impose a condition on M. We first recall the Definition 3.3.1 in [NS1] which introduced the uniformity of finite type of the vector fields on manifold M.

Definition 3.6. *[NS1] Vector fields $\mathbb{X}_1, \mathbb{X}_2, \mathbb{T}$ are uniformly of finite type m on an open set $U \subset \mathbb{R}^3$ if the derivatives of all coefficients of the vector fields are uniformly bounded on U and if the quantity $\sum_{j=2}^m \Lambda_j(q)$ is uniformly bounded and uniformly bounded away from zero on U. The vector fields $\mathbb{Y}, \mathbb{X}_1, \mathbb{X}_2, \mathbb{T}$ are uniformly of finite type m on an open set $V \subset \mathbb{R}^4$ if the derivatives of all coefficients of the vector fields are uniformly bounded on U and if the quantity $\sum_{j=2}^m \Lambda_j(q)$ is uniformly bounded and uniformly bounded away from zero on V.*

Now we assume that $\overline{Z}$, $\mathbb{X}_1$ and $\mathbb{X}_2$ are uniformly of finite type m on M. Thus we have

$$\mu(B(x,\delta)) \approx \mu(B(y,\delta)) \qquad \text{for all } x, y \in M,$$

and

$$\mu(B(x,\delta)) \approx \delta^{m+2} \quad \text{for } \delta \geq 1; \qquad \mu(B(x,\delta)) \approx \delta^4 \quad \text{for } \delta \leq 1.$$

With this restriction on M, we then give the definition of BMO space on M via the sequence of operators $\{Q_j\}_{j\in\mathbb{Z}}$ as follows.

Definition 3.7. *For $0 < \vartheta < 1$, $\frac{2(m+2)}{2(m+2)+\vartheta} < p \leq 1$ and $0 < \beta, \gamma < \vartheta$, we define the Carleson measure space $CMO^p(M)$ to be the set of all $f \in (\overset{\circ}{G}_\vartheta(\beta,\gamma))'$ such that*

$$\|f\|_{CMO^p(M)} = \sup_P \left\{ \frac{1}{|P|^{\frac{2}{p}-1}} \int_P \sum_k \sum_{I:\ I\subseteq P} |Q_k[f](x)|^2\ \chi_I(x)\ dx \right\}^{\frac{1}{2}} < \infty,$$

where P ranges over all dyadic cubes with finite measures and for each k, I ranges over all the dyadic cubes with length $\ell(I) = 2^{-k-N_0}$.

First we can see that the definition of $CMO^p(M)$ is independent of the choice of distribution space $(\overset{\circ}{G}_\vartheta(\beta,\gamma))'$ with $0<\beta,\gamma<\vartheta$.

Now we introduce the Min-Max inequality for $CMO^p(M)$ as follows.

Theorem 3.8. *Let all the notation be the same as above. For* $\frac{2(m+2)}{2(m+2)+\vartheta}<p\le 1$ *and all* $f\in CMO^p(M)$,

$$\sup_P\Big\{\frac{1}{|P|^{\frac{2}{p}-1}}\int_P\sum_k\sum_{I:\ I\subseteq P}\sup_{u\in I}\big|Q_k[f](u)\big|^2\ \chi_I(x)dx\Big\}^{\frac12}$$
$$\approx\sup_P\Big\{\frac{1}{|P|^{\frac{2}{p}-1}}\int_P\sum_k\sum_{I:\ I\subseteq P}\inf_{u\in I}\big|Q_k[f](u)\big|^2\ \chi_I(x)\ dx\Big\}^{\frac12}.$$

To show the duality of $H^p(M)$, where $H^p(M)$ was introduced in [HMY], with $CMO^p(M)$ for $\frac{2(m+2)}{2(m+2)+\vartheta}<p\le 1$, we follow the idea and skills used in [HLL2]. Now we define the sequence spaces s^p and c^p as follows.

Definition 3.9. *Let* $\widetilde{\chi}_I(x)=|I|^{-\frac12}\chi_I(x)$ *for any dyadic cube* I. *The sequence space* s^p, $0<p\le 1$, *is defined as the collection of all complex-value sequences* $s=\{s_I\}_I$ *such that*

$$\|s\|_{s^p}=\big\|\big\{\sum_I|s_I\widetilde{\chi}_I(x)|^2\big\}^{\frac12}\big\|_{L^p(M)}<\infty.$$

Similarly, c^p, $0<p\le 1$, *is defined as the collection of all complex-value sequences* $t=\{t_I\}_I$ *such that*

$$\|t\|_{c^p}=\sup_P\big(\frac{1}{|P|^{\frac{2}{p}-1}}\sum_{I\subseteq P}|t_I|^2\big)^{\frac12}<\infty,$$

where P *ranges over all dyadic cubes in* M.

The basic result of these sequence spaces is as follows.

Theorem 3.10. $\big(s^p\big)'=c^p$ *for* $0<p\le 1$.

For the detail of the proof, we refer it to Theorem 2.44 in part 2.

We also need to introduce the lifting and projection operators.

Definition 3.11. *For any* $f\in(\overset{\circ}{G}_\vartheta(\beta,\gamma))'$ *with* $0<\beta,\gamma<1$, *define the lifting operator* S_Q *by*

$$S_Q(f)=\Big\{|I|^{\frac12}Q_k[f](x_I)\Big\}_{k\in\mathbb{Z},\ I:\ \ell(I)\approx 2^{-k}},$$

where $k\in\mathbb{Z}$, I *range over all dyadic cubes with length* $\ell(I)=2^{-k-N_0}$ *for each* k *and* x_I *is the center of* I.

Definition 3.12. *For any complex-value sequence* λ, *define the projection operator* $T_{\widetilde{Q}}$ *by*

$$T_Q(\lambda)(x) = \sum_k \sum_I |I|^{\frac{1}{2}} \widetilde{q}_k(x, x_I) \cdot \lambda_{k,I},$$

where k, I *are the same as in the above definition and the function* $\widetilde{q}_k$ *is similar to* q_k *given in* (a').

Moreover,

$$T_Q\big(S_Q(f)\big)(x) = \sum_k \sum_I |I| \widetilde{q}_k(x, x_I) Q_k[f](x_I)$$

and

$$\big\langle S_Q(f), S_Q(g) \big\rangle = \sum_k \sum_I |I| Q_k[f](x_I) Q_k[g](x_I).$$

For the above lifting and projection operators, we have the following basic results.

Theorem 3.13. *Let* $0 < \vartheta < 1$. *For any* $f \in H^p(M)$ *with* $\dfrac{m+2}{m+2+\vartheta} < p \leq 1$, *we have*

$$\|S_Q(f)\|_{s^p} \lesssim \|f\|_{H^p(M)}.$$

Conversely, for any $s \in s^p$,

$$\|T_Q(s)\|_{H^p(M)} \lesssim \|s\|_{s^p}.$$

Theorem 3.14. *Let* $0 < \vartheta < 1$. *For any* $f \in CMO^p(M)$ *with* $\dfrac{2(m+2)}{2(m+2)+\vartheta} < p \leq 1$, *we have*

$$\|S_Q(f)\|_{c^p} \lesssim \|f\|_{CMO^p(M)}.$$

Conversely, for any $t \in c^p$,

$$\|T_Q(t)\|_{CMO^p(M)} \lesssim \|t\|_{c^p}.$$

The first estimate in Theorem 3.13 and 3.14 follows directly from the definitions of s^p and c^p, $H^p(M)$ and $CMO^p(M)$ and the Min-Max inequalities for $H^p(M)$ and $CMO^p(M)$, respectively. The second estimate in Theorem 3.13 and 3.14 follow from the proofs of the Min-Max inequalities with only minor changes, respectively. For the detail, we omit it here and refer the reader to [HLL2]. Moreover, from the Min-Max inequalities, we can obtain that the above two theorems also hold when operator Q is replaced by $\widetilde{Q}$. And from the discrete reproducing identity, we can see that $T_{\widetilde{Q}} \circ S_Q$ equals the identity operator on space of distributions.

Using Theorems 3.10, 3.13 and 3.14, we prove the duality of $H^p(M)$ with $CMO^p(M)$.

Theorem 3.15. *For* $0 < \vartheta < 1$, $\dfrac{2(m+2)}{2(m+2)+\vartheta} < p \leq 1$,

$$\big(H^p(M)\big)' = CMO^p(M).$$

3.5 The H^p theory on product space $\widetilde{M} = M_1 \times \cdots \times M_n$

In this subsection, we would like to introduce the H^p theory on product space $\widetilde{M} = M_1 \times \cdots \times M_n$. Without lost of generality, we first show all our results on the product space of two factors, namely $\widetilde{M} = M_1 \times M_2$. And for the sake of simplicity, we assume that $M_1 = M_2$. Hence $\widetilde{M} = M \times M$, dropping the subscript. For all the results on product space $\widetilde{M}$, we will only give detailed description of the Carleson measure space $CMO^p(\widetilde{M})$ since there are fundamental differences between the proof of $CMO^p(\widetilde{M})$ and $CMO^p(M)$. Roughly speaking, the other results can be obtained from the single factor case by "iteration".

3.5.1 Test function spaces on $\widetilde{M}$

Now we introduce the test function space on $\widetilde{M}$

Definition 3.16. *Let $(x_0, y_0) \in \widetilde{M}$, $\gamma_1, \gamma_2, r_1, r_2 > 0$, $0 < \beta_1, \beta_2 \leq 1$. A function on $\widetilde{M}$ is said to be a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ if there exists a constant $C \geq 0$ such that*

(i) $|f(x,y)| \leq C \dfrac{1}{V_{r_1}(x_0) + V(x_0,x)} \left(\dfrac{r_1}{r_1 + d(x,x_0)}\right)^{\gamma_1} \dfrac{1}{V_{r_2}(y_0) + V(y_0,y)} \left(\dfrac{r_2}{r_2 + d(y,y_0)}\right)^{\gamma_2}$ *for all $(x,y) \in \widetilde{M}$;*

(ii) $$|f(x,y) - f(x',y)| \leq C \left(\frac{d(x,x')}{r_1 + d(x,x_0)}\right)^{\beta_1} \frac{1}{V_{r_1}(x_0) + V(x_0,x)} \left(\frac{r_1}{r_1 + d(x,x_0)}\right)^{\gamma_1} \times \frac{1}{V_{r_2}(y_0) + V(y_0,y)} \left(\frac{r_2}{r_2 + d(y,y_0)}\right)^{\gamma_2}$$ *for all $x, x' \in \widetilde{M}$ satisfying that $d(x,x') \leq (r_1 + d(x,x_0))/2$;*

(iii) Property (ii) also holds with x and y interchanged;

(iv) $$|f(x,y) - f(x',y) - f(x,y') + f(x',y')| \leq C \left(\frac{d(x,x')}{r_1 + d(x,x_0)}\right)^{\beta_1} \frac{1}{V_{r_1}(x_0) + V(x_0,x)} \times \left(\frac{r_1}{r_1 + d(x,x_0)}\right)^{\gamma_1} \left(\frac{d(y,y')}{r_2 + d(y,y_0)}\right)^{\beta_2} \frac{1}{V_{r_2}(y_0) + V(y_0,y)} \left(\frac{r_2}{r_2 + d(y,y_0)}\right)^{\gamma_2}$$ *for all $x, x', y, y' \in \widetilde{M}$ satisfying that $d(x,x') \leq (r_1 + d(x,x_0))/2$ and $d(y,y') \leq (r_2 + d(y,y_0))/2$;*

(v) $\int_{\widetilde{M}} f(x,y)dx = 0$ *for all $y \in \widetilde{M}$;*

(vi) $\int_{\widetilde{M}} f(x,y)dy = 0$ *for all $x \in \widetilde{M}$.*

If f is a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$, we write $f \in G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and we define the norm of f by

$$\|f\|_{G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C : \ (i), (ii), (iii) \text{ and } (iv) \ \text{ hold}\}.$$

We denote by $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $G(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$ for any fixed $(x_0, y_0) \in \widetilde{M}$. We can check that $G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with an equivalent norms for all $(x_0, y_0) \in \widetilde{M}$ and $r_1, r_2 > 0$. Furthermore, it is easy to check that $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space with respect to the norm in $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$.

Now for $\vartheta_1, \vartheta_2 \in (0,1)$, let $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ be the completion of the space $G(\vartheta_1, \vartheta_2; \vartheta_1, \vartheta_2)$ in $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ when $0 < \beta_i, \gamma_i < \vartheta_i$ with $i = 1, 2$. Moreover, $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ if and only if $f \in G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and there exists $\{f_n\}_{n\in\mathbb{N}} \subset G(\vartheta_1, \vartheta_2; \vartheta_1, \vartheta_2)$ such that $\|f - f_n\|_{G(\beta_1,\beta_2;\gamma_1,\gamma_2)} \to 0$ as $n \to \infty$. If $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, we then define $\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)} = \|f\|_{G(\beta_1,\beta_2;\gamma_1,\gamma_2)}$. Then obviously $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space and we also have $\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)} = \lim_{n\to\infty} \|f_n\|_{G(\beta_1,\beta_2;\gamma_1,\gamma_2)}$ for the above chosen $\{f_n\}_{n\in\mathbb{N}}$.

We define the dual space $\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$ to be the set of all linear functionals L from $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{G}_{\vartheta}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

$$|L(f)| \leq C\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

3.5.2 Continuous and discrete reproducing identity

We will establish the continuous and discrete reproducing identities on the product test function space and its dual space.

To establish the continuous Calderón reproducing formula on $\widetilde{M}$, from the result in subsection 3.1, we have

$$I = \sum_{k=-\infty}^{\infty} Q_k \qquad \text{in} \ \ L^2(M).$$

Let $N \in \mathbb{N}$, Coifman's idea (see [**?**]) is to rewrite (3.1) into

$$I = \Big(\sum_{k=-\infty}^{\infty} Q_k\Big)\Big(\sum_{j=-\infty}^{\infty} Q_j\Big) = T_N + R_N,$$

where

$$R_N = \sum_{k=-\infty}^{\infty} \sum_{|j|>N} Q_{k+j} Q_k$$

and

$$T_N = \sum_{k=-\infty}^{\infty} \sum_{|j|\leq N} Q_{k+j} Q_k = \sum_{k=-\infty}^{\infty} Q_k^N Q_k$$

with $Q_k^N = \sum_{|j|\leq N} Q_{k+j}$.

Then we have that R_N is bounded on spaces of test functions with a small operator norm, namely

$$\|R_N(f)\|_{G(\beta,\gamma)} \leq C2^{-N\delta}\|f\|_{G(\beta,\gamma)}$$

for all $f \in G(\beta, \gamma)$ with $0 < \beta, \gamma < \vartheta$, where C and δ are constants independent of N. Moreover, by choosing N so large that $C2^{-N\delta} < 1$, we can see that T_N^{-1} exists and maps any space of test functions to itself. More precisely, there exists a constant $C > 0$ such that for all $f \in G(\beta, \gamma)$ with $0 < \beta, \gamma < \vartheta$,

$$\|T_N^{-1}(f)\|_{G(\beta,\gamma)} \leq C2^{-N\delta}\|f\|_{G(\beta,\gamma)}.$$

For such chosen N, letting

$$\tilde{Q}_k = T_N^{-1} Q_k^N,$$

we then obtain the following

Theorem 3.17. *Let $0 < \vartheta_1, \vartheta_2 < 1$. There exists a family of operators $\{\widetilde{Q}_j\}_{j\in\mathbb{Z}}$ such that for all $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $0 < \beta_1, \gamma_1 < \vartheta_1$, $0 < \beta_2, \gamma_2 < \vartheta_2$, we have*

$$\sum_{k_1} \sum_{k_2} \widetilde{Q}_{k_1} \widetilde{Q}_{k_2} Q_{k_1} Q_{k_2}[f](x) = f(x),$$

where the series converges in the norm of $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and the norm of $L^p(\widetilde{M})$ for $1 < p < \infty$. Moreover, for $f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$, (3.1) also holds in $\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$.

Using a similar idea, namely discretizing the continuous version of Calderón's identity given in Theorem 3.17, we have

Theorem 3.18. *Let $0 < \vartheta_1, \vartheta_2 < 1$. There exists a family of operators $\{\widetilde{Q}_j\}_{j\in\mathbb{Z}}$ such that for all $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $0 < \beta_1, \gamma_1 < \vartheta_1$, $0 < \beta_2, \gamma_2 < \vartheta_2$, we have*

$$f(x_1, x_2) = \sum_{k_1,k_2} \sum_{I,J} |I||J| \widetilde{q}_{k_1} \widetilde{q}_{k_2}(x_1, x_2, x_I, y_J) Q_{k_1} Q_{k_2}[f](x_I, y_J),$$

where $\widetilde{q}_{k_i} \in \overset{\circ}{G}_{\vartheta}(\beta_i, \gamma_i)$ for $i = 1, 2$, $I, J \subset M$ are dyadic cubes with length $2^{-k_1-N_0}$ and $2^{-k_2-N_0}$ for a fixed integer N_0, and x_I, x_J are any fixed points in I and J, respectively. The series in (3.1) converges in the norm of $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and the norm of $L^p(\widetilde{M})$ for $1 < p < \infty$. Moreover, for $f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$, (3.1) holds in the distribution space $\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\big)'$.

