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Multi-parameter local Hardy spaces

Wei Ding [a], Guozhen Lu [b,*], YuePing Zhu [c]

[a] School of Sciences, Nantong University, Nantong 226007, PR China
[b] Department of Mathematics University of Connecticut, Storrs, CT 06269, USA
[c] Department of Mathematics, Nantong Normal College, Nantong 226010, PR China

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ABSTRACT

Though multi-parameter Hardy space theory has been well developed in the past half century, not much has been studied for a local Hardy space theory in the multi-parameter settings. Such multi-parameter local Hardy spaces can play an important role in studying the boundedness of multi-parameter pseudo-differential operators, multi-parameter singular integrals of non-convolution type, and applications to partial differential equations, etc. By establishing a bi-parameter local reproducing formula, bi-parameter local Hardy space $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is introduced in this paper. This space coincides with $L^p(\mathbb{R}^{n_1+n_2})$ when $p > 1$. While $p \leq 1$, this space is substantially different from the classical bi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. We will establish its atomic decomposition of the bi-parameter local Hardy spaces and as an application, we prove the boundedness from $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1+n_2})$ of the bi-parameter singular integral operators in inhomogeneous Journé class for p near 1. For the simplicity, we have chosen to present all the results in the bi-parameter setting. Nevertheless, all of them hold for arbitrary number of parameters. The multi-parameter local theory developed in this paper can serve as a model case for similar theory in other multi-parameter settings.

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1. Introduction

The classical theory of harmonic analysis may be described as centering around the Hardy–Littlewood maximal operator and its relationship with certain singular integral operators which commute with the usual dilations on $\mathbb{R}^n$, given by $\delta : x \to \delta x, \delta > 0$. The multi-parameter pure product theory corresponds to the dilations $\delta : x \to (\delta_1 x_1, \ldots, \delta_k x_k), x = (x_1, \ldots, x_k) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}, \delta = (\delta_1, \ldots, \delta_k), \delta_1, \ldots, \delta_k > 0$ and the strong maximal function. The multi-parameter function spaces and boundedness of Fourier multipliers, singular integral operators, pseudo-differential operators, Fourier integral operators and singular Radon transforms on such spaces have been extensively studied by many authors over the past decades. We refer the

* Corresponding author.
E-mail addresses: dingwei@ntu.edu.cn (W. Ding), guozhen.lu@uconn.edu (G. Lu), zhuyueping@ntu.edu.cn (Y. Zhu).

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reader to the work in [1–6,8–10,13,14,18,21–23,25,26,29–32,34–36,38–42,44–54,56–58,60–63], just to name a few.

In 1915, Hardy [37] considered the averages

$$M_p(r,F)=\Big(\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^p d\theta\Big)^{1/p}, 1<p<\infty$$

and

$$M_\infty(r,F)=\max_{0\le\theta\le 2\pi}|F(re^{i\theta})|$$

for $0<r<1$ to study the behavior of holomorphic functions on the unit disk. That paper was the starting point for the theory of Hardy spaces $H^p(\mathbb{R}^n)$ which can be defined by a purely real method [11,12,24]. That is, a tempered distribution $f\in H^p(\mathbb{R}^n)$ if and only if $M_\varphi(f)\in L^p(\mathbb{R}^n)$ where

$$M_\varphi(f)(x)=\sup_{0<t<\infty}|\varphi_t * f(x)|$$

and $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\int\varphi\neq 0$. This space coincides with $L^p(\mathbb{R}^n)$ when $1<p<\infty$ and is contained in $L^p(\mathbb{R}^n)$ when $p\le 1$. Much classical theory implies that $H^p(\mathbb{R}^n)$ space is a well substitute of $L^p(\mathbb{R}^n)$ when $p\le 1$ in pure Fourier analysis. One of the most basic examples is the Riesz transform $\widehat{(R_if)}(x)=i\frac{x_i}{|x|}\hat{f}(x)$. When $1<p<\infty$ these operators are bounded on $L^p(\mathbb{R}^n)$. But it is easy to see that if $\int f\neq 0$, R_if cannot be in $L^1(\mathbb{R}^n)$. Nevertheless, $\|R_if\|_{H^p}\lesssim\|f\|_{H^p}$, $0<p<\infty$. The above symbol $\mathcal{S}(\mathbb{R}^n)$ has its usual meaning, a Schwartz class of rapidly decreasing test functions. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.

In the fundamental work [29], Gundy and Stein introduce the product Hardy space $H^p(\mathbb{R}\times\mathbb{R})$. They define this space via area integrals and maximal functions as well as by the Hilbert transform. Let $\varphi\in\mathcal{S}(\mathbb{R}^2)$ with $\int\varphi\neq 0$. For $t_1,t_2>0$, set $\varphi_{t_1,t_2}(x_1,x_2)=t_1^{-1}t_2^{-1}\varphi(\frac{x_1}{t_1},\frac{x_2}{t_2})$, and define the maximal function

$$\mathcal{M}_\varphi(f)(x_1,x_2)=\sup_{t_1,t_2>0}|\varphi_{t_1,t_2}*f(x)|.$$

Then $f\in H^p(\mathbb{R}\times\mathbb{R})$ if and only if $\mathcal{M}_\varphi(f)\in L^p(\mathbb{R}^2)$. Subsequently, Chang and R. Fefferman [3–5] developed a staisfactory theory of multi-parameter Hardy spaces. Recently, using discrete Calderón's identity of the type in [27], the authors of [33] proved that this product Hardy space can also be characterized by a discrete Littlewood–Paley–Stein square function. Let $\psi^{(i)}\in\mathcal{S}(\mathbb{R})$ satisfying $\sum_{j\in\mathbb{Z}}\big|\widehat{\psi^{(i)}}(2^{-j}\xi)\big|^2=1$ for all $\xi\in\mathbb{R}\setminus\{0\}$, $i=1,2$. Then $f\in H^p(\mathbb{R}\times\mathbb{R})$ if and only if

$$\Big\|\Big(\sum_{j,k\in\mathbb{Z}}\sum_{I,J}|\psi_{j,k}*f(x_I,x_J)|^2\chi_I(x_1)\chi_J(x_2)\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^2)}<\infty,$$

where $\psi_{j,k}(x_1,x_2)=2^j\psi^{(1)}(2^jx_1)2^k\psi^{(2)}(2^kx_2)$, I and J are dyadic intervals in $\mathbb{R}$ with the side length $l(I)=2^{-j}$ and $l(J)=2^{-k}$, and the left end points of I and J are x_I and x_J, respectively.

As Goldberg pointed out in [28] that H^p space is well suited only to partial differential equations with constant coefficients, but not stable under multiplication by test functions which associated with PDEs with variable coefficients. One reason is that H^p does not contain $\mathcal{S}(\mathbb{R}^n)$. To circumvent those drawbacks, Goldberg [28] introduced the class of localizable Hardy spaces $h^p(\mathbb{R}^n), 0<p<\infty$. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\int\varphi\neq 0$ and

$$m_\varphi(f)(x)=\sup_{0<t<1}|\varphi_t*f(x)|.$$

Then $h^p(\mathbb{R}^n)=\{f\in\mathcal{S}'(\mathbb{R}^n), m_\varphi(f)\in L^p(\mathbb{R}^n)\}$. He proved that $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^p(\mathbb{R}^n)$ space and $h^p(\mathbb{R}^n)$ is preserved by pseudo differential operators $Tf(x)=\int\sigma(x,\xi)e^{2\pi ix\xi}\hat{f}(\xi)d\xi$ whose symbol σ satisfies $|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|\le c_{\alpha,\beta}(1+|\xi|)^{-|\beta|}$, that is $\sigma\in S^0_{1,0}$. There is a natural question that whether $h^p(\mathbb{R}^n)$ can be characterized by a Littlewood–Paley–Stein square function. The answer is positive, which was given by

Rychkov in [59]. Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ with nonzero integral and $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$, then $f \in h^p$ if and only if

$$\|(\sum_{j=0}^{\infty} |\varphi_j * f|^2)^{1/2}\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\mathcal{D}(\mathbb{R}^n)$ is the set of all smooth functions with compact support on $\mathbb{R}^n$.

Motivated by the classical theory of H^p and h^p spaces, in this paper, we investigate bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ via bi-parameter local reproducing formula. Actually, all results in this paper can be extended to $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})$ with arbitrary $m \in \mathbb{Z}^+$.

We use the symbol $\mathbb{N}$ to denote the class of all natural numbers, that is, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Given a function φ on $\mathbb{R}^n$, we denote $M_\varphi = \max\{N \in \mathbb{N} : \int x^\alpha \varphi(x)dx = 0, |\alpha| \le N\}$. If no moments of φ vanish, then we put $M_\varphi = -1$. For $s \in \mathbb{R}$, $[s]$ means the largest integer less than or equal to s.

For $i = 1, 2$, let φ_0^i be functions defined on $\mathbb{R}^{n_i}$ satisfying

$$\varphi_0^i \in \mathcal{D}(\mathbb{R}^{n_i}), \text{ and } \int \varphi_0^i = 1. \tag{1.1}$$

Our first main result is the following local Calderón's reproducing formula.

Theorem 1.1. *Assume that functions $\varphi_0^i, i = 1, 2$ satisfy conditions (1.1), and let $\varphi^i(x_i) = \varphi_0^i(x_i) - 2^{-n_i}\varphi_0^i(\frac{x_i}{2}), i = 1, 2$. Then for any given integers $M_i \ge 0$, there exist $\psi_0^i, \psi^i \in \mathcal{D}(\mathbb{R}^{n_i})$ with $M_{\psi^i} \ge M_i, i = 1, 2$, such that*

$$f(x) = \sum_{j,k=0}^{\infty} \psi_{j,k} * \varphi_{j,k} * f(x), \tag{1.2}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ and $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

In the above bi-parameter local reproducing formula (1.2), $\psi_{j,k}$ is constructed as follows. For $j, k \in \mathbb{N}, j, k \ge 1$, set $\psi_j^1(x_1) = 2^{jn_1}\psi^1(2^j x_1)$, $\psi_k^2(x_2) = 2^{kn_2}\psi^2(2^k x_2)$ and $\psi_{j,k}(x_1, x_2) = \psi_j^1(x_1)\psi_k^2(x_2)$, $\psi_{0,k}(x_1, x_2) = \psi_0^1(x_1)\psi_k^2(x_2)$, $\psi_{j,0}(x_1, x_2) = \psi_j^1(x_1)\psi_0^2(x_2)$, and $\psi_{0,0}(x_1, x_2) = \psi_0^1(x_1)\psi_0^2(x_2)$. $\varphi_{j,k}$ is constructed similarly.

With the above bi-parameter local reproducing formula, we give the definition of bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 1.1. Let $0 < p < \infty$. For $i = 1, 2$, suppose that φ_0^i satisfy conditions (1.1), and $\varphi^i(x) = \varphi_0^i(x) - 2^{-n_i}\varphi_0^i(\frac{x}{2})$. Then the bi-parameter local Hardy space $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2}) : \ \|f\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \|S(f)\|_{L^p(\mathbb{R}^{n_1+n_2})} < \infty\},$$

where

$$S(f)(x) = \Big(\sum_{j,k \in \mathbb{N}} |\varphi_{j,k} * f(x)|^2\Big)^{\frac{1}{2}}. \tag{1.3}$$

Remark 1.2. As pointed out in [59] in the one-parameter case, for any positive integer m, φ^i in Definition 1.1 can be chosen such that $M_{\varphi^i} \ge m$.

To see that this space is well defined, we need to show that the above definition is independent of the choice of the functions φ_0^1 and φ_0^2. This will directly follow from the following theorem.

Theorem 1.2. *Let $0 < p < \infty$. Suppose that φ_0^i, ψ_0^i satisfy conditions (1.1), and $\varphi^i(x) = \varphi_0^i(x) - 2^{-n}\varphi_0^i(\frac{x}{2})$, $\psi^i(x) = \psi_0^i(x) - 2^{-n_i}\psi_0^i(\frac{x}{2}), i = 1, 2$. Then*

$$\|\Big(\sum_{j,k\in\mathbb{N}} |\varphi_{j,k} * f|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})} \approx \|\Big(\sum_{j,k\in\mathbb{N}} |\psi_{j,k} * f|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})},$$

for every $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

In [28], Goldberg pointed out that $\mathcal{S}(\mathbb{R}^n)$ is dense in one-parameter local Hardy spaces $h^p(\mathbb{R}^n)$. Detailed proof can also be seen in [55]. This result also holds for $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Theorem 1.3. *$\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for $0 < p < \infty$.*

Atomic decomposition in multi-parameter Hardy spaces is much more complicated. It was conjectured that multi-parameter Hardy spaces can be decomposed into rectangle atoms. However, this conjecture was disproved by constructing a counter-example of a measure satisfying the product form of the Carleson measure. Carleson [2] showed that the measure he constructed is not bounded on the product Hardy space $H^1(\mathbb{R} \times \mathbb{R})$. This led to the role of cubes in the classical atomic decomposition of $H^p(\mathbb{R}^n)$ being replaced by arbitrary open sets of finite measure in the product $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ [3–5]. As to atomic decomposition of one parameter local Hardy spaces $h^p(\mathbb{R}^n)$, it was given by Goldberg in [28]. Different from the classical result, there are no vanishing moment conditions for those one parameter local atoms a with $\operatorname{supp} a \subseteq Q$ and $\ell(Q) \geq 1$, where Q is the smallest cube containing the support of a. We now give the definition of atoms of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 1.3. Let $0 < p \leq 1$. An $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atom is a function $a(x_1, x_2)$ supported in an open set $\Omega \subseteq \mathbb{R}^{n_1+n_2}$ satisfying

$$\|a\|_{L^2(\mathbb{R}^{n_1+n_2})} \leq \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{2}}}.$$

Moreover, a can be further decomposed into some rectangle-atoms a_R associated with the rectangle $R = I \times J$ which are supported in τR for some positive integer $\tau > 1$ independent of a and a_R, and such that

$$a = \sum_{R\in\mathcal{M}(\Omega)} a_R$$

with

$$\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})} \lesssim |\Omega|^{1-\frac{2}{p}},$$

where $\mathcal{M}(\Omega)$ is the set of all dyadic rectangles in Ω which are maximal in both directions of x_1 and x_2. Moreover, there exists a positive constant ϱ, for rectangle-atoms a_R associated with the rectangle $R = I \times J$, one has

$$\text{for a.e. } x_2 \in \tau J, \int a_R(x_1, x_2) x_1^\alpha dx_1 = 0, 0 < |\alpha| \leq N_p, \text{ if } \ell(I) < \varrho;$$

and

$$\text{for a.e. } x_1 \in \tau I, \int a_R(x_1, x_2) x_2^\beta dx_2 = 0, 0 < |\beta| \leq N_p, \text{ if } \ell(J) < \varrho,$$

where $N_p = [\frac{n_1}{p} - n_1] + [\frac{n_2}{p} - n_2]$.

