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Research Article

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Multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular integrals and their atomic decomposition

Abstract: Atomic decomposition plays an important role in establishing the boundedness of operators on function spaces. Let $0 < p, q < \infty$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. In this paper, we introduce multi-parameter Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ associated with different homogeneities arising from the composition of two singular integral operators whose weak $(1,1)$ boundedness was first studied by Phong and Stein [32]. We then establish its atomic decomposition which is substantially different from that for the classical one-parameter Triebel–Lizorkin spaces. As an application of our atomic decomposition, we obtain the necessary and sufficient conditions for the boundedness of an operator T on the multi-parameter Triebel–Lizorkin type spaces. In the special case of $\alpha_1 = \alpha_2 = 0$, $q = 2$ and $0 < p \le 1$, our spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ coincide with the Hardy spaces H^p_{com} associated with the composition of two different singular integrals (see [19]). Therefore, our results also give an atomic decomposition of H^p_{com}. Our work appears to be the first result of atomic decomposition in the Triebel–Lizorkin spaces in the multi-parameter setting.

Keywords: Triebel–Lizorkin spaces, atomic decomposition, discrete Calderón's identity, Littlewood–Paley analysis, restriction estimates

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1 Introduction

The classical theory of harmonic analysis may be described as centering around the Hardy–Littlewood maximal operator and its relationship with certain singular integral operators which commute with the usual dilations on $\mathbb{R}^m$, given by

$$\delta : x \to \delta x, \quad \delta > 0.$$

The above isotropic dilations can be replaced by more general non-isotropic groups of dilations. This modification produces many non-isotropic variants of the classical theories, such as, the multi-parameter pure product theory corresponding to the dilations

$$\delta : x \to (\delta_1 x_1, \delta_2 x_2), \quad x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m,\ \delta = (\delta_1, \delta_2),\ \delta_1 > 0,\ \delta_2 > 0,$$

which has been developed by many authors over the past decades. Similar to the classical theory, this theory includes the boundedness of multi-parameter singular integral operators on the L^p spaces $(1 < p < \infty)$ and multi-parameter Hardy spaces H^p $(0 < p \le 1)$. Another interesting feature of this multi-parameter theory also includes the atomic decomposition of multi-parameter Hardy spaces, duality and interpolation theorems on product spaces, and maximal function characterizations, etc. We refer the reader to the works in [1–4, 10–13, 17, 18, 20, 23, 25, 26, 28, 33].

To familiarize the reader with the background and the motivation of our study, we will explain how this multi-parameter structure arises in other aspects of harmonic analysis, in particular, from the composition of two singular integral operators with different homogeneities. To be more precise, for $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$, then we can consider two kinds of homogeneities

$$\delta : (x', x_m) \to (\delta x', \delta x_m), \quad \delta > 0,$$
$$\delta : (x', x_m) \to (\delta x', \delta^2 x_m), \quad \delta > 0.$$

The first is the classical isotropic dilations occurring in the classical Calderón–Zygmund singular integrals, while the second is non-isotropic and related to the heat equations (similar to that on the Heisenberg groups). For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$. Then there are two different types of singular integrals associated with these dilations.

Definition 1.1. A locally integrable function $\mathcal{K}_1$ on $\mathbb{R}^m \setminus \{0\}$ is said to be a Calderón–Zygmund kernel associated with the isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{K}_1(x)\right| \le A|x|_e^{-m-|\alpha|} \quad \text{for all } |\alpha| \ge 0 \tag{1.1}$$

and

$$\int\limits_{r_1<|x|_e<r_2} \mathcal{K}_1(x)\,dx = 0 \quad \text{for all } 0 < r_1 < r_2 < \infty. \tag{1.2}$$

An operator T_1 is said to be a Calderón–Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = \text{p.v.}(\mathcal{K}_1 * f)(x)$, where $\mathcal{K}_1$ satisfies conditions in (1.1) and (1.2).