3.5.3 Hardy space $H^p(\widetilde{M})$

For $f \in L^2(\widetilde{M})$ we define the square function $\widetilde{S}(f)$ by

$$\widetilde{S}[f](x) = \bigg(\sum_j \sum_k \big|Q_j Q_k[f](x)\big|^2 \bigg)^{\frac{1}{2}}.$$

Definition 3.19. *Let* $0 < \vartheta_1, \vartheta_2 < 1$, $\max\left(\frac{m+2}{m+2+\vartheta_1}, \frac{m+2}{m+2+\vartheta_2}\right) < p \le 1$ *and*

$$(m+2)(\frac{1}{p} - 1)_+ < \beta_i, \gamma_i < \vartheta_i$$

for $i = 1, 2$. *We define the Hardy space* $H^p(\widetilde{M})$ *to be the set of all* $f \in (\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ *such that* $\|\widetilde{S}[f]\|_p < \infty$. *And define*

$$\|f\|_{H^p} = \|\widetilde{S}[f]\|_{L^p},$$

where $\widetilde{S}[f]$ *is the product Littlewood-Paley square function.*

Just like the step in section 3.3, we can first see that the Hardy space $H^p(\widetilde{M})$ is independent of the choice of the spaces of distributions $(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ with β_i, γ_i satisfying the conditions in Definition 3.19 for $i = 1, 2$. We can also obtain the Min-Max inequalities for $H^p(\widetilde{M})$.

Theorem 3.20. *Let* $0 < \vartheta_1, \vartheta_2 < 1$, $\max\left(\frac{m+2}{m+2+\vartheta_1}, \frac{m+2}{m+2+\vartheta_2}\right) < p \le 1$ *and* $(m+2)(\frac{1}{p} - 1)_+ < \beta_i, \gamma_i < \vartheta_i$. *for* $i = 1, 2$. *For all* $f \in (\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$,

$$\left\|\left\{\sum_{k_1,k_2}\sum_{I,J}\sup_{u\in I, v\in J}|Q_{k_1}Q_{k_2}[f](u,v)|^2\ \chi_I(x)\chi_J(y)\right\}^{\frac{1}{2}}\right\|_{L^p(\widetilde{M})}$$
$$\approx \left\|\left\{\sum_{k_1,k_2}\sum_{I,J}\inf_{u\in I, v\in J}|Q_{k_1}Q_{k_2}[f](u,v)|^2\chi_I(x)\chi_J(y)\right\}^{\frac{1}{2}}\right\|_{L^p(\widetilde{M})},$$

where I, J *are the same as in Theorem 3.18.*

3.5.4 Product Carleson measure space and duality

To introduce the product Carleson measure space on $\widetilde{M} = M \times M$, we need to add the same condition on M as in section 3.4. Then we have

Definition 3.21. *For* $0 < \vartheta_1, \vartheta_2 < 1$, $\max\left(\frac{2(m+2)}{2(m+2)+\vartheta_1}, \frac{2(m+2)}{2(m+2)+\vartheta_2}\right) < p \le 1$ *and* $0 < \beta_i, \gamma_i < \vartheta_i$ *for* $i = 1, 2$, *we define the Carleson measure space* $CMO^p(\widetilde{M})$ *to be the set of all* $f \in \left(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ *such that*

$$\|f\|_{CMO^p(\widetilde{M})} = \sup_{\Omega}\left\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\int_\Omega \sum_{k_1,k_2}\sum_{I\times J\subseteq P}\left|Q_{k_1}Q_{k_2}[f](x,y)\right|^2\chi_I(x)\chi_J(y)dxdy\right\}^{\frac{1}{2}} < \infty,$$

where Ω *ranges over all open sets in* $\widetilde{M}$ *with finite measures and for each* k_1 *and* k_2, I, J *range over all the dyadic cubes with length* $\ell(I) = 2^{-k_1-N_0}$ *and* $\ell(J) = 2^{-k_2-N_0}$, *respectively.*

Now we will introduce the Min-Max inequalities for $CMO^p(\widetilde{M})$, whose proof is different from Min-Max inequalities in single factor case.

Theorem 3.22. *Let all the notation be the same as above. Then for all $f \in CMO^p(\widetilde{M})$,*

$$\sup_{\Omega}\Big\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2}\sum_{I\times J\subseteq P}\sup_{u\in I,v\in J}\big|Q_{k_1}Q_{k_2}[f](u,v)\big|^2\chi_I(x)\chi_J(y)dxdy\Big\}^{\frac{1}{2}}$$
$$\lesssim \sup_{\Omega}\Big\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2}\sum_{I\times J\subseteq P}\inf_{u\in I,v\in J}\big|Q_{k_1}Q_{k_2}[f](u,v)\big|^2\chi_I(x)\chi_J(y)dxdy\Big\}^{\frac{1}{2}}.$$

To show the duality of $H^p(\widetilde{M})$ with $CMO^p(\widetilde{M})$ for $\max\big(\frac{2(m+2)}{2(m+2)+\vartheta_1},\frac{2(m+2)}{2(m+2)+\vartheta_2}\big) < p \le 1$, we use the same skills as in the single factor case with only minor changes. Similarly we define the product sequence spaces s^p and c^p as follows

Definition 3.23. *Let $\tilde{\chi}_I(x) = \mu(I)^{-1/2}\chi_I(x)$. For $0 < p \le 1$, the product sequence space s^p is defined as the collection of all complex-value sequences $s = \{s_R\}_R$ such that*

$$\|s\|_{s^p} = \big\|\big\{\sum_R (s_R\tilde{\chi}_R(x_1,x_2))^2\big\}^{1/2}\big\|_{L^p},$$

similarly, for $0 < p \le 1$, the product sequence space c^p is defined as the collection of all complex-value sequences $t = \{t_R\}_R$ such that

$$\|t\|_{c^p} = \sup_{\Omega}\big(\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{R\subseteq\Omega}|t_R|^2\big)^{1/2},$$

where the sup *is taken over all open sets $\Omega \in \widetilde{M}$ with finite measure and R ranges over all the dyadic rectangles in $\widetilde{M}$.*

Then we have the following duality theorem.

Theorem 3.24. $\big(s^p\big)' = c^p$.

We also need to introduce the lifting and projection operators as follows.

Definition 3.25. *Suppose $\vartheta_i \in (0,1)$ and $0 < \beta_i,\gamma_i < \vartheta_i$ for $i = 1,2$. For any $f \in (\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$, define the lifting operator S_Q by*

$$S_Q(f) = \Big\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}Q_{k_1}Q_{k_2}[f](x_I,y_J)\Big\}_{k_1,k_2,I,J},$$

where $k_1,k_2 \in \mathbb{Z}$, I,J are the same as in Theorem 3.18 and $R = I\times J$, x_I and y_J are the centers of I and J, respectively.

Definition 3.26. *For any complex-value sequence* $\lambda = \{\lambda_{k_1,k_2,I,J}\}_{k_1,k_2,I,J}$, *define the projection operator* $T_{\widetilde{Q}}$ *by*

$$T_{\widetilde{Q}}(\lambda)(x,y) = \sum_{j,k}\sum_{I,J}|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\widetilde{q}_{k_1}\widetilde{q}_{k_2}(x,x_I,y,y_J)\cdot\lambda_{j,k,I,J}$$

where $\widetilde{q}_{s_1}\widetilde{q}_{s_2}(x,x_I,y,y_J)$ *are the same as in Theorem 3.18, and* k_1,k_2; I,J; x_I,y_J *are the same as in the above definition. Moreover,*

$$T_{\widetilde{Q}}\big(S_Q(f)\big)(x,y)=\sum_{k_1,k_2}\sum_{I,J}|I||J|\widetilde{q}_{k_1}\widetilde{q}_{k_2}(x,x_I,y,y_J)Q_{k_1}Q_{k_2}[f](x_I,y_J).$$

For the above lifting and projection operators, we have the following basic results.

Theorem 3.27. *Let* $0<\vartheta_1,\vartheta_2<1$. *For any* $f\in H^p(\widetilde{M})$ *with* $\max\big(\frac{2(m+2)}{2(m+2)+\vartheta_1},\frac{2(m+2)}{2(m+2)+\vartheta_2}\big)<p\le 1$, *we have*

$$\|S_Q(f)\|_{s^p}\lesssim\|f\|_{H^p(\widetilde{M})}.$$

Conversely, for any $s\in s^p$,

$$\|T_Q(s)\|_{H^p(\widetilde{M})}\lesssim\|s\|_{s^p}.$$

Theorem 3.28. *Let* $0<\vartheta_1,\vartheta_2<1$. *For any* $f\in CMO^p(\widetilde{M})$ *with* $\max\big(\frac{2(m+2)}{2(m+2)+\vartheta_1},\frac{2(m+2)}{2(m+2)+\vartheta_2}\big)<p\le 1$, *we have*

$$\|S_Q(f)\|_{c^p}\lesssim\|f\|_{CMO^p(\widetilde{M})}.$$

Conversely, for any $t\in c^p$,

$$\|T_Q(t)\|_{CMO^p(\widetilde{M})}\lesssim\|t\|_{c^p}.$$

The above results follow from the same routine as in the single factor case, see also [HLL2]. Then we introduce the main theorem in this section.

Theorem 3.29. *For* $0<\vartheta_1,\vartheta_2<1$, $\max\big(\frac{2(m+2)}{2(m+2)+\vartheta_1},\frac{2(m+2)}{2(m+2)+\vartheta_2}\big)<p\le 1$,

$$\big(H^p(\widetilde{M})\big)'=CMO^p(\widetilde{M}).$$

3.5.5 Endpoint estimates of singular integral operators on product space

We can formulate the results as follows.

Theorem 3.30. *For* $0<\vartheta_1,\vartheta_2<1$, $\max\big(\frac{2(m+2)}{2(m+2)+\vartheta_1},\frac{2(m+2)}{2(m+2)+\vartheta_2}\big)<p\le 1$, *each product singular integral satisfying conditions (II-1) to (II-6) extends to a bounded operator on* $H^p(\widetilde{M})$ *to itself.*

Next we will show that T is bounded from $H^p(M)$ to $L^p(M)$. To do this, we need the following result.

Theorem 3.31. *Let $0 < \vartheta < 1$ and $\frac{m+2}{m+2+\vartheta} < p \leq 1$. If $f \in L^2(M) \bigcap H^p(M)$, then $f \in L^p(M)$ and there exists a constant $C_p > 0$ which is independent of the L^2 norm of f such that*

$$\|f\|_{L^p(M)} \leq C_p \|f\|_{H^p(M)}.$$

From Theorem 3.30 and 3.31, we can easily obtain the boundedness of T from $H^p(M)$ to $L^p(M)$ since $L^2(M) \bigcap H^p(M)$ is dense in $H^p(M)$. More precisely, we have

Theorem 3.32. *Let $0 < \vartheta < 1$ and $\frac{m+2}{m+2+\vartheta} < p \leq 1$. Suppose T is a singular integral operator as defined in Section 2.4, then T is bounded from $H^p(M)$ to $L^p(M)$. Namely, there exists a constant C_p such that*

$$\|T(f)\|_{L^p(M)} \leq C_p \|f\|_{H^p(M)}.$$

Now, for the manifold M satisfying the conditions mentioned at the beginning of this section, for $0 < \vartheta < 1$ and $\frac{2(m+2)}{2(m+2)+\vartheta} < p \leq 1$, Theorem 3.30, together with the duality of $H^p(M)$ with $CMO^p(M)$, yields that T is bounded on $CMO^p(M)$. Particularly, when $p = 1$, we obtain that T is bounded on $BMO(M)$. Moreover, Theorem 3.32 yields that T is bounded from $H^1(M)$ to $L^1(M)$ and hence from $L^\infty(M)$ to $BMO(M)$. These provide the endpoint estimates for the L^p boundedness of singular integral operators of Nagel-Stein.

4 Multiparameter Hardy spaces $H^p_Z(\mathbb{R}^3)$ associated with the Zygmund dilation

This section discusses some recent results on multiparameter Hardy space theory developed by the authors in [HL4]. We first recall Zygmund's conjecture that if the rectangles in R^n had n side lengths which involve only k independent variables, then the resulting strong maximal operator should behave like M_k, the k-parameter strong maximal operator. The first (and probably the only) non-trivial case of this conjecture was demonstrated by A. Cordoba [Cod] who showed that for Q the unit cube in R^3,

$$|\{(x, y, z) \in Q : \mathcal{M}_{st} f(x, y, z) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{LlogL(Q)}$$

where $\mathcal{M}_{st}(f)$ is the strong maximal function on R^3 defined by

$$\mathcal{M}_{st} f(x, y, z) = \sup_{(x,y,z)\in R} \frac{1}{|R|} \int_R |f(x, y, z)| dxdydz$$

where the supremum is taken over rectangles whose sides are parallel to the axes and have side lengths of the form s, t, and $\phi(s, t)$.

Cordoba's result was generalized to the case of $\phi_1(s,t), \phi_2(s,t), \phi_3(s,t)$ by Soria in [So] when some assumptions are given on ϕ_1, ϕ_2, ϕ_3. Moreover, F. Soria showed that the Zygmund's conjecture is not true even when $\phi_1(s,t) = s$, $\phi_2(s,t) = s\phi(t)$ and $\phi_3(s,t) = s\psi(t)$, and ϕ and ψ are some positive and increasing functions.

It has been widely considered that the next simplest multi-parameter group of dilations after the product multi-parameter dilations is the so-called Zygmund dilation, e.g., in $\mathbb{R}^3$, defined by $\rho_{s,t}(x,y,z) = (sx, ty, stz)$ for $s,t > 0$. Indeed, R. Fefferman in his 1996 survey article pointed out the future direction of research on multi-parameter analysis:

"The eventual goal of the program is to extend harmonic analysis past the realm of product spaces consisting other dilation groups, and the operators associated to them. This theory is just at its start, and it seems very difficult indeed at this point. ..., the setting will be the next simplest after product space dilations, and those are as follows: In $\mathbb{R}^3$ consider the family of dilations $\{\rho_{\delta_1,\delta_2}\}_{\delta_1>0,\delta_2>0}$ given by $\rho_{\delta_1,\delta_2}(x,y,z) = (\delta_1 x, \delta_2 y, \delta_1\delta_2 z)$. ..., "

There are two operators intimately associated to this dilation. There is a maximal operator $\mathcal{M}_{\mathcal{Z}}$ (first considered by Zygmund, a special case of $\mathcal{M}_{st}$ when $\phi(s,t) = st$) and singular integral operator $T_{\mathcal{Z}}$ which commutes with this dilation (introduced by Ricci and Stein [RS]).

A class of singular integrals associated to the dilation ρ_{st} was introduced in [RS] by Ricci and Stein. In [RS], Ricci and Stein considered the mappings

$$\tau^{\Lambda}(x_1, ..., x_n) = (\delta_1^{\lambda_{11}} \cdots \delta_k^{\lambda_{1k}} x_1, \cdots, \delta_1^{\lambda_{n1}} \cdots \delta_k^{\lambda_{nk}} x_n),$$

where $\tau = (\delta_1, \cdots, \delta_k) \in \mathbb{R}_+^k, \Lambda = \{\lambda_{ij}\}$, and convolution type operators of the form $Tf = f * K$, where K is given by

$$K(x) = \sum_{I\in\mathbb{Z}^k} \mu_I^{(I)}(x),$$

where $\mu^{(I)}$ are appropriate distributions and $\mu_I^{(I)}(x) = det(2^{-\Lambda I})\mu^{(I)}(2^{-\Lambda I}x)$ with $2^{\Lambda I} = \tau^{\Lambda}$ when $\tau = (2^{i_1}, \cdots, 2^{i_k})$ and $I = (i_1, ..., i_k)$. Then Ricci-Stein proved that T is bounded on L^p for $1 < p < \infty$ under some assumptions.

A special class of singular integral operators $T_{\mathcal{Z}}$ considered by Ricci and Stein is of the form defined by $T_{\mathcal{Z}} f = f * K$ where

$$K(x,y,z) = \sum_{k,j\in\mathbb{Z}} 2^{-2(k+j)} \phi^{k,j}\left(\frac{x}{2^k}, \frac{y}{2^j}, \frac{z}{2^{k+j}}\right),$$

where the functions $\phi^{k,j}$ are supported in the unit cube of $\mathbb{R}^3$ and have a certain amount of uniform smoothness and each satisfies the cancellation conditions

$$\int_{\mathbb{R}^2} \phi^{k,j}(x,y,z)dxdy = \int_{\mathbb{R}^2} \phi^{k,j}(x,y,z)dydz = \int_{\mathbb{R}^2} \phi^{k,j}(x,y,z)dzdx = 0. \tag{4.1}$$

It was shown in [RS] that $T_{\mathcal{Z}} f = K * f$ is bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$. Moreover, they have shown that for $T_{\mathcal{Z}}$ to be $L^2(\mathbb{R}^3)$ bounded (if, say, $\phi_{k,j} = \phi$ for all k and j), (4.1) must hold. It is easy to see that if the dyadic Zygmund dilation is given by

$$(\delta_{2^j,2^k} f)(x,y,z) = 2^{2(j+k)} f(2^j x, 2^k y, 2^{(j+k)} z),$$

then

$$(\delta_{2^j,2^k} T_{\mathcal{Z}}(f))(x,y,z) = T_{\mathcal{Z}}(\delta_{2^j,2^k} f)(x,y,z).$$

This means that the operators studied by Ricci and Stein commute with the Zygmund dilation of dyadic form.