Similarly to the classical results, every $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atom belongs to $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ due to the following theorem.

Theorem 1.4. *Let $0 < p \leq 1$. Then there exists a constant c_{p,n_1,n_2}, such that for all $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atoms a,*

$$\|a\|_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \leq c_{p,n_1,n_2}.$$

Theorem 1.5. *Let $0 < p \leq 1$. $f \in h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if and only if $f(x) = \sum_{k\in\mathbb{Z}} \lambda_k a_k(x)$ in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$, where $\sum_{k\in\mathbb{Z}} |\lambda_k|^p \leq C\|f\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$ and a_k are $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atoms. Moreover, if $f \in L^2(\mathbb{R}^{n_1+n_2})$, the series $\sum_{k\in\mathbb{Z}} \lambda_k a_k(x)$ also converges to f in $L^2(\mathbb{R}^{n_1+n_2})$.*

It is known that the theory of one parameter singular integral operators has been generalized in two ways. First, the convolution singular integral operators were replaced by non-convolution ones associated with a kernel in the following sense. A locally integral function defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ is called a one-parameter Calderón–Zygmund kernel with regularity exponent $\varepsilon > 0$ if there exists a constant $C > 0$ such that

$$|\mathcal{K}(x, y)| \leq C\frac{1}{|x - y|^n}, \text{ for } x \neq y,$$

and

$$|\mathcal{K}(x, y) - \mathcal{K}(x, y')| \leq C\frac{|y - y'|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

whenever $|y - y'| \leq \frac{1}{2}|x - y|$, and

$$|\mathcal{K}(x, y) - \mathcal{K}(x', y)| \leq C\frac{|x - x'|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

whenever $|x - x'| \leq \frac{1}{2}|x - y|$. The smallest such constant C is denoted by $\|\mathcal{K}\|_{CZ}$. We call an operator T one-parameter Calderón–Zygmund operator if T is a singular integral operator associated with a one-parameter Calderón–Zygmund kernel $\mathcal{K}(x, y)$ given by $\langle T(f), g\rangle = \int \mathcal{K}(x, y)f(y)g(x)dxdy$ for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ with disjoint supports. Define $\|T\|_{CZ}$ by $\|T\|_{CZ} = \|\mathcal{K}\|_{CZ} + \|T\|_{L^2\to L^2}$. Then T is bounded on $L^p(\mathbb{R}^n)(1 < p < +\infty)$, if $\|T\|_{CZ} < +\infty$. Moreover, to ensure the boundedness of T on local Hardy spaces h^p, $0 < p \leq 1$, additional condition

$$|\mathcal{K}(x, y)| \leq C\frac{1}{|x - y|^{n/p+\delta}},$$

for some $\delta > 0$, should be included [28].

Secondly, taking the space $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ along with two-parameter family of dilations $\delta : x \to (\delta_1 x_1, \delta_2 x_2), x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \delta = (\delta_1, \delta_2), \delta_1, \delta_2 > 0$, instead of the classical one-parameter dilation, R. Fefferman and Stein initiated the study of the product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on $\mathbb{R} \times \mathbb{R}$ [25]. After Journé introduced the non-convolution product singular integral operator in [42], R. Fefferman developed a boundedness criterion from Hardy spaces H^p to L^p in the two parameter case by considering its action on rectangle atoms. As Journe realized [44], R. Fefferman's criterion cannot apply to three or more parameter cases. Pipher [56] found a direct way to establish the Hardy H^p to L^p boundedness of Journé type singular integral operator by considering its action on atoms directly. More works about operators in Journé class and duality theory of product Hardy spaces were obtained [17,20–22,34,35,44,45].

We now give the definition of the singular integral operators in the inhomogeneous Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ suitable for our bi-parameter local Hardy spaces.

Definition 1.4. A singular integral operator T is said to be in inhomogeneous Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with regularity exponents $\varepsilon \in (0, 1]$ and $\delta > 0$ if

$$T(f)(x_1, x_2) = \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \mathcal{K}(x_1, y_1, x_2, y_2)f(y_1, y_2)dy_1dy_2,$$

when (x_1, x_2) is outside the support of function f, where the kernel $\mathcal{K}$ satisfies the following conditions:

For fixed $x_1, y_1 \in \mathbb{R}^{n_1}$, set $\tilde{\mathcal{K}}^1(x_1, y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^{n_2}$ with the kernel

$$\tilde{\mathcal{K}}^1(x_1, y_1)(x_2, y_2) = \mathcal{K}(x_1, y_1, x_2, y_2),$$

and similarly,

$$\tilde{\mathcal{K}}^2(x_2, y_2)(x_1, y_1) = \mathcal{K}(x_1, y_1, x_2, y_2),$$

then there exists a constant $C > 0$ such that

(1) T is bounded on $L^2(\mathbb{R}^{n_1+n_2})$;
(2) $\|\tilde{\mathcal{K}}^1(x_1, y_1)\|_{CZ} \le C \min\{\frac{1}{|x_1-y_1|^{n_1}}, \frac{1}{|x_1-y_1|^{n_1+\delta}}\}$, if $|x_1 - y_1| > 0$;
$\|\tilde{\mathcal{K}}^1(x_1, y_1) - \tilde{\mathcal{K}}^1(x_1, y_1')\|_{CZ} \le C \frac{|y_1-y_1'|^{\varepsilon}}{|x_1-y_1|^{n_1+\varepsilon}}$, if $|y_1 - y_1'| \le \frac{1}{2}|x_1 - y_1|$;
$\|\tilde{\mathcal{K}}^1(x_1, y_1) - \tilde{\mathcal{K}}^1(x_1', y_1)\|_{CZ} \le C \frac{|x_1-x_1'|^{\varepsilon}}{|x_1-y_1|^{n_1+\varepsilon}}$, if $|x_1 - x_1'| \le \frac{1}{2}|x_1 - y_1|$;
(3) $\|\tilde{\mathcal{K}}^2(x_2, y_2)\|_{CZ} \le C \min\{\frac{1}{|x_2-y_2|^{n_2}}, \frac{1}{|x_2-y_2|^{n_2+\delta}}\}$, if $|x_2 - y_2| > 0$;
$\|\tilde{\mathcal{K}}^2(x_2, y_2) - \tilde{\mathcal{K}}^2(x_2, y_2')\|_{CZ} \le C \frac{|y_2-y_2'|^{\varepsilon}}{|x_2-y_2|^{n_2+\varepsilon}}$, if $|y_2 - y_2'| \le \frac{1}{2}|x_2 - y_2|$;
$\|\tilde{\mathcal{K}}^2(x_2, y_2) - \tilde{\mathcal{K}}^2(x_2', y_2)\|_{CZ} \le C \frac{|x_2-x_2'|^{\varepsilon}}{|x_2-y_2|^{n_2+\varepsilon}}$, if $|x_2 - x_2'| \le \frac{1}{2}|x_2 - y_2|$.

The smallest such constant C is denoted by $\|\mathcal{K}\|_{CZ}$. The norm of the operator T is defined by $\|T\|_{CZ} = \|T\|_{L^2\to L^2} + \|\mathcal{K}\|_{CZ}$.

Theorem 1.6. *Suppose that T is a singular integral operator in inhomogeneous Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with regularity exponents $\varepsilon \in (0, 1]$ and $\delta > 0$. Then T is bounded from $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1+n_2})$ if* $\max\{\frac{n_1}{n_1+\varepsilon}, \frac{n_1}{n_1+\delta}, \frac{n_2}{n_2+\varepsilon}, \frac{n_2}{n_2+\delta}\} < p \le 1$.

We end this introduction by the following remarks. As further applications, in the subsequent papers we have completed, we will establish the boundedness of multi-parameter pseudo-differential operators and inhomogeneous Journé type singular integral operators on multi-parameter local Hardy spaces [7,15]. In [15], we establish a necessary and sufficient condition of the boundedness of inhomogeneous Journe's type operators on multi-parameter local Hardy spaces from $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. We will also give another characterization of the multi-parameter local Hardy spaces using discrete Littlewood–Paley–Stein square functions [19].

The organization of this paper is as follows. In Section 2, we prove bi-parameter local Calderón identity, that is Theorem 1.1. Using this local reproducing formula, in Section 3, we prove that bi-parameter local Hardy space $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is well defined. Some propositions of bi-parameter local Hardy space $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, such as $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = L^p(\mathbb{R}^{n_1+n_2}), p > 1$, are investigated in Section 4. In Section 5, we give the atomic decomposition of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. In the last section, as an application, we prove the boundedness from $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1+n_2})$ of the singular integral operators in inhomogeneous Journé class for p near 1.

Finally, we make some conventions. Throughout the paper, C denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as C_1, do not change in different occurrences. We denote $f \le Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

2. Local bi-parameter Calderón reproducing formula

In this section, we will establish the local bi-parameter Calderón reproducing formula, namely we will give the proof of Theorem 1.1. Such a local reproducing formula plays an important role in the development of the local bi-parameter Hardy space theory.

Proof of Theorem 1.1. Let δ^i be the Dirac function defined on $\mathbb{R}^{n_i}$, $i=1,2$. As [59], for $i=1,2$ and any $N_i\in\mathbb{N}$, denote $g_0^i=\varphi_0^i*\varphi_0^i$, $g^i=g_0^i-2^{-n}g_0^i(\frac{\cdot}{2})$ and

$$\psi_0^i=\varphi_0^i*\sum_{m=1}^{N_i}\binom{N_i}{m}(g_0^i)^{*(m-1)}*(\delta^i-g_0^i)^{*(N_i-m)},$$

$$\psi^i=(\varphi_0^i+2^{-n_i}\varphi_0^i(\frac{\cdot}{2}))*\sum_{m=1}^{N_i}\binom{N_i}{m}(g^i)^{*(m-1)}*(\delta^i-g_0^i)^{*(N_i-m)},$$

where $(\cdot)^{*k}$ is the k-fold convolution. It is easy to check that $\psi_0^i,\psi^i\in\mathcal{D}$, $M_{\psi^i}\geq N_i-2$, and $\int\psi_0^i=\int g_0^i=1$. Moreover, one has

$$\delta^1(x_1)=\sum_{j=0}^{\infty}\psi_j^1*\varphi_j^1(x_1),\quad \delta^2(x_2)=\sum_{k=0}^{\infty}\psi_k^2*\varphi_k^2(x_2),$$

and

$$\psi^i*\varphi^i(x_i)=\psi_0^i*\varphi_0^i(x_i)-2^{-n_i}\psi_0^i*\varphi_0^i(\frac{x_i}{2}),\ i=1,2. \tag{2.1}$$

Details can be seen in [59].

We now prove that the series in (1.2) converges in $\mathcal{S}(\mathbb{R}^{n_1+n_2})$.

For $f\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$, by (2.1), for any $N_1,N_2\in\mathbb{N}$, one has

$$\sum_{j=0}^{N_1}\sum_{k=0}^{N_2}\psi_{j,k}*\varphi_{j,k}*f=\big((\psi_0^1*\varphi_0^1)_{N_1}(\psi_0^2*\varphi_0^2)_{N_2}\big)*f. \tag{2.2}$$

Hence for any $L>0,|\beta|\geq 0$, since $\int\psi_0^i=\int\varphi_0^i=1,i=1,2$,

$$\begin{aligned}&\Big|(1+|x|)^L\partial^\beta\{\sum_{j=0}^{N_1}\sum_{k=0}^{N_2}\psi_{j,k}*\varphi_{j,k}*f-f\}\Big|\\&=\Big|(1+|x|)^L\int(\partial^\beta f(x-y)-\partial^\beta f(x))(\psi_0^1*\varphi_0^1)_{N_1}(y_1)(\psi_0^2*\varphi_0^2)_{N_2}(y_2)dy\Big|\\&\leq C|(1+|x|)^L\int\frac{1}{(1+|x-\varsigma|)^L}|y||(\psi_0^1*\varphi_0^1)_{N_1}(y_1)||(\psi_0^2*\varphi_0^2)_{N_2}(y_2)|dy,\end{aligned}$$

where $\varsigma\in\mathbb{R}^{n_1+n_2}$ satisfies $|\varsigma|\leq|y|$. For any fixed $x\in\mathbb{R}^{n_1+n_2}$, without loss of generality, we may assume $|x|>0$, then the compact supports of $\psi_0^1*\varphi_0^1$ and $\psi_0^2*\varphi_0^2$ imply that $|x|\geq 2|y|$ if N_1,N_2 are big enough, which implies that

$$\begin{aligned}&\Big|(1+|x|)^L\partial^\beta\{\sum_{j=0}^{N_1}\sum_{k=0}^{N_2}\psi_{j,k}*\varphi_{j,k}*f-f\}\Big|\\&\leq C\int|y||(\psi_0^1*\varphi_0^1)_{N_1}(y_1)||(\psi_0^2*\varphi_0^2)_{N_2}(y_2)|dy\to 0,\end{aligned}$$

as $N_1,N_2\to+\infty$. By the duality argument, we obtain the series in (1.2) converges in $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

Similarly, for the convergence in $L^2(\mathbb{R}^{n_1+n_2})$, by (2.2),

$$\begin{aligned}&\|\sum_{j=0}^{N_1}\sum_{k=0}^{N_2}\psi_{j,k}*\varphi_{j,k}*f-f\|_{L^2(\mathbb{R}^{n_1+n_2})}\\&=\|\big((\psi_0^1*\varphi_0^1)_{N_1}(\psi_0^2*\varphi_0^2)_{N_2}\big)*f-f\|_{L^2(\mathbb{R}^{n_1+n_2})}\\&=\|\big(1-\widehat{\varphi_0^1}(2^{-N_1}x_1)\widehat{\psi_0^1}(2^{-N_1}x_1)\widehat{\varphi_0^2}(2^{-N_2}x_2)\widehat{\psi_0^2}(2^{-N_2}x_2)\big)\hat{f}(x_1,x_2)\|_{L^2(\mathbb{R}^{n_1+n_2})}\to 0\end{aligned}$$

as $N_1,N_2\to+\infty$, by Lebesgue dominated convergence theorem, since $\int\psi_0^i=\int\varphi_0^i=1,i=1,2$.

