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A Relationship between Poincaré-Type Inequalities and Representation Formulas in Spaces of Homogeneous Type

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The purpose of this note is to study the relationship between the validity of L^1 versions of Poincaré's inequality and the existence of representation formulas for functions as (fractional) integral transforms of first-order vector fields.

The simplest example of a representation formula of the type we have in mind is the following familiar inequality for a smooth, real-valued function $f(x)$ defined on a ball B in N-dimensional Euclidean space $\mathbb{R}^N$:

$$|f(x) - f_B| \le C \int_B \frac{|\nabla f(y)|}{|x-y|^{N-1}}\, dy, \qquad x \in B,$$

where ∇f denotes the gradient of f, f_B is the average $|B|^{-1} \int_B f(y)\, dy$, $|B|$ is the Lebesgue measure of B, and C is a constant which is independent of f, x, B.

We are primarily interested in showing that various analogues of the formula above for more general systems of first-order vector fields $Xf = (X_1 f, \dots, X_m f)$ are simple corollaries of (and, in fact, often equivalent to) appropriate L^1 Poincaré inequalities of the form

$$\frac{1}{\nu(B)} \int_B |f - f_{B,\nu}|\, d\nu \le C r(B) \frac{1}{\mu(B)} \int_B |Xf|\, d\mu. \tag{1}$$

Here ν and μ are measures, B is a ball of radius $r(B)$ with respect to a metric that is naturally associated with the vector fields, and $f_{B,\nu} = \nu(B)^{-1} \int_B f\, d\nu$.

Recently, representation formulas in $\mathbb{R}^N$ were derived for Hörmander vector fields in [FLW] (see also [L1]) as well as for some nonsmooth vector fields of Grushin type in [FGW]. In the case of Hörmander vector fields, if $\rho(x, y)$ denotes the associated metric (see [FP], [NSW], [San]) and $B(x, r)$ denotes the metric ball with center x and radius r, then we

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have the representation formula (derived in [FLW])

$$|f(x) - f_B| \leq C \int_{\tau B} |Xf(y)| \frac{\rho(x,y)}{|B(x,\rho(x,y))|}\, dy \tag{2}$$

for $x \in B$ and $\tau > 1$, where τB is the ball concentric with B of radius $\tau r(B)$.

If a representation formula like (2) holds, then we can repeat our arguments in [FLW] to obtain two-weight L^p, L^q Sobolev-Poincaré inequalities. As is shown in [FLW, Section 4], these inequalities lead to relative isoperimetric estimates for Hörmander vector fields, or even for nonsmooth vector fields as in [FL], [F], [FGW].

The proof of (2) in [FLW] consists of an elaborate argument relying directly on the lifting procedure introduced by Rothschild and Stein [RS], whereas it will follow from our main result that (2) is equivalent to the Poincaré estimate (1) when both ν and μ are chosen to be Lebesgue measure. Since this form of (1) is known to be true (the argument given in Jerison [J] for the L^2 version of Poincaré's inequality for Hörmander vector fields and Lebesgue measure also works for the L^1 version), we thus obtain a proof of (2) that is shorter than the one given in [FLW].

As we shall see, the equivalence between estimates like (1) and (2) holds in a very general context and can be applied to different situations. This fact stresses once more the central role played by Poincaré's inequality in many problems. The simple technique we will use to show that Poincaré's inequality leads to a representation formula is based on some modifications in an argument due to Lotkowski and the third author [LW]. The method works in any homogeneous space in the sense of Coifman and Weiss [CW], i.e., in any quasimetric space equipped with a doubling measure. Moreover, the method does not require the presence of a derivation operator Xf on the right-hand side of either (1) or the representation formula: any function can be used, as we will see in Remark 1 below.

After this paper was submitted for publication, the authors received a preprint of the note [CDG2] containing another proof of the representation formula (2), relying on Jerison's Poincaré inequality, on the notion of "subelliptic mollifiers" introduced in [CDG1], and on the estimates for the fundamental solution and its derivatives for sums-of-squares operators proved in [NSW] and [San]. Our present results do not require any differential structure or vector field context, and so also do not require estimates for the fundamental solutions of differential operators.

For a homogeneous space $(\mathcal{S}, \rho, m)$, ρ denotes a quasimetric with quasimetric constant K; i.e., for all $x, y, z \in \mathcal{S}$,

$$\rho(x,y) \leq K\left[\rho(x,z) + \rho(z,y)\right],$$

and m is a doubling measure; i.e., there is a constant c such that

$$m(B(x,2r)) \leq c\, m(B(x,r)), \qquad x \in \mathcal{S},\ r > 0,$$

where, by definition, $B(x,r) = \{y \in \mathcal{S} : \rho(x,y) < r\}$, and $m(B(x,r))$ denotes the m-measure of $B(x,r)$. As usual, we refer to $B(x,r)$ as the ball with center x and radius r, and if B is a ball, we write x_B for its center, $r(B)$ for its radius, and cB for the ball of radius $cr(B)$ having the same center as B.