R. Fefferman and Pipher in [FP] further showed that $T_{\mathcal{Z}}$ is bounded in weighted L^p_w spaces for $1 < p < \infty$ when the weights w satisfy an analogous condition of Muckenhoupt associated to the Zygmund dilation. Such a weighted result can not be obtained or reduced to the pure product case through iteration argument. In fact, they proved that if $K_{\mathcal{Z}}$ is the kernel of the Ricci-Stein operator $T_{\mathcal{Z}}$ satisfying (1.1), then $K_{\mathcal{Z}}$ can be decomposed into $K_{\mathcal{Z}} = K^{(1)}_{\mathcal{Z}} + K^{(2)}_{\mathcal{Z}}$ such that

$$\int_{\mathbb{R}} K^{(1)}_{\mathcal{Z}}(x,y,z)dx = 0, \int_{\mathbb{R}^2} K^{(1)}_{\mathcal{Z}}(x,y,z)dydz = 0$$

$$\int_{\mathbb{R}} K^{(2)}_{\mathcal{Z}}(x,y,z)dy = 0, \int_{\mathbb{R}^2} K^{(2)}_{\mathcal{Z}}(x,y,z)dxdz = 0.$$

Subsequently, they proved that each of the operators with the kernels $K^{(1)}_{\mathcal{Z}}$ and $K^{(2)}_{\mathcal{Z}}$ are bounded on L^p_w for $1 < p < \infty$. Weighted boundedness for Cordoba's maximal functions were derived earlier by R. Fefferman, see [F3].

Related to the theory of operators like $\mathcal{M}_{\mathcal{Z}}$ and $T_{\mathcal{Z}}$, several authors have considered the issue of singular integrals along surfaces in $\mathbb{R}^n$ and this has introduced operators like $T_{\mathcal{Z}}$ (for example, Nagel-Wainger [NW]). As far as $\mathcal{M}_{\mathcal{Z}}$ is concerned, E. M. Stein was the first to link the properties of this type of maximal operator to boundary value problems for Poisson integrals on symmetric spaces, such as the Siegel generalized upper half space.

In [NW], Nagel and Wainger first considered the L^2 boundedness of certain singular integral operators on $\mathbb{R}^n$ whose kernel has the appropriate homogeneity with respect to a multi-parameter group of dilations, generated by a finite number of diagonal matrices. In particular, they considered the two-parameter dilation group

$$\delta(s,t)(x,y,z) = (sx, ty, s^\alpha t^\beta z)$$

acting on $\mathbb{R}^3$ for $s, t, \alpha, \beta > 0$. They defined a singular kernel K_1 by

$$K_1(x,y,z) = sgn(xy)\{\frac{|x|^{\alpha-1}|y|^{\beta-1}}{|x|^{2\alpha}|y|^{2\beta} + z^2}\}$$

and proved that convolution with K_1 is bounded in $L^2(\mathbb{R}^3)$.

They also considered multiple Hilbert transforms along surfaces given by

$$f \to Tf(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s, y-t, z-\{st\})\frac{ds}{s}\frac{dt}{t}$$

and showed that T is not bounded on $L^2(\mathbb{R}^3)$ when $\{st\} = st$ and is bounded on L^p for all $1 < p < \infty$ when $\{st\} = |st|$. Moreover, T is also bounded on L^2 when $\{st\} = |s|^\alpha |t|^\beta$ for $\alpha > 0, \beta > 0$.

We now state and describe our main results on multiparameter Hardy spaces $H^p_Z(\mathbb{R}^3)$ associated with the Zygmund dilation. Results described here can be found in the works of the authors [HL4]. We start with some preliminaries. Let $\mathcal{S}(\mathbb{R}^n)$ denote Schwartz functions in $\mathbb{R}^n$. We first construct a test function defined on $\mathbb{R}^3$, given by

$$\psi(x,y,z) = \psi^{(1)}(x)\psi^{(2)}(y,z)$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}), \psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, and satisfy

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi_1)|^2 = 1 \text{ for all } \xi_1 \in \mathbb{R}\backslash\{(0)\},$$

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}^{(2)}(2^{-k}\xi_2, 2^{-k}\xi_3)|^2 = 1 \text{ for all } (\xi_2,\xi_3) \in \mathbb{R}^2\backslash\{(0,0)\},$$

and the moment conditions

$$\int\limits_{\mathbb{R}} x^\alpha \psi^{(1)}(x)dx = \int\limits_{\mathbb{R}^2} y^\beta z^\gamma \psi^{(2)}(y,z)dydz = 0$$

for all nonnegative integers α, β, and γ.

Let $f \in L^p, 1 < p < \infty$. Thus $g_Z(f)$, the Littlewood-Paley-Stein square function of f associated to the Zygmund dilation, is defined by

$$g_Z(f)(x,y,z) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(x,y,z)|^2 \right\}^{\frac{1}{2}}$$

where functions

$$\psi_{j,k}(x,y,z) = 2^{2(j+k)}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y, 2^{j+k} z).$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón's identity holds on $L^2(\mathbb{R}^3)$,

$$f(x,y,z) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y,z).$$

Using the L^p boundedness of Ricci-Stein operator for $1 < p < \infty$ in [RS] together with Calderón's identity on L^2 allows us to obtain the L^p estimates of g_Z for $1 < p < \infty$. Namely, there exist constants C_1 and C_2 such that for $1 < p < \infty$,

$$C_1 \|f\|_p \le \|g_Z(f)\|_p \le C_2 \|f\|_p.$$

We now introduce the product test function space on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$.

Definition 4.1. *A Schwartz test function $f(x,y,z)$ defined on $\mathbb{R}^3$ is said to be a product test function on $\mathbb{R} \times \mathbb{R}^2$ if $f \in \mathcal{S}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}} f(x,y,z)x^\alpha dx = \int_{\mathbb{R}^2} f(x,y,z)y^\beta z^\gamma dydz = 0$$

for all indices α, β, γ of nonnegative integers.

If f is a product test function on $\mathbb{R} \times \mathbb{R}^2$ we denote $f \in \mathcal{S}_Z(\mathbb{R}^3)$ and the norm of f is defined by the norm of Schwartz test function.

We denote by $(\mathcal{S}_Z(\mathbb{R}^3))'$ the dual of $\mathcal{S}_Z(\mathbb{R}^3)$.

We also denote $(\mathcal{S}_{Z,M}(\mathbb{R}^3)$ by the collection of Schwartz test functions $f(x,y,z)$ defined on $\mathbb{R}^3$ with

$$\|f\|_{\mathcal{S}_{Z,M}} = \sup_{x\in\mathbb{R},y\in\mathbb{R},z\in\mathbb{R}} (1+|x|+|y|+|z|)^M \sum_{|\alpha|\le M,|\beta|\le M,|\gamma|\le M} |\frac{\partial^\alpha}{\partial^\alpha x}\frac{\partial^\beta}{\partial^\beta y}\frac{\partial^\gamma}{\partial^\gamma z} f(x,y,z)| < \infty,$$

and

$$\int_{\mathbb{R}} f(x,y,z)x^\alpha dx = \int_{\mathbb{R}^2} f(x,y,z)y^\beta z^\gamma dydz = 0$$

for all indices $\alpha, \beta, \gamma \le M$.

Similarly, we denote $(\mathcal{S}_{Z,M}(\mathbb{R}^3)'$ the dual of $\mathcal{S}_{Z,M}(\mathbb{R}^3)$.

Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_Z(\mathbb{R}^3)$, so the Littlewood-Paley-Stein square function g_Z can be defined for all distributions in $(\mathcal{S}_Z(\mathbb{R}^3))'$. Formally, we can define the multi-parameter Hardy space associated to the Zygmund dilation as follows.

Definition 4.2. *Let $0 < p < \infty$. The multi-parameter Hardy space associated with the Zygmund dilation is defined as $H^p_Z(\mathbb{R}^3) = \left\{f \in (\mathcal{S}_Z)' : g_Z(f) \in L^p(\mathbb{R}^3)\right\}$. If $f \in H^p_Z(\mathbb{R}^3)$, the norm of f is defined by $\|f\|_{H^p_Z} = \|g_Z(f)\|_p$.*

Clearly, it follows that $H^p_Z(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ for $1 < p < \infty$.

We will show the Min-Max comparison principle of first kind (Theorem 4.11) from which it follows that the definition of $H^p_Z(\mathbb{R}^3)$ is independent of the choice of functions $\psi_{j,k}$. The main tool to derive such a Min-Max comparison principle is the discrete Calderón's identity.

The main theorems concerning the Hardy space $H^p_Z(\mathbb{R}^3)$ are the following.

Theorem 4.3. *Let $T_{\mathcal{Z}} = K * f$ be the Ricci-Stein singular integral operator on $\mathbb{R}^3$ where K is defined*

$$K(x,y,z) = \sum_{k,j\in Z} 2^{-2(k+j)}\psi_{k,j}(\frac{x}{2^k}, \frac{y}{2^j}, \frac{z}{2^{k+j}}),$$

where the functions $\psi_{k,j}$ are test functions in $\mathcal{S}_Z(\mathbb{R}^3)$. Then T is bounded on $H^p_Z(\mathbb{R}^3)$ for all $0 < p \le 1$.

Moreover, we can show the $H^p_Z(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ boundedness of the singular integrals.

Theorem 4.4. *Let $0 < p \le 1$. If T is a linear operator which is bounded on $L^2(\mathbb{R}^3)$ and $H^p_Z(\mathbb{R}^3)$, then T can be extended to a bounded operator from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. As a consequence of Theorem 4.3, the Ricci-Stein operator $T_{\mathcal{Z}}$ is bounded from $H^p_Z(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$.*

To study the duality of $H^p_Z(\mathbb{R}^3)$, we introduce the function space $CMO^p_Z(\mathbb{R}^3)$, namely, the Carleson measure spaces.

Definition 4.5. *Let $\psi_{j,k}$ be the same as in Definition 4.1. We say that $f \in CMO_Z^p(\mathbb{R}^3)$ if $f \in (\mathcal{S}_Z(\mathbb{R}^3))'$ with finite norm $\|f\|_{CMO_Z^p}$ defined by*

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \int_{\Omega} \sum_{I\times J\times R\subseteq\Omega} |\psi_{j,k} * f(x,y,z)|^2 \chi_I(x)\chi_J(y)\chi_R(z)dxdydz \right\}^{\frac{1}{2}}$$

for all open sets Ω in $\mathbb{R}^3$ with finite measures, and $I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ and $\ell(R) = 2^{-j-k-2N}$ for a fixed large positive integer N respectively.

Theorem 4.6. *Let $0 < p \leq 1$. Then $(H_Z^p(\mathbb{R}^3))^* = CMO_Z^p(\mathbb{R}^3)$, namely the dual space of $H_Z^p(\mathbb{R}^3)$ is $CMO_Z^p(\mathbb{R}^3)$. More precisely, if $g \in CMO_Z^p(\mathbb{R}^3)$, the map ℓ_g given by $\ell_g(f) =< f, g >$, defined initially for $f \in \mathcal{S}_Z(\mathbb{R}^3)$, extends to a continuous linear functional on $H_Z^p(\mathbb{R}^3)$ with $\|\ell_g\| \approx \|g\|_{CMO_Z^p(\mathbb{R}^3)}$. Conversely, for every $\ell \in (H_F^p)^*$ there exists some $g \in CMO_Z^p(\mathbb{R}^3)$ so that $\ell = \ell_g$. In particular, $(H_Z^1(\mathbb{R}^3))^* = BMO_Z(\mathbb{R}^3)$.*

As a consequence of the duality of $H_Z^1(\mathbb{R}^3)$ with $BMO_Z(\mathbb{R}^3)$ and the $H_Z^1(\mathbb{R}^3)$-boundedness of the singular integral operator $T_{\mathcal{Z}}$, we obtain the $BMO_Z(\mathbb{R}^3)$-boundedness of $T_{\mathcal{Z}}$. Furthermore, we will prove that $L^\infty(\mathbb{R}^3) \subseteq BMO_Z(\mathbb{R}^3)$ and, hence, the $L^\infty(\mathbb{R}^3) \to BMO_Z(\mathbb{R}^3)$ boundedness of Ricci-Stein singular integrals follows. These provide the endpoint results of those in [RS] and can be stated as

Theorem 4.7. *The operator $T_{\mathcal{Z}}$ as defined in Theorem 4.3 is bounded on $BMO_Z(\mathbb{R}^3)$.*

The above theorems can be extended in several directions. First of all, we can extend the Ricci-Stein operator to the nonconvolution type. To state these extensions, we need to introduce some more preliminaries.

For a fixed large positive integer N, we define $\mathcal{S}_N(\mathbb{R}^3 \times \mathbb{R}^3)$ to be the collection of functions $\psi(x, y, z, u, v, w) \in C^N(\mathbb{R}^3 \times \mathbb{R}^3)$ with finite norm $||\psi||_{\mathcal{S}_N}$ defined by

$$\sup_{(x,y,z)\in\mathbb{R}^3,(u,v,w)\in\mathbb{R}^3} (1 + |(x-u, y-v, z-w)|)^N \sum_{|\alpha|,|\beta|\leq N} |\partial_{x,y,z}^\alpha \partial_{u,v,w}^\beta \psi(x, y, z, u, v, w)|$$

where $\partial_{x,y,z}^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}$ and $\partial_{u,v,w}^\beta = \partial_u^{\beta_1}\partial_v^{\beta_2}\partial_w^{\beta_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $|\beta| = \beta_1 + \beta_2 + \beta_3$. We further assume that the following cancellation conditions on ψ:

$$\int_{\mathbb{R}} \psi(x, y, z, u, v, w)x^{\alpha_1}dx = \int_{\mathbb{R}} \psi(x, y, z, u, v, w)u^{\alpha_1}du = 0$$

and

$$\int_{\mathbb{R}^2} \psi(x, y, z, u, v, w)y^{\beta_1}z^{\gamma_1}dydz = \int_{\mathbb{R}^2} \psi(x, y, z, u, v, w)v^{\beta_2}w^{\gamma_2}dvdw = 0$$

for all $0 \leq \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \leq N$. We also use the notation $\mathcal{S}_\infty(\mathbb{R}^3\times\mathbb{R}^3) = \cap_{N>1}\mathcal{S}_N(\mathbb{R}^3\times\mathbb{R}^3)$.

Thus, we can extend the operator considered by Ricci-Stein to nonconvolution type as follows:

$$T_{\mathcal{NC}}f(x,y,z)=\int_{\mathbb{R}^3}K(x,y,z,u,v,w)f(u,v,w)dudvdw$$

where

$$K(x,y,z,u,v,w)=\sum_{j,k\in\mathbb{Z}}2^{-2(j+k)}\psi_{j,k}(\frac{x}{2^k},\frac{y}{2^j},\frac{z}{2^{k+j}},\frac{u}{2^k},\frac{v}{2^j},\frac{w}{2^{k+j}})$$

and $\psi_{j,k}\in\mathcal{S}_N(\mathbb{R}^3\times\mathbb{R}^3)$.

We then have

Theorem 4.8. *The nonconvolution type Ricci-Stein operator $T_{\mathcal{NC}}$ defined for $\psi\in\mathcal{S}_N(\mathbb{R}^3\times\mathbb{R}^3)$ is bounded on $H^p_Z(\mathbb{R}^3)$ and $BMO_Z(\mathbb{R}^3)$ and from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for $p_0(N)<p\le 1$, where $p_0(N)\to 0$ as $N\to\infty$. In particular, $T_{\mathcal{NC}}$ is bounded on $H^p_Z(\mathbb{R}^3)$ and from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for all $0<p\le 1$ when $\psi\in\mathcal{S}_\infty(\mathbb{R}^3\times\mathbb{R}^3)$.*

We should point out that all the above boundedness results are for Ricci-Stein type operators $T_{\mathcal{Z}}$ when $\psi_{j,k}$ in the kernels satisfy the condition in Theorem 4.3 and for nonconvolution type operators $T_{\mathcal{NC}}$. A more refined result with minimal (but most likely not optimal) assumption is the following

Theorem 4.9. *$T_{\mathcal{NC}}$ is bounded on $H^p_Z(\mathbb{R}^3)$ and $BMO_Z(\mathbb{R}^3)$ and from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ when $p_0<p\le 1$ for some $p_0<1, 0\le\alpha_1\le 2$ and $0\le\beta_1+\gamma_1\le 1$.*

By formulating and proving a Journe's type covering lemma associated with the Zygmund dilation, Pipher and the authors are working on the H^p_Z to L^p boundedness when the Ricci-Stein kenel satisfies the optimal cancellation condition as that used in [RS] for L^p boundedness.