Then we complete the proof. □

3. Well-definedness of multi-parameter local Hardy spaces

As we can see, the definition of the bi-parameter local Hardy spaces can depend on the choices of $\psi_0^i \in \mathcal{D}(\mathbb{R}^{n_i})$ with $\int \psi_0^i = 1$. The main purpose of this section is to show that the definition of the bi-parameter local Hardy spaces does not depend on the choices of such ψ_0^i. Namely, we will prove Theorem 1.2.

Lemma 3.1. *Suppose that $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $M_\varphi = s \geq 0$, then*

$$|\hat{\varphi}(x)| \leq O(|x|^{s+1}).$$

Proof. By the identity $\partial^\beta \hat{\varphi}(x) = \int (-2\pi i\xi)^\beta \varphi(\xi) e^{-2\pi i x\xi} d\xi$,

$$\partial^\beta \hat{\varphi}(0) = \int (-2\pi i\xi)^\beta \varphi(\xi) d\xi = 0, |\beta| \leq s,$$

since $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $M_\varphi = s \geq 0$. Together with Taylor's formula,

$$\hat{\varphi}(x) = \sum_{|\beta|\leq s} \frac{\partial^\beta \hat{\varphi}(0)}{\beta!} x^\beta + \sum_{|\beta|=s+1} \frac{\partial^\beta \hat{\varphi}(\zeta)}{\beta!} x^\beta,$$

which implies our desired result. □

Fix a φ. For $A_1, A_2 \geq 0$, set

$$m_{j,k,A_1,A_2}(x) = (1 + 2^j|x_1|)^{A_1}(1 + 2^k|x_2|)^{A_2},$$

and

$$M_{j,k,A_1,A_2}(x) = \sup_{j'\geq j,k'\geq k, y\in\mathbb{R}^{n_1+n_2}} 2^{(j-j')A_1} 2^{(k-k')A_2} \frac{|\varphi_{j',k'} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}.$$

Lemma 3.2. *Assume that $j, k \in \mathbb{N}$ and $\varphi_0^i \in \mathcal{D}(\mathbb{R}^{n_i}), \varphi^i \in \mathcal{D}(\mathbb{R}^{n_i})$, $i = 1, 2$. Then for any $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$, there exists a positive constant c_0 such that if $A_1, A_2 \geq c_0$, one has*

$$M_{j,k,A_1,A_2}(x) < \infty.$$

Proof. Rewrite $M_{j,k,A_1,A_2}(x)$ as

$$M_{j,k,A_1,A_2}(x) = \sup_{j'\geq j,k'\geq k, y\in\mathbb{R}^{n_1+n_2}} 2^{(j-j')A_1} 2^{(k-k')A_2} \frac{|\varphi_{j',k'} * f(y)|}{m_{j,k,A_1,A_2}(x-y)}.$$

For any $x = (x_1, x_2), z = (z_1, z_2) \in \mathbb{R}^{n_1+n_2}$, it can be seen that $m_{j,k,A_1,A_2}(z-y) \leq m_{j,k,A_1,A_2}(x-y) m_{j,k,A_1,A_2}(x-z)$. Hence

$$M_{j,k,A_1,A_2}(x) \leq M_{j,k,A_1,A_2}(z) m_{j,k,A_1,A_2}(x-z).$$

It implies that $M_{j,k,A_1,A_2}(x)$ is finite for all x or identically infinite. Therefore, we only need to consider the case $x = 0$.

$f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ implies that there exists some positive integer N such that

$$\begin{aligned} |\varphi_{j',k'} * f(y)| &= |\langle f, \varphi_{j',k'}(y-\cdot)\rangle| \\ &\leq c \sum_{|\alpha|\leq N, |\beta|\leq N} \sup_t |t^\alpha \partial_t^\beta \varphi_{j',k'}(y-t)| \end{aligned}$$

$$\begin{aligned}
&\le c \sum_{|\alpha|\le N,|\beta|\le N} \sup_t |(y-t)^\alpha \partial_t^\beta \varphi_{j',k'}(t)| \\
&\lesssim c 2^{j'n_1+j'N} 2^{k'n_2+k'N} (1+|y|)^N \\
&\le c 2^{j'n_1+j'N} 2^{k'n_2+k'N} (1+|y_1|)^N (1+|y_2|)^N.
\end{aligned}$$

Therefore

$$\begin{aligned}
&2^{(j-j')A_1} 2^{(k-k')A_2} \frac{|\varphi_{j',k'} * f(y)|}{m_{j,k,A_1,A_2}(y)} \\
&\lesssim 2^{(j-j')A_1} 2^{(k-k')A_2} 2^{j'n_1+j'N} 2^{k'n_2+k'N} \frac{(1+|y_1|)^N (1+|y_2|)^N}{(1+2^j|y_1|)^{A_1}(1+2^k|y_2|)^{A_2}} \\
&\le 2^{-j'A_1} 2^{-k'A_2} 2^{j'n_1+j'N} 2^{k'n_2+k'N} \frac{(1+|y_1|)^N (1+|y_2|)^N}{(2^{-j}+|y_1|)^{A_1}(2^{-k}+2^k|y_2|)^{A_2}},
\end{aligned}$$

which yields our desired results when $A_1, A_2 \ge n_1+n_2+N$, since j,k are fixed.

Thus we have completed the proof. □

Lemma 3.3. *Let $0<r<\infty$, $\varphi_0^i \in \mathcal{D}(\mathbb{R}^{n_i})$ with nonzero integrals and $\varphi^i = \varphi_0^i - 2^{-n_i}\varphi_0^i(\frac{\cdot}{2})$, $i=1,2$. Then for every $A_1, A_2 \ge 0$, there exists a constant c depending only on $n_1, n_2, r, \varphi_0^1, \varphi_0^2, A_1$ and A_2 such that, for all $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$, $x \in \mathbb{R}^{n_1+n_2}$ and $j,k \in \mathbb{N}$, we have*

$$|\varphi_{j,k} * f(x)|^r \le c \sum_{j'=j}^{\infty} \sum_{k'=k}^{\infty} 2^{(j-j')A_1 r} 2^{j'n_1} 2^{(k-k')A_2 r} 2^{k'n_2} \int \frac{|\varphi_{j',k'} * f(x-y)|^r}{m_{j,k,A_1 r,A_2 r}(y)} dy. \tag{3.1}$$

Proof. By Theorem 1.1, there exist $\psi_0^i, \psi^i \in \mathcal{D}(\mathbb{R}^{n_i})$ with $M_{\psi^i} \ge 2A_i, i=1,2$ such that

$$f(x) = \sum_{j',k'=0}^{\infty} \psi_{j',k'} * \varphi_{j',k'} * f(x).$$

Decompose f as follows

$$\begin{aligned}
f(x) &= \Big(\sum_{j'=0}^{j}\sum_{k'=0}^{k} + \sum_{j'=0}^{j}\sum_{k'=k+1}^{\infty} + \sum_{j'=j+1}^{\infty}\sum_{k'=0}^{k} + \sum_{j'=j+1}^{\infty}\sum_{k'=k+1}^{\infty}\Big)\psi_{j',k'} * \varphi_{j',k'} * f(x) \\
&= (\psi_0)_{j,k} * (\varphi_0)_{j,k} * f(x) + \sum_{k'=k+1}^{\infty} \Big([(\psi_0^1)_j * (\varphi_0^1)_j][\psi_{k'}^2 * \varphi_{k'}^2]\Big) * f(x) \\
&\quad + \sum_{j'=j+1}^{\infty} \Big([\psi_{j'}^1 * \varphi_{j'}^1][(\psi_0^2)_k * (\varphi_0^2)_k]\Big) * f(x) + \sum_{j'=j+1}^{\infty}\sum_{k'=k+1}^{\infty} \psi_{j',k'} * \varphi_{j',k'} * f(x) \\
&= f_1 + f_2 + f_3 + f_4,
\end{aligned}$$

by (2.1), where $\psi_0(x_1,x_2) = \psi_0^1(x_1)\psi_0^2(x_2)$, $\varphi_0(x_1,x_2) = \varphi_0^1(x_1)\varphi_0^2(x_2)$.

We first estimate $\varphi_{j,k} * f_1$ and $\varphi_{j,k} * f_2$. It is easy to see that

$$\begin{aligned}
\varphi_{j,k} * f_2 &= \sum_{k'=k+1}^{\infty} \Big([(\psi_0^1)_j * (\varphi_0^1)_j][\psi_{k'}^2 * \varphi_{k'}^2]\Big) * \varphi_{j,k} * f(x) \\
&= \sum_{k'=k+1}^{\infty} \Big([(\psi_0^1)_j * (\varphi_0^1)_j][\psi_{k'}^2 * \varphi_k^2]\Big) * \varphi_{j,k'} * f(x).
\end{aligned}$$

Note that when $k' \ge k$, the function $\psi_{k'}^2 * \varphi_k^2$ has compact support $\{x_2 : |x_2| \le c2^{-k}\}$ and satisfies

$$\|\psi_{k'}^2 * \varphi_k^2\|_{L^\infty} \le c 2^{(k-k')2A_2} 2^{kn_2}$$

by the vanishing moment condition on ψ^2. At the same time, one also has $\mathrm{supp}((\psi_0^1)_j * (\varphi_0^1)_j) \subseteq \{x_1 : |x_1| \leq c2^{-j}\}$ and

$$\|(\psi_0^1)_j * (\varphi_0^1)_j\|_{L^\infty} \leq c2^{jn_1}.$$

By above estimates, one has

$$|([(\psi_0^1)_j * (\varphi_0^1)_j][\psi_{k'}^2 * \varphi_k^2])(y)| \leq c\frac{2^{jn_1}}{m_{j,A_1}(y_1)}\frac{2^{(k-k')A_2}2^{kn_2}}{m_{k,A_2}(y_2)},$$

and

$$|(\psi_0)_{j,k} * (\varphi_0)_{j,k}(y)| \leq c\frac{2^{jn_1}}{m_{j,A_1}(y_1)}\frac{2^{kn_2}}{m_{k,A_2}(y_2)},$$

where $m_{j,A_1}(y_1) = (1 + 2^j|y_1|)^{A_1}$, $m_{k,A_2}(y_2) = (1 + 2^k|y_2|)^{A_2}$. Therefore

$$\begin{aligned}|\varphi_{j,k} * f_2(x)| &= |\sum_{k'=k+1}^{\infty}\int([(\psi_0^1)_j * (\varphi_0^1)_j][\psi_{k'}^2 * \varphi_k^2])(y)\varphi_{j,k'} * f(x-y)dy| \\ &\leq c\sum_{k'=k+1}^{\infty}2^{(k-k')2A_2}2^{kn_2}2^{jn_1}\int_{B_{j,k}}\frac{|\varphi_{j,k'} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}dy, \end{aligned} \tag{3.2}$$

and

$$|\varphi_{j,k} * f_1(x)| \leq 2^{jn_1}2^{kn_2}\int_{B_{j,k}}\frac{|\varphi_{j,k} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}dy, \tag{3.3}$$

where $B_{j,k} = \{x \in \mathbb{R}^{n_1+n_2} : |x_1| \leq c2^{-j}, |x_2| \leq c2^{-k}\}$.

Similar processes for f_3 and f_4 respectively give that

$$\begin{aligned}|\varphi_{j,k} * f_3(x)| &= |\sum_{j'=j+1}^{\infty}\Big([\psi_{j'}^1 * \varphi_{j'}^1][(\psi_0^2)_k * (\varphi_0^2)_k]\Big) * \varphi_{j,k} * f(x)| \\ &= |\sum_{j'=j+1}^{\infty}\Big([\psi_{j'}^1 * \varphi_j^1][(\psi_0^2)_k * (\varphi_0^2)_k]\Big) * \varphi_{j',k} * f(x)| \\ &\leq c\sum_{j'=j+1}^{\infty}2^{(j-j')2A_1}2^{jn_1}2^{kn_2}\int_{B_{j,k}}\frac{|\varphi_{j',k} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}dy \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}|\varphi_{j,k} * f_4(x)| &= |\sum_{j'=j+1}^{\infty}\sum_{k'=k+1}^{\infty}\psi_{j',k'} * \varphi_{j',k'} * \varphi_{j,k} * f(x)| \\ &= |\sum_{j'=j+1}^{\infty}\sum_{k'=k+1}^{\infty}\psi_{j',k'} * \varphi_{j,k} * \varphi_{j',k'} * f(x)| \\ &\leq c\sum_{j'=j+1}^{\infty}\sum_{k'=k+1}^{\infty}2^{(j-j')2A_1}2^{jn_1}2^{(k-k')2A_2}2^{kn_2}\int_{B_{j,k}}\frac{|\varphi_{j',k'} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}dy. \end{aligned} \tag{3.5}$$

It can be seen that (3.2)–(3.5) yield that

$$\begin{aligned}&|\varphi_{j,k} * f(x)| \\ &\leq c\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}2^{(j-j')2A_1}2^{jn_1}2^{(k-k')2A_2}2^{kn_2}\int_{B_{j,k}}\frac{|\varphi_{j',k'} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}dy \qquad (3.6) \\ &\leq c\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}2^{(j-j')A_1}2^{j'n_1}2^{(k-k')A_2}2^{k'n_2}\int_{B_{j,k}}\frac{|\varphi_{j',k'} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}dy. \qquad (3.7)\end{aligned}$$