We now state and prove our main result, showing when Poincaré's inequality implies a representation formula. The reverse implication, namely, results showing when representation formulas lead to L^1 Poincaré estimates, can be easily derived by using Fubini's theorem. We briefly discuss this in Theorem 2 below; see also the remarks at the end of Section 3 of [FLW]. Finally, some examples of applications of the theorems are listed at the end of the paper. In these examples, $\rho(x,y)$ is actually a metric, and so K may be taken to be 1.

Let us state now our main result.

Theorem 1. Let ν, μ be doubling measures on a homogeneous space $(\mathcal{S}, \rho, m)$. Let B_0 be a ball, $\tau > 1$, $\epsilon > 0$, and assume for all balls $B \subset \tau K B_0$ (K is the quasimetric constant) that

$$\frac{1}{\nu(B)} \int_B |f - f_{B,\nu}|\, d\nu \le C r(B) \frac{1}{\mu(B)} \int_B |Xf|\, d\mu, \tag{1.a}$$

where f is a given function on $\tau K B_0$, and that for all balls $\tilde{B}$, B with $\tilde{B} \subset B \subset \tau K B_0$,

$$\frac{r(B)}{\mu(B)} \le C \left(\frac{r(\tilde{B})}{r(B)}\right)^{\epsilon} \frac{r(\tilde{B})}{\mu(\tilde{B})}, \tag{1.b}$$

or equivalently,

$$\mu(B) \ge c \left(\frac{r(B)}{r(\tilde{B})}\right)^{1+\epsilon} \mu(\tilde{B}).$$

Then, for ν–almost every $x \in B_0$,

$$|f(x) - f_{B_0,\nu}| \le C \int_{\tau K B_0} |Xf(y)| \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\, d\mu(y) \tag{1.c}$$

with $f_{B_0,\nu} = \nu(B_0)^{-1} \int_{B_0} f\, d\nu$. □

On the right sides of (1.a) and (1.c), we have used the vector field notation Xf, but any function will do; see the first remark which follows the proof for slightly more general forms of Theorem 1.

Proof. By hypothesis, for fixed $\tau > 1$ and all balls $B \subset \tau K B_0$,

$$\frac{1}{\nu(B)} \int_B |f - f_B| \, d\nu \le C \frac{r(B)}{\mu(B)} \int_B |Xf| \, d\mu,$$

where for simplicity we have written f_B for $f_{B,\nu}$. Let $x \in B_0$. There is a constant $\eta > 0$ independent of x and B_0 such that $B(x, \eta r(B_0)) \subset \tau K B_0$: in fact, it is enough to choose η such that $1 + \eta < \tau$, since if $y \in B(x, \eta r(B_0))$, then

$$\begin{aligned}
\rho(x_{B_0}, y) &\le K \left(\rho(x_{B_0}, x) + \rho(x, y)\right) \\
&\le K \left(r(B_0) + \eta r(B_0)\right) = K(1 + \eta) r(B_0) \\
&< K\tau r(B_0).
\end{aligned}$$

Denote $B(x, \eta r(B_0)) = B_1$. Then $r = r(B_1) = \eta r(B_0)$. Now

$$|f(x) - f_{B_0}| \le |f(x) - f_{B_1}| + |f_{B_1} - f_{B_0}|. \tag{3}$$

For the second term on the right of (3), we have

$$\begin{aligned}
|f_{B_1} - f_{B_0}| &\le |f_{B_1} - f_{\tau K B_0}| + |f_{B_0} - f_{\tau K B_0}| \\
&\le \frac{1}{\nu(B_1)} \int_{B_1} |f(y) - f_{\tau K B_0}| \, d\nu(y) + \frac{1}{\nu(B_0)} \int_{B_0} |f(y) - f_{\tau K B_0}| \, d\nu(y) \\
&\le \left(\frac{1}{\nu(B_1)} + \frac{1}{\nu(B_0)} \right) \int_{\tau K B_0} |f(y) - f_{\tau K B_0}| \, d\nu(y) \\
&\quad \text{since } B_1, B_0 \subset \tau K B_0 \\
&\le \frac{C}{\nu(\tau K B_0)} \int_{\tau K B_0} |f - f_{\tau K B_0}| \, d\nu \\
&\quad \text{since } \nu \text{ is doubling and } r(B_1), r(B_0) \approx r(\tau K B_0) \\
&\le C \frac{r(\tau K B_0)}{\mu(\tau K B_0)} \int_{\tau K B_0} |Xf| \, d\mu \qquad \text{by hypothesis (1.a)} \\
&\le C \int_{\tau K B_0} |Xf(y)| \frac{\rho(x, y)}{\mu(B(x, \rho(x, y)))} \, d\mu(y)
\end{aligned}$$