We remark here that if we define the test function on $\mathbb{R}^3$, given by

$$\psi(x,y,z)=\psi^{(1)}(y)\psi^{(2)}(x,z)$$

where $\psi^{(1)}\in\mathcal{S}(\mathbb{R}),\psi^{(2)}\in\mathcal{S}(\mathbb{R}^2)$. Thus $\tilde{g_Z}(f)$, the Littlewood-Paley-Stein square function of f, is defined by

$$\tilde{g_Z}(f)(x,y,z)=\left\{\sum_j\sum_k|\psi_{j,k}*f(x,y,z)|^2\right\}^{\frac{1}{2}}$$

where functions

$$\psi_{j,k}(x,y,z)=2^{2(j+k)}\psi^{(1)}(2^jy)\psi^{(2)}(2^kx,2^{j+k}z).$$

Thus, it is easy to see that the L^p boundedness holds with $g_Z(f)$ replaced by $\tilde{g_Z}(f)$ for $p>1$.

We can also use $\tilde{g_Z}(f)$ to define Hardy spaces $\widetilde{H^p_Z}(\mathbb{R}^3)$ for $0<p\le 1$. Thus we have proved that the original Ricci-Stein operator plus some extra cancellation conditions is bounded on $H^p_Z(\mathbb{R}^3)\cap\widetilde{H^p_Z}(\mathbb{R}^3)$ for $0<p\le 1$ by using the kernel decompositions of R. Fefferman and Pipher [FP].

We point out that our result in this paper can be extended to the high dimension dilations given by

$$(x_1, x_2, ..., x_n) \to (\delta_1 x_1, \delta_2 x_2, ..., \delta_{n-1} x_{n-1}, \delta_1 \delta_2 \cdots \delta_{n-1} x_n).$$

To carry out the theory of multi-parameter Hardy spaces associated with the Zygmund dilation, we begin with establishing the discrete Calderón's identity associated with this dilation.

Theorem 4.10. *Suppose that* $\psi_{j,k}$ *are the same as in Definition 4.1. Then*

$$f(x,y,z) = \sum_{j,k} \sum_{I,J,R} |I||J||R| \widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R) \left(\psi_{j,k} * f\right)(x_I, y_J, z_R)$$

where $\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R) \in \mathcal{S}_{Z,M}(\mathbb{R}^3), I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ *are dyadic intervals with interval-length* $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ *and* $\ell(R) = 2^{-j-k-2N}$ *for a fixed large integer* N, x_I, y_J, z_R *are any fixed points in* I, J, R*, respectively, and the above series converges in the norm of* $\mathcal{S}_{Z,M}(\mathbb{R}^3)$ *and in the dual space* $(\mathcal{S}_{Z,M}(\mathbb{R}^3))'$.

The above discrete Calderón's identity enables us to derive the following Min-Max comparison principle of first kind.

Theorem 4.11. *Suppose* $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R}), \psi^{(2)}, \phi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ *and* ψ_{jk}, ϕ_{jk} *satisfy the conditions as in Definition 4.1. Then for* $f \in (\mathcal{S}_{Z,M}(\mathbb{R}^3))'$ *where* M *depends on* p *and* $0 < p < \infty$,

$$\| \left\{ \sum_{j,k} \sum_{I,J,R} \sup_{u \in I, v \in J, w \in R} |\psi_{j,k} * f(u,v,w)|^2 \chi_I(\cdot) \chi_J(\cdot) \chi_R(\cdot) \right\}^{\frac{1}{2}} \|_p$$

$$\approx \| \left\{ \sum_{j,k} \sum_{I,J,R} \inf_{u \in I, v \in J, w \in R} |\phi_{j,k} * f(u,v,w)|^2 \chi_I(\cdot) \chi_J(\cdot) \chi_R(\cdot) \right\}^{\frac{1}{2}} \|_p$$

where $\psi_{j,k}(x,y)$ *is defined using* $\psi^{(1)}$ *and* $\psi^{(2)}$ *and* $\phi_{j,k}(x,y)$ *is defined using* $\phi^{(1)}$ *and* $\phi^{(2)}$ *as in Definition 4.1,* $I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ *are dyadic intervals with interval-length* $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ *and* $\ell(R) = 2^{-j-k-2N}$ *for a fixed large positive integer* N, χ_I, χ_J *and* χ_R *are indicator functions of* I, J *and* R*, respectively.*

The Min-Max comparison principle in Theorem 4.11 leads us to define the discrete Littlewood-Paley-Stein square function

$$g_Z^d(f)(x,y,z) = \left\{ \sum_{j,k} \sum_{I,J,R} |(\psi_{j,k} * f)(x_I, y_J, z_R)|^2 \chi_I(x) \chi_J(y) \chi_R(z) \right\}^{\frac{1}{2}}$$

where I, J, R, x_I, y_J and z_R are the same as in Theorem 4.11.

It is easy to see from Theorem 4.11 that the Hardy space H^p_Z given in Definition 4.2 is well defined and the H^p_Z norm of f is equivalent to the L^p norm of g^d_Z.

To derive the discrete Calderón identity, we have the following almost orthogonal arguments.

Corollary 4.12. *If we allow N_1, N_2, M_1, M_2 to be any positive numbers less than ∞, that is, $\psi, \phi \in \mathcal{S}_Z(\mathbb{R}^3)$ with moment condition of any order, then for any positive integers L, M there exists $C = C(L, M)$ such that*

$$\int_{\mathbb{R}^3} \psi_{ts}(x,y,z,u,v,w)\phi_{t's'}(u,v,w,x_0,y_0,z_0)dudvdw$$
$$\leq C(\frac{t'}{t} \wedge \frac{t}{t'})^L(\frac{s'}{s} \wedge \frac{s}{s'})^L \frac{(t\vee t')^M}{(t\vee t' + |x-x_0|)^{1+M}} \frac{(s\vee s')^M}{t^*(s\vee s' + |y-y_0| + \frac{|z-z_0|}{t^*})^{2+M}}$$

where $t^* = t$ if $s > s'$ and $t^* = t'$ if $s \leq s'$.

Corollary 4.13. *If f and $g \in \mathcal{S}_Z(\mathbb{R}^3)$ and $f_{ts}(x,y,z) = t^{-2}s^{-2}f(\frac{x}{t}, \frac{y}{s}, \frac{z}{ts})$ and g_{ts} is defined similarly. Then for any positive integers L and M there exists a constant $C = C(L, M)$ such that*

$$|f_{ts} * g_{t's'}(x,y,z)| \leq C(\frac{t'}{t} \wedge \frac{t}{t'})^L(\frac{s'}{s} \wedge \frac{s}{s'})^L \frac{(t\vee t')^M}{(t\vee t' + |x|)^{1+M}} \frac{(s\vee s')^M}{t^*(s\vee s' + |y| + \frac{|z|}{t^*})^{2+M}}$$

where $t^* = t$ if $s > s'$ and $t^* = t'$ if $s \leq s'$.

Next, we will show that the operator $T_{\mathcal{Z}}$ is actually bounded from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for all $0 < p \leq 1$, and $T_{\mathcal{NC}}$ for $\psi \in \mathcal{S}_N(\mathbb{R}^3 \times \mathbb{R}^3)$ is bounded for $p_0(N) < p \leq 1$ with $p_0(N) \to 0$ as $N \to \infty$. To this end, we need to give several properties of $H^p_Z(\mathbb{R}^3)$.

Proposition 4.14. *$\mathcal{S}_Z(\mathbb{R}^3)$ is dense in $H^p_Z(\mathbb{R}^3)$.*

Since $S_Z(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$, as a consequence of Proposition 4.14, it is immediate that

Proposition 4.15. *$L^q(\mathbb{R}^3), 1 \leq q < \infty$, is dense in $H^p_Z(\mathbb{R}^3)$ for $0 < p \leq 1$.*

Proposition 4.16. *$L^2(\mathbb{R}^3) \cap H^p_Z(\mathbb{R}^3) \subseteq L^p(\mathbb{R}^3)$ for $0 < p \leq 1$, and moreover, if $f \in L^2(\mathbb{R}^3) \cap H^p_Z(\mathbb{R}^3))$, then*

$$\|f\|_p \leq C\|f\|_{H^p_Z}$$

where the constant C is independent of the L^2 norm of f.

As a consequence, we obtain the following result:

Theorem 4.17. *If T is bounded on $L^2(\mathbb{R}^3)$ and $H^p_Z(\mathbb{R}^3)$, then T extends to a bounded operator from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. Moreover,*

$$\|Tf\|_p \leq C\|f\|_{H^p_Z}$$

where the constant C is independent of the L^2 norm of f.

Proof: If $f \in L^2(\mathbb{R}^3) \cap H^p_Z(\mathbb{R}^3)$, then $Tf \in L^2(\mathbb{R}^3) \cap H^p_Z(\mathbb{R}^3)$. Thus, by proposition,

$$\|Tf\|_p \leq C\|Tf\|_{H^p_Z} \leq C\|f\|_{H^p_Z}.$$

Since $L^2(\mathbb{R}^3) \cap H^p_Z(\mathbb{R}^3)$ is dense in $H^p_Z(\mathbb{R}^3)$, the theorem follows.

Q.E.D

The boundedness of Ricci-Stein type operators from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ follows immediately from Theorem 4.17. This completes the proof of Theorem 4.8.

We now establish the duality theory of $H^p_Z(\mathbb{R}^3)$, namely, the dual of $H^p_Z(\mathbb{R}^3)$ is $CMO^p_Z(\mathbb{R}^3)$. This is exactly Theorem 4.6. To see spaces $CMO^p_Z(\mathbb{R}^3)$ are well defined, we need to prove the Min-Max comparison principle of second kind with respect to the norm of $CMO^p_z(\mathbb{R}^3)$. This is the following theorem.

Theorem 4.18. *Suppose ψ, ϕ satisfy the same conditions as in Theorem 4.10. Then for $f \in (\mathcal{S}_{Z,M}(\mathbb{R}^3))'$,*

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I\times J\times R \subseteq \Omega} \sup_{u\in I, v\in J, w\in R} |\psi_{j,k} * f(u,v,w)|^2 |I||J||R| \right\}^{\frac{1}{2}} \approx$$

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I\times J\times R \subseteq \Omega} \inf_{u\in I, v\in J, w\in R} |\phi_{j,k} * f(u,v,w)|^2 |I||J||R| \right\}^{\frac{1}{2}}$$

where $I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ are dyadic intervals with interval-length $\ell(I) = 2^{-j-N}, \ell(J) = 2^{-k-N}$ and $\ell(R) = 2^{-j-k-2N}$ for a fixed large integer N respectively, and Ω are all open sets in $\mathbb{R}^3$ with finite measures.

We end this section by remarking that Calderón-Zygmund decomposition and interpolation theorems on the Hardy space $H^p_Z(\mathbb{R}^3)$ hold. Nevertheless, we have decided to skip the description here and refer the reader to next section since the proofs are similar to those interpolation theorems established in the flag multiparameter Hardy spaces in next section. This concurs that the definition of Hardy spaces associated to the Zygmund dilation is canonical and intrinsic to the underlying multiparameter structures.

5 Multiparameter flag Hardy spaces $H^p_F(R^n \times R^m)$

We begin this section by recalling two instances of implicit multiparameter structures which are of interest to us. We begin with reviewing one of these cases first. In the work of Muller-Ricci-Stein [MRS1,2], by considering an implicit multi-parameter structure on Heisenberg(-type) groups, the Marcinkiewicz multipliers on the Heisenberg groups yield a new class of flag singular integrals. To be more precise, let $m(\mathcal{L}, iT)$ be the Marcinkiewicz multiplier operator, where $\mathcal{L}$ is the sub-Laplacian, T is the central element of the Lie algebra on the Heisenberg group $\mathbb{H}^n$, and m satisfies the Marcinkiewicz conditions. It was proved in [MRS1,2] that the kernel of $m(\mathcal{L}, iT)$ satisfies the standard one-parameter Calderón-Zygmund type estimates associated with automorphic dilations in the region where $|t| < |z|^2$, and the multi-parameter product kernel in the region where $|t| \geq |z|^2$ on the space $\mathbb{C}^n \times R$. The proof of the $L^p, 1 < p < \infty$, boundedness of $m(\mathcal{L}, iT)$ given in [MRS1] requires lifting the operator to a larger group, $\mathbb{H}^n \times R$. This lifts K, the kernel of $m(\mathcal{L}, iT)$ on $\mathbb{H}^n$, to a product kernel $\widetilde{K}$ on $\mathbb{H}^n \times R$. The lifted kernel $\widetilde{K}$ is constructed so that it projects to K by

$$K(z,t) = \int_{-\infty}^{\infty} \widetilde{K}(z, t-u, u)du$$

taken in the sense of distributions.

The operator $\widetilde{T}$ corresponding to product kernel $\widetilde{K}$ can be dealt with in terms of tensor products of operators, and one can obtain their $L^p, 1 < p < \infty$, boundedness by the known pure product theory. Finally, the $L^p, 1 < p < \infty$, boundedness of operator with kernel K follows from transference method of Coifman and Weiss ([CW2]), using the projection $\pi : \mathbb{H}^n \times R \to \mathbb{H}^n$ by $\pi((z,t),u) = (z, t+u)$.

Another example of implicit multi-parameter structure is the flag singular integrals on $R^n \times R^m$ studied by Nagel-Ricci-Stein [NRS]. The simplest form of flag singular integral kernel $K(x,y)$ on $R^n \times R^m$ is defined through a projection of a product kernel $\widetilde{K}(x,y,z)$ defined on $R^{n+m} \times R^m$ given by

$$K(x,y) = \int_{R^m} \widetilde{K}(x, y-z, z)dz.$$

A more general definition of flag singular kernel was introduced in [NRS], see more details of definitions and applications of flag singular integrals there. We will also briefly recall them later in the introduction. Note that convolution with a flag singular kernel is a special case of product singular kernel. As a consequence, the $L^p, 1 < p < \infty$, boundedness of flag singular integral follows directly from the product theory on $R^n \times R^m$. We note the regularity satisfied by flag singular kernels is better than that of the product singular kernels. More precisely, the singularity of the standard pure product kernel on $R^n \times R^m$, is sets $\{(x,0)\} \cup \{(0,y)\}$ while the singularity of $K(x,y)$, the flag singular kernel on $R^n \times R^m$ defined above, is a flag set given by $\{(0,0)\} \subseteq \{(0,y)\}$. For example, $K_1(x,y) = \frac{1}{xy}$ is a product kernel on R^2 and $K_2(x,y) = \frac{1}{x(x+iy)}$ is a flag kernel on R^2.

Though the L^p theory has been established for the aforementioned two cases, the multiparameter Hardy space theory in the second case above has been still absent till recently developed by the authors in [HL3]. In this part, we describe some recent works of multiparameter Hardy space theory associated with the implicit flag structure on $R^n \times R^m$. In a forthcoming article, in joint work with Eric Sawyer, we have established the Hardy space theory associated to the implicit flag structure on the Heisenberg group $\mathbb{H}^n$ and proved the H^p-boundedness of the Marcinkiewcz multipliers on $\mathbb{H}^n$. We will, however, not describe these results here. We would also like to mention that as an extension of results in [HL3] to the non-isotropic dilation on $R^{n+m} \times R^m$ given by $\delta(x,y,z) = (\delta x, \delta y, \delta^2 z)$, the multiparameter Hardy space theory associated to this non-isotropic flag singular integrals has been carried out in [R]. Multiparameter Treibel-Lizorkin and Besov space theory has been done in [DLM].

We also remark here that we shall provide some ideas and outlines of proofs of some main theorems here since this is the multiparameter setting in Euclidean spaces. These methods employed here also apply to the cases considered in the past three sections 2, 3 and 4.

5.1 Hardy space theory associated with the implicit flag singular integral operators: Preliminaries and main results

The works of [NRS], [MRS1-2], [CF1], [CF2] suggest that a satisfactory Hardy space theory associated with implicit flag structure should be developed and boundedness of flag singular integrals on such spaces should be established. Thus some natural questions arise. From now on, we will use the subscript "F" to express function spaces or functions associated with the multi-parameter flag structure without further explanation.

We will consider in this section the following questions:

Question 1: What is the analogous estimate when $p = 1$? Namely, do we have a satisfactory flag Hardy space $H^1_F(R^n \times R^m)$ theory associated with the flag singular integral operators? More generally, can we develop the flag Hardy space $H^p_F(R^n \times R^m)$ theory for all $0 < p \leq 1$ such that the flag singular integral operators are bounded on such spaces?

Question 2: Do we have a boundedness result on a certain type of BMO_F space for flag singular integral operators considered in [NRS]? Namely, does an endpoint estimate of the result by Nagel-Ricci-Stein hold when $p = \infty$?

Question 3: What is the duality theory of so defined flag Hardy space? More precisely, do we have an analogue of BMO and Carleson measure type function spaces which are dual spaces of the flag Hardy spaces as Chang and R. Fefferman did in pure product setting?

Question 4: Is there a Calderón-Zygmund decomposition in terms of functions in flag Hardy spaces $H^p_F(R^n \times R^m)$? Furthermore, is there a satisfactory theory of interpolation on such spaces as Chang and R. Fefferman established in pure product setting?

Question 5: What is the difference and relationship between the Hardy space $H^p(R^n \times R^m)$ in the pure product setting and $H^p_F(R^n \times R^m)$ in flag multiparameter setting?