Definition 1.2. Suppose $\mathcal{K}_2 \in L^1_{\text{loc}}(\mathbb{R}^m \setminus \{0\})$. Then $\mathcal{K}_2$ is said to be a Calderón–Zygmund kernel associated with the non-isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial (x')^\alpha}\frac{\partial^\beta}{\partial (x_m)^\beta}\mathcal{K}_2(x', x_m)\right| \le B|x|_h^{-m-1-|\alpha|-2\beta} \quad \text{for all } |\alpha| \ge 0,\ \beta \ge 0, \tag{1.3}$$

and

$$\int\limits_{r_1<|x|_h<r_2} \mathcal{K}_2(x)\,dx = 0 \quad \text{for all } 0 < r_1 < r_2 < \infty, \tag{1.4}$$

An operator T_2 is said to be a Calderón–Zygmund singular integral operator associated with the non-isotropic homogeneity if $T_2(f)(x) = \text{p.v.}(\mathcal{K}_2 * f)(x)$, where $\mathcal{K}_2$ satisfies the conditions in (1.3) and (1.4).

Both the classical Calderón–Zygmund theory and theory of singular integral operators associated with the non-isotropic dilations indicate that both the operators T_1 and T_2 are bounded on L^p for $1 < p < \infty$ and of weak type $(1, 1)$. Nevertheless, it is showed by Phong and Stein in [32] that in general the composition operator $T_1 \circ T_2$ is not of weak-type $(1, 1)$. Moreover, the authors of [32] gave a necessary and sufficient condition such that the composition operator $T_1 \circ T_2$ is of weak-type $(1, 1)$. This answers the question raised by Rivieré in [35]. In fact, the operators studied in [32] are compositions with different homogeneities and such a composition operator arises naturally in the study of $\bar{\partial}$-Neumann problem (see also Folland–Stein [14]).

It is well known that any Calderón–Zygmund singular integral operator associated with the isotropic homogeneity is bounded on the classical Hardy space $H^p(\mathbb{R}^m)$ with $0 < p \le 1$. A Calderón–Zygmund singular integral operator associated with the non-isotropic homogeneity is not bounded on the classical Hardy space but bounded on the non-isotropic Hardy space (see [15]). However, the composition operator is bounded on neither the classical Hardy space nor the non-isotropic Hardy space. This motivates the authors of [19] to introduce a new Hardy space associated with the different homogeneities and establish the boundedness of composition singular integrals on such Hardy spaces. It is interesting to note that such Hardy spaces are of multi-parameter setting in nature. Recently, the duality theory of the multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular integral operators has been established by the first two authors [8].

Inspired by those works, and recent study of multi-parameter Triebel–Lizorkin spaces in the pure product setting and as well as in the setting associated with the flag singular integrals [9, 29], we consider multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular integral operators of different homogeneities. In particular, we are interested in the atomic decomposition of these spaces. Such spaces associated with the different homogeneities were introduced earlier in [7] along with the boundedness of the composition of two singular integral operators with different homogeneities on these spaces.

For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, we recall two different norms: $|x|_e = (|x'|^2 + |x_m|^2)^{1/2}$ and $|x|_h = (|x'|^2 + |x_m|)^{1/2}$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$. Denote

$$\mathcal{S}_0(\mathbb{R}^m) = \Big\{ f \in \mathcal{S}(\mathbb{R}^m) : \int_{\mathbb{R}^m} f(x) x^\alpha \, dx = 0 \text{ for any multi-index } \alpha \text{ with } |\alpha| \geq 0 \Big\},$$

and for a positive integer L,

$$\mathcal{S}_L(\mathbb{R}^m) = \Big\{ f \in \mathcal{S}(\mathbb{R}^m) : \int_{\mathbb{R}^m} f(x) x^\alpha \, dx = 0 \text{ for any multi-index } \alpha \text{ with } |\alpha| \leq L-1 \Big\}.$$

Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\operatorname{supp} \widehat{\psi^{(1)}} \subseteq \Big\{ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2 \Big\}, \tag{1.5}$$

and

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)|^2 = 1 \quad \text{for all } (\xi', \xi_m) \in \mathbb{R}^m \setminus \{0\}. \tag{1.6}$$

Let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\operatorname{supp} \widehat{\psi^{(2)}} \subseteq \Big\{ (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq \sqrt{2} \Big\}, \tag{1.7}$$

and

$$\sum_{k \in \mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \quad \text{for all } (\xi', \xi_m) \in \mathbb{R}^m \setminus \{0\}. \tag{1.8}$$

Set

$$\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x),$$

where $\psi_j^{(1)}(x', x_m) = 2^{jm} \psi^{(1)}(2^j x', 2^j x_m)$, $\psi_k^{(2)}(x', x_m) = 2^{k(m+1)} \psi^{(2)}(2^k x', 2^{2k} x_m)$. The following discrete Calderón reproducing formula is from [19].