as desired, since if $y \in \tau K B_0$, then $\rho(x, y) < 2K r(\tau K B_0) \approx r(B_0)$, and then we can apply (1.b), with $B = \tau K B_0$ and $\tilde{B} = B(x, (\tau - 1)\rho(x, y)/2K^2\tau)$ (the fact that $\tilde{B} \subset \tau K B_0$ follows by using the "triangle" inequality as before), together with the doubling of μ to get

$$\frac{r(\tau K B_0)}{\mu(\tau K B_0)} \le C \left(\frac{\rho(x, y)}{r(B_0)} \right)^{\epsilon} \frac{\rho(x, y)}{\mu(B(x, \rho(x, y)))} \le C \frac{\rho(x, y)}{\mu(B(x, \rho(x, y)))}.$$

For the first term on the right of (3), we may assume that $\lim_{s\to 0} f_{B(x,s)} = f(x)$ since ν–almost every x has this property. Thus, we have

$$\begin{aligned}
|f(x) - f_{B_1}| &= |f(x) - f_{B(x,r)}| \\
&\le \sum_{k=0}^{\infty} |f_{B(x,r2^{-k-1})} - f_{B(x,r2^{-k})}| \\
&\le \sum_{k=0}^{\infty} \frac{1}{\nu(B(x,r2^{-k-1}))} \int_{B(x,r2^{-k-1})} |f(y) - f_{B(x,r2^{-k})}|\, d\nu(y) \\
&\le \sum_{k=0}^{\infty} \frac{C}{\nu(B(x,r2^{-k}))} \int_{B(x,r2^{-k})} |f(y) - f_{B(x,r2^{-k})}|\, d\nu(y) \\
&\le C \sum_{k=0}^{\infty} \frac{r2^{-k}}{\mu(B(x,r2^{-k}))} \int_{B(x,r2^{-k})} |Xf(y)|\, d\mu(y) \\
&\quad \text{by hypothesis (1.a)} \\
&= C \int_B |Xf(y)| \left\{ \sum_{k=0}^{\infty} \frac{r2^{-k}}{\mu(B(x,r2^{-k}))} \chi_{\{y:\rho(x,y)<r2^{-k}\}}(y) \right\} d\mu(y).
\end{aligned}$$

If $\rho(x,y) < r2^{-k}$, then by hypothesis (1.b) we have

$$\frac{r2^{-k}}{\mu(B(x,r2^{-k}))} \le C \left(\frac{\rho(x,y)}{r2^{-k}}\right)^{\epsilon} \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))},$$

and so the sum above in curly brackets is at most

$$\sum_{k\ge 0: 2^k \le r\rho(x,y)^{-1}} \left(\frac{2^k \rho(x,y)}{r}\right)^{\epsilon} \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))} \le C \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))},$$

with C independent of x and y. Combining the estimates, we obtain the theorem. ■

Remark 1. As the proof of Theorem 1 shows, we can replace (1.a) by

$$\frac{1}{\nu(B)} \int_B |f - f_{B,\nu}|\, d\nu \le C\phi(B)\sigma(B) \tag{1.a$'$}$$

and (1.b) by

$$\phi(B) \le C \left(\frac{r(\tilde{B})}{r(B)}\right)^{\epsilon} \phi(\tilde{B}), \qquad \tilde{B} \subset c_1 B \text{ and } x_{\tilde{B}} \in B, \tag{1.b$'$}$$

where ϕ is any nonnegative function of balls B, c_1 is sufficiently large depending only on τ and K, and σ is any measure, obtaining as a conclusion that for ν–almost every $x \in B_0$,

$$|f(x) - f_{B_0,\nu}| \le C \int_{\tau K B_0} \phi(B(x,\rho(x,y)))\, d\sigma(y). \tag{1.c$'$}$$

If we choose $\phi(B) = r(B)/\mu(B)$ and $d\sigma = |Xf|\,d\mu$, we obtain Theorem 1. The hypothesis (1.b′) (which is slightly stronger than (1.b) since $c_1 > 1$) is needed in order to handle the first term on the right of (3); we were able to take $c_1 = 1$ in Theorem 1 due to the special form of ϕ there and the fact that μ is doubling.