The original goal of our work [HL1] is to address these questions. As in the L^p theory for $p > 1$ considered in [MRS], one is naturally tempted to establish the Hardy space theory under the implicit multi-parameter structure associated with the flag singular kernel by lifting method to the pure product setting together with the transference method in [CW]. However, this direct lifting method is not adaptable directly to the case of $p \leq 1$ because the transference method is not known to be valid when $p \leq 1$. This suggests that a different approach in dealing with the Hardy $H^p(R^n \times R^m)$ space associated with this implicit multi-parameter structure is necessary. This motivated our work in this paper. In fact, we will develop a unified approach to study multi-parameter Hardy space theory. Our approach will be carried out in the order of the following steps as we have seen in the previous sections.

(1) We first establish the theory of Littlewood-Paley-Stein square function g_F associated with the implicit multi-parameter structure and the L^p estimates of g_F $(1 < p < \infty)$. We then develop a discrete Calderón reproducing formula and a Min-Max type inequality in a test

function space associated to this structure. As in the classical case of pure product setting, these L^p estimates can be used to provide a new proof of Nagel-Ricci-Stein's $L^p(1 < p < \infty)$ boundedenss of flag singular integral operators.

(2) We next develop the theory of Hardy spaces H^p_F associated to the multi-parameter flag structures and the boundedness of flag singular integrals on these spaces; We then establish the boundedness of flag singular integrals from H^p_F to L^p. We refer to the reader the work of product multi-parameter Hardy space theory by Chang-R. Fefferman [CF1-3], R. Fefferman [F1-3], Journe [J1-2] and Pipher [P].

(3) We then establish the duality theory of the flag Hardy space H^p_F and introduce the dual space CMO^p_F, in particular, the duality of H^1_F and the space BMO_F. We then establish the boundedness of flag singular integrals on BMO_F. It is worthwhile to point out that in the classical one-parameter or pure product case, $BMO(R^n)$ or $BMO(R^n \times R^m)$ is related to the Carleson measure. The space CMO^p_F for all $0 < p \leq 1$, as the dual space of H^p_F introduced in this paper, is then defined by a generalized Carleson measure.

(4) We further establish a Calderón-Zygmund decomposition lemma for any $H^p_F(R^n \times R^m$ function $(0 < p < \infty)$ in terms of functions in $H^{p_1}_F(R^n \times R^m)$ and $H^{p_2}_F(R^n \times R^m)$ with $0 < p_1 < p < p_2 < \infty$. Then an interpolation theorem is established between $H^{p_1}_F(R^n \times R^m)$ and $H^{p_2}_F(R^n \times R^m)$ for any $0 < p_2 < p_1 < \infty$ (it is noted that $H^p_F(R^n \times R^m) = L^p(R^{n+m})$ for $1 < p < \infty$).

In the present section, we will use the above approach to study the Hardy space theory associated with the implicit multi-parameter structures induced by the flag singular integrals. We now describe our approach and results in more details.

We first introduce the continuous version of the Littlewood-Paley-Stein square function g_F. Inspired by the idea of lifting method of proving the $L^p(R^n \times R^m)$ boundedness given in [MRS1], we will use a lifting method to construct a test function defined on $R^n \times R^m$, given by the non-standard convolution $*_2$ on the second variable only:

$$\psi(x, y) = \psi^{(1)} *_2 \psi^{(2)}(x, y) = \int\limits_{R^m} \psi^{(1)}(x, y - z)\psi^{(2)}(z)dz,$$

where $\psi^{(1)} \in \mathcal{S}(R^{n+m}), \psi^{(2)} \in \mathcal{S}(R^m)$, and satisfy

$$\sum_j |\widehat{\psi^{(1)}}(2^{-j}\xi_1, 2^{-j}\xi_2)|^2 = 1$$

for all $(\xi_1, \xi_2) \in R^n \times R^m \backslash \{(0, 0)\}$, and

$$\sum_k |\widehat{\psi}^{(2)}(2^{-k}\eta)|^2 = 1$$

for all $\eta \in R^m \backslash \{0\}$, and the moment conditions

$$\int\limits_{R^{n+m}} x^\alpha y^\beta \psi^{(1)}(x, y)dxdy = \int\limits_{R^m} z^\gamma \psi^{(2)}(z)dz = 0$$

for all multi-indices α, β, and γ. We remark here that it is this subtle convolution $*_2$ which provides a rich theory for the implicit multi-parameter analysis.

For $f \in L^p, 1 < p < \infty, g_F(f)$, the Littlewood-Paley-Stein square function of f, is defined by

$$g_F(f)(x,y) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(x,y)|^2 \right\}^{\frac{1}{2}}$$

where functions

$$\psi_{j,k}(x,y) = \psi_j^{(1)} *_2 \psi_k^{(2)}(x,y), \tag{5.1}$$

$$\psi_j^{(1)}(x,y) = 2^{(n+m)j}\psi^{(1)}(2^j x, 2^j y) \text{ and } \psi_k^{(2)}(z) = 2^{mk}\psi^{(2)}(2^k z).$$

We remark here that the terminology "implicit multi-parameter structure" is clear from the fact that the dilation $\psi_{j,k}(x,y)$ is not induced from $\psi(x,y)$ explicitly.

By taking the Fourier transform, it is easy to see the following continuous version of the Calderón reproducing formula holds on $L^2(R^{n+m})$,

$$f(x,y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y).$$

Note that if one considers the above summation on the right hand side as an operator then, by the construction of function ψ, it is a flag singular integral and has the implicit multi-parameter structure as mentioned before. Using iteration and the vector-valued Littlewood-Paley-Stein estimate together with the Calderón reproducing formula on L^2 allows us to obtain the $L^p, 1 < p < \infty$, estimates of g_F.

Theorem 5.1. *Let $1 < p < \infty$. Then there exist constants C_1 and C_2 depending on p such that for*

$$C_1\|f\|_p \le \|g_F(f)\|_p \le C_2\|f\|_p.$$

In order to state our results for flag singular integrals, we need to recall some definitions given in [NRS]. Following closely from [NRS], we begin with the definitions of a class of distributions on an Euclidean space R^N. A $k-$*normalized* bump function on a space R^N is a C^k-function supported on the unit ball with C^k-norm bounded by 1. As pointed out in [NRS], the definitions given below are independent of the choices of k, and thus we will simply refer to "normalized bump function" without specifying k.

For the sake of simplicity of presentations, we will restrict our considerations to the case $R^N = R^{n+m} \times R^m$. We will rephrase Definition 2.1.1 in [NRS] of product kernel in this case as follows:

Definition 5.2. *A product kernel on $R^{n+m} \times R^m$ is a distribution K on R^{n+m+m} which coincides with a C^∞ function away from the coordinate subspaces $(0,0,z)$ and $(x,y,0)$, where $(0,0) \in R^{n+m}$ and $(x,y) \in R^{n+m}$, and satisfies*

(1) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$ and $\gamma_m = (\gamma_1, \cdots, \gamma_m)$

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| \leq C_{\alpha,\beta,\gamma}(|x| + |y|)^{-n-m-|\alpha|-|\beta|} \cdot |z|^{-m-|\gamma|}$$

for all $(x, y, z) \in R^n \times R^m \times R^m$ with $|x| + |y| \neq 0$ and $|z| \neq 0$.

(2) (Cancellation Condition)

$$|\int_{R^m} \partial_x^\alpha \partial_y^\beta K(x, y, z)\phi_1(\delta z)dz| \leq C_{\alpha,\beta}(|x| + |y|)^{-n-m-|\alpha|-|\beta|}$$

for all multi-indices α, β and every normalized bump function ϕ_1 on R^m and every $\delta > 0$;

$$|\int_{R^m} \partial_z^\gamma K(x, y, z)\phi_2(\delta x, \delta y)dxdy| \leq C_\gamma |z|^{-m-|\gamma|}$$

for every multi-index γ and every normalized bump function ϕ_2 on R^{n+m} and every $\delta > 0$; and

$$|\int_{R^{n+m+m}} K(x, y, z)\phi_3(\delta_1 x, \delta_1 y, \delta_2 z)dxdydz| \leq C$$

for every normalized bump function ϕ_3 on R^{n+m+m} and every $\delta_1 > 0$ and $\delta_2 > 0$.

Definition 5.3. *A flag kernel on $R^n \times R^m$ is a distribution on R^{n+m} which coincides with a C^∞ function away from the coordinate subspace $\{(0, y)\} \subset R^{n+m}$, where $0 \in R^n$ and $y \in R^m$ and satisfies*

(1) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\alpha,\beta}|x|^{-n-|\alpha|} \cdot (|x| + |y|)^{-m-|\beta|}$$

for all $(x, y) \in R^n \times R^m$ with $|x| \neq 0$.

(2) (Cancellation Condition)

$$|\int_{R^m} \partial_x^\alpha K(x, y)\phi_1(\delta y)dy| \leq C_\alpha |x|^{-n-|\alpha|}$$

for every multi-index α and every normalized bump function ϕ_1 on R^m and every $\delta > 0$;

$$|\int_{R^n} \partial_y^\beta K(x, y)\phi_2(\delta x)dx| \leq C_\gamma |y|^{-m-|\beta|}$$

for every multi-index β and every normalized bump function ϕ_2 on R^n and every $\delta > 0$; and

$$|\int_{R^{n+m}} K(x, y)\phi_3(\delta_1 x, \delta_2 y)dxdy| \leq C$$

for every normalized bump function ϕ_3 on R^{n+m} and every $\delta_1 > 0$ and $\delta_2 > 0$.

By a result in [MRS], we may assume first that a flag kernel K lies in $L^1(R^{n+m})$. Thus, there exists a product kernel $K^\sharp$ on $R^{n+m}\times R^m$ such that

$$K(x,y)=\int_{R^m}K^\sharp(x,y-z,z)dz.$$

Conversely, if a product kernel $K^\sharp$ lies in $L^1(R^{n+m}\times R^m)$, then $K(x,y)$ defined as above is a flag kernel on $R^n\times R^m$. As pointed out in [MRS], we may always assume that $K(x,y)$, a flag kernel, is integrable on $R^n\times R^m$ by using a smooth truncation argument.

As a consequence of Theorem 5.1, we give a new proof of the $L^p, 1<p<\infty$, boundedness of flag singular integrals due to Nagel, Ricci and Stein in [NRS]. More precisely, let $T(f)(x,y)=K*f(x,y)$ be a flag singular integral on $R^n\times R^m$. Then K is a projection of a product kernel $K^\sharp$ on $R^{n+m}\times R^m$.

Theorem 5.4. *Suppose that T is a flag singular integral defined on $R^n\times R^m$ with the flag kernel $K(x,y)=\int\limits_{R^m}K^\sharp(x,y-z,z)dz$, where the product kernel $K^\sharp$ satisfies the conditions of Definition 5.2 above. Then T is bounded on L^p for $1<p<\infty$. Moreover, there exists a constant C depending on p such that for $f\in L^p, 1<p<\infty$,*

$$\|T(f)\|_p\le C\|f\|_p.$$

In order to use the Littlewood-Paley-Stein square function g_F to define the Hardy space, one needs to extend the Littlewood-Paley-Stein square function to be defined on a suitable distribution space. For this purpose, we first introduce the product test function space on $R^{n+m}\times R^m$.

Definition 5.5. *A Schwartz test function $f(x,y,z)$ defined on $R^n\times R^m\times R^m$ is said to be a product test function on $R^{n+m}\times R^m$ if*

$$\int f(x,y,z)x^\alpha y^\beta dxdy=\int f(x,y,z)z^\gamma dz=0$$

for all multi-indices α,β,γ of nonnegative integers.

If f is a product test function on $R^{n+m}\times R^m$ we denote $f\in\mathcal{S}_\infty(R^{n+m}\times R^m)$ and the norm of f is defined by the norm of Schwartz test function.

We also denote $(\mathcal{S}_M(\mathbb{R}^{n+m}\times\mathbb{R}^m))$ by the collection of Schwartz test functions $f(x,y,z)$ defined on $\mathbb{R}^{n+m}\times\mathbb{R}^m$ with

$$\|f\|_{\mathcal{S}_M}=\sup_{x\in\mathbb{R}^n,y\in\mathbb{R}^m,z\in\mathbb{R}^m}(1+|x|+|y|+|z|)^M\sum_{|\alpha|\le M,|\beta|\le M,|\gamma|\le M}|\frac{\partial^\alpha}{\partial^\alpha x}\frac{\partial^\beta}{\partial^\beta y}\frac{\partial^\gamma}{\partial^\gamma z}f(x,y,z)|<\infty,$$

and

$$\int_{\mathbb{R}^{n+m}}f(x,y,z)x^\alpha y^\beta dxdy=\int_{\mathbb{R}^m}f(x,y,z)z^\gamma dz=0$$

for all indices $\alpha,\beta,\gamma\le M$.

Similarly, we denote $(\mathcal{S}_M(\mathbb{R}^{n+m}\times\mathbb{R}^m))'$ the dual of $\mathcal{S}_M(\mathbb{R}^{n+m}\times\mathbb{R}^m)$.

We now define the test function space $\mathcal{S}_F$ on $R^n\times R^m$ associated with the flag structure.

Definition 5.6. *A function $f(x,y)$ defined on $R^n \times R^m$ is said to be a test function in $\mathcal{S}_F(R^n \times R^m)$ if there exists a function $f^\sharp \in \mathcal{S}_\infty(R^{n+m} \times R^m)$ such that*

$$f(x,y) = \int\limits_{R^m} f^\sharp(x, y-z, z)dz.$$

If $f \in \mathcal{S}_F(R^n \times R^m)$, then the norm of f is defined by

$$\|f\|_{\mathcal{S}_F(R^n \times R^m)} = \inf\{\|f^\sharp\|_{\mathcal{S}_\infty(R^{n+m}\times R^m)} : \textit{ for all representations of } f \textit{ given above}\}.$$

We denote by $(\mathcal{S}_F)'$ the dual space of $\mathcal{S}_F$.

We also denote that a function $f(x,y)$ defined on $R^n \times R^m$ is said to be a test function in $\mathcal{S}_{F,M}(R^n \times R^m)$ if there exists a function $f^\sharp \in \mathcal{S}_M(R^{n+m} \times R^m)$ such that

$$f(x,y) = \int\limits_{R^m} f^\sharp(x, y-z, z)dz.$$

If $f \in \mathcal{S}_{F,M}(R^n \times R^m)$, then the norm of f is defined by

$$\|f\|_{\mathcal{S}_{F,M}(R^n \times R^m)} = \inf\{\|f^\sharp\|_{\mathcal{S}_M(R^{n+m}\times R^m)} : \text{ for all representations of } f \text{ given above}\}.$$

We denote by $(\mathcal{S}_{F,M})'$ the dual space of $\mathcal{S}_{F,M}$.

We would like to point out that the implicit multi-parameter structure is involved in $\mathcal{S}_F$. Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_F(R^n \times R^m)$, so the Littlewood-Paley-Stein square function g_F can be defined for all distributions in $(\mathcal{S}_F)'$. Formally, we can define the flag Hardy space as follows.

Definition 5.7. *Let $0 < p \le 1$. $H^p(R^n \times R^m) = \{f \in (\mathcal{S}_F)' : g_F(f) \in L^p(R^n \times R^m)\}$.*

If $f \in H^p(R^n \times R^m)$, the norm of f is defined by

$$\|f\|_{H^p_F} = \|g_F(f)\|_p.$$

A natural question arises whether this definition is independent of the choice of functions $\psi_{j,k}$. Moreover, to study the H^p_F-boundedness of flag singular integrals and establish the duality result of H^p_F, this formal definition is not sufficiently good. We need to discretize the norm of H^p_F. In order to obtain such a discrete H^p_F norm we will prove the Min-Max-type inequalities. The main tool to provide such inequalities is the Calderón reproducing formula given below. To be more specific, we will prove that such a formula still holds on test function space $\mathcal{S}_F(R^n \times R^m)$ and its dual space $(\mathcal{S}_F)'$. Furthermore, using an approximation procedure and the almost orthogonality argument, we prove the following discrete Calderón reproducing formula.

Theorem 5.8. *Suppose that $\psi_{j,k}$ are the same as before. Then*

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J)\psi_{j,k} * f(x_I,y_J)$$

where $\widetilde{\psi}_{j,k}(x,y,x_I,y_J) \in \mathcal{S}_{F,M}(R^n \times R^m), I \subset R^n, J \subset R^m$, *are dyadic cubes with side-length* $\ell(I) = 2^{-j-N}$ *and* $\ell(J) = 2^{-k-N} + 2^{-j-N}$ *for a fixed large integer* N, x_I, y_J *are any fixed points in* I, J, *respectively, and the above series converges in the norm of* $\mathcal{S}_{F,M}(R^n \times R^m)$ *and in the dual space* $(\mathcal{S}_{F,M})'$.

The above discrete Calderón reproducing formula provides the following Min-Max type inequalities. We use the notation $A \approx B$ to denote that two quantities A and B are comparable independent of other substantial quantities involved in the context.