If $r > 1$, applying Hölder's inequality and the Cauchy–Schwarz inequality to (3.6) one has

$$\begin{aligned}
&|\varphi_{j,k} * f(x)| \\
&\le c\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{(j-j')2A_1}2^{jn_1}2^{(k-k')2A_2}2^{kn_2}|B_{j,k}|^{1/r'}\Big(\int_{B_{j,k}}\frac{|\varphi_{j',k'} * f(x-y)|^r}{m_{j,k,A_1r,A_2r}(y)}dy\Big)^{1/r} \\
&= c\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{(j-j')2A_1}2^{(k-k')2A_2}2^{(kn_2)/r}2^{(jn_1)/r}\Big(\int_{B_{j,k}}\frac{|\varphi_{j',k'} * f(x-y)|^r}{m_{j,k,A_1r,A_2r}(y)}dy\Big)^{1/r} \\
&\le c\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{(j-j')A_1r}2^{(k-k')A_2r}2^{jn_1}2^{kn_2}\int\frac{|\varphi_{j',k'} * f(x-y)|^r}{m_{j,k,A_1r,A_2r}(y)}dy\Big)^{1/r} \\
&\le c\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{(j-j')A_1r}2^{(k-k')A_2r}2^{j'n_1}2^{k'n_2}\int\frac{|\varphi_{j',k'} * f(x-y)|^r}{m_{j,k,A_1r,A_2r}(y)}dy\Big)^{1/r}.
\end{aligned}$$

To deal with the case $r < 1$, we use the technique of Strömberg–Torchinsky in [64], and recall that

$$M_{j,k,A_1,A_2}(x) = \sup_{j'\ge j,k'\ge k,y\in\mathbb{R}^{n_1+n_2}} 2^{(j-j')A_1}2^{(k-k')A_2}\frac{|\varphi_{j',k'} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}.$$

By (3.7), for every $j', k' \in \mathbb{N}$ with $j' \ge j, k' \ge k$, one has

$$\begin{aligned}
&2^{(j-j')A_1}2^{(k-k')A_2}|\varphi_{j',k'} * f(x-y)| \\
&\le c\sum_{l=j'}^{\infty}\sum_{i=k'}^{\infty} 2^{(j-l)A_1}2^{ln_1}2^{(k-i)A_2}2^{in_2}\int\frac{|\varphi_{l,i} * f(x-y-z)|}{m_{j',k',A_1,A_2}(z)}dz \\
&\le c\sum_{l=j'}^{\infty}\sum_{i=k'}^{\infty} 2^{(j-l)A_1}2^{ln_1}2^{(k-i)A_2}2^{in_2}\int\frac{|\varphi_{l,i} * f(x-y-z)|}{m_{j,k,A_1,A_2}(z)}dz \\
&\le c\sum_{l=j}^{\infty}\sum_{i=k}^{\infty} 2^{(j-l)A_1}2^{ln_1}2^{(k-i)A_2}2^{in_2}\int\frac{|\varphi_{l,i} * f(x-z)|}{m_{j,k,A_1,A_2}(z-y)}dz.
\end{aligned}$$

Combining with

$$m_{j,k,A_1,A_2}(z) \le m_{j,k,A_1,A_2}(y)m_{j,k,A_1,A_2}(z-y), \tag{3.8}$$

we have

$$M_{j,k,A_1,A_2}(x) \le c\sum_{l=j}^{\infty}\sum_{i=k}^{\infty} 2^{(j-l)A_1}2^{ln_1}2^{(k-i)A_2}2^{in_2}\int\frac{|\varphi_{l,i} * f(x-z)|}{m_{j,k,A_1,A_2}(z)}dz.$$

Therefore

$$\begin{aligned}
&M_{j,k,A_1,A_2}(x) \\
&\le c\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{j'n_1}2^{k'n_2}\int\Big\{\Big(\frac{2^{(j-j')A_1}2^{(k-k')A_2}|\varphi_{j',k'} * f(x-z)|}{m_{j,k,A_1,A_2}(z)}\Big)^r \\
&\qquad\qquad \times\Big(\frac{2^{(j-j')A_1}2^{(k-k')A_2}|\varphi_{j',k'} * f(x-z)|}{m_{j,k,A_1,A_2}(z)}\Big)^{1-r}\Big\}dz \\
&\le c(M_{j,k,A_1,A_2}(x))^{1-r}\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{j'n_1}2^{k'n_2}\int\frac{(2^{(j-j')A_1}2^{(k-k')A_2}|\varphi_{j',k'} * f(x-z)|)^r}{(m_{j,k,A_1,A_2}(z))^r}dz.
\end{aligned}$$

By Lemma 3.2, if A_1, A_2 are big enough, $M_{j,k,A_1,A_2}(x) < \infty$. Hence, we have

$$(M_{j,k,A_1,A_2}(x))^r \le c\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty} 2^{j'n_1}2^{k'n_2}\int\frac{(2^{(j-j')A_1}2^{(k-k')A_2}|\varphi_{j',k'} * f(x-z)|)^r}{(m_{j,k,A_1,A_2}(z))^r}dz,$$

which gives (3.1), since $\varphi_{j,k} * f(x) \le M_{j,k,A_1,A_2}(x)$. Note that the right side of (3.1) increases as A_1, A_2 decrease, then we have completed the proof. □

Lemma 3.4. *For any $\theta, \eta, \tau > 0$, there exists a constant $c = c_{\theta,\eta,\tau}$ such that*

$$\sum_{j,k=0}^{\infty} 2^{j\theta}2^{k\eta}\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}|b_{j',k'}|\Big)^{\tau} \le c\sum_{j,k=0}^{\infty} 2^{j\theta}2^{k\eta}|b_{j,k}|^{\tau},$$

where $\{b_{j,k}\}$ is any bi-indexed sequence of numbers.

Proof. The case $0 < \tau \le 1$ is immediate. When $\tau > 1$, by Cauchy Schwartz inequality, one has

$$\begin{aligned}
&\sum_{j,k=0}^{\infty} 2^{j\theta}2^{k\eta}\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}|b_{j',k'}|\Big)^{\tau}\\
&= \sum_{j,k=0}^{\infty} 2^{j\theta}2^{k\eta}\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}2^{(j-j')\theta/(2\tau)}2^{(k-k')\eta/(2\tau)}|b_{j',k'}|2^{(j'-j)\theta/(2\tau)}2^{(k'-k)\eta/(2\tau)}\Big)^{\tau}\\
&\le \sum_{j,k=0}^{\infty} 2^{j\theta}2^{k\eta}\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}2^{(j-j')\theta\tau'/(2\tau)}2^{(k-k')\eta\tau'/(2\tau)}\Big)^{\tau/\tau'}\Big(\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}|b_{j',k'}|^{\tau}2^{(j'-j)\theta/2}2^{(k'-k)\eta/2}\Big)\\
&\le c_{\theta,\eta,\tau}\sum_{j,k=0}^{\infty} 2^{j\theta}2^{k\eta}\sum_{j'=j}^{\infty}\sum_{k'=k}^{\infty}|b_{j',k'}|^{\tau}2^{(j'-j)\theta/2}2^{(k'-k)\eta/2}\\
&= c_{\theta,\eta,\tau}\sum_{j',k'=0}^{\infty} 2^{j'\theta/2}2^{k'\eta/2}|b_{j',k'}|^{\tau}\sum_{j=0}^{j'}\sum_{k=0}^{k'}2^{j\theta/2}2^{k\eta/2}\\
&\le c_{\theta,\eta,\tau}\sum_{j',k'=0}^{\infty} 2^{j'\theta}2^{k'\eta}|b_{j',k'}|^{\tau},
\end{aligned}$$

where $\tau' = \frac{\tau}{\tau-1}$. Thus we have completed the proof. □

The following result is obvious.

Lemma 3.5. *Assume that ϕ^i are radially decreasing, nonnegative integrable functions on $\mathbb{R}^{n_i}$, $i = 1, 2$. Let $\phi_{j,k}(x) = 2^{jn_1}2^{kn_2}\phi^1(2^j x_1)\phi^2(2^k x_2)$, and*

$$\Phi^* f(x) = \sup_{j,k\in\mathbb{Z}} |\phi_{j,k} * f(x)|.$$

Then there exists a constant $c > 0$ such that

$$\Phi^* f(x) \le c\|\phi^1\|_{L^1}\|\phi^2\|_{L^1} M_s f(x),$$

where M_s is the strong maximal operator.

Denote $\varphi^*_{j,k,A_1,A_2} f(x)$ by

$$\varphi^*_{j,k,A_1,A_2} f(x) = \sup_{y\in\mathbb{R}^{n_1+n_2}} \frac{|\varphi_{j,k} * f(x-y)|}{m_{j,k,A_1,A_2}(y)}.$$

Lemma 3.6. *Let $r, \varphi_0^i, \varphi^i, i = 1, 2$ be as in Lemma 3.3 with $A_1 > n_1/r, A_2 > n_2/r$. Then there is a constant c depending only on n_1, n_2, r, A_1, A_2 such that for all $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$, $j, k \in \mathbb{N}$, we have*

$$\varphi^*_{j,k,A_1,A_2} f(x)^r \le c \sum_{j'=j}^{\infty} \sum_{k'=k}^{\infty} 2^{(j-j')(A_1 r - n_1)} 2^{(k-k')(A_2 r - n_2)} M_s(|\varphi_{j',k'} * f|^r)(x).$$

Proof. Without loss of generality, we assume $r \ge 1$. By (3.1) and (3.8) it is easy to have

$$\begin{aligned}
\varphi^*_{j,k,A_1,A_2} f(x)^r &= \sup_{y \in \mathbb{R}^{n_1+n_2}} \frac{|\varphi_{j,k} * f(x-y)|^r}{m_{j,k,A_1,A_2}(y)} \\
&\le c \sup_{y \in \mathbb{R}^{n_1+n_2}} \sum_{j'=j}^{\infty} \sum_{k'=k}^{\infty} 2^{(j-j')A_1 r} 2^{j' n} 2^{(k-k')A_2 r} 2^{k' n} \\
&\qquad \frac{1}{m_{j,k,A_1,A_2}(y)} \int \frac{|\varphi_{j',k'} * f(x-z)|^r}{m_{j,k,A_1 r, A_2 r}(z-y)} dz \\
&\le c \sup_{y \in \mathbb{R}^{n_1+n_2}} \sum_{j'=j}^{\infty} \sum_{k'=k}^{\infty} 2^{(j-j')A_1 r} 2^{j' n_1} 2^{(k-k')A_2 r} 2^{k' n_2} \int \frac{|\varphi_{j',k'} * f(x-z)|^r}{m_{j,k,A_1,A_2}(z)} dz.
\end{aligned}$$

Then Lemma 3.5 gives that

$$\varphi^*_{j,k,A_1,A_2} f(x)^r \le c \sum_{j'=j}^{\infty} \sum_{k'=k}^{\infty} 2^{(j-j')(A_1 r - n_1)} 2^{(k-k')(A_2 r - n_2)} M_s(|\varphi_{j',k'} * f|^r)(x).$$

Thus we have completed the proof. □

Proof of Theorem 1.2. For $\psi_0^i \in \mathcal{D}(\mathbb{R}^{n_i})$ with $\int \psi_0^i = 1$, by Theorem 1.1, we can take ϕ_0^i, ϕ^i with large $M_{\phi^i}, i = 1, 2$, such that (1.2) holds. Hence

$$\begin{aligned}
|\varphi_{j,k} * f(x)| &\le \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} \int |\varphi_{j,k} * \phi_{j',k'}(y)| |\psi_{j',k'} * f(x-y)| dy \\
&\le \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} \psi^*_{j',k',A_1,A_2} f(x) \int |\varphi_{j,k} * \phi_{j',k'}(y)| m_{j',k',A_1,A_2}(y) dy.
\end{aligned}$$

We now estimate $\int |\varphi_j^1 * \phi_{j'}^1(y_1)| m_{j',A_1}(y_1) dy_1$. As before, one has

$$\|\varphi_j^1 * \phi_{j'}^1\|_{L^\infty} \le \begin{cases} c 2^{(j-j')(M_{\phi^1}+1)} 2^{j n_1}, & j < j', \\ 2^{(j'-j)(M_{\varphi^1}+1)} 2^{j' n_1}, & j \ge j', \end{cases}$$

and $\mathrm{supp}(\varphi_j^1 * \phi_{j'}^1) \subseteq \{x_1 : |x_1| \le c \max\{2^{-j}, 2^{-j'}\}\}$. Therefore

$$\int |\varphi_j^1 * \phi_{j'}^1(y_1)| m_{j',A_1}(y_1) dy_1 \le \begin{cases} c 2^{(j-j')(M_{\phi^1}+1)} 2^{j n_1} 2^{(j'-j)A_1} 2^{-j n_1}, & j < j', \\ c 2^{(j'-j)(M_{\varphi^1}+1)} 2^{j' n_1} 2^{-j' n_1}, & j \ge j', \end{cases}$$

that is

$$\int |\varphi_j^1 * \phi_{j'}^1(y_1)| m_{j',A_1}(y_1) dy_1 \le \begin{cases} c 2^{(j-j')(M_{\phi^1}+1-A_1)}, & j < j', \\ c 2^{(j'-j)(M_{\varphi^1}+1)}, & j \ge j'. \end{cases}$$

Similar estimates can apply to $\int |\varphi_k^2 * \phi_{k'}^2(y_2)| m_{k',A_2}(y_2) dy_2$. Choosing $M_{\phi^i} \ge A_i$, and taking into account $M_{\varphi^i} \ge 0$, we have

$$|\varphi_{j,k} * f(x)| \le c \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|\varepsilon} 2^{-|k-k'|\varepsilon} \psi^*_{j',k',A_1,A_2} f(x)$$

with some constant $\varepsilon > 0$. Then by Hölder's inequality, we have

$$\|\Big(\sum_{j,k\in\mathbb{N}} |\varphi_{j,k} * f|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})} \lesssim \|\Big(\sum_{j',k'\in\mathbb{N}} (\psi^*_{j',k',A_1,A_2} f)^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})}. \tag{3.9}$$