Theorem A. *Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in* (1.5)–(1.6) *and* (1.7)–(1.8), *respectively. Then*

$$\begin{aligned} f(x', x_m) = \sum_{j,k \in \mathbb{Z}} \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} & 2^{-(m-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_m) \\ & \times \psi_{j,k}(x' - 2^{-(j \wedge k)} \ell', x_m - 2^{-(j \wedge 2k)} \ell_m), \end{aligned} \tag{1.9}$$

where the series converges in $L^2(\mathbb{R}^m)$, $\mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}_0'(\mathbb{R}^m)$.

Remark 1.3. In the proof of Theorem A, the authors use the additional assumption that $\psi^{(1)}$ and $\psi^{(2)}$ are real and radial Schwartz functions. If dispensing with this assumption, (1.9) should be

$$f(x', x_m) = \sum_{j,k \in \mathbb{Z}} \sum_{(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}} 2^{-(m-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\widetilde{\psi}_{j,k} * f)(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_m) \psi_{j,k}(x' - 2^{-(j \wedge k)} \ell', x_m - 2^{-(j \wedge 2k)} \ell_m),$$

where $\widetilde{\psi}_{j,k}(x) = \overline{\psi_{j,k}(-x)}$, which is a generation of [16, Lemma 2.1].

With the discrete Calderón reproducing formula, Triebel–Lizorkin spaces associated with different homogeneities were introduced in [7] as follows.

Definition 1.4. Let $0 < p, q < \infty$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The Triebel–Lizorkin type space with different homogeneities $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is defined by

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^m) = \{f \in \mathcal{S}_0'(\mathbb{R}^m) : \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} = \bigg\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^q \chi_I(x')\chi_J(x_m)\Big)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)},$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $l(I) = 2^{-(j\wedge k)}$ and $l(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}\ell'$ and $2^{-(j\wedge 2k)}\ell_m$, respectively.

This new Triebel–Lizorkin type space is well defined, since it has been proved in [7] that $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is independent of the choice of the functions ψ^1 and ψ^2. Nevertheless, it is not clear if such a space is equivalent to the one defined by using the continuous form. One of the main purpose in this paper is to show that they are indeed equivalent using both the discrete and continuous forms. This is described in the following theorem.

Theorem 1.5. *Let $0 < p, q < \infty$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ satisfy conditions in* (1.5)–(1.6) *and* (1.7)–(1.8)*, respectively. Then*

$$\begin{aligned}\bigg\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^q \chi_I(x')\chi_J(x_m)\Big)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)} \\ \approx \bigg\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}|\psi_{j,k} * f|^q\Big)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)}.\end{aligned}$$

In the case of the pure product structure, the equivalence was proved in [29]. However, our multi-parameter structure associated with the composition of two different homogeneities is more complicated. Therefore, such an equivalence is not evident. Nevertheless, establishing such an equivalence is not just interesting but also necessary to justify the definition of such multi-parameter Triebel–Lizorkin spaces. Otherwise the spaces would seem to depend on the choice of the points used in Definition 1.4, namely, depending on the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}\ell'$ and $2^{-(j\wedge 2k)}\ell_m$, respectively, where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side length $l(I) = 2^{-(j\wedge k)}$ and $l(J) = 2^{-(j\wedge 2k)}$.

When $\alpha_1 = \alpha_2 = 0$, $q = 2$ and $0 < p \le 1$, our spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m) = H^p_{\mathrm{com}}(\mathbb{R}^m)$. Hence, our Theorem 1.5 also verifies that the Hardy spaces $H^p_{\mathrm{com}}(\mathbb{R}^m)$ given in [19] in the discrete form is actually equivalent to the one defined in the continuous form.

Using the discrete Calderón reproducing formulae in Theorem A above, and an argument similar to that in [20, 29], etc., one can obtain one direction,

$$\begin{aligned}\bigg\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}|\psi_{j,k} * f|^q\Big)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)} \\ \lesssim \bigg\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^q \chi_I(x')\chi_J(x_m)\Big)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)}.\end{aligned}$$

However, the other direction is harder. We will apply a similar argument in one-parameter setting as in the work of Frazier and Jawerth [16] to prove the inequality in our multi-parameter setting under study.