Theorem 5.9. *Suppose* $\psi^{(1)}, \phi^{(1)} \in \mathcal{S}(R^{n+m}), \psi^{(2)}, \phi^{(2)} \in \mathcal{S}(R^m)$ *and*

$$\psi(x,y) = \int\limits_{R^m} \psi^{(1)}(x, y-z)\psi^{(2)}(z)dz,$$

$$\phi(x,y) = \int\limits_{R^m} \phi^{(1)}(x, y-z)\psi^{(2)}(z)dz,$$

and ψ_{jk}, ϕ_{jk} *satisfy the conditions in Theorem 5.8. Then for* $f \in (\mathcal{S}_{F,M})'$ *where* M *depends on* p *and* $0 < p < \infty$,

$$\|\{\sum_j \sum_k \sum_J \sum_I \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p$$

$$\approx \|\{\sum_j \sum_k \sum_J \sum_I \inf_{u \in I, v \in J} |\phi_{j,k} * f(u,v)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p$$

where $I \subset R^n, J \subset R^m$, *are dyadic cubes with side-length* $\ell(I) = 2^{-j-N}$ *and* $\ell(J) = 2^{-k-N} + 2^{-j-N}$ *for a fixed large integer* N, χ_I *and* χ_J *are indicator functions of* I *and* J, *respectively.*

The Min-Max type inequalities in Theorem 5.9 give the discrete Littlewood-Paley-Stein square function

$$g^d_F(f)(x,y) = \left\{ \sum_j \sum_k \sum_J \sum_I |\psi_{j,k} * f(x_I, y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}}$$

where I, J, x_I, and y_J are the same as before.

From this it is easy to see that the Hardy space H^p_F in Definition 5.7 is well defined and the H^p_F norm of f is equivalent to the L^p norm of g^d_F. By use of the Min-Max type inequalities, we will prove the boundedness of flag singular integrals on H^p_F.

Theorem 5.10. *Suppose that* T *is a flag singular integral with the kernel* $K(x,y)$ *satisfying the same conditions as in Theorem 5.4. Then* T *is bounded on* H^p_F, *for* $0 < p \le 1$. *Namely, for all* $0 < p \le 1$ *there exists a constant* C_p *such that*

$$\|T(f)\|_{H^p_F} \le C_p \|f\|_{H^p_F}.$$

To obtain the $H^p_F \to L^p$ boundedness of flag singular integrals, we prove the following general result:

Theorem 5.11. *Let $0 < p \le 1$. If T is a linear operator which is bounded on $L^2(R^{n+m})$ and $H^p_F(R^n \times R^m)$, then T can be extended to a bounded operator from $H^p_F(R^n \times R^m)$ to $L^p(R^{n+m})$.*

From the proof, we can see that this general result holds in a very broad setting, which includes the classical one-parameter and product Hardy spaces and the Hardy spaces on spaces of homogeneous type. Our method in proving this result offers an alternative approach of R. Fefferman's criterion on boundedness of a singular integral operator by restricting its action on rectangle atoms [F4], combining with Journe's geometric lemma (see [J1], [J2] and [P]).

In particular, for flag singular integral we can deduce from this general result the following

Corollary 5.12. *Let T be a flag singular integral as in Theorem 5.10. Then T is bounded from $H^p_F(R^n \times R^m)$ to $L^p(R^{n+m})$ for $0 < p \le 1$.*

To study the duality of H^p_F, we introduce the space CMO^p_F.

Definition 5.13. *Let $\psi_{j,k}$ be the same as in (5.1). We say that $f \in CMO^p_F$ if $f \in (\mathcal{S}_F)'$ and it has the finite norm $\|f\|_{CMO^p_F}$ defined by*

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \int_{\Omega} \sum_{I,J: I\times J \subseteq \Omega} |\psi_{j,k} * f(x,y)|^2 \chi_I(x)\chi_J(y) dx dy \right\}^{\frac{1}{2}}$$

for all open sets Ω in $R^n \times R^m$ with finite measures, and $I \subset R^n, J \subset R^m$, are dyadic cubes with side-length $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-k} + 2^{-j}$ respectively.

Note that the Carleson measure condition is used and the implicit multi-parameter structure is involved in CMO^p_F space. When $p = 1$, as usual, we denote by BMO_F the space CMO^1_F. To see the space CMO^p_F is well defined, one needs to show the definition of CMO^p_F is independent of the choice of the functions $\psi_{j,k}$. This can be proved, again as in the Hardy space H^p_F, by the following Min-Max type inequality.

Theorem 5.14. *Suppose $\psi_{j,k}, \phi_{j,k}$ satisfy the same condition (5.1). Then for $f \in (\mathcal{S}_{F,M})'$ where M depends on p,*

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j} \sum_{k} \sum_{I\times J \subseteq \Omega} \sup_{u\in I, v\in J} |\psi_{j,k} * f(u,v)|^2 |I||J| \right\}^{\frac{1}{2}} \approx$$

$$\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j} \sum_{k} \sum_{I\times J \subseteq \Omega} \inf_{u\in I, v\in J} |\phi_{j,k} * f(u,v)|^2 |I||J| \right\}^{\frac{1}{2}}$$

where $I \subset R^n, J \subset R^m$, are dyadic cubes with side-length $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N respectively, and Ω are all open sets in $R^n \times R^m$ with finite measures.

To show that space CMO_F^p is the dual space of H_F^p, we also need to introduce the sequence spaces.

Definition 5.15. *Let s^p be the collection of all sequences $s = \{s_{I\times J}\}$ such that*

$$\|s\|_{s^p} = \left\| \left\{ \sum_{j,k} \sum_{I,J} |s_{I\times J}|^2 |I|^{-1}|J|^{-1}\chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}} \right\|_{L^p} < \infty,$$

where the sum runs over all dyadic cubes $I \subset R^n, J \subset R^m$ with side-length $\ell(I) = 2^{-j-N}$ and $\ell(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N, and χ_I, and χ_J are indicator functions of I and J respectively.

Let c^p be the collection of all sequences $s = \{s_{I\times J}\}$ such that

$$\|s\|_{c^p} = \sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k} \sum_{I,J: I\times J \subseteq \Omega} |s_{I\times J}|^2 \right\}^{\frac{1}{2}} < \infty,$$

where Ω are all open sets in $R^n \times R^m$ with finite measures and the sum runs over all dyadic cubes $I \subset R^n, J \subset R^m$, with side-length $l(I) = 2^{-j-N}$ and $l(J) = 2^{-k-N} + 2^{-j-N}$ for a fixed large integer N.

We would like to point out again that certain dyadic rectangles used in s^p and c^p reflect the implicit multi-parameter structure. Moreover, the Carleson measure condition is used in the definition of c^p. Next, we obtain the following duality theorem.

Theorem 5.16. *Let $0 < p \le 1$. Then we have $(s^p)^* = c^p$. More precisely, the map which maps $s = \{s_{I\times J}\}$ to $< s, t > \equiv \sum_{I\times J} s_{I\times J}\bar{t}_{I\times J}$ defines a continuous linear functional on s^p with operator norm $\|t\|_{(s^p)^*} \approx \|t\|_{c^p}$, and moreover, every $\ell \in (s^p)^*$ is of this form for some $t \in c^p$.*

When $p = 1$, this theorem in the one-parameter setting on R^n was proved in [FJ]. The proof given in [FJ] depends on estimates of certain distribution functions, which seems to be difficult to apply to the multi-parameter case. For all $0 < p \le 1$ we give a simple and more constructive proof of Theorem 5.8, which uses the stopping time argument for sequence spaces. Theorem 5.8 together with the discrete Calderón reproducing formula and the Min-Max type inequalities yields the duality of H_F^p.

Theorem 5.17. *Let $0 < p \le 1$. Then $(H_F^p)^* = CMO_F^p$. More precisely, if $g \in CMO_F^p$, the map ℓ_g given by $\ell_g(f) =< f, g >$, defined initially for $f \in \mathcal{S}_F$, extends to a continuous linear functional on H_F^p with $\|\ell_g\| \approx \|g\|_{CMO_F^p}$. Conversely, for every $\ell \in (H_F^p)^*$ there exists some $g \in CMO_F^p$ so that $\ell = \ell_g$. In particular, $(H_F^1)^* = BMO_F$.*

As a consequence of the duality of H_F^1 and the H_F^1-boundedness of flag singular integrals, we obtain the BMO_F-boundedness of flag singular integrals. Furthermore, we will see that $L^\infty \subseteq BMO_F$ and, hence, the $L^\infty \to BMO_F$ boundedness of flag singular integrals is also obtained. These provide the endpoint results of those in [MRS1] and [NRS]. These can be summarized as follows:

Theorem 5.18. *Suppose that T is a flag singular integral as in Theorem 5.2. Then T is bounded on BMO_F. Moreover, there exists a constant C such that*

$$\|T(f)\|_{BMO_F} \leq C\|f\|_{BMO_F}.$$

Next we have the Calderón-Zygmund decomposition and interpolation theorems on the flag Hardy spaces. We note that $H^p(R^n \times R^m) = L^p(R^{n+m})$ for $1 < p < \infty$.

Theorem 5.19. *(Calderón-Zygmund decomposition for flag Hardy spaces) Let $0 < p_2 \leq 1, p_2 < p < p_1 < \infty$ and let $\alpha > 0$ be given and $f \in H^p(R^n \times R^m)$. Then we may write $f = g + b$ where $g \in H_F^{p_1}(R^n \times R^m)$ with $p < p_1 < \infty$ and $b \in H_F^{p_2}(R^n \times R^m)$ with $0 < p_2 < p$ such that $||g||^{p_1}_{H_F^{p_1}} \leq C\alpha^{p_1-p}||f||^p_{H_F^p}$ and $||b||^{p_2}_{H_F^{p_2}} \leq C\alpha^{p_2-p}||f||^p_{H_F^p}$, where C is an absolute constant.*

Theorem 5.20. *(Interpolation theorem on flag Hardy spaces) Let $0 < p_2 < p_1 < \infty$ and T be a linear operator which is bounded from $H_F^{p_2}$ to L^{p_2} and bounded from $H_F^{p_1}$ to L^{p_1}, then T is bounded from H_F^p to L^p for all $p_2 < p < p_1$. Similarly, if T is bounded on $H_F^{p_2}$ and $H_F^{p_1}$, then T is bounded on H_F^p for all $p_2 < p < p_1$.*

We point out that the Calderón-Zygmund decomposition in pure product domains for all L^p functions $(1 < p < 2)$ into H^1 and L^2 functions and interpolation theorem was established by Chang and R. Fefferman ([CF1], [CF2]) (see for more precise statement in Section 6).

We end the introduction of this subsection with the following remarks. As we can see from the definition of flag kernels, the regularity satisfied by flag singular kernels is better than that of the product singular kernels. It is thus natural to conjecture that the Hardy space associated with flag singular integrals should be larger than the classical pure product Hardy space. This is indeed the case. In fact, if we define the flag kernel on $R^n \times R^m$ by

$$K(x, y) = \int_{R^n} \tilde{\tilde{K}}(x - z, z, y)dz,$$

where $\tilde{\tilde{K}}(x, z, y)$ is a pure product kernel on $R^n \times R^{n+m}$, and let $\widetilde{H_F^p}$ be the flag Hardy space associated with this structure, thus we have shown in a forthcoming paper that $H^p(R^n \times R^m) = H_F^p(R^n \times R^m) \cap \widetilde{H_F^p}(R^n \times R^m)$. Results in [MRS1] and [NRS] together with those in this section demonstrate that the implicit multi-parameter structure, the geometric property of sets of singularities and regularities of singular kernels and multipliers are closely related.

5.2 Test function spaces, almost orthogonality estimates and discrete Calderón reproducing formula

In this section, we develop the discrete Calderón reproducing formula and the Min-Max inequalities on test function spaces. These are crucial tools in establishing the theory of Hardy spaces associated with the flag type multi-parameter dilation structure. The key ideas to provide the discrete Calderón reproducing formula and the Min-Max-Pôlya-type inequalities are the continuous version of the Calderón reproducing formula on test function spaces and the almost orthogonality estimates.

If $\psi^\sharp(x,y,z,u,v,w)$ for $(x,y,z),(u,v,w) \in R^n \times R^m \times R^m$ is a smooth function and satisfies the differential inequalities

$$|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\partial_u^{\alpha_2}\partial_v^{\beta_2}\partial_w^{\gamma_2}\psi^\sharp(x,y,z,u,v,w)|$$
$$\le A_{N,M,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2}(1+|x-u|+|y-v|)^{-N}(1+|z-w|)^{-M}$$

and the cancellation conditions

$$\int \psi^\sharp(x,y,z,u,v,w)x^{\alpha_1}y^{\beta_1}dxdy = \int \psi^\sharp(x,y,z,u,v,w)z^{\gamma_1}dz$$
$$= \int \psi^\sharp(x,y,z,u,v,w)u^{\alpha_2}v^{\beta_2}dudv = \int \psi^\sharp(x,y,z,u,v,w)w^{\gamma_2}dw = 0,$$

and for fixed $x_0 \in R^n, y_0 \in R^m, \phi^\sharp(x,y,z,x_0,y_0) \in \mathcal{S}_\infty(R^{n+m}\times R^m)$ and satisfies

$$|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\phi^\sharp(x,y,z,x_0,y_0)|$$
$$\le B_{N,M,\alpha_1,\beta_1,\gamma_1,}(1+|x-x_0|+|y-y_0|)^{-N}(1+|z|)^{-M},$$

for all positive integers N,M and multi-indices $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ of nonnegative integers. Then we have the following almost orthogonality estimate:

Lemma 5.21. *For any given positive integers L_1,L_2 and K_1,K_2, there exists a constant $C = C(L_1,L_2,K_1,K_2)$ depending only on L_1,L_2,K_1,K_2 and the constants given above such that for all positive t,s,t',s' we have*

$$|\int\limits_{R^{n+m+m}} \psi^\sharp_{t,s}(x,y,z,u,v,w)\phi^\sharp_{t',s'}(u,v,w,x_0,y_0)dudvdw|$$
$$\le C(\frac{t}{t'}\wedge\frac{t'}{t})^{L_1}(\frac{s}{s'}\wedge\frac{s'}{s})^{L_2}\frac{(t\vee t')^{K_1}}{(t\vee t'+|x-x_0|+|y-y_0|)^{(n+m+K_1)}}\frac{(s\vee s')^{K_2}}{(s\vee s'+|z|)^{(m+K_2)}},$$

where $\psi^\sharp_{t,s}(x,y,z,u,v,w) = t^{-n-m}s^{-m}\psi^\sharp(\frac{x}{t},\frac{y}{t},\frac{z}{s},\frac{u}{t},\frac{v}{t},\frac{w}{s})$ *and*

$$\phi^\sharp_{t,s}(x,y,z,x_0,y_0) = t^{-n-m}s^{-m}\phi^\sharp(\frac{x}{t},\frac{y}{t},\frac{z}{s},\frac{x_0}{t},\frac{y_0}{t},).$$

Lemma 5.22. *Let $\psi,\phi \in \mathcal{S}_F(R^n\times R^m)$, and $\psi^\sharp,\phi^\sharp \in \mathcal{S}_\infty(R^{n+m}\times R^m)$ such that*

$$\psi(x,y) = \int_{R^m}\psi^\sharp(x,y-z,z)dz,\ \phi(x,y) = \int_{R^m}\phi^\sharp(x,y-z,z)dz.$$

Then

$$(\psi*\phi)(x,y) = \int_{R^m}\left(\psi^\sharp*\phi^\sharp\right)(x,y-z,z)dz.$$

Lemma 5.22 can be proved very easily. Using this lemma and the almost orthogonality estimates on $R^{n+m}\times R^m$, we can get the following

Lemma 5.23. *For any given positive integers L_1, L_2 and K_1, K_2, there exists a constant $C = C(L_1, L_2, K_1, K_2)$ depending only on L_1, L_2, K_1, K_2 such that if $t \vee t' \leq s \vee s'$, then*

$$|\psi_{t,s} * \phi_{t',s'}(x,y)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}},$$

and if $t \vee t' \geq s \vee s'$, then

$$|\psi_{t,s} * \phi_{t',s'}(x,y)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \frac{(t \vee t')^{K_2}}{(t \vee t' + |y|)^{(m+K_2)}}.$$

We can use these estimates to prove the following continuous version of the Calderón reproducing formula on test function space $\mathcal{S}_F(R^n \times R^m)$ and its dual space $(\mathcal{S}_F)'$.

Theorem 5.24. *Suppose that $\psi_{j,k}$ are the same as in Lemma 5.1. Then*

$$f(x,y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x,y), \tag{5.2}$$

where the series converges in the norm of $\mathcal{S}_F$ and in dual space $(\mathcal{S}_F)'$.

Proof: Suppose $f \in \mathcal{S}_F$ and $f(x,y) = \int_{R^m} f^\sharp(x, y-z, z)dz$, where $f^\sharp \in \mathcal{S}_\infty(R^{n+m} \times R^m)$. Then, by the classical Calderón reproducing formula as mentioned in the first section, for all $f^\sharp \in L^2$,

$$f^\sharp(x,y,z) = \sum_j \sum_k \psi^\sharp_{j,k} * \psi^\sharp_{j,k} * f^\sharp(x,y,z),$$

where $\psi^\sharp_{j,k}(x,y,z) = \psi^{(1)}_j(x,y)\psi^{(2)}_k(z)$.