By Lemma 3.6, for every $r > 0$ satisfying $A_1, A_2 > n/r$, one has

$$\begin{aligned}
&\sum_{j',k'\in\mathbb{N}} (\psi^*_{j',k',A_1,A_2} f)^2 \\
&= \sum_{j',k'\in\mathbb{N}} \Big((\psi^*_{j',k',A_1,A_2} f)^r\Big)^{2/r} \\
&\le c \sum_{j',k'\in\mathbb{N}} \Big(\sum_{l=j'}^{\infty}\sum_{i=k'}^{\infty} 2^{(j'-l)(A_1 r-n_1)} 2^{(k'-i)(A_2 r-n_2)} M_s(|\psi_{l,i} * f|^r)(x)\Big)^{2/r} \\
&= c \sum_{j',k'\in\mathbb{N}} 2^{j'(A_1 r-n_1)2/r + k'(A_2 r-n_2)2/r} \Big(\sum_{l=j'}^{\infty}\sum_{i=k'}^{\infty} 2^{-l(A_1 r-n_1)} 2^{-i(A_2 r-n_2)} M_s(|\psi_{l,i} * f|^r)(x)\Big)^{2/r},
\end{aligned}$$

which, combined with Lemma 3.4, gives that

$$\begin{aligned}
&\sum_{j',k'\in\mathbb{N}} (\psi^*_{j',k',A_1,A_2} f)^2 \\
&\le c \sum_{j',k'\in\mathbb{N}} 2^{j'(A_1 r-n_1)2/r + k'(A_2 r-n_2)2/r} \Big(2^{-j'(A_1 r-n_1)} 2^{-k'(A_2 r-n_2)} M_s(|\psi_{j',k'} * f|^r)(x)\Big)^{2/r} \\
&= c \sum_{j',k'\in\mathbb{N}} \Big(M_s(|\psi_{j',k'} * f|^r)(x)\Big)^{2/r}.
\end{aligned}$$

Putting it into (3.9), and applying Fefferman–Stein's vector-valued strong maximal function inequality on $L^{p/r}(\ell^{2/r})$, provided $r < \max\{p, 2\}$, we obtain that

$$\|\Big(\sum_{j,k\in\mathbb{N}} |\varphi_{j,k} * f|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})} \lesssim \|\Big(\sum_{j,k\in\mathbb{N}} |\psi_{j,k} * f|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})}.$$

Thus we have completed the proof. □

Actually, from the above proof, one can obtain the discrete characterization of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. For $i = 1, 2$ and any $j \in \mathbb{Z}$, let

$$\Pi_j^{n_i} = \{\prod_{i=1}^{n_i} 2^{-j}[\ell_i, \ell_i + 1) : (\ell_1, \dots, \ell_{n_i}) \in \mathbb{Z}^{n_i}\}.$$

That is, $\Pi_j^{n_i}$ is the set of dyadic cubes in $\mathbb{R}^{n_i}$ with the side length $l(I) = 2^{-j}$. For convenience, the left lower corners of $I \in \Pi_j^{n_i}$ are denoted by x_I. At last, set $\Pi_{j,k} = \Pi_j^{n_1} \times \Pi_k^{n_2}$, and $\Pi = \cup_{j,k\in\mathbb{N}} \Pi_{j,k}$.

Corollary 3.7. *Let $0 < p < \infty$. Suppose that $\varphi_0^i \in \mathcal{D}(\mathbb{R}^{n_i})$ with $\int \varphi_0^i = 1$, and $\varphi^i = \varphi_0^i - 2^{-n}\varphi_0^i(\frac{\cdot}{2})$, $i = 1, 2$. Then*

$$\|\Big(\sum_{j,k\in\mathbb{N}} |\varphi_{j,k} * f|^2\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})} \approx \|\Big(\sum_{j,k\in\mathbb{N}} \sum_{R\in\Pi_{j,k}} \max_{z\in R} |\varphi_{j,k} * f(z)|^2 \chi_R\Big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^{n_1+n_2})}.$$

Proof. We only need to prove " $\gtrsim$ ". For $\varphi_0^i \in \mathcal{D}(\mathbb{R}^{n_i})$, by Theorem 1.1, we can take ϕ_0^i, ϕ^i with large $M_{\phi^i}, i = 1, 2$ such that (1.2) holds. Hence for any $x, z \in R \in \Pi_{j,k}$, one has

$$\begin{aligned}
|\varphi_{j,k} * f(z)| &\le \sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty}\int |\varphi_{j,k} * \phi_{j',k'}(y)||\varphi_{j',k'} * f(z-y)|dy \\
&= \sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty}\int |\varphi_{j,k} * \phi_{j',k'}(y)| m_{j',k',A_1,A_2}(y-z+x)\frac{|\varphi_{j',k'} * f(z-y)|}{m_{j',k',A_1,A_2}(y-z+x)}dy \\
&\le \sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty}\varphi^*_{j',k',A_1,A_2}f(x)\int |\varphi_{j,k} * \phi_{j',k'}(y)| m_{j',k',A_1,A_2}(y-z+x)dy.
\end{aligned}$$

It is fortunate that same estimates in the proof of Theorem 1.2 hold for $\int |\varphi_j^1 * \phi_{j'}^1(y_1)| m_{j',A_1}(y_1 - z_1 + x_1)dy_1$. Indeed, when $j < j'$,

$$\begin{aligned}
&\int |\varphi_j^1 * \phi_{j'}^1(y_1)| m_{j',A_1}(y_1 - z_1 + x_1)dy_1 \\
&\lesssim 2^{(j-j')(M_{\phi^1}+1)}2^{jn_1}\int_{|y_1|\le c2^{-j}}(1 + 2^{j'}|y_1 - z_1 + x_1|)^{A_1}dy_1 \\
&\lesssim 2^{(j-j')(M_{\phi^1}+1)}2^{jn_1}2^{(j'-j)A_1}2^{-jn_1},
\end{aligned}$$

since $|x_1 - z_1| \lesssim 2^{-j}$. At the same time, if $j \ge j'$ one has

$$\int |\varphi_j^1 * \phi_{j'}^1(y_1)| m_{j',A_1}(y_1 - z_1 + x_1)dy_1 \lesssim 2^{(j'-j)(M_{\varphi^1}+1)}.$$

Therefore

$$\begin{aligned}
&\sum_{R\in\Pi_{j,k}}\max_{z\in R}|\varphi_{j,k} * f(z)|\chi_R(x) \\
&\lesssim \sum_{R\in\Pi_{j,k}}\max_{z\in R}\sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty}\varphi^*_{j',k',A_1,A_2}f(x)2^{-|j-j'|\theta_1}2^{-|k-k'|\theta_2}\chi_R(x) \\
&= \sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty}\varphi^*_{j',k',A_1,A_2}f(x)2^{-|j-j'|\theta_1}2^{-|k-k'|\theta_2},
\end{aligned}$$

for some $\theta_i > 0, i = 1, 2$. Hence,

$$\begin{aligned}
\sum_{R\in\Pi_{j,k}}\max_{z\in R}|\varphi_{j,k} * f(z)|^2\chi_R(x) &= \Big(\sum_{R\in\Pi_{j,k}}\max_{z\in R}|\varphi_{j,k} * f(z)|\chi_R(x)\Big)^2 \\
&\lesssim \Big(\sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty}\varphi^*_{j',k',A_1,A_2}f(x)2^{-|j-j'|\theta_1}2^{-|k-k'|\theta_2}\Big)^2
\end{aligned}$$

Repeating the rest of the proof for Theorem 1.2, we can complete the proof. □

4. Density of Schwartz functions in bi-parameter local Hardy spaces

In this section, we will show that the bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are the same as the Lebesgue spaces $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if $p > 1$ and the Schwartz function space $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ are dense in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}), 0 < p < \infty$.

Proposition 4.1. *Under the same assumptions of Definition 1.1, $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}), 0<p<\infty$.*

Proof. Firstly, if $f\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$, by classical one-parameter almost orthogonality, for any large positive integer L,

$$|\varphi_{j,k}*f(x,y)|\le C\frac{2^{-j}}{(1+|x|)^L}\frac{2^{-k}}{(1+|y|)^L} \tag{4.1}$$

using the vanishing moment conditions of φ^1,φ^2. Then

$$\begin{aligned}\|f\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} &= \|\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*f(x,y)|^2\Big)^{\frac12}\|^p_{L^p}\\ &\le C\int\frac{1}{(1+|x_1|)^{pL}}\frac{1}{(1+|x_2|)^{pL}}dx_1dx_2\le C,\end{aligned}$$

that is $f\in h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}), 0<p<\infty$.

Let $f\in h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. For any fixed $N>0$, denote

$$E=\{(j,k):0\le j\le N, 0\le k\le N\},$$

and

$$f_N(x_1,x_2):=\sum_{(j,k)\in E}\psi_{j,k}*\varphi_{j,k}*f(x),$$

where $\psi_{j,k},\varphi_{j,k}$ are the same as in Theorem 1.1. It is easy to see that $f_N\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$. Repeating the proof of Theorem 1.2, we can conclude that

$$\|f_N\|_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\lesssim\|f\|_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Moreover, it is easy to see that

$$|\varphi_{j',k'}*(f-f_N)(x)|\le\sum_{(j,k)\in E^c}\int|\varphi_{j',k'}*\psi_{j,k}(y)||\varphi_{j,k}*f(x-y)|dy,$$

then one can repeat the proof of Theorem 1.2 again to obtain

$$\|f-f_N\|_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\lesssim\|\Big(\sum_{j,k\in E^c}|\varphi_{j,k}*f|^2\Big)^{\frac12}\|_{L^p(\mathbb{R}^{n_1+n_2})}\to0$$

as N tends to infinity. This implies that f_N tends to f in $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ as N tends to infinity. Thus we have completed the proof. □

By iteration method, one can use vector-valued Calderón–Zygmund theorem to obtain the following result.

Proposition 4.2. *Let $1<p<\infty$. Then*

$$\|S(f)\|_{L^p(\mathbb{R}^{n_1+n_2})}\approx\|f\|_{L^p(\mathbb{R}^{n_1+n_2})},$$

that is, $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})=L^p(\mathbb{R}^{n_1+n_2})$.

Proposition 4.3. *Let $0<p\le1$, then we have*

(1) Convergence in $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ implies convergence in $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

(2)$h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is a complete normed metrizable space.

Proof. Let $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ and $g \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$. By (1.2),

$$\begin{aligned}
|\langle f, g\rangle| &= |\sum_{j,k\in\mathbb{N}} \langle \psi_{j,k} * \varphi_{j,k} * f, g\rangle| \\
&= |\sum_{j,k\in\mathbb{N}} \int \varphi_{j,k} * f(y)\psi_{j,k} * \bar{g}(y)dy| \\
&= |\sum_{j,k\in\mathbb{N}} \sum_{R\in\Pi_{j,k}} \int_R \varphi_{j,k} * f(y)\psi_{j,k} * \bar{g}(y)dy| \\
&\le \sum_{j,k\in\mathbb{N}} \sum_{R\in\Pi_{j,k}} |R| \max_{z\in R} |\varphi_{j,k} * f(z)| \max_{z\in R} |\psi_{j,k} * \bar{g}(z)|.
\end{aligned}$$

Set $\tilde{S}(f)(x) = \Big(\sum_{j,k\in\mathbb{N}} \sum_{R\in\Pi_{j,k}} \max_{z\in R} |\varphi_{j,k} * f(z)|^2 \chi_R\Big)^{\frac{1}{2}}$. For any $i \in \mathbb{Z}$, denote

$$\Omega_i = \{x \in \mathbb{R}^{n_1+n_2} : \tilde{S}(f)(x) > 2^i\}, \tilde{\Omega}_i = \{x \in \mathbb{R}^{n_1+n_2} : M_s(\chi_{\Omega_i}) > \frac{1}{10^{n_1+n_2}}\},$$

and

$$\mathcal{B}_i = \{R : R \in \Pi,\ |R \cap \Omega_i| > \frac{1}{2}|R|,\ \ |R \cap \Omega_{i+1}| \le \frac{1}{2}|R|\}.$$

It is easy to see that $\cup_{R\in\mathcal{B}_i} R \subseteq \tilde{\Omega}_i$.

For $R \in \Pi_{j,k}$, denote $\varphi_R = \varphi_{j,k}$ and $\psi_R = \varphi_{j,k}$. Then

$$\begin{aligned}
|\langle f, g\rangle| &\le \Big\{ \sum_{i=-\infty}^{+\infty} \Big(\sum_{R\in\mathcal{B}_i} |R| \max_{z\in R} |\varphi_{j,k} * f(z)|^2 \Big)^{p/2} \\
&\qquad \Big(\sum_{R\in\mathcal{B}_i} |R| \max_{z\in R} |\psi_{j,k} * \bar{g}(z)|^2 \Big)^{p/2} \Big\}^{1/p} \\
&= \Big\{ \sum_{i=-\infty}^{+\infty} \big(|\tilde{\Omega}_i|^{\frac{2}{p}-1} \sum_{R\in\mathcal{B}_i} |R| \max_{z\in R} |\varphi_{j,k} * f(z)|^2\big)^{p/2} \\
&\qquad \Big(\frac{1}{|\tilde{\Omega}_i|^{\frac{2}{p}-1}} \sum_{R\in\mathcal{B}_i} |R| \max_{z\in R} |\psi_{j,k} * \bar{g}(z)|^2 \Big)^{p/2} \Big\}^{1/p} \\
&\le \|\bar{g}\|_{cmo_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \Big\{ \sum_{i=-\infty}^{+\infty} |\tilde{\Omega}_i|^{1-\frac{p}{2}} \Big(\sum_{R\in\mathcal{B}_i} |R| \max_{z\in R} |\varphi_{j,k} * f(z)|^2 \Big)^{p/2} \Big\}^{1/p},
\end{aligned}$$

where

$$\|g\|_{cmo_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \sup_{\Omega} \Big(\frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_\Omega \sum_{j,k\in\mathbb{N}} \sum_{R\in\Pi_j^{n_1}\times\Pi_k^{n_2}, R\subseteq\Omega} (|\psi_{j,k} * g(x_I, x_J)|^2 \chi_R(x))dx \Big)^{1/2},$$

for all open sets $\Omega \subseteq \mathbb{R}^{n_1+n_2}$ with finite measure (see [16]).