Moreover, we may replace (1.b′) by a weaker condition of Dini type; i.e., if $\delta(t)$ is any nonnegative, bounded, monotone function on the interval $0 < t < c_1$ which satisfies

$$\int_0^1 \frac{\delta(t)}{t}\,dt < \infty,$$

then we may replace the factor $(r(\tilde{B})/r(B))^\epsilon$ in (1.b′) by $\delta(r(\tilde{B})/r(B))$.

Remark 2. The conclusions of Theorem 1 can be modified as follows: if $\beta \in (0,1)$ is such that

$$\nu(E_f) := \nu(\{y \in B_0 \,:\, f(y) = 0\}) \geq \beta\nu(B_0), \tag{1.d}$$

then for ν–almost every $x \in B_0$ we have

$$|f(x)| \leq C_\beta \int_{\tau K B_0} |Xf(y)| \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,d\mu(y). \tag{1.e}$$

Indeed, by (1.a),

$$\begin{aligned}\frac{r(B_0)}{\mu(B_0)} \int_{B_0} |Xf|\,d\mu &\geq c\frac{1}{\nu(B_0)} \int_{B_0} |f - f_{B_0,\nu}|\,d\nu \\ &\geq c\frac{1}{\nu(B_0)} \int_{E_f} |f - f_{B_0,\nu}|\,d\nu = c\frac{\nu(E_f)}{\nu(B_0)}|f_{B_0,\nu}| \geq c\beta|f_{B_0,\nu}|,\end{aligned}$$

and then (1.e) follows from (1.c) and (1.b) by the sort of reasoning we used in the last part of the argument for the second term on the right of (3).

Suppose now that f is supported in B_0. Then we can argue as follows: if (1.a) and (1.b) hold, then they also hold if we replace B_0 by $\sqrt{\tau}B_0$ and τ by $\sqrt{\tau}$. On the other hand, in this case, (1.d) is satisfied by doubling if we again replace B_0 by $\sqrt{\tau}B_0$. Hence (1.e) holds. Analogous remarks can be made for (1.a′), (1.b′), and (1.c′).

As mentioned earlier, the implication opposite to Theorem 1 is easy to derive by using the Fubini-Tonelli theorem. In fact, we have the following result.

Theorem 2. Let $\phi(B)$ be a nonnegative function of balls B in a quasimetric space (S,ρ), and let μ,ν be measures. Given $\tau \geq 1$ and a fixed ball B, suppose there is a constant c_B such that

$$|f(x) - c_B| \leq c \int_{\tau B} \phi(B(x,\rho(x,y)))\,d\sigma(y), \tag{2.a}$$

for ν–almost every $x \in B$, and that ϕ satisfies

$$\frac{1}{\nu(B)} \int_B \phi(B(x, \rho(x, y)))\, d\nu(x) \le c\phi(B), \tag{2.b}$$

for σ–almost every $y \in \tau B$. Then

$$\frac{1}{\nu(B)} \int_B |f(x) - c_B|\, d\nu(x) \le c\phi(B)\sigma(\tau B). \tag{2.c}$$

□

Before giving the proof, we make two additional comments. First, by a standard argument, we can always replace c_B in (2.c) by $f_{B,\nu}$. Second, even when $\tau > 1$, we may often replace $\sigma(\tau B)$ by $\sigma(B)$ on the right side of (2.c). For example, if $(\mathcal{S}, \rho)$ is a metric space and ν is doubling, let B_0 be a fixed ball in $\mathcal{S}$ which satisfies the Boman $\mathcal{F}(\tau, M)$ condition (see, e.g., [FGW]). Then if (2.c) holds for all balls B with $\tau B \subset B_0$ and if also

$$\frac{\nu(B)}{\nu(B_0)} \le c\frac{\phi(B_0)}{\phi(B)} \qquad \text{for all } B \text{ with } \tau B \subset B_0, \tag{4}$$

we may conclude that

$$\frac{1}{\nu(B_0)} \int_{B_0} |f(x) - f_{B_0,\nu}|\, d\nu(x) \le c\phi(B_0)\sigma(B_0). \tag{5}$$

We refer to [FGW, Theorems (5.2) and (5.4)] for a further discussion, noting here only that (2.c) and (4) imply

$$\int_B |f(x) - c_B|\, d\nu(x) \le A\sigma(\tau B) \qquad \text{with} \qquad A = c\phi(B_0)\nu(B_0)$$

for all balls B satisfying $\tau B \subset B_0$; this leads to (5) by Theorem (5.2) of [FGW].