Atomic decomposition is important in proving boundedness of operators on various function spaces both in one-parameter and multi-parameter settings. Since the atoms and molecules were introduced in the one-parameter setting by Coifman, Weiss, and Latter in [5, 6, 27], they have played a very important role in harmonic and wavelet analysis [30, 31, 34]. The study of the operators acting on a space of functions or distributions becomes easier when the elements in the space admit atomic decompositions. It is often the case that it suffices to prove the uniform boundedness of an operator on atoms of the function space in order to establish the boundedness of an operator on such a space. In the multi-parameter situations, atomic decomposition on multi-parameter Hardy spaces was established by Chang and Fefferman [1–3, 10], and was applied to prove the boundedness of multi-parameter singular integral operators on Hardy spaces, using the atomic decomposition together with the Journé's covering lemma, by Fefferman [11], Journé [25, 26], Pipher [33], Han, Lu and Ruan [21, 22] etc. A more refined and improved version of atomic decomposition in the multi-parameter Hardy spaces was carried out in [24] and a boundedness criterion was established using the atomic decomposition. Therefore, it is interesting and useful to establish the atomic decomposition and the boundedness criterion on the multi-parameter Triebel–Lizorkin spaces associated with the composition of two dilations of different homogeneities.

To obtain the atomic characterization of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, more precisely, of $L^2(\mathbb{R}^m)\cap\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, firstly we introduce the following atoms.

Definition 1.6. Let $0<p\le q<\infty$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. A distribution $a\in\mathcal{S}_0(\mathbb{R}^m)'$ is said to be a (p,q,α)-atom of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)(\cap L^2(\mathbb{R}^m))$ if
(i) $\operatorname{supp} a\subset\Omega$, where $\Omega\subseteq\mathbb{R}^m$ is an open set with finite measure,
(ii) $\|a\|_{\dot{F}_q^{\alpha,q}(\mathbb{R}^m)}\le|\Omega|^{\frac{1}{q}-\frac{1}{p}}$,
(iii) $\|a\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}\le C_0$ for some constant C_0.
Moreover a can be further decomposed into some rectangle-atoms a_R associated to the rectangle $R=I\times J$ which is supported in τR for some positive integer τ independent of a, and such that
(iv) $a=\sum_{R\subset m(\Omega)}a_R$ with

$$\Big(\sum_{R\in m(\Omega)}\|a_R\|^q_{\dot{F}_q^{\alpha,q}(\mathbb{R}^m)}\Big)^{1/q}\le|\Omega|^{\frac{1}{q}-\frac{1}{p}},$$

(iv′) for every $g\in\mathcal{S}(\mathbb{R}^m)$, every polynomial P of degree at most $N=[(m+1)(1/\min\{p,q,1\}-1)]$, and any smooth cut off function $\eta_R\in\mathcal{S}(\mathbb{R}^m)$ such that $\eta_R\equiv1$ on τR, and $\eta_R\equiv0$ outside $2\tau R$, we have

$$\langle a,g\rangle=\langle a,(g-P)\eta_R\rangle.$$

Here and in the sequel, $m(\Omega)$ is the set of all maximal dyadic rectangles contained in Ω.

We are now ready to give some remarks here. Firstly, when $0<p\le1, q=2, \alpha=(0,0)$, we then obtain the definition of atoms in H^p_{com}. Different from classical definitions of atoms in pure product Hardy spaces [11, 24], an additional good condition: $\|a\|_{H^p_{\mathrm{com}}}\le C_0$ is involved. This condition can be obtained from the remaining conditions in pure product spaces (see the appendix of [24]). But it seems hard to do so in H^p_{com}. Secondly, if (p,q,α)-atoms are locally integrable, one can see that (iv′) is the usual cancellation condition.

Theorem 1.7. *Suppose $0<p\le q<\infty$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. Let $f\in L^2(\mathbb{R}^m)\cap\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$. Then there exists a sequence of (p,q,α)-atoms $\{a_i\}$ of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ and a sequence of scalars $\{\lambda_i\}$ with $(\sum_i|\lambda_i|^p)^{1/p}\le C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}$ such that*

$$f=\sum_i\lambda_ia_i$$

and the series converge to f in both $L^2(\mathbb{R}^m)$ and $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, where C is a positive constant independent of f.