Note that $x \in R \in \mathcal{B}_i$ implies $M_s(\chi_{R\cap(\tilde{\Omega}_i\setminus\Omega_{i+1})})(x) > \frac{1}{2}$, then

$$\chi_R(x) = \chi_R^2(x) \le 2^2 M_s^2(\chi_{R\cap(\tilde{\Omega}_i\setminus\Omega_{i+1})})(x),$$

which implies that,

$$\| \Big(\sum_{R\in\mathcal{B}_i} \max_{z\in R} |\varphi_{j,k} * f(z)|^2 \chi_R(x) \Big)^{\frac{1}{2}} \|^2_{L^2(\mathbb{R}^{n_1+n_2})}$$

$$\lesssim \|\Big(\sum_{R\in\mathcal{B}_i} M_s^2(\max_{z\in R}|\varphi_{j,k}*f(z)|\chi_{R\cap(\tilde{\Omega}_i\setminus\Omega_{i+1})})(x)\Big)^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^{n_1+n_2})}$$
$$\lesssim \int_{\tilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in B_i}\max_{z\in R}|\varphi_{j,k}*f(z)|^2\chi_R(x)dx$$

by the Fefferman–Stein vector-valued strong maximal inequality. Since $\tilde{S}(f)(x)\le 2^{i+1}$ in the domain $\tilde{\Omega}_i\setminus\Omega_{i+1}$, one has

$$\|\Big(\sum_{R\in\mathcal{B}_i}\max_{z\in R}|\varphi_{j,k}*f(z)|^2\chi_R(x)\Big)^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\lesssim 2^{2i}|\tilde{\Omega}_i|.$$

Hence,

$$|\langle f,g\rangle|\lesssim \|\bar{g}\|_{cmo_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\Big\{\sum_{i=-\infty}^{+\infty}|\tilde{\Omega}_i|^{1-\frac{p}{2}}(2^{2i}|\tilde{\Omega}_i|)^{p/2}\Big\}^{1/p}$$
$$\lesssim \|\bar{g}\|_{cmo_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\|f\|_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\lesssim\|f\|_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})},$$

since $g\in\mathcal{S}(\mathbb{R}^{n_1+n_2})\subseteq cmo_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for all $0<p\le 1$ [19], which implies the first statement.

As to the second statement, first observe that the map $(f,g)\to\|f-g\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$ is a metric on $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. For any sequence $\{f_j\}\subseteq h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ satisfying $\sum_j\|f_j\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}<\infty$, it is easy to see that the partial sums of this series are Cauchy in $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and hence are Cauchy in $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$ by part (1). It implies that the partial sums of this series converge to some $f\in\mathcal{S}'(\mathbb{R}^{n_1+n_2})$. Sublinearity gives

$$\|f\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=\|\sum_j f_j\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le\sum_j\|f_j\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}<\infty.$$

Moreover,

$$\|f-\sum_{0\le j<N}f_j\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=\|\sum_{j\ge N}f_j\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le\sum_{j\ge N}\|f_j\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\to 0$$

as $N\to+\infty$. □

5. Atomic decomposition of bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$

In this section, we establish the atomic decomposition of the bi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $0<p\le 1$. Namely, we will give the proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4. This proof is similar to the process of proving Theorem 1.6. Notations in the proof of Theorem 1.6 are retained. Details can also be seen in [19].

By Proposition 4.2,

$$\int_{\widetilde{\widetilde{\Omega}}}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a(x)|^2\Big)^{p/2}dx\le|\widetilde{\widetilde{\Omega}}|^{1-p/2}\|\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a|^2\Big)^{1/2}\|^p_{L^2(\mathbb{R}^{n_1+n_2})}$$
$$\le C|\Omega|^{1-p/2}\|a\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\le C.$$

Now we are going to prove

$$\sum_{R\in\mathcal{M}(\Omega)}\int_{\widetilde{\widetilde{\Omega}}^c}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\le C.$$

It is easy to see that

$$\begin{aligned}
&\int_{\widetilde{\widetilde{\Omega}}^c}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\\
&\quad\le\int_{(10\tau\tilde{I})^c\times\mathbb{R}^{n_2}}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx+\int_{\mathbb{R}^{n_1}\times(10\tau\tilde{J})^c}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\\
&\quad= I+II.
\end{aligned}$$

To estimate I, we write

$$\begin{aligned}
I&=\int_{(10\tau\tilde{I})^c\times\mathbb{R}^{n_2}}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\\
&=\int_{(10\tau\tilde{I})^c\times(10\tau J)^c}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx+\int_{(10\tau\tilde{I})^c\times(10\tau J)}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\\
&=U(R)+V(R).
\end{aligned}$$

According to side length of R, there are four cases: case 1: $\ell(I)\ge\varrho,\ell(J)\ge\varrho$; case 2: $\ell(I)<\varrho,\ell(J)\ge\varrho$; case 3: $\ell(I)\ge\varrho,\ell(J)<\varrho$; case 4: $\ell(I)<\varrho,\ell(J)<\varrho$. We claim that, in all cases,

$$U(R)\le C\Big(\frac{\ell(\tilde{I})}{\ell(I)}\Big)^{-(n_1+\varepsilon)p+n_1}|R|^{1-p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})},$$

and

$$V(R)\le C\Big(\frac{\ell(\tilde{I})}{\ell(I)}\Big)^{-(n_1+\varepsilon)p+n_1}|R|^{1-p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}$$

for some positive constant ε such that $(n_1+\varepsilon)p-n_1>0$. Indeed, for example, in case 2: $\ell(I)<\varrho,\ell(J)\ge\varrho$, for $U(R)$, let $x\in(10\tau\tilde{I})^c\times(10\tau J)^c$, and denote $N=[\frac{n_1}{p}-n_1]+1$. Then the cancellation condition of a_R in x_1 direction implies that, for any positive integer L,

$$\begin{aligned}
&|\varphi_{j,k}*a_R(x)|\\
&\quad=\Big|\int[\varphi_j(x_1-y_1)-\sum_{|\alpha|\le N-1}2^{j|\alpha|}(y_1-x_0)^\alpha(D^\alpha\varphi)_j(x_1-x_0)]\varphi_k(x_2-y_2)a_R(y)dy\Big|\\
&\quad\le C\frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}}\frac{2^{kn_2}}{(1+2^j|x_2-y_0|)^{n_2+L}}\int|a_R(y)|dy\\
&\quad\le C\frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}}\frac{2^{-kL}}{|x_2-y_0|^{n_2+L}}|R|^{1/2}\|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})}.
\end{aligned}$$

Hence

$$\begin{aligned}
U(R)&\le C\int_{(10\tau\tilde{I})^c\times(10\tau J)^c}\frac{\ell(I)^{Np}}{|x_1-x_0|^{(n_1+N)p}}\frac{1}{|x_2-y_0|^{(n_2+L)p}}|R|^{p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\le C\ell(I)^{Np}\ell(\tilde{I})^{-(n_1+N)p+n_1}\ell(J)^{-(n_2+L)p+n_2}|R|^{p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\le C\Big(\frac{\ell(\tilde{I})}{\ell(I)}\Big)^{-(n_1+\varepsilon)p+n_1}|R|^{1-p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})},
\end{aligned}$$

since $(n_1+N)p-n_1>0$.

For $V(R)$, let $x\in(10\tau\tilde{I})^c\times(10\tau J)$, then for any positive integer L,

$$|\psi_{j,k}*a_R(x)|=\Big|\int\varphi_{j,k}(x-y)a_R(y)dy\Big|$$

$$\leq \Big(\int_{\tau I} |\varphi_j(x_1-y_1) - \sum_{|\alpha|\leq N-1} 2^{j|\alpha|}(y_1-x_0)^\alpha (D^\alpha\varphi)_j(x_1-x_0)\Big|^2 dy_1\Big)^{1/2}$$
$$\Big(\int \Big|\int \varphi_k(x_2-y_2)a_R(y_1,y_2)dy_2\Big|^2 dy_1\Big)^{1/2}$$
$$\leq C|I|^{1/2}\frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}}\Big(\int \Big|\int \varphi_k(x_2-y_2)a_R(y_1,y_2)dy_2\Big|^2 dy_1\Big)^{1/2}.$$

Hence

$$\begin{aligned}
V(R) &= \int_{(10\tau\tilde{I})^c\times(10\tau J)} \Big(\sum_{j,k\in\mathbb{N}} |\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\\
&\leq C|J|^{1-p/2}\int_{(10\tau\tilde{I})^c}\Big(\sum_{j,k\in\mathbb{N}}\int_{10\tau J}|\varphi_{j,k}*a_R(x)|^2dx_2\Big)^{p/2}dx_1\\
&\leq C|J|^{1-p/2}|I|^{p/2}\int_{(10\tau\tilde{I})^c}\frac{\ell(I)^{Np}}{|x_1-x_0|^{(n_1+N)p}}dx_1\\
&\quad\Big\{\int\Big(\int_{10\tau J}\sum_{k\in\mathbb{N}}\Big|\int\varphi_k(x_2-y_2)a_R(y_1,y_2)dy_2\Big|^2dx_2\Big)dy_1\Big\}^{p/2}\\
&\leq C|J|^{1-p/2}|I|^{p/2}\ell(I)^{Np}\ell(\tilde{I})^{-(n_1+N)p+n_1}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\leq C\Big(\frac{\ell(\tilde{I})}{\ell(I)}\Big)^{-(n_1+\varepsilon)p+n_1}|R|^{1-p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}.
\end{aligned}$$

Similarly,

$$II \leq C\Big(\frac{\ell(\tilde{J})}{\ell(J)}\Big)^{-(n_2+\varepsilon')p+n_2}|R|^{1-p/2}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})},$$

for some positive constant ε' such that $(n_2+\varepsilon')p-n_2>0$.

Then the above two estimates yield that

$$\begin{aligned}
&\sum_{R\in\mathcal{M}(\Omega)}\int_{\widetilde{\widetilde{\Omega}}^c}\Big(\sum_{j,k\in\mathbb{N}}|\varphi_{j,k}*a_R(x)|^2\Big)^{p/2}dx\\
&\lesssim \sum_{R\in\mathcal{M}(\Omega)}\gamma(R)^{-[(n_1+\varepsilon)p-n_1]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\quad+\sum_{R\in\mathcal{M}(\Omega)}\gamma'(R)^{-[(n_2+\varepsilon')p+n_2]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\leq \Big(\sum_{R\in\mathcal{M}_2(\Omega)}\gamma(R)^{-[(n_1+\varepsilon)p+n_1]p'}|R|\Big)^{1/p'}\Big(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\Big)^{p/2}\\
&\quad+\Big(\sum_{R\in\mathcal{M}_1(\Omega)}\gamma'(R)^{-[(n_2+\varepsilon')p+n_2]p'}|R|\Big)^{1/p'}\Big(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\Big)^{p/2}\\
&\lesssim |\Omega|^{1-p/2}\Big(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\Big)^{p/2}\leq C.
\end{aligned}$$

by Hölder's inequality and Journé's lemma [43,56]. □

Proof of Theorem 1.5. For each fixed $f\in h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, consider $\tilde{S}(f)(x) = \Big(\sum_{j,k\in\mathbb{N}}\sum_{R\in\Pi_{j,k}}\max_{z\in R}|\varphi_{j,k}*f(z)|^2\chi_R\Big)^{\frac{1}{2}}$, then by Corollary 3.7,

$$\|S(f)\|_{L^p(\mathbb{R}^{n_1+n_2})}\approx\|\tilde{S}(f)\|_{L^p(\mathbb{R}^{n_1+n_2})}.$$

For any $i \in \mathbb{Z}$, set

$$\Omega_i = \{x \in \mathbb{R}^{n_1+n_2} : \tilde{S}(f)(x) > 2^i\}.$$

By (1.2), we decompose f as follows:

$$\begin{aligned} f(x) &= \sum_{j,k=0}^{\infty} \sum_{R\in \mathit{\Pi}_{j,k}} \int_R \psi_{j,k}(x-y)\varphi_{j,k} * f(y)dy \\ &= \sum_{j,k=0} \sum_{R\in \mathit{\Pi}_{j,k}} \int_R \psi_{j,k}(x-y)\varphi_{j,k} * f(y)dy + \sum_{j=0,k\geq 1} \sum_{R\in \mathit{\Pi}_{j,k}} \int_R \psi_{j,k}(x-y)\varphi_{j,k} * f(y)dy \\ &\quad + \sum_{j\geq 1,k=0} \sum_{R\in \mathit{\Pi}_{j,k}} \int_R \psi_{j,k}(x-y)\varphi_{j,k} * f(y)dy + \sum_{j\geq 1,k\geq 1} \sum_{R\in \mathit{\Pi}_{j,k}} \int_R \psi_{j,k}(x-y)\varphi_{j,k} * f(y)dy \\ &= f_1(x) + f_2(x) + f_3(x) + f_4(x), \end{aligned}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$ and $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$. By Theorem 1.1, we can choose ψ^1, ψ^2 such that $M_{\psi^1}, M_{\psi^2} \geq [\frac{n_1}{p} - n_1] + [\frac{n_2}{p} - n_2]$.

Rewrite f_1 as

$$f_1 = \sum_{R\in \mathit{\Pi}_{0,0}} \int_R \psi_{0,0}(x-y)\varphi_{0,0} * f(y)dy = \sum_{R\in \mathit{\Pi}_{0,0}} \lambda_R a_R,$$

where

$$a_R = \frac{1}{\lambda_R} \int_R \psi_{0,0}(x-y)\varphi_{0,0} * f(y)dy$$

and

$$\lambda_R = C \max_{z\in R} |\varphi_{0,0} * f(z)|.$$

It is easy to see that $\operatorname{supp} a_R \subseteq \tau R$ for some constant $\tau > 1$ depending only on ψ_0^1, ψ_0^2. Moreover,

$$\begin{aligned} \|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})} &= \frac{1}{\lambda_R} \Big(\int |\int_R \psi_{0,0}(x-y)\varphi_{0,0} * f(y)dy|^2 dx \Big)^{1/2} \\ &\leq \frac{1}{\lambda_R} \max_{z\in R} |\varphi_{0,0} * f(z)| \Big(\int_{\tau R} (\int |\psi_{0,0}(x-y)|dy)^2 dx \Big)^{1/2} \\ &\leq C \frac{1}{\lambda_R} \max_{z\in R} |\varphi_{0,0} * f(z)||R|^{1/2} = |\tau R|^{\frac{1}{2}-\frac{1}{p}}, \end{aligned}$$

since $|R| = 1$. Furthermore,

$$\begin{aligned} \sum_{R\in \mathit{\Pi}_{0,0}} \lambda_R^p &= C \sum_{R\in \mathit{\Pi}_{0,0}} |R| \max_{z\in R} |\varphi_{0,0} * f(z)|^p \\ &= C \sum_{R\in \mathit{\Pi}_{0,0}} \int_R \max_{z\in R} |\varphi_{0,0} * f(z)|^p dy \\ &= C \sum_{R\in \mathit{\Pi}_{0,0}} \int \max_{z\in R} |\varphi_{0,0} * f(z)|^p \chi_R(y) dy \\ &= C \int (\sum_{R\in \mathit{\Pi}_{0,0}} \max_{z\in R} |\varphi_{0,0} * f(z)|^2 \chi_R(y))^{p/2} dy \\ &\lesssim \|f\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}. \end{aligned}$$

It is easy to see that the atoms generated from $f_1(x)$ have no vanishing moment, since we have no vanishing moment assumption on ψ_0^1, ψ_0^2.