Proof of Theorem 2. Integrating (2.a) with respect to ν over B and changing the order of integration, we obtain

$$\begin{aligned} \frac{1}{\nu(B)} \int_B |f(x) - c_B|\, d\nu(x) &\le c \int_{\tau B} \left(\frac{1}{\nu(B)} \int_B \phi(B(x, \rho(x, y)))\, d\nu(x) \right) d\sigma(y) \\ &\le c\phi(B)\sigma(\tau B), \end{aligned}$$

by (2.b), and the proof is complete. ■

We now list some examples and applications related to Theorems 1 and 2.

Example 1. If the homogeneous space is $(\mathbb{R}^N, \rho, dx)$, where ρ is the metric associated with a collection $X_1 f, \dots, X_m f$ of Hörmander vector fields, then (1.a) holds with $|Xf| = \sum_{i=1}^m |X_i f|$ for $d\mu = d\nu = dx$ by the work of Jerison [J]. Also, (1.b) holds with $d\mu = dx$ and $\epsilon = N-1$ by [NSW] (see (2.1) of [FLW]). Moreover, the quasimetric constant $K = 1$, since ρ is a metric. Thus, we obtain the representation formula in [FLW].

Conversely, with regard to Theorem 2, note that if we take $\phi(B) = r(B)/|B|$, $d\sigma = |Xf|dx$, and $d\nu = d\mu = dx$, then (2.b) holds as in [FLW], (4) is obvious, and (5) takes the form

$$\frac{1}{|B_0|}\int_{B_0} |f(x) - f_{B_0}|\, dx \le c r(B_0) \left(\frac{1}{|B_0|}\int_{B_0} |Xf(x)|\, dx \right).$$

The fact that B_0 satisfies the Boman condition is discussed in [FLW], [FGW]. Thus, the Poincaré estimate above follows from the representation formula in [FLW].

Example 2. Let the homogeneous space be $(\mathbb{R}^N, \rho, dx)$, where ρ is the metric associated with Grushin vector fields

$$X = \nabla_\lambda = (\partial/\partial x_1, \dots, \partial/\partial x_n, \lambda(x)\partial/\partial x_{n+1}, \dots, \lambda(x)\partial/\partial x_N)$$

as described in [FGW]. Pick $d\mu = w^{1-1/N} dx$, where w is a strong A_∞ weight as in [FGW]. Then by formula (4.s), page 586, of [FGW],

$$\frac{r(B)}{\mu(B)} = \frac{r(B)}{\int_B w^{1-1/N}\, dx} \approx \left(\frac{1}{\int_B w\lambda^{m/(N-1)}\, dx} \right)^{1-1/N}, \tag{6}$$

and if $\tilde{B} \subset B$, then by the reverse doubling of $w\lambda^{m/(N-1)}$, there exists a $\delta > 0$ such that

$$\int_B w\lambda^{m/(N-1)}\, dx \ge C \left(\frac{r(B)}{r(\tilde{B})} \right)^{\delta} \int_{\tilde{B}} w\lambda^{m/(N-1)}\, dx. \tag{7}$$

Thus, (1.b) follows with $\epsilon = \delta(1 - 1/N)$ by combining (7) with the equivalence (6) for both B and $\tilde{B}$. If we assume (1.a) holds with $X = \nabla_\lambda$ for some doubling ν, we obtain

$$|f(x) - f_{B_0,\nu}| \le C \int_{\tau B_0} |\nabla_\lambda f(y)| \frac{\rho(x,y)}{\int_{B(x,\rho(x,y))} w^{1-1/N}\, dz} w^{1-1/N}(y)\, dy$$

for ν–almost every $x \in B_0$. Using (6) again, we obtain

$$|f(x) - f_{B_0,\nu}| \le C \int_{\tau B_0} \frac{|\nabla_\lambda f(y)|}{\left(\int_{B(x,\rho(x,y))} w\lambda^{m/(N-1)}\, dz \right)^{1-1/N}} w^{1-1/N}(y)\, dy$$

for ν–almost every $x \in B_0$. This is the representation formula in [FGW]. We have assumed the Poincaré estimate (1.a) for $d\mu = w^{1-1/N}\, dx$ and some $d\nu$. The representation formula in [FGW] was derived without prior knowledge of any Poincaré estimate.