Since when $0<p\le q<\infty$ and $p\le1$, $f\to\|f\|^p_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}$ is subadditive, we have the following corollary.

Corollary 1.8. *Suppose $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$, $0<p\le1$, $p\le q<\infty$ and $f\in L^2$. Then $f\in\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ if and only if f can be written as $f=\sum_i\lambda_ia_i$ in $L^2(\mathbb{R}^m)$ and in $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, where a_i are (p,q,α)- atoms of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ and $\{\lambda_i\}$ satisfies $(\sum_i|\lambda_i|^p)^{1/p}<\infty$. Moreover,*

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}\approx\inf\Big\{\Big(\sum_i|\lambda_i|^p\Big)^{1/p}:f=\sum_i\lambda_ia_i\text{ where }a_i\text{ are }(p,q,\alpha)\text{-atoms of }\dot{F}_p^{\alpha,q}(\mathbb{R}^m)\Big\}.$$

In the multi-parameter setting, a boundedness criterion has been established in [24]. In the setting of multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular integrals, we will confirm such a boundedness criterion as well. As an application of Corollary 1.8, we obtain a boundedness criterion for linear operators from $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ to $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$.

Theorem 1.9. *Suppose $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $0 < p \le 1$, $p \le q < \infty$. If T is a linear operator bounded on $L^2(\mathbb{R}^m)$, then T is bounded on $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ if and only if*

$$\sup\{\|T(a)\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} : a \text{ is any } (p,q,\alpha)\text{-atom of } \dot{F}_p^{\alpha,q}(\mathbb{R}^m)\} < \infty.$$

The organization of this paper is as follows. In Section 2, we will prove that the discrete definition of Triebel–Lizorkin type space is equivalent to its continuous form. Section 3 gives $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ norm estimates of a function restricted in a domain. Section 4 contains the proofs of Theorem 1.7 (the atomic decomposition) and Theorem 1.9 (boundedness criterion).

Throughout this paper, C is a positive constant which is independent of essential parameters and not necessarily the same at each occurrence. Constants with subscript, such as C_1, do not change in different occurrences. We denote $f \le Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

2 Comparison principle

In this section, for $j, k \in \mathbb{Z}$, denote $\Pi_{j,k}$ to be the set of all $R = I \times J$ such that I are dyadic cubes in $\mathbb{R}^{m-1}$, J are dyadic intervals in $\mathbb{R}$, with the side length $l(I) = 2^{-(j\wedge k)}$ and $l(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $x_I = 2^{-(j\wedge k)}\ell'$ and $x_J = 2^{-(j\wedge 2k)}\ell_m$, respectively, $(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}$; and we set

$$\mathcal{D} = \bigcup_{j,k} \Pi_{j,k}.$$

One should note that, for any $\mu \in \mathbb{Z}$, there exist $j, k \in \mathbb{Z}$ and $j', k' \in \mathbb{Z}$ such that $2^{-(j\wedge k)} = 2^{-\mu}$ and $2^{-(j'\wedge 2k')} = 2^{-\mu}$ respectively. But, for some $\mu, \nu \in \mathbb{Z}$, there may be no j, k such that $2^{-(j\wedge k)} = 2^{-\mu}$, $2^{-(j\wedge 2k)} = 2^{-\nu}$ since $j\wedge k \le j\wedge 2k$ if $j, k \ge 0$. So $\mathcal{D} \subsetneq \{R = I \times J : I$ are dyadic cubes in $\mathbb{R}^{m-1}$, J are dyadic intervals in $\mathbb{R}\}$. For $R \in \Pi_{j,k}$, setting

$$\psi_R(x) = |R|^{1/2}\psi_{j,k}(x' - x_I, x_m - x_J),$$

then by (1.9), it is easy to have

$$f(x) = \sum_{R\in\mathcal{D}} \langle f, \psi_R\rangle \psi_R(x).$$

For $0 < p, q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, corresponding to discrete multi-parameter Triebel–Lizorkin spaces associated with the composition of two homogeneities, we shall define $\dot{f}_p^{\alpha,q}(\mathbb{R}^m)$ which is the collection of all complex-valued sequences $s = \{s_R\}_{R\in\mathcal{D}}$ such that

$$\|s\|_{\dot{f}_p^{\alpha,q}} = \Big\|\Big(\sum_{R=I\times J\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\widetilde{\chi}_R(x))^q\Big)^{1/q}\Big\|_{L^p}.$$