As to f_4, set

$$\mathcal{B}_i = \{R : R \in \cup_{j,k\geq 1} \mathit{\Pi}_{j,k},\ |R \cap \Omega_i| > \frac{1}{2}|R|,\ \ |R \cap \Omega_{i+1}| \leq \frac{1}{2}|R|\},$$

and

$$\tilde{\Omega}_i^{\tau} = \{x \in \mathbb{R}^{n_1+n_2} : M_s(\chi_{\Omega_i}) > \frac{1}{(10\tau)^{n_1+n_2}}\}.$$

It is easy to have that $\cup_{R\in\mathcal{B}_i} R \subseteq \tilde{\Omega}_i^1$.

For $R \in \Pi_{j,k}$, denote $\psi_R = \psi_{j,k}, \varphi_R = \varphi_{j,k}$. Rewrite f_4 as

$$\begin{aligned} f_4(x) &= \sum_{j,k\geq 1}\sum_{R\in\Pi_{j,k}} \int_R \psi_{j,k}(x-y)\varphi_{j,k} * f(y)dy \\ &= \sum_{i=-\infty}^{+\infty}\sum_{R\in\mathcal{B}_i}\int_R \psi_R(x-y)\varphi_R * f(y)dy = \sum_i \lambda_i a_i(x), \end{aligned}$$

where

$$a_i(x) = \frac{1}{\lambda_i}\sum_{R\in\mathcal{B}_i}\int_R \psi_R(x-y)\varphi_R * f(y)dy,$$

and

$$\lambda_i = C|\tilde{\Omega}_i^{\tau_0}|^{\frac{1}{p}-\frac{1}{2}} \| \Big(\sum_{R\in\mathcal{B}_i} |\varphi_R * f(x)\chi_R(x)|^2\Big)^{\frac{1}{2}} \|_{L^2(\mathbb{R}^{n_1+n_2})}$$

where $\tau_0 > 0$ is a constant big enough and dependent only on ψ^1, ψ^2 such that $\operatorname{supp}\int_R \psi_R(x-y)\varphi_R * f(y)dy \subseteq \tau_0 R$, hence $\operatorname{supp} a_i \subseteq \tilde{\Omega}_i^{\tau_0}$. This can be realized since $\psi^i \in \mathcal{D}(\mathbb{R}^{n_i}), i = 1, 2$. For convenience, denote $\tilde{\Omega}_i^{\tau_0} = \tilde{\Omega}_i$. For $a_i(x)$, one has

$$\begin{aligned} \|a_i\|_{L^2(\mathbb{R}^{n_1+n_2})} &= \frac{1}{\lambda_i}\sup_{\|h\|_{L^2}\leq 1} |\int (\sum_{R\in\mathcal{B}_i}\int_R \psi_R(x-y)\varphi_R * f(y)dy)h(x)dx| \\ &= \frac{1}{\lambda_i}\sup_{\|h\|_{L^2}\leq 1} |\int \sum_{R\in\mathcal{B}_i} (\int \psi_R(x-y)h(x)dx)\varphi_R * f(y)\chi_R dy| \\ &= \frac{1}{\lambda_i}\sup_{\|h\|_{L^2}\leq 1} |\int \sum_{R\in\mathcal{B}_i} \tilde{\psi}_R * h(y)\varphi_R * f(y)\chi_R dy| \\ &= \frac{1}{\lambda_i}\sup_{\|h\|_{L^2}\leq 1} \| \Big(\sum_{R\in\mathcal{B}_i} |\varphi_R * f|^2\chi_R\Big)^{\frac{1}{2}} \|_{L^2} \| \Big(\sum_{R\in\mathcal{B}_i} |\tilde{\psi}_R * h|^2\chi_R\Big)^{\frac{1}{2}} \|_{L^2(\mathbb{R}^{n_1+n_2})} \\ &\leq |\tilde{\Omega}_i|^{\frac{1}{2}-p} \end{aligned}$$

by Proposition 4.2. We now estimate $\sum_i \lambda_i^p$. First of all, $x \in R \in \mathcal{B}_i$ implies that $M_s(\chi_{R\cap(\tilde{\Omega}_i\setminus\Omega_{i+1})})(x) > \frac{1}{2}$, then

$$\chi_R(x) = \chi_R^2(x) \leq 2^2 M_s^2(\chi_{R\cap\tilde{\Omega}_i\setminus\Omega_{i+1}})(x),$$

which implies that,

$$\begin{aligned} &\| \Big(\sum_{R\in\mathcal{B}_i} |\varphi_R * f(x)\chi_R(x)|^2\Big)^{\frac{1}{2}} \|^2_{L^2(\mathbb{R}^{n_1+n_2})} \\ &\lesssim \int \sum_{R\in\mathcal{B}_i} \max_{z\in R} |\varphi_R * f(z)|^2 M_s^2(\chi_{R\cap(\tilde{\Omega}_i\setminus\Omega_{i+1})})(x)dx \\ &\lesssim \int_{\tilde{\Omega}_i\setminus\Omega_{i+1}} \sum_{R\in B_i} \max_{z\in R} |\varphi_R * f(z)|^2\chi_R(x)dx, \end{aligned}$$

by the Fefferman–Stein vector-valued strong maximal inequality. Since $\tilde{S}(f)(x) \leq 2^{i+1}$ in the domain $\tilde{\Omega}_i \setminus \Omega_{i+1}$, one has

$$\| \Big(\sum_{R\in\mathcal{B}_i} |\varphi_R * f(x)\chi_R(x)|^2\Big)^{\frac{1}{2}} \|^2_{L^2(\mathbb{R}^{n_1+n_2})} \lesssim 2^{2i}|\tilde{\Omega}_i|.$$

Hence,

$$\sum_i |\lambda_i|^p = \sum_i C|\tilde{\Omega}_i|^{1-\frac{p}{2}} \|\Big(\sum_{R\in\mathcal{B}_i} |\varphi_R * f(x)\chi_R(x)|^2\Big)^{\frac{1}{2}}\|^p_{L^2(\mathbb{R}^{n_1+n_2})}$$
$$\lesssim \sum_i |\tilde{\Omega}_i|^{1-\frac{p}{2}} 2^{pi}|\tilde{\Omega}_i|^{\frac{p}{2}} \lesssim \sum_i 2^{pi}|\tilde{\Omega}_i| \lesssim \|\tilde{S}(f)\|^p_{L^p(\mathbb{R}^{n_1+n_2})} \approx \|f\|^p_{h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Moreover, note that $a_i = \sum_{R\in\mathcal{M}(\tilde{\Omega}_i)} a_{i,R}$, where

$$a_{i,R}(x) = \frac{1}{\lambda_i} \sum_{Q\in\mathcal{B}_i, Q\subset R} \int_Q \psi_Q(x-y)\varphi_Q * f(y)dy.$$

It is easy to see that $\operatorname{supp} a_{i,R} \subseteq \tau_0 R$, and $a_{i,R}$ has vanishing moments up to $[\frac{n_1}{p} - n_1] + [\frac{n_2}{p} - n_2]$ in two directions. Moreover by the similar estimates for $\|a_i\|_{L^2}$, one has

$$\|a_{i,R}\|_{L^2(\mathbb{R}^{n_1+n_2})} \lesssim \frac{1}{\lambda_i} \|\Big(\sum_{Q\in\mathcal{B}_i, Q\subset R} |\varphi_Q * f|^2 \chi_Q\Big)^{\frac{1}{2}}\|_{L^2(\mathbb{R}^{n_1+n_2})},$$

which implies that

$$\sum_{R\in\mathcal{M}(\tilde{\Omega}_i)} \|a_{i,R}\|^2_{L^2(\mathbb{R}^{n_1+n_2})} \lesssim \sum_{R\in\mathcal{M}(\tilde{\Omega}_i)} \frac{1}{\lambda_i^2} \|\Big(\sum_{Q\in\mathcal{B}_i, Q\subset R} |\varphi_Q * f|^2\Big)^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^{n_1+n_2})}$$
$$= \frac{1}{\lambda_i^2} \|\Big(\sum_{Q\in\mathcal{B}_i} |\varphi_Q * f|^2 \chi_Q\Big)^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^{n_1+n_2})}$$
$$\lesssim |\tilde{\Omega}_i|^{1-\frac{2}{p}},$$

by the definition of λ_i.

$\sum_i \lambda_i a_i(x) \to f$ in $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$ is easy to obtain. We now prove that, if $f \in L^2(\mathbb{R}^{n_1+n_2})$, the series $\sum_i \lambda_i a_i(x)$ converges in $L^2(\mathbb{R}^{n_1+n_2})$. Firstly, by duality argument,

$$\| \sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \int_R \psi_R(x-y)\varphi_R * f(y)dy\|_{L^2(\mathbb{R}^{n_1+n_2})}$$
$$= \sup_{\|g\|_{L^2}\le 1} |\langle \sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \int_R \psi_R(x-y)\varphi_R * f(y)dy, g\rangle|$$
$$= \sup_{\|g\|_{L^2}\le 1} |\sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \int_R \tilde{\psi}_R * g(y)\varphi_R * f(y)dy|$$
$$\le \sup_{\|g\|_{L^2}\le 1} \sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \max_{z\in R} |\varphi_R * f(z)| \max_{z\in R} |\tilde{\psi}_R * g(z)||R|$$
$$\le \sup_{\|g\|_{L^2}\le 1} \Big(\sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \max_{z\in R} |\varphi_R * f(z)|^2 |R|\Big)^{1/2} \Big(\sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \max_{z\in R} |\tilde{\psi}_R * g(z)|^2 |R|\Big)^{1/2}$$
$$= \sup_{\|g\|_{L^2}\le 1} \Big(\sum_{j,k\ge 1}^{\infty} \sum_{R\in\Pi_{j,k}} \max_{z\in R} |\varphi_R * f(z)|^2 |R|\Big)^{1/2} \Big(\sum_{j,k\ge 1}^{\infty} \sum_{R\in\Pi_{j,k}} \max_{z\in R} |\tilde{\psi}_R * g(z)|^2 |R|\Big)^{1/2}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^{n_1+n_2})}$$

by Proposition 4.2 and Corollary 3.7. Moreover, $\sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \max_{z\in R} |\varphi_R * f(z)|^2 |R| \le \|f\|^2_{L^2(\mathbb{R}^{n_1+n_2})} < +\infty$ implies that

$$\sum_{|i|>N} \sum_{R\in\mathcal{B}_i} \max_{z\in R} |\varphi_R * f(z)|^2 |R| \to 0,$$

as $N \to \infty$, which yields that $\sum_{i=-\infty}^{+\infty} \lambda_i a_i(x) = \sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} \int_R \psi_R(x-y)\varphi_R * f(y)dy$ converges to f in $L^2(\mathbb{R}^{n_1+n_2})$ by above duality argument.

Repeating the same process to obtain atoms from f_4, we can obtain the atomic decomposition of $f_i, i = 2, 3$, respectively. Note that there are some tiny differences in the propositions of those rectangle atoms. We denote A_i by the classes of the rectangle atoms obtained from $f_i, i = 1, 2, 3, 4$, respectively. In class A_2, the rectangle atoms $a_{i,R}$ associated with $R = I \times J$ satisfy $\ell(I) \geq 1$ and have vanishing moments up to $[\frac{n_1}{p} - n_1] + [\frac{n_2}{p} - n_2]$ only in x_2 direction. Similar differences hold for those rectangle atoms in A_3. □

Remark 5.1. In one-parameter case, for $f \in h^p(\mathbb{R}^n), 0 < p \leq 1$, one has

$$f(x) = \sum_{j=0}^{\infty} \psi_j * \varphi_j * f(x) = \psi_0 * \varphi_0 * f(x) + \sum_{j=1}^{\infty} \psi_j * \varphi_j * f(x) = f_1(x) + f_2(x).$$

We may choose $\varphi_0 \in \mathcal{D}$ satisfying $\hat{\varphi}_0(x) = 1 + |x|^{[s]+1}$ near the origin for any fixed $s > 0$. Then it is easy to see that $\int \varphi_0 \neq 0$ and $\int \varphi_0(x)x^\beta dx = 0, \forall\ 0 < |\beta| \leq s$. From the Lemma 4 of [28], $f_2 \in H^p(\mathbb{R}^n)$. Then we can obtain atomic decomposition of $f \in h^p(\mathbb{R}^n)$ as above.

6. Boundedness of bi-parameter singular integral operators on $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

In this section, as an application of the atomic decomposition of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we obtain the $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to L^p(\mathbb{R}^{n_1+n_2})$ boundedness of T for p near 1.

For $f \in h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1+n_2})$, by Theorem 1.5, we have $f(x_1, x_2) = \sum \lambda_k a_k(x_1, x_2)$ in $L^2(\mathbb{R}^{n_1+n_2})$, then $T(f)(x_1, x_2) = \sum \lambda_k T(a_k)(x_1, x_2)$, since T is bounded on $L^2(\mathbb{R}^{n_1+n_2})$, which implies that the series (subsequence) converges almost everywhere. Hence,

$$\|T(f)\|_{L^p(\mathbb{R}^{n_1+n_2})} \leq \sum |\lambda_k| \|T(a_k)\|_{L^p(\mathbb{R}^{n_1+n_2})}.$$

If we can obtain $\|T(a)\|_{L^p(\mathbb{R}^{n_1+n_2})} \leq C$ for any atom a of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then Theorem 1.6 is proved by Theorem 1.5.