Conversely, we will show by using Theorem 2 that the representation formula above implies (1.a) with $d\mu = w^{1-1/N}dx$ and $d\nu$ chosen to be either $w^{1-1/N}dx$ or $w\lambda^{m/(N-1)}dx$. In fact, first let

$$d\mu = w^{1-1/N}dx, \qquad \phi(B) = \frac{r(B)}{\mu(B)}, \qquad \text{and} \qquad d\sigma = |\nabla_\lambda f|\,d\mu.$$

Then, by (6), the representation formula in [FGW] for a ball B is the same as (2.a). We will now show that (2.b) and (4) hold if $d\nu$ is either $d\mu$ or $w\lambda^{m/(N-1)}dx$. The estimate (4) is obvious if $d\nu = d\mu$, while if $d\nu = w\lambda^{m/(N-1)}dx$, then by using (6), we see that (4) is equivalent to

$$\frac{\nu(B)}{\nu(B_0)} \le c\frac{\nu(B)^{1-1/N}}{\nu(B_0)^{1-1/N}}, \qquad B \subset B_0,$$

which is obvious since $\nu(B) \le \nu(B_0)$ if $B \subset B_0$.

It remains to show (2.b) for either choice of ν. Since μ is a doubling measure, we have

$$\phi(B(x, \rho(x, y))) \approx \phi(B(y, \rho(x, y)))$$

and

$$\int_B \phi(B(x, \rho(x, y)))\, d\nu(x) \approx \int_B \phi(B(y, \rho(x, y)))\, d\nu(x).$$

Since $y \in \tau B$, by enlarging the domain B of integration proportionally, we may assume that y is the center of B. The last integral is then at most

$$\sum_{k=0}^{\infty} \int_{\{x:\rho(x,y)\approx 2^{-k}r(B)\}} \phi(B(y, \rho(x, y)))\, d\nu(x) \le c\sum_{k=0}^{\infty} \phi(2^{-k}B)\nu(2^{-k}B).$$

If $\nu = \mu$, this sum is

$$c\sum_{k=0}^{\infty} r(B)2^{-k} = cr(B) = c\phi(B)\nu(B),$$

which proves (2.b). If $d\nu = w\lambda^{m/(N-1)}dx$, then $\phi(B) \approx \nu(B)^{1/N-1}$ by (6), and the sum above is at most

$$c\sum_{k=0}^{\infty} \nu(2^{-k}B)^{1/N} \le c\sum_{k=0}^{\infty} \left(2^{-k\epsilon}\nu(B)\right)^{1/N}$$

for some $\epsilon > 0$ by reverse doubling of ν, which is equal to

$$c\nu(B)^{1/N} = c\nu(B)^{1/N-1}\nu(B) \le c\phi(B)\nu(B).$$

This proves (2.b) in every case. Finally, since balls satisfy the Boman condition by Theorem 5.4 of [FGW], the representation formula in [FGW] then implies (1.a) for either choice of ν, with $d\mu = w^{1-1/N}dx$.

Example 3. In the general setting, if we assume that $r(B)/\mu(B) \approx \eta(B)^{-\alpha}$ for some $\alpha > 0$ and some measure η that satisfies a reverse doubling condition, then (1.b) is clearly satisfied, and so (1.a) implies by Theorem 1 that

$$|f(x) - f_{B_0,\nu}| \leq C \int_{\tau K B_0} |Xf(y)| \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))} \, d\mu(y)$$

for ν–almost every $x \in B_0$. This example includes the class of nonsmooth vector fields considered in [FL] and [F1] by picking $d\mu = dx$, since we know there is a compensation couple, i.e., a doubling measure ηdx and $s > 1$ such that

$$\frac{r(B)}{|B|} \approx \eta(B)^{-1/s}.$$

In fact, by [FW], we may choose $s = N/(N-1)$ and $d\eta = \left(\Pi\lambda_j\right)^{1/(N-1)} dx$, where $X_j = \lambda_j(x)\partial_j$, $j = 1, \dots, N$, are assumed to satisfy the conditions in [F1]. Arguing as in Example 2 with $\phi(B) = r(B)/|B|$, $d\sigma = \left(\sum |X_j f|\right) dx$, and $d\mu = dx$, we obtain by Theorem 2 that the representation formula in [F2] implies (1) with $d\mu = dx$ and $d\nu$ taken to be either dx or $d\eta$. We omit the details.