Note that, for $R \in \Pi_{j,k}$, setting $s_R(x) = |R|^{1/2}(\psi_{j,k} * f)(x_I, x_J)$, one has

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \|s\|_{\dot{f}_p^{\alpha,q}}.$$

Next for a sequence $s = \{s_R\}_{R\in\mathcal{D}}$, $0 < r < +\infty$, and a fixed $\lambda > 0$, define the sequence $s_r^* = \{(s_r^*)_R\}_{R\in\mathcal{D}}$ by

$$(s_r^*)_R = \Big(\sum_{P:l(P)=l(R)} \frac{|s_P|^r}{(1 + l(I)^{-1}|x_{I'} - x_I|)^\lambda (1 + l(J)^{-1}|x_{J'} - x_J|)^\lambda}\Big)^{1/r}$$

where $P = I' \times J' \in \mathcal{D}$, $R = I \times J$.

Lemma 2.1. *Suppose $0 < a \le r < +\infty$ and $\lambda > (m-1)r/a$. Fix a dyadic rectangle $R = I \times J \in \mathcal{D}$, denote Π as a collection of dyadic rectangles $P = I' \times J'$ with $l(P) = l(R)$. Then for each $x \in R$,*

$$\Big(\sum_{P\in\Pi} \frac{|s_P|^r}{(1 + l(I)^{-1}|x_{I'} - x_I|)^\lambda (1 + l(J)^{-1}|x_{J'} - x_J|)^\lambda}\Big)^{1/r} \le C\Big(\mathcal{M}_s\Big(\sum_{P\in\Pi} |s_P|^a \chi_P\Big)(x)\Big)^{1/a},$$

where C depends only on r, λ, m, and $\mathcal{M}_s$ is the strong maximal function.

Remark 2.2. We want to point out, if the side-lengths of R are $l(I) = 2^{-j\wedge k}$ and $l(J) = 2^{-j\wedge 2k}$, respectively, then Π may be a subset of $\Pi_{j,k}$.

Proof. Set

$$A_0 = \{I' : l(I)^{-1}|x_{I'} - x_I| \le 1\}, \quad B_0 = \{J' : l(J)^{-1}|x_{J'} - x_J| \le 1\},$$

and for $t \ge 1$ and $s \ge 1$,

$$A_t = \{I' : 2^{t-1} < l(I)^{-1}|x_{I'} - x_I| \le 2^t\}, \quad B_s = \{J' : 2^{s-1} < l(J)^{-1}|x_{J'} - x_J| \le 2^s\}.$$

For any fixed $t, s \ge 0$, denote

$$E = \{(w', w_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : |w' - x_I| \le (2^t l(I) + l(I))\sqrt{m-1},\ |w_m - x_J| \le 2^s l(J) + l(J)\}.$$

Then $A_t \times B_s \subseteq E$ and $I \times J \subseteq E$. Obviously,

$$|E| \lesssim 2^{t(m-1)}|I|2^s|J|.$$

Thus for any $x \in R$,

$$\begin{aligned}
\sum_{P\in\Pi} \frac{|s_P|^r}{(1 + l(I)^{-1}|x_{I'} - x_I|)^\lambda (1 + l(J)^{-1}|x_{J'} - x_J|)^\lambda} &\lesssim \sum_{t,s\ge0} 2^{-t\lambda}2^{-s\lambda} \sum_{I'\times J'\in(A_t\times B_s)\cap\Pi} |s_P|^r \\
&\lesssim \sum_{t,s\ge0} 2^{-t\lambda}2^{-s\lambda}\Big(\sum_{I'\times J'\in(A_t\times B_s)\cap\Pi} |s_P|^a\Big)^{r/a} \\
&= \sum_{t,s\ge0} 2^{-t\lambda}2^{-s\lambda}|R|^{-r/a}\Big(\int \sum_{I'\times J'\in(A_t\times B_s)\cap\Pi} |s_P|^a \chi_P dx\Big)^{r/a} \\
&\le \sum_{t,s\ge0} 2^{-t\lambda}2^{-s\lambda}|R|^{-r/a}|E|^{r/a}\Big(\mathcal{M}_s\Big(\sum_{P\in\Pi} |s_P|^a \chi_P\Big)(x)\Big)^{r/a} \\
&= \sum_{t,s\ge0} 2^{-t[\lambda-(m-1)r/a]}2^{-s(\lambda-r/a)}\Big(\mathcal{M}_s\Big(\sum_{P\in\Pi} |s_P|^a \chi_P\Big)(x)\Big)^{r/a} \\
&\lesssim \Big(\mathcal{M}_s\Big(\sum_{P\in\Pi} |s_P|^a \chi_P\Big)(x)\Big)^{r/a},
\end{aligned}$$