Suppose that a is an $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atom supported on an open set $\Omega \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Furthermore, a can be decomposed as $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, and each a_R is supported in τR for some positive integer $\tau > 1$. Denote

$$\widetilde{\Omega} = \{x \in \mathbb{R}^{n_1+n_2} : M_s(\chi_\Omega)(x) > \frac{1}{(10\tau)^{n_1+n_2}}\},$$

and similarly for $\widetilde{\widetilde{\Omega}}$, $\widetilde{\widetilde{\widetilde{\Omega}}}$. It follows from the strong maximal theorem that $|\widetilde{\widetilde{\widetilde{\Omega}}}| \leq C|\Omega|$. By the $L^2(\mathbb{R}^{n_1+n_2})$ boundedness, it is easy to have

$$\int_{\widetilde{\widetilde{\widetilde{\Omega}}}} |T(a)(x)|^p dx \leq |\widetilde{\widetilde{\widetilde{\Omega}}}|^{1-p/2} \|T(a)\|^p_{L^2(\mathbb{R}^{n_1+n_2})} \leq C|\Omega|^{1-p/2} \|a\|^p_{L^2(\mathbb{R}^{n_1+n_2})} \leq C.$$

Therefore it remains to prove

$$\int_{\widetilde{\widetilde{\widetilde{\Omega}}}^c} |T(a)(x)|^p dx \leq C,$$

which follows from

$$\sum_{R\in\mathcal{M}(\Omega)} \int_{\widetilde{\widetilde{\widetilde{\Omega}}}^c} |T(a_R)(x)|^p dx \leq C.$$

Let $\mathcal{M}_1(\Omega)$ denote the collection of all dyadic rectangles $R = I \times J \subseteq \Omega$ that are maximal in the x_1 direction. Define $\mathcal{M}_2(\Omega)$ similarly. It is easy to see that $\mathcal{M}(\Omega) \subseteq \mathcal{M}_1(\Omega)$ and $\mathcal{M}(\Omega) \subseteq \mathcal{M}_2(\Omega)$. For

$R = I \times J \in \mathcal{M}(\Omega)$, we set a rectangle $\tilde{R} = \tilde{I} \times J$ such that $\tilde{I}$ is the largest dyadic cube containing I and $\tilde{I} \times J \subseteq \widetilde{\Omega}$. Then $\tilde{R} \in \mathcal{M}_1(\widetilde{\Omega})$. Let $\tilde{J}$ be largest dyadic cube containing J such that $\tilde{\tilde{R}} = \tilde{I} \times \tilde{J} \subseteq \widetilde{\widetilde{\Omega}}$. Then $\tilde{\tilde{R}} \in \mathcal{M}_2(\widetilde{\widetilde{\Omega}})$. Note that $10\tau\tilde{\tilde{R}} \subseteq \widetilde{\widetilde{\Omega}}$.

Let $a_R(x_1, x_2)$ be a rectangle atom of a associated with R centered at (x_0, y_0). Define $\gamma = \gamma(R) = \frac{\ell(\tilde{I})}{\ell(I)}$ and $\gamma' = \gamma'(R) = \frac{\ell(\tilde{J})}{\ell(J)}$. Then

$$\begin{aligned}\int_{\widetilde{\widetilde{\Omega}}^c} |T(a_R)(x)|^p dx &\le \int_{(10\tau\tilde{\tilde{R}})^c} |T(a_R)(x)|^p dx \\ &\le \int_{(10\tau\tilde{I})^c \times \mathbb{R}^{n_2}} |T(a_R)(x)|^p dx + \int_{\mathbb{R}^{n_1} \times (10\tau\tilde{J})^c} |T(a_R)(x)|^p dx \\ &= I + II.\end{aligned}$$

For I, denote $E_{k,j} = \{(x_1, x_2) \in \mathbb{R}^{n_1+n_2} : 2^k\gamma\ell(I) \le |x_1 - x_0| < 2^{k+1}\gamma\ell(I), 2^j\ell(J) \le |x_2 - y_0| < 2^{j+1}\ell(J)\}$ and $E_k = \{(x_1, x_2) \in \mathbb{R}^{n_1+n_2} : 2^k\gamma\ell(I) \le |x_1 - x_0| < 2^{k+1}\gamma\ell(I), |x_2 - y_0| \le 2\ell(J)\}$. Decompose I as follows,

$$\int_{(10\tau\tilde{I})^c \times \mathbb{R}^{n_2}} |T(a_R)(x)|^p dx \le \sum_{k,j\ge 1} \int_{E_{k,j}} |T(a_R)(x)|^p dx + \sum_{k\ge 1} \int_{E_k} |T(a_R)(x)|^p dx.$$

According to side length of R, there are four cases: case 1: $\ell(I) \ge \varrho, \ell(J) \ge \varrho$; case 2: $\ell(I) < \varrho, \ell(J) \ge \varrho$; case 3: $\ell(I) \ge \varrho, \ell(J) < \varrho$; case 4: $\ell(I) < \varrho, \ell(J) < \varrho$. The proof for case 4 can be seen in [21], and we omit it.

Without loss of generality, we assume $\delta \le \varepsilon$. We first discuss case 1: $\ell(I) \ge \varrho, \ell(J) \ge \varrho$. Using the size conditions of kernel, one has

$$\begin{aligned}&\iint_{E_{k,j}} |T(a_R)(x_1, x_2)| dx_1 dx_2 \\ &\le \iint_{E_{k,j}} \iint |\mathcal{K}(x_1, y_1, x_2, y_2)||a_R(y_1, y_2)| dy_1 dy_2 dx_1 dx_2 \\ &\le C \iint_{E_{k,j}} \iint \frac{1}{|x_1 - y_1|^{n_1+\delta}} \frac{1}{|x_2 - y_2|^{n_2+\delta}} |a_R(y_1, y_2)| dy_1 dy_2 dx_1 dx_2 \\ &\le C 2^{-k\delta}\gamma^{-\delta} 2^{-j\delta} \ell(I)^{-\delta} \ell(J)^{-\delta} \iint |a_R(y_1, y_2)| dy_1 dy_2 \\ &\le C 2^{-k\delta}\gamma^{-\delta} 2^{-j\delta} |R|^{\frac{1}{2}} \|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})} \le C 2^{-k\delta}\gamma^{-\delta} 2^{-j\delta} |R|^{\frac{1}{2}} \|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})},\end{aligned}$$

since $\ell(I), \ell(J) \ge \varrho$.

On the other hand,

$$\begin{aligned}&\iint_{E_k} |T(a_R)(x_1, x_2)| dx_1 dx_2 \\ &\le \int_{|x_1 - x_0| \sim 2^k\gamma\ell(I)} \int_{|x_2 - y_0| \le 2\ell(J)} \int |\int \mathcal{K}(x_1, y_1)(x_2, y_2) a_R(y_1, y_2) dy_2| dy_1 dx_1 dx_2 \\ &\le C \int_{|x_1 - x_0| \sim 2^k\gamma\ell(I)} \int_{\tau I} |J|^{1/2} \Big(\int_{|x_2 - y_0| \le 2\ell(J)} |\int \mathcal{K}(x_1, y_1)(x_2, y_2) a_R(y_1, y_2) dy_2|^2 dx_2 \Big)^{1/2} dy_1 dx_1 \\ &\le C \int_{|x_1 - x_0| \sim 2^k\gamma\ell(I)} \int_{\tau I} |J|^{1/2} \|\mathcal{K}(x_1, y_1)\|_{CZ} \|a_R(y_1, \cdot)\|_{L^2(\mathbb{R}^{n_2})} dy_1 dx_1 \\ &\le C \int_{|x_1 - x_0| \sim 2^k\gamma|I|} \int_{\tau I} |J|^{1/2} \frac{1}{|x_1 - y_1|^{n_1+\delta}} \|a_R(y_1, \cdot)\|_{L^2(\mathbb{R}^{n_2})} dy_1 dx_1 \\ &\le C 2^{-k\delta}\gamma^{-\delta} |I|^{-\delta} |R|^{\frac{1}{2}} \|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})} \le C 2^{-k\delta}\gamma^{-\delta} |R|^{\frac{1}{2}} \|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})}.\end{aligned}$$

We now turn to case 2: $\ell(I) < \varrho, \ell(J) \geq \varrho$. Firstly, the cancellation condition of a_R in $\mathbb{R}^{n_1}$ yields

$$\begin{aligned}
&\iint_{E_{k,j}} |T(a_R)(x_1,x_2)|dx_1dx_2\\
&= \iint_{E_{k,j}} |\iint [\mathcal{K}(x_1,y_1,x_2,y_2) - \mathcal{K}(x_1,x_0,x_2,y_2)]a_R(y_1,y_2)dy_1dy_2|dx_1dx_2\\
&\leq \iint_{E_{k,j}} \iint_{\tau R} \frac{|y_1-x_0|^{\varepsilon}}{|x_1-x_0|^{n_1+\varepsilon}} \frac{1}{|x_2-y_2|^{n_2+\delta}} |a_R(y_1,y_2)|dy_1dx_1dy_2dx_2\\
&\leq C2^{-k\delta}\gamma^{-\delta}2^{-j\delta}|J|^{-\delta}|R|^{\frac{1}{2}}\|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})} \leq C2^{-k\delta}\gamma^{-\delta}2^{-j\delta}|R|^{\frac{1}{2}}\|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})}.
\end{aligned}$$

At the same time,

$$\begin{aligned}
&\iint_{E_k} |T(a_R)(x_1,x_2)|dx_1dx_2\\
&\leq \int_{|x_1-x_0|\sim 2^k\gamma\ell(I)} \int_{|x_2-y_0|\leq 2\ell(J)} \int |\int [\mathcal{K}(x_1,y_1)(x_2,y_2) - \mathcal{K}(x_1,x_0)(x_2,y_2)]\\
&\quad \times a_R(y_1,y_2)dy_2|dy_1dx_1dx_2\\
&\leq C\int_{|x_1-x_0|\sim 2^k\gamma\ell(I)} \int_{\tau I} |J|^{1/2}\|\mathcal{K}(x_1,y_1) - \mathcal{K}(x_1,x_0)\|_{CZ}\|a_R(y_1,\cdot)\|_{L^2(\mathbb{R}^{n_2})}dy_1dx_1\\
&\leq C2^{-k\delta}\gamma^{-\delta}|R|^{\frac{1}{2}}\|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})}.
\end{aligned}$$

Similarly, using the cancellation condition of a_R in $\mathbb{R}^{n_2}$ to integral on $E_{k,j}$, and L^2 boundedness in x^2 direction to integral on E_k, one can deal with case 3 to obtain the same estimates.

All the above estimates imply that, in any case, one has

$$\iint_{E_{k,j}} |T(a_R)(x_1,x_2)|dx_1dx_2 \leq C2^{-k\delta}\gamma^{-\delta}2^{-j\delta}|R|^{\frac{1}{2}}\|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})},$$

and

$$\iint_{E_k} |T(a_R)(x_1,x_2)|dx_1dx_2 \leq C2^{-k\delta}\gamma^{-\delta}|R|^{\frac{1}{2}}\|a_R\|_{L^2(\mathbb{R}^{n_1+n_2})}.$$

Hence by Hölder's inequality, one has

$$\begin{aligned}
&\iint_{E_{k,j}} |T(a_R)(x_1,x_2)|^p dx_1dx_2\\
&\leq C2^{-kp\delta}\gamma^{-p\delta}2^{-jp\delta}|R|^{\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}|E_{k,j}|^{1-p}\\
&\leq 2^{-k[p\delta-n_1(1-p)]}\gamma^{-[p\delta-n_1(1-p)]}2^{-j[p\delta-n_2(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})},
\end{aligned}$$

and

$$\iint_{E_k} |T(a_R)(x_1,x_2)|^p dx_1dx_2 \leq 2^{-k[p\delta-n_1(1-p)]}\gamma^{-[p\delta-n_1(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})},$$

which yields

$$\begin{aligned}
&\int_{(10\tau\tilde{I})^c\times\mathbb{R}^{n_2}} |T(a_R)(x)|^p dx\\
&\leq \sum_{k,j\geq 1} 2^{-k\delta}2^{-k[p\delta-n_1(1-p)]}\gamma^{-[p\delta-n_1(1-p)]}2^{-j[p\delta-n_2(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\quad + \sum_{k\geq 1} 2^{-k[p\delta-n_1(1-p)]}\gamma^{-[p\delta-n_1(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}\\
&\leq C\gamma^{-[p\delta-n_1(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})},
\end{aligned}$$

provided $p > \max\{\frac{n_1}{n_1+\delta}, \frac{n_2}{n_2+\delta}\}$. With a similar proof,

$$\int_{\mathbb{R}^{n_1}\times(10\tau\tilde{J})^c} |T(a_R)(x)|^p dx \le C\gamma'(R)^{-[p\delta-n_2(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})}.$$

Denote $p' = \frac{2}{2-p}$, then $\frac{p}{2} + \frac{1}{p'} = 1$. Therefore by Hölder's inequality and Journé's lemma [43,56]

$$\begin{aligned}
&\sum_{R\in\mathcal{M}(\Omega)} \int_{\widetilde{\Omega}^c} |T(a_R)(x)|^p dx \\
&\lesssim \sum_{R\in\mathcal{M}(\Omega)} \gamma(R)^{-[p\delta-n_1(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})} \\
&\quad + \sum_{R\in\mathcal{M}(\Omega)} \gamma'(R)^{-[p\delta-n_2(1-p)]}|R|^{1-\frac{p}{2}}\|a_R\|^p_{L^2(\mathbb{R}^{n_1+n_2})} \\
&\le \Big(\sum_{R\in\mathcal{M}_2(\Omega)} \gamma(R)^{-[p\delta-n_1(1-p)]p'}|R|\Big)^{1/p'}\Big(\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\Big)^{p/2} \\
&\quad + \Big(\sum_{R\in\mathcal{M}_1(\Omega)} \gamma'(R)^{-[p\delta-n_2(1-p)]p'}|R|\Big)^{1/p'}\Big(\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\Big)^{p/2} \\
&\lesssim |\Omega|^{1-p/2}\Big(\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|^2_{L^2(\mathbb{R}^{n_1+n_2})}\Big)^{p/2} \le C,
\end{aligned}$$

provided $p > \max\{\frac{n_1}{n_1+\delta}, \frac{n_2}{n_2+\delta}\}$. □

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