Example 4. Let G be a connected Lie group endowed with its left-invariant Haar measure μ. For the sake of simplicity, let us suppose that G is unimodular, and let $\{X_1, \dots, X_m\}$ be a family of left-invariant vector fields on G which generate its Lie algebra. Then we can define a natural left-invariant metric ρ on G (see [VSC] for precise definitions) so that there exists a positive integer d such that, for any ρ-ball $B(x,r)$, we have that $\mu(B(x,r)) \approx r^d$ as $r \to 0$. If, in addition, G has polynomial growth at infinity, then there exists a nonnegative integer D such that $\mu(B(x,r)) \approx r^D$ as $r \to \infty$. Thus, if we assume that $\min\{d, D\} > 1$, then (1.b) holds with $\epsilon = \min\{d,D\} - 1$ for all balls $B_0 \subset G$, so that the hypotheses of Theorem 1 are satisfied, since the Poincaré inequality (1.a) with $\nu = \mu$ holds by [V] and [MS]. (The doubling property of the measure of balls follows in a straightforward way from the existence of d and D.) Thus, a representation formula in G follows for all balls B_0.

Analogous arguments can be carried out for the balls of $M = G \setminus H$, where H is a closed subgroup of G: see again [MS, Example 4], and the literature quoted therein. For further information on these subjects, see [VSC].

Example 5. As is well known, representation formulas like (1.c) and (1.c′), when used in conjunction with known facts about operators of potential type, lead to two-weight L^p, L^q Poincaré inequalities, $1 \leq p \leq q < \infty$, where the L^p norm appears on the right side of the inequality and the L^q norm on the left. We shall not explicitly recall any results of this kind here, but refer to [SW], [FGW], [FLW], [L1], [L2] and the references listed in these papers for precise statements. In particular, it then follows from Theorem 1, by using the representation formula, that (1.a) and (1.b) also lead to such weighted L^p, L^q Poincaré estimates.

After Saloff-Coste's paper [Sal], many results in the same spirit (see also, e.g., [MS], [BM], and [HK]) have been proved in different settings, stating (roughly speaking) that if an L^1, L^1 Poincaré inequality like (1.a) holds with $\mu = \nu$, together with a doubling property of the measure, then (1.a) can be improved by replacing the L^1-norm on the left side by an L^q-norm for some $q > 1$ depending on the doubling order. As long as the value of q is the best possible, Poincaré inequalities with $p = 1$ contain deep geometric information since they imply suitable relative isoperimetric inequalities (see [FLW]). Now, if (1.a) and (1.b) hold, then the weighted L^p, L^q Poincaré inequalities which we obtain by using the representation formula extend some of the results in the papers listed above. The results in these papers are obtained without using a representation formula. They include analogues when the initial Poincaré hypothesis is an L^p, L^p estimate for some $p > 1$, rather than just when $p = 1$. Many such analogues, including two-weight versions, can also be obtained by using representation formulas which are similar to (1.c), but which are instead based on an initial L^p, L^p Poincaré hypothesis. In fact, analogues of Theorems 1 and 2 for this situation will be discussed in a sequel to this paper.

Example 6. In [Ha], the author defines a class of first-order Sobolev spaces on a generic metric space $(\mathcal{S}, \rho)$ endowed with a Borel measure μ as follows: if $1 < p \leq \infty$, then $W^{1,p}(\mathcal{S}, \rho, \mu)$ denotes the set of all $f \in L^p(\mathcal{S}, \mu)$ for which there exist $E \subset \mathcal{S}$, $\mu(E) = 0$, and $g \in L^p(\mathcal{S}, \mu)$ such that

$$|f(x) - f(y)| \leq \rho(x, y)\left(g(x) + g(y)\right) \tag{8}$$

for all $x, y \in \mathcal{S} \setminus E$. Sobolev spaces associated with a family of Hörmander vector fields are examples of such spaces, as well as weighted Sobolev spaces associated with a weight function belonging to Muckenhoupt's A_p classes. Now, it follows from our Theorem 1 that, if μ is a doubling measure such that (1.b) holds, then the pointwise condition (8) is a corollary of the weaker integral condition

$$\int_B |f(y) - f_B|\, d\mu(y) \leq Cr \int_B |g(y)|\, d\mu(y) \tag{9}$$

for all ρ-balls $B = B(x, r)$, where $g \in L^p(\mathcal{S}, \mu)$ is a given function depending on f but independent of B. Indeed, by Theorem 1, (9) implies that for μ–almost every $y \in B$, we have

$$
\begin{aligned}
|f(y) - f_{B,\mu}| &\le C \int_{\tau B} \frac{|g(z)|\, \rho(y,z)}{\mu(B(y,\rho(y,z)))}\, d\mu(z) \le C \int_{(\tau+1)B(y,r)} \frac{|g(z)|\, \rho(y,z)}{\mu(B(y,\rho(y,z)))}\, d\mu(z) \\
&\le \sum_{k=-\infty}^{0} \int_{2^{k-1}(\tau+1)r \le \rho(z,y) < 2^k(\tau+1)r} \frac{|g(z)|\, \rho(y,z)}{\mu(B(y,\rho(y,z)))}\, d\mu(z) \\
&\le Cr \sum_{k=-\infty}^{0} 2^k ⨍_{B(y,2^k(\tau+1)r)} |g(z)|\, d\mu(z) \\
&\le Cr\, Mg(y),
\end{aligned}
$$