since $\lambda > (m-1)r/a$. □

Lemma 2.3. *Suppose $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $0 < p, q < \infty$, and $\lambda > m-1$. Then*

$$\|s\|_{\dot f_p^{\alpha,q}} \approx \|s^*_{\min(p,q)}\|_{\dot f_p^{\alpha,q}}.$$

Proof. We only need to prove

$$\|s^*_{\min(p,q)}\|_{\dot f_p^{\alpha,q}} \lesssim \|s\|_{\dot f_p^{\alpha,q}},$$

since the converse estimates is trivial. Let $r = \min(p,q)$, $\varepsilon = -1 + \lambda/(m-1)$, and $a = r/(1+\varepsilon/2)$. Then $0 < a < r$ and $\lambda > (m-1)r/a$. For all $j, k \in \mathbb{Z}$, by Lemma 2.1, one has

$$\sum_{\substack{R=I\times J\\ l(I)=2^{-j\wedge k},\, l(J)=2^{-j\wedge 2k}}} (s_r^*)_R \widetilde\chi_R(x) \lesssim \Big(\mathcal{M}_s\Big(\sum_{\substack{P=I'\times J'\\ l(I')=2^{-j\wedge k},\, l(J')=2^{-j\wedge 2k}}} |s_P|\widetilde\chi_P\Big)^a(x)\Big)^{1/a}.$$

Hence,

$$\|s_r^*\|_{\dot f_p^{\alpha,q}} \lesssim \Big\|\Big\{\sum_{j,k\in\mathbb{Z}}\Big(\mathcal{M}_s\Big(\sum_{\substack{P=I'\times J'\\ l(I')=2^{-j\wedge k},\, l(J')=2^{-j\wedge 2k}}} |I'|^{s_1/(m-1)}|J'|^{s_2}|s_P|\widetilde\chi_P\Big)^a(x)\Big)^{q/a}\Big\}^{1/q}\Big\|_{L^p} \lesssim \|s\|_{\dot f_p^{\alpha,q}}$$

by applying Fefferman–Stein's vector-valued strong maximal inequality on the space $L^{p/a}(\ell^{q/a})$ since we have $a < r = \min(p,q)$. □

For any $R \in \Pi_{j,k}$, we consider its generations. With the following Lemma 2.4, one can see that for a positive integer γ, a dyadic rectangle $\widetilde R = \widetilde I \times \widetilde J$ with $l(\widetilde I) = 2^{-(j\wedge k)-\gamma}$ and $l(\widetilde J) = 2^{-(j\wedge 2k)-\gamma}$ may not be in $\mathcal{D}$. So we should consider its generations with side length $l(\widetilde I) = 2^{-(j+\gamma)\wedge(k+\gamma)}$ and $l(\widetilde J) = 2^{-(j+\gamma)\wedge 2(k+\gamma)}$. Of course, for any integer γ,

$$\{\widetilde R = \widetilde I \times \widetilde J : l(\widetilde I) = 2^{-(j+\gamma)\wedge(k+\gamma)}, l(\widetilde J) = 2^{-(j+\gamma)\wedge 2(k+\gamma)},\ j,k \in \mathbb{Z}\} = \mathcal{D}.$$

Lemma 2.4. *For every $\gamma \in \mathbb{Z}$, $\gamma > 0$, there exist $j, k \in \mathbb{Z}$ such that*

$$(j \wedge k + \gamma, j \wedge 2k + \gamma) \notin \Lambda = \{(j \wedge k, j \wedge 2k) : j, k \in \mathbb{Z}\}.$$