We **claim** that the above series converges in $\mathcal{S}_\infty(R^{n+m} \times R^m)$. This claim yields

$$\begin{aligned} &\|f(x,y) - \sum_{-N \leq j \leq N} \sum_{-M \leq k \leq M} \psi_{j,k} * \psi_{j,k} * f(x,y)\|_{\mathcal{S}_F} \\ &= \|\int_{R^m} [f^\sharp(x, y-z, z) - \sum_{-N \leq j \leq N} \sum_{-M \leq k \leq M} \psi^\sharp_{j,k} * \psi^\sharp_{j,k} * f^\sharp(x, y-z, z)]dz\|_{\mathcal{S}_F} \\ &\leq \|f^\sharp(x,y,z) - \sum_{-N \leq j \leq N} \sum_{-M \leq k \leq M} \psi^\sharp_{j,k} * \psi^\sharp_{j,k} * f^\sharp(x,y,z)\|_{\mathcal{S}_\infty} \end{aligned}$$

where the last term above goes to zero as N and M tend to infinity by the above claim.

The convergence in dual space follows from the duality argument. The proof of Theorem 5.13 is complete. **Q.E.D.**

Using Theorem 5.24, we prove the discrete Calderón reproducing formula.

Proof of Theorem 5.8: We first discretize (5.2) as follows. For $f \in \mathcal{S}_F$, by (5.2) and using an idea similar to that of decomposition of the identity operator due to Coifman, we can rewrite

$$f(x,y) = \sum_{j,k} \sum_{I,J} \int_J \int_I \psi_{j,k}(x-u, y-w) \left(\psi_{j,k} * f\right)(u,w) du dw$$

$$= \sum_{j,k}\sum_{I,J}\left[\int_J\int_I \psi_{j,k}(x-u,y-w)dudw\right]\left(\psi_{j,k}*f\right)(x_I,y_J)+\mathcal{R}(f)(x,y).$$

We shall show that $\mathcal{R}$ is bounded on $\mathcal{S}_F$ with the small norm when I and J are dyadic cubes in R^n and R^m with side length 2^{-j-N} and $2^{-k-N}+2^{-j-N}$ for a large given integer N, and x_I, y_J are any fixed points in I, J, respectively.

We now set

$$\begin{aligned}
&\mathcal{R}(f)(x,y)\\
=&\sum_{j,k}\sum_{I,J}\int_J\int_I \psi_{j,k}(x-u,y-w)\left[\left(\psi_{j,k}*f\right)(u,w)-\left(\psi_{j,k}*f\right)(x_I,y_J)\right]dudw\\
=&\int\int\int\int \mathcal{R}^\sharp(x,y-z,z,u',v',w')f^\sharp(u',v',w')du'dv'dw'dz\\
=&\int_{R^m}\mathcal{R}^\sharp(f^\sharp)(x,y-z,z)dz,
\end{aligned}$$

where $\mathcal{R}^\sharp(x,y,z,u',v',w')$ is the kernel of $\mathcal{R}^\sharp$.

Thus we can show that for any M, $\mathcal{R}^\sharp(f^\sharp)(x,y,z)\in\mathcal{S}_M(R^{n+m}\times R^m)$ and

$$||\mathcal{R}^\sharp(f^\sharp)||_{\mathcal{S}_M(R^{n+m}\times R^m)}\le C2^{-N}||f^\sharp||_{\mathcal{S}_M(R^{n+m}\times R^m)},$$

which implies

$$||\mathcal{R}(f)||\le C2^{-N}||f||.$$

Details can be found in [HL3].

Using the boundedness of $\mathcal{R}$ on $\mathcal{S}_F$ with the norm at most $C2^{-N}$, if N is chosen large enough, then we obtain

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I\left[\sum_{i=0}^{\infty}\mathcal{R}^i\int_J\int_I\psi_{j,k}(\cdot-u,\cdot-v)dudv\right](x,y)\left(\psi_{j,k}*f\right)(x_I,y_J).$$

Set

$$\left[\sum_{i=0}^{\infty}\mathcal{R}^i\int_J\int_I\psi_{j,k}(\cdot-u,\cdot-v)dudv\right](x,y)=|I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J).$$

It remains to show $\widetilde{\psi}_{j,k}(x,y,x_I,y_J)\in\mathcal{S}_{F,M}$. This, however, follows easily.

Q.E.D

Remark 5.25. *If we begin with discretizing (5.2)) by*

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I\psi_{j,k}(x-x_I,y-y_J)\int_J\int_I\left(\psi_{j,k}*f\right)(u,v)dudv+\widetilde{\widetilde{\mathcal{R}}}(f)(x,y),$$

and repeating the similar proof, then the discrete Calderón reproducing formula can also be given by the following form

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I |I||J|\psi_{j,k}(x-x_I,y-y_J)\widetilde{\widetilde{\psi}}_{j,k}(f)(x_I,y_J),$$

where $|I||J|\widetilde{\widetilde{\psi}}_{j,k}(f)(x_I,y_J)=\sum\limits_{i=0}^{\infty}\int\limits_J\int\limits_I \psi_{j,k}*(\widetilde{\widetilde{\mathcal{R}}})^i(f)(u,v)dudv$. *We leave the details of these proofs to the reader.*

Before we prove the Min-Max type inequality, we first prove the following lemma.

Lemma 5.26. *Let* $I,I',\ J,J'$ *be dyadic cubes in* R^n *and* R^m *respectively such that* $\ell(I)=2^{-j-N}$, $\ell(J)=2^{-j-N}+2^{-k-N}$, $\ell(I')=2^{-j'-N}$ *and* $\ell(J')=2^{-j'-N}+2^{-k'-N}$. *Thus for any* $u,u^*\in I$ *and* $v,v^*\in J$, *we have when* $j\wedge j'\geq k\wedge k'$

$$\sum_{I',J'}\frac{2^{-|j-j'|L_1-|k-k'|L_2}2^{-(j\wedge j')K_1-(k\wedge k')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{n+K_1}(2^{-k\wedge k'}+|v-y_{J'}|)^{m+K_2}}|\phi_{j',k'}*f(x_{I'},y_{J'})|$$

$$\leq C_1(N,r,j,j',k,k')2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2}\left\{M_s\left[\left(\sum_{J'}\sum_{I'}|\phi_{j',k'}*f(x_{I'},y_{J'})|\chi_{J'}\chi_{I'}\right)^r\right]\right\}^{\frac{1}{r}}(u^*,v^*)$$

and when $j\wedge j'\leq k\wedge k'$

$$\sum_{I',J'}\frac{2^{-|j-j'|L_1-|k-k'|L_2}2^{-(j\wedge j')K_1-(j\wedge j')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{n+K_1}(2^{-j\wedge j'}+|v-y_{J'}|)^{m+K_2}}|\phi_{j',k'}*f(x_{I'},y_{J'})|$$

$$\leq C_2(N,r,j,j',k,k')2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2}\left\{M\left[\left(\sum_{J'}\sum_{I'}|\phi_{j',k'}*f(x_{I'},y_{J'})|\chi_{J'}\chi_{I'}\right)^r\right]\right\}^{\frac{1}{r}}(u^*,v^*)$$

where M *is the Hardy-Littlewood maximal function on* R^{n+m}, M_s *is the strong maximal function on* $R^n\times R^m$, *and* $\max\left\{\frac{n}{n+K_1},\frac{m}{m+K_2}\right\}<r$ *and*

$$C_1(N,r,j,j',k,k')=2^{(\frac{1}{r}-1)N(n+m)}\cdot 2^{[n(j\wedge j'-j')+m(k\wedge k'-k')](1-\frac{1}{r})}$$

$$C_2(N,r,j,j',k,k')=2^{(\frac{1}{r}-1)N(n+m)}\cdot 2^{[n(j\wedge j'-j')+m(j\wedge j'-j'\wedge k')](1-\frac{1}{r})}.$$

We now are ready to give the

Proof of Theorem 5.9: By Theorem 5.8, $f\in\mathcal{S}_{F,M}$ can be represented by

$$f(x,y)=\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|J'||I'|\widetilde{\phi}_{j',k'}(x,y,x_{I'},y_{J'})\left(\phi_{j',k'}*f\right)(x_{I'},y_{J'}).$$

We write

$$\begin{aligned}&\left(\psi_{j,k}*f\right)(u,v)\\ =&\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|I'||J'|\left(\psi_{j,k}*\widetilde{\phi}_{j',k'}(\cdot,\cdot,x_{I'},y_{J'})\right)(u,v)\left(\phi_{j',k'}*f\right)(x_{I'},y_{J'}).\end{aligned}$$

Using the almost orthogonality estimates by choosing $t = 2^{-j}$, $s = 2^{-k}$, $t' = 2^{-j'}$, $s' = 2^{-k'}$, and for any given positive integers L_1, L_2, K_1, K_2 we have if $j \wedge j' \geq k \wedge k'$,

$$
\begin{aligned}
&|\left(\psi_{j,k} * \widetilde{\phi}_{j',k'}(\cdot,\cdot,x_{I'},y_{J'})\right)(u,v)| \\
&\leq \frac{2^{-|j-j'|L_1-|k-k'|L_2}2^{-(j\wedge j')K_1-(k\wedge k')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{n+K_1}(2^{-k\wedge k'}+|v-y_{J'}|)^{m+K_2}}|\phi_{j',k'} * f(x_{I'},y_{J'})|
\end{aligned}
$$

and when $j \wedge j' \leq k \wedge k'$ we have

$$
\begin{aligned}
&|\left(\psi_{j,k} * \widetilde{\phi}_{j',k'}(\cdot,\cdot,x_{I'},y_{J'})\right)(u,v)| \\
&\leq \frac{2^{-|j-j'|L_1-|k-k'|L_2}2^{-(j\wedge j')K_1-(j\wedge j')K_2}|I'||J'|}{(2^{-j\wedge j'}+|u-x_{I'}|)^{n+K_1}(2^{-j\wedge j'}+|v-y_{J'}|)^{m+K_2}}|\phi_{j',k'} * f(x_{I'},y_{J'})|
\end{aligned}
$$

Using Lemma 5.26 for any $u, u^* \in I$, $x_{I'} \in I'$, $v, v^* \in J$ and $y_{J'} \in J'$, we have

$$
\begin{aligned}
&|\psi_{j,k} * f(u,v)| \\
&\leq C_1 \sum_{j',k':j\wedge j'\geq k\wedge k'} 2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2}\left\{M_s\left[\left(\sum_{J'}\sum_{I'}|\phi_{j',k'} * f(x_{I'},y_{J'})|\chi_{J'}\chi_{I'}\right)^r\right]\right\}^{\frac{1}{r}}(u^*,v^*) \\
&+ C_2 \sum_{j',k':j\wedge j'\leq k\wedge k'} 2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2}\left\{M\left[\left(\sum_{J'}\sum_{I'}|\phi_{j',k'} * f(x_{I'},y_{J'})|\chi_{J'}\chi_{I'}\right)^r\right]\right\}^{\frac{1}{r}}(u^*,v^*) \\
&\leq C\sum_{j',k'} 2^{-|j-j'|L_1}\cdot 2^{-|k-k'|L_2}\left\{M_s\left[\left(\sum_{J'}\sum_{I'}|\phi_{j',k'} * f(x_{I'},y_{J'})|\chi_{J'}\chi_{I'}\right)^r\right]\right\}^{\frac{1}{r}}(u^*,v^*)
\end{aligned}
$$

where M is the Hardy-Littlewood maximal function on R^{n+m}, M_s is the strong maximal function on $R^n \times R^m$, and $\max\{\frac{n}{n+K_1}, \frac{m}{m+K_2}\} < r < p$.

Applying the Holder's inequality and summing over j, k, I, J yields

$$
\begin{aligned}
&\left\{\sum_{j,k}\sum_{I,J}\sup_{u\in I, v\in J}|\psi_{j,k} * f(u,v)|^2\chi_I\chi_J\right\}^{\frac{1}{2}} \\
&\leq C\left\{\sum_{j',k'}\left\{M_s(\sum_{I',J'}|\phi_{j',k'} * f(x_{I'},y_{J'})|\chi_{I'}\chi_{J'})^r\right\}^{\frac{2}{r}}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Since $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J', respectively, we have

$$
\begin{aligned}
&\left\{\sum_{j,k}\sum_{I,J}\sup_{u\in I, v\in J}|\psi_{j,k} * f(u,v)|^2\chi_I\chi_J\right\}^{\frac{1}{2}} \\
&\leq C\left\{\sum_{j',k'}\left\{M_s(\sum_{I',J'}\inf_{u\in I', v\in J'}|\phi_{j',k'} * f(u,v)|\chi_{I'}\chi_{J'})^r\right\}^{\frac{2}{r}}\right\}^{\frac{1}{2}},
\end{aligned}
$$

and hence, by the Fefferman-Stein vector-valued maximal function inequality [FS1] with $r < p$, we get

$$\| \left\{ \sum_j \sum_k \sum_J \sum_I \sup_{u \in I, v \in J} |\psi_{j,k} * f(u,v)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \|_p$$
$$\leq C \| \left\{ \sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} \inf_{u \in I', v \in J'} |\phi_{j',k'} * f(u,v)|^2 \chi_{I'} \chi_{J'} \right\}^{\frac{1}{2}} \|_p.$$

This ends the proof of Theorem 5.9. **Q.E.D.**

5.3 Discrete Littlewood-Paley-Stein square function, boundedness of flag singular integrals on Hardy spaces H_F^p, from H_F^p to L^p

The main purpose of this section is to establish the Hardy space theory associated with the flag multi-parameter structure using the results we have proved in the previous subsections. As a consequence of Theorem 5.9, it is easy to see that the Hardy space H_F^p is independent of the choice of the functions ψ. Moreover, we have the following characterization of H_F^p using the discrete norm.

Proposition 5.27. *Let $0 < p \leq 1$. Then we have*

$$\|f\|_{H_F^p} \approx \| \left\{ \sum_j \sum_k \sum_J \sum_I |\psi_{j,k} * f(x_I, y_J)|^2 \chi_I(x) \chi_J(y) \right\}^{\frac{1}{2}} \|_p$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are same as in Theorem 5.9.

Before we give the proof of the boundedness of flag singular integrals on H_F^p, we show several properties of H_F^p.

Proposition 5.28. *$\mathcal{S}_F(R^n \times R^m)$ is dense in H_F^p.*

Theorem 5.29. *If $f \in L^2(R^{n+m}) \cap H^p(R^n \times R^m), 0 < p \leq 1$, then $f \in L^p(R^{n+m})$ and there is a constant $C_p > 0$ which is independent of the L^2 norm of f such that*

$$\|f\|_p \leq C \|f\|_{H_F^p}.$$

To show theorem 5.29, we need a discrete Calderón reproducing formula on $L^2(R^{n+m})$. To be more precise, take $\phi^{(1)} \in C_0^\infty(R^{n+m})$ with

$$\int_{R^{n+m}} \phi^{(1)}(x,y) x^\alpha y^\beta dxdy = 0, \text{ for all } \alpha, \beta \text{ satisfying } 0 \leq |\alpha| \leq M_0,\ 0 \leq |\beta| \leq M_0, \tag{5.3}$$

where M_0 is a large positive integer which will be determined later, and

$$\sum_j |\widehat{\phi^{(1)}}(2^{-j}\xi_1, 2^{-j}\xi_2)|^2 = 1, \text{ for all } (\xi_1, \xi_2) \in R^{n+m} \backslash \{(0,0)\},$$

and take $\phi^{(2)} \in C_0^\infty(R^m)$ with

$$\int_{R^m} \phi^{(2)}(z) z^\gamma dz = 0 \text{ for all } 0 \leq |\gamma| \leq M_0,$$

and $\sum_k |\widehat{\phi^{(2)}}(2^{-k}\xi_2)^2 = 1$ for all $\xi_2 \in R^m \backslash \{0\}$.

Furthermore, we may assume that $\phi^{(1)}$ and $\phi^{(2)}$ are radial functions and supported in the unit balls of R^{n+m} and R^m respectively. Set again

$$\phi_{jk}(x,y) = \int_{R^m} \phi_j^{(1)}(x, y-z)\phi_k^{(2)}(z)dz.$$

By taking the Fourier transform, it is easy to see the following continuous version of Calderón reproducing formula on L^2: for $f \in L^2(R^{n+m})$,

$$f(x,y) = \sum_j \sum_k \phi_{jk} * \phi_{jk} * f(x,y).$$

For our purpose, we need the discrete version of the above reproducing formula.

Theorem 5.30. *There exist functions $\widetilde{\phi}_{jk}$ and an operator T_N^{-1} such that*

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J| \widetilde{\phi}_{j,k}(x - x_I, y - y_J) \phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

where functions $\widetilde{\phi}_{jk}(x - x_I, y - y_J)$ satisfy the conditions in (5.3) with $\alpha_1, \beta_1, \gamma_1, N, M$ depending on M_0, $x_0 = x_I$ and $y_0 = y_J$. Moreover, T_N^{-1} is bounded on $L^2(R^{n+m})$ and $H^p(R^n \times R^m)$, and the series converges in $L^2(R^{n+m})$.