where Mg denotes the Hardy-Littlewood maximal function of g with respect to the measure μ, which belongs to $L^p(\mathcal{S}, \mu)$ if $p > 1$. Thus, if $x, y \in \mathcal{S}$, we can apply the above estimate to the ball $B = B(x, 2\rho(x,y))$, and we get

$$
\begin{aligned}
|f(x) - f(y)| &\le |f(x) - f_{B,\mu}| + |f(y) - f_{B,\mu}| \\
&\le C\rho(x,y)\, \{Mg(x) + Mg(y)\}\,,
\end{aligned}
$$

and then (8) holds. Thus, since (8) implies (9) [Ha, Lemma 2], (9) provides us with an alternative definition of Sobolev spaces associated with a metric.

More generally, we can easily modify the argument above to show that (9) implies

$$
|f(x) - f(y)| \le C \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))^{1-\alpha}} \{M_\alpha g(x) + M_\alpha g(y)\} \tag{10}
$$

for any α satisfying $1 - 1/d < \alpha \le 1$, where d is the doubling order of μ (i.e., where $\mu(B(y,r)) \le c(r/s)^d \mu(B(y,s))$ if $s < r$) and $M_\alpha g$ is the fractional maximal function of g defined by

$$
M_\alpha g(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))^\alpha} \int_{B(x,r)} |g(z)|\, d\mu(z).
$$

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References

[BM] M. Biroli and U. Mosco, *Proceedings of the Conference Potential Theory and Partial Differential Operators with Nonnegative Characteristic Form*, *Parma*, *February 1994*, Kluwer, Amsterdam, to appear.

[CDG1] L. Capogna, D. Danielli, and N. Garofalo, *Subelliptic mollifiers and a characterization of Rellich and Poincaré domains*, Rend. Sem. Mat. Univ. Politec. Torino **54** (1993), 361–386.

[CDG2] ———, *Subelliptic mollifiers and a basic pointwise estimate of Poincaré type*, preprint, 1995.

[CW] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. **242**, Springer-Verlag, Berlin, 1971.

[FP] C. Fefferman and D. H. Phong, "Subelliptic eigenvalue problems" in *Conference on Harmonic Analysis in Honor of Antoni Zygmond*, ed. by W. Beckner et al., Wadsworth, Belmont, Calif., 1981, 590–606.

[F1] B. Franchi, *Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations*, Trans. Amer. Math. Soc. **327** (1991), 125–158.

[F2] ———, *Inégalités de Sobolev pour des champs de vecteurs lipschitziens*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 329–332.

[FGW] B. Franchi, C. Gutierrez, and R. L. Wheeden, *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (1994), 523–604.

[FL] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 523–541.

[FLW] B. Franchi, G. Lu, and R. L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier (Grenoble) **45** (1995), 577–604.

[FW] B. Franchi and R. L. Wheeden, *Compensation couples and isoperimetric estimates for vector fields*, preprint, 1995.

[Ha] P. Hajlasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal., to appear.

[HK] P. Hajlasz and P. Koskela, *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 1211–1215.

[H] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[J] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), 503–523.

[LW] E. P. Lotkowski and R. L. Wheeden, *The equivalence of various Lipschitz conditions on the weighted mean oscillation of a function*, Proc. Amer. Math. Soc. **61** (1976), 323–328.

[L1] G. Lu, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoamericana **8** (1992), 367–439.

[L2] ———, *The sharp Poincaré inequality for free vector fields: an endpoint result*, Rev. Mat. Iberoamericana **10** (1994), 453–466.

[MS] P. Maheux and L. Saloff-Coste, *Analyse sur les boules d'un opérateur sous-elliptique*, to appear in Math. Ann.

[NSW] A. Nagel, E. M. Stein, and S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985), 103–147.

[RS] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[Sal] L. Saloff-Coste, *A note on Poincaré*, *Sobolev and Harnack inequalities*, Internat. Math.

Res. Notices **1992**, 27–38.

[San] A. Sánchez-Calle, *Fundamental solutions and geometry of the sums of squares of vector fields*, Invent. Math. **78** (1984), 143–160.

[SW] E. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874.

[V] N. Varopoulos, *Fonctions harmoniques sur les groupes de Lie*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 519–521.

[VSC] N. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Univ. Press, Cambridge, 1992.

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