Remark 5.31. *The difference between Theorem 5.8 and Theorem 5.30 are that our $\widetilde{\phi}_{jk}$ in Theorem 5.30 has compact support. The price we pay here is that $\widetilde{\phi}_{jk}$ only satisfies the moment condition of finite order, unlike that in Theorem 5.8 where the moment condition of infinite order is satisfied. Moreover, the formula in Theorem 5.30 only holds on $L^2(R^{n+m})$ while the formula in Theorem 5.8 holds in test function space $\mathcal{S}_{F,M}$ and its dual space $(\mathcal{S}_{F,M})'$.*

Proof of Theorem 5.30: Following the proof of Theorem 5.8, we have

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I [\int_J \int_I \phi_{j,k}(x-u, y-v)dudv] \left(\phi_{j,k} * f\right)(x_I, y_J) + \mathcal{R}(f)(x,y).$$

where I, J, j, k and $\mathcal{R}$ are the same as in Theorem 5.8.

Thus,

Lemma 5.32. *Let $0 < p \leq 1$. Then the operator $\mathcal{R}$ is bounded on $L^2(R^{n+m}) \cap H^p(R^n \times R^m)$ whenever M_0 is chosen to be a large positive integer. Moreover, there exists a constant $C > 0$ such that*

$$||\mathcal{R}(f)||_2 \leq C 2^{-N} ||f||_2$$

and

$$||\mathcal{R}(f)||_{H^p(R^n \times R^m)} \leq C 2^{-N} ||f||_{H^p(R^n \times R^m)}.$$

Proof of Lemma 5.32: Following the proofs of Theorems 5.8 and 5.9 and using the discrete Calderón reproducing formula for $f \in L^2(R^{n+m})$, we have

$$||g_F(\mathcal{R}(f))||_p$$

$$\leq \| \left\{ \sum_j \sum_k \sum_J \sum_I |\left(\psi_{j,k} * \mathcal{R}(f)\right)|^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \|_p$$

$$= \left\| \left\{ \sum_{j,k,J,I} \sum_{j',k',J',I'} |J'||I'|| \left(\psi_{j,k} * \mathcal{R}\left(\widetilde{\psi_{j',k'}}(\cdot, x_{I'}, \cdot, y_{J'}) \cdot \psi_{j'k'} * f(x_{I'}, y_{J'})\right)\right) |^2 \chi_I \chi_J \right\}^{\frac{1}{2}} \right\|_p$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are the same as in Theorem 5.9.

We **claim:**

$$| \left(\psi_{j,k} * \mathcal{R}\left(\widetilde{\psi}_{j',k'}(\cdot, x_{I'}, \cdot, y_{J'})\right)\right)(x, y)|$$

$$\leq C2^{-N} 2^{-|j-j'|K} 2^{-|k-k'|K} \cdot \int_{R^m} \frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')} + |x - x_{I'}| + |y - z - y_{J'})^{n+m+K}} \frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')} + |z|)^{m+K}} dz$$

where we have chosen for simplicity $L_1 = L_2 = K_1 = K_2 = K < M_0, max(\frac{n}{n+K}, \frac{m}{m+K}) < p$, and M_0 is chosen to be a lager integer later.

Assuming the claim for the moment, repeating a similar proof in Lemma 5.26 and then Theorem 5.9, we obtain

$$\||g_F(\mathcal{R}f)\|_p \leq C2^{-N} \|\{\sum_{j'} \sum_{k'} \{M_s(\sum_{J'} \sum_{I'} |\psi_{j',k'} * f(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'})^r\}^{\frac{2}{r}}\}^{\frac{1}{2}}\|_p$$

$$\leq C2^{-N} \|\{\sum_{j'} \sum_{k'} \sum_{J'} \sum_{I'} |\psi_{j',k'} * f(x_{I'}, y_{J'})|^2 \chi_{I'}\chi_{J'}\}^{\frac{1}{2}}\|_p \leq C2^{-N}\|f\|_{H^p_F(R^n\times R^m)}.$$

It is clear that the above estimates still hold when p is replaced by 2. These imply the assertion of Lemma 5.32.

We now prove the Claim. Again, by the proof of Theorem 5.8,

$$\mathcal{R}\left(\widetilde{\psi}_{j',k'}(\cdot, x_{I'}, \cdot, y_{J'})\right)(x, y) = \int_{R^m} \mathcal{R}^\sharp(x, y - z, z, u', v', w')\widetilde{\psi}_{j',k'}(\cdot, x_{I'}, \cdot, y_{J'}) du' dv' dw' dz$$

where $\mathcal{R}^\sharp(x, y, z, u', v', w')$ is similar to $\mathcal{R}^\sharp$ as given in the proof of Theorem 5.8 but, as we pointed out in Remark 5.31, that the difference between $\mathcal{R}^\sharp$ here and $\mathcal{R}^\sharp$ given in the proof of Theorem 5.8 is the moment conditions. However, the almost orthogonality estimate still holds if we only require sufficiently high order of moment conditions. More precisely, if we replace the moment conditions in (5.3) "for all $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$" by "for all $|\alpha_1|, |\beta_1|, |\gamma_1|, |\alpha_2|, |\beta_2|, |\gamma_2| \leq M_0$ where M_0 is a large integer, then the orthogonal estimate still holds with L_1, L_2, K_1, K_2 depending on M_0. Thus, the claim follows by applying the same proof as that of Theorem 5.8, and the proof of Lemma 5.32 is complete. **Q. E. D.**

We now return to the proof of Theorem 5.30.

Denote $(T_N)^{-1} = \sum_{i=1}^{\infty} \mathcal{R}^i$, where

$$T_N(f) = \sum_j \sum_k \sum_J \sum_I [\int_J \int_I \phi_{j,k}(x-u, y-v)dudvd]\,(\phi_{j,k} * f)\,(x_I, y_J).$$

Lemma 4.5 shows that if N is large enough, then both of T_N and $(T_N)^{-1}$ are bounded on $L^2(R^{n+m}) \cap H^p(R^n \times R^m)$. Hence, we can get the following reproducing formula

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J|\widetilde{\phi}_{j,k}(x-x_I, y-y_J)\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

where $\widetilde{\phi}_{jk}(x-x_I, y-y_J) = \frac{1}{|I|}\frac{1}{|J|}\int_J\int_I \phi_{jk}(x-x_I-(u-x_I), y-y_J-(v-y_J))dudv$ satisfies the estimate in (5.3) and the series converges in $L^2(R^{n+m})$.

This completes the proof of Theorem 5.30. **Q.E.D.**

As a consequence of Theorem 5.30, we obtain the following

Corollary 5.33. *If $f \in L^2(R^{n+m}) \cap H^p(R^n \times R^m)$ and $0 < p \leq 1$, then*

$$\|f\|_{H_F^p} \approx \|\{(\sum_j \sum_k \sum_J \sum_I |\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2 \chi_I(x)\chi_J(y)\})^{\frac{1}{2}}\|_p$$

where the constants are independent of the L^2 norm of f.

To see the proof of Corollary 5.33, note that if $f \in L^2(R^{n+m})$, one can apply the Calderoń reproducing formulas in Theorem 5.8 and 5.30 and then repeat the same proof as in Theorem 5.9. We leave the details to the reader.

As a consequence of Theorem 5.29, we have the following

Corollary 5.34. *$H_F^1(R^n \times R^m)$ is a subspace of $L^1(R^n \times R^m)$.*

Proof: Given $f \in H_F^1(R^{n+m})$, by Proposition 5.28, there is a sequence $\{f_n\}$ such that $f_n \in L^2(R^{n+m}) \cap H_F^1(R^{n+m})$ and f_n converges to f in the norm of $H_F^1(R^{n+m})$. By Theorem 5.29, f_n converges to g in $L^1(R^{n+m})$ for some $g \in L^1(R^{n+m})$. Therefore, $f = g$ in $(\mathcal{S}_F)'$. **Q.E.D.**

We now turn to the

Proof of Theorem 5.10: We assume that K is the kernel of T. Applying the discrete Calderón reproducing formula in Theorem 5.30 implies that for $f \in L^2(R^{n+m}) \cap H^p(R^n \times R^m)$,

$$\begin{aligned}&\|\{\sum_{j,k}\sum_{I,J} |\phi_{j,k} * K * f(x,y)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p \\ =&\|\{\sum_{j,k}\sum_{I,J} |\sum_{j',k'}\sum_{I',J'} |J'||I'|\phi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y)\phi_{j',k'} * \left(T_N^{-1}(f)\right)(x_{I'}, y_{J'})|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_p,\end{aligned}$$

where the discrete Calderón reproducing formula in $L^2(R^{n+m})$ is used.

Note that ϕ_{jk} are dilations of bump functions, by estimates similar to the those of orthogonal estimates, one can easily check that

$$|\phi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y)| \le C2^{-|j-j'|K}2^{-|k-k'|K}$$

$$\int_{R^m} \frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')} + |x - x_{I'}| + |y - z - y_{J'}|)^{n+m+K}} \cdot \frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')} + |z|)^{m+K}} dz,$$

where K depends on M_0 given in Theorem 5.30 and M_0 is chosen to be large enough. Repeating a similar proof in Theorem 5.9 together with Corollary 5.33, we obtain

$$\|Tf\|_{H^p_F} \le C\|\{\sum_{j'}\sum_{k'}\{M_s(\sum_{J'}\sum_{I'}|\phi_{j',k'} * \left(T_N^{-1}(f)\right)(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'})^r\}^{\frac{2}{r}}(x,y)\}^{\frac{1}{2}}\|_p$$

$$\le C\|\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|\phi_{j',k'} * \left(T_N^{-1}(f)\right)(x_{I'}, y_{J'})|^2\chi_{J'}(y)\chi_{I'}(x)\}^{\frac{1}{2}}\|_p \le C\|f\|_{H^p_F},$$

where the last inequality follows from Corollary 5.33.

Since $L^2(R^{n+m})$ is dense in $H^p_F(R^n \times R^m)$, T can extend to a bounded operator on $H^p_F(R^n \times R^m)$. This ends the proof of Theorem 5.10.

Proof of H^p_F to L^p boundedness We note that $H^p_F \cap L^2$ is dense in H^p_F, so we only have to show this for $f \in H^p_F \cap L^2$. This follows from Theorems 5.29 and 5.10 immediately. **Q.E.D.**

5.4 Calderón-Zygmund decomposition and interpolation on flag Hardy spaces $H^p(R^n \times R^m)$

The main purpose of this section is to derive a Calderón-Zygmund decomposition using functions in flag Hardy spaces. Furthermore, we will prove an interpolation theorem on $H^p(R^n \times R^m)$.

We first recall that Chang and R. Fefferman established the following Calderón-Zygmund decomposition on the pure product domains $R^2_+ \times R^2_+$ ([CF2]).

Calderón-Zygmund Lemma: Let $\alpha > 0$ be given and $f \in L^p(R^2)$, $1 < p < 2$. Then we may write $f = g + b$ where $g \in L^2(R^2)$ and $b \in H^1(R^2_+ \times R^2_+)$ with $||g||_2^2 \le \alpha^{2-p}||f||_p^p$ and $||b||_{H^1(R^2_+\times R^2_+)} \le C\alpha^{1-p}||f||_p^p$, where c is an absolute constant.

We now prove the Calderón-Zygmund decomposition in the setting of flag Hardy spaces, namely we give the

Proof of Theorem 5.19 We first assume $f \in L^2(R^{n+m}) \cap H^p(R^n \times R^m)$. Let $\alpha > 0$ and $\Omega_\ell = \{(x,y) \in R^n \times R^m : S(f)(x,y) > \alpha 2^\ell\}$, where, as in Corollary 5.33,

$$S(f)(x,y) = \left\{\sum_{j,k}\sum_{I,J}|\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2\chi_I(x)\chi_J(y)\right\}^{\frac{1}{2}}.$$

It has been shown in Corollary 5.33 that $f \in L^2(R^{n+m}) \cap H^p(R^n \times R^m)$ then $||f||_{H^p_F} \approx ||S(f)||_p$.

In the following we take $R = I\times J$ as all dyadic rectangles in $R^n\times R^m$ with $|I| = 2^{-j-N}$, $|J| = 2^{-j-N} + 2^{-k-N}$, where j, k are integers and N is large enough.

Let

$$\mathcal{R}_0 = \left\{R = I\times J, \text{ such that } |R\cap\Omega_0| < \frac{1}{2}|R|\right\}$$

and for $\ell\ge 1$

$$\mathcal{R}_\ell = \left\{R = I\times J, \text{ such that } |R\cap\Omega_{\ell-1}| \ge \frac{1}{2}|R| \text{ but } |R\cap\Omega_\ell| < \frac{1}{2}|R|\right\}.$$

By the discrete Calderón reproducing formula in Theorem 5.30,

$$\begin{aligned} f(x,y) = & \sum_{j,k}\sum_{I,J}|I||J|\widetilde{\phi}_{jk}(x-x_I, y-y_J)\phi_{jk}*\left(T_N^{-1}(f)\right)(x_I,y_J)\\ =&\sum_{\ell\ge 1}\sum_{I\times J\in\mathcal{R}_\ell}|I||J|\widetilde{\phi}_{jk}(x-x_I, y-y_J)\phi_{jk}*\left(T_N^{-1}(f)\right)(x_I,y_J)\\ &+\sum_{I\times J\in\mathcal{R}_0}|I||J|\widetilde{\phi}_{jk}(x-x_I, y-y_J)\phi_{jk}*\left(T_N^{-1}(f)\right)(x_I,y_J)\\ =& b(x,y)+g(x,y)\end{aligned}$$

Such $b(x,y)$ and $g(x,y)$ satisfy the desired properties. Details can be found in [HL3].

We are now ready to prove the interpolation theorem on Hardy spaces H_F^p for all $0 < p < \infty$.

Proof of Theorem 5.20: Suppose that T is bounded from $H_F^{p_2}$ to L^{p_2} and from $H_F^{p_1}$ to L^{p_1}. For any given $\lambda > 0$ and $f\in H_F^p$, by the Calderón-Zygmund decomposition,

$$f(x,y) = g(x,y) + b(x,y)$$

with

$$||g||^{p_1}_{H_F^{p_1}} \le C\lambda^{p_1-p}||f||^p_{H_F^p} \text{ and } ||b||^{p_2}_{H_F^{p_2}} \le C\lambda^{p_2-p}||f||^p_{H_F^p}.$$

Moreover, we have proved the estimates

$$||g||^{p_1}_{H_F^{p_1}} \le C\int_{S(f)(x,y)\le\alpha} S(f)^{p_1}(x,y)dxdy$$

and

$$||b||^{p_2}_{H_F^{p_2}} \le C\int_{S(f)(x,y)>\alpha} S(f)^{p_2}(x,y)dxdy$$

which implies that

$$\begin{aligned} ||Tf||_p^p = & p\int_0^\infty \alpha^{p-1}|\{(x,y): |Tf(x,y)|>\lambda\}|d\alpha\\ \le & p\int_0^\infty \alpha^{p-1}\left|\left\{(x,y): |Tg(x,y)|>\frac{\lambda}{2}\right\}\right|d\alpha + p\int_0^\infty \alpha^{p-1}\left|\left\{(x,y): |Tb(x,y)|>\frac{\lambda}{2}\right\}\right|d\alpha\\ \le & p\int_0^\infty \alpha^{p-1}\int_{S(f)(x,y)\le\alpha} S(f)^{p_1}(x,y)dxdyd\alpha + p\int_0^\infty\alpha^{p-1}\int_{S(f)(x,y)>\alpha}S(f)^{p_2}(x,y)dxdyd\alpha\\ \le & C||f||^p_{H_F^p}\end{aligned}$$

Thus,

$$||Tf||_p \leq C||f||_{H^p_F}$$

for any $p_2 < p < p_1$. Hence, T is bounded from H^p_F to L^p.

To prove the second assertion that T is bounded on H^p_F for $p_2 < p < p_1$, for any given $\lambda > 0$ and $f \in H^p_F$, by the Calderón-Zygmund decomposition again

$$\begin{aligned}
&|\{(x,y) : |g(Tf)(x,y)| > \alpha\}| \\
\leq &|\left\{(x,y) : |g(Tg)(x,y)| > \frac{\alpha}{2}\right\}| + |\left\{(x,y) : |g(Tb)(x,y)| > \frac{\alpha}{2}\right\}| \\
\leq &C\alpha^{-p_1}||Tg||^{p_1}_{H^{p_1}_F} + C\alpha^{-p_2}||Tb||^{p_2}_{H^{p_2}_F} \\
\leq &C\alpha^{-p_1}||g||^{p_1}_{H^{p_1}_F} + C\alpha^{-p_2}||b||^{p_2}_{H^{p_2}_F} \\
\leq &C\alpha^{-p_1}\int_{S(f)(x,y)\leq\alpha} S(f)^{p_1}(x,y)dxdy + C\alpha^{-p_2}\int_{S(f)(x,y)>\alpha} S(f)^{p_2}(x,y)dxdy
\end{aligned}$$

which, as above, shows that $||Tf||_{H^p_F} \leq C||g(TF)||_p \leq C||f||_{H^p_F}$ for any $p_2 < p < p_1$. **Q.E.D.**

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