Proof. Let $\Lambda_1 = \{(m, n) : j, k \in \mathbb{Z},\ m > n \geq 0\}$. Note that $\Lambda_1 \cap \Lambda = \emptyset$ since $j \wedge k \leq j \wedge 2k$ when $j, k \geq 0$. By choosing $j_0 = k_0 = -1$, we see that for all $\gamma \geq 2$,

$$(j_0 \wedge k_0 + \gamma, j_0 \wedge 2k_0 + \gamma) \in \Lambda_1.$$

It follows that $(j_0 \wedge k_0 + \gamma, j_0 \wedge 2k_0 + \gamma) \notin \Lambda$. For $\gamma = 1$, we have $(j_0 \wedge k_0 + 1, j_0 \wedge 2k_0 + 1) = (0, -1) \notin \Lambda$. □

Lemma 2.5. *Let $\gamma \in \mathbb{Z}$ with $\gamma \geq 0$ be fixed. For any $j, k \in \mathbb{Z}$, we have*

$$2^{-(j+\gamma)\wedge(k+\gamma)} = 2^{-j\wedge k} 2^{-\gamma} \tag{2.1}$$

and

$$2^{-2\gamma} \leq \frac{2^{-(j+\gamma)\wedge 2(k+\gamma)}}{2^{-j\wedge 2k}} \leq 2^{-\gamma}. \tag{2.2}$$

Proof. Formula (2.1) is trivial. As to (2.2), if $j \leq 2k$, obviously, we obtain

$$2^{-j\wedge 2k} = 2^{-j} \quad \text{and} \quad j + \gamma \leq 2(k + \gamma)$$

which gives

$$2^{-(j+\gamma)\wedge 2(k+\gamma)} = 2^{-(j+\gamma)} = 2^{-j} 2^{-\gamma} = 2^{-j\wedge 2k} 2^{-\gamma}.$$

When $j > 2k$, one has $2^{-j\wedge 2k} = 2^{-2k}$. If $j + \gamma > 2(k + \gamma)$, one has $2^{-(j+\gamma)\wedge 2(k+\gamma)} = 2^{-2(k+\gamma)} = 2^{-2k} 2^{-2\gamma} = 2^{-j\wedge 2k} 2^{-2\gamma}$. If $j + \gamma \leq 2(k + \gamma)$, one has $2^{-2k} 2^{-2\gamma} \leq 2^{-(j+\gamma)} \leq 2^{-2k} 2^{-\gamma}$. Hence

$$\frac{2^{-(j+\gamma)\wedge 2(k+\gamma)}}{2^{-j\wedge 2k}} = \frac{2^{-(j+\gamma)}}{2^{-2k}} \in [2^{-2\gamma}, 2^{-\gamma}].$$ □

Remark 2.6. When $\gamma < 0$, formula (2.1) also holds and (2.2) is replaced by

$$2^{-\gamma} \leq \frac{2^{-(j+\gamma)\wedge 2(k+\gamma)}}{2^{-j\wedge 2k}} \leq 2^{-2\gamma}.$$

Lemma 2.7. *Suppose $f \in \mathcal{S}'(\mathbb{R}^m)$ and $\operatorname{supp} \hat{f} \subseteq \{\xi : |\xi| \leq 2\}$. Let $\gamma \in \mathbb{Z}$ with $\gamma \geq 0$ and let $j, k \in \mathbb{Z}$. For $R \in \Pi_{j,k}$, let $a_R = \sup_{y\in R} |f(y)|$ and $b_{R,\gamma} = \max\{\inf_{y\in\hat{R}} |f(y)| : \hat{R} = \hat{I} \times \hat{J},\ l(\hat{I}) = 2^{-(j+\gamma)\wedge(k+\gamma)},\ l(\hat{J}) = 2^{-(j+\gamma)\wedge 2(k+\gamma)},\ \hat{R} \subseteq R\}$. Let $a = \{a_R\}_R$ and $b = \{b_{R,\gamma}\}_R$. If $0 < r < \infty$, $l(R) = 1$ and γ is sufficiently large, then*

$$(a_r^*)_R \approx (b_r^*)_R.$$

Proof. From the definition it is easy to see

$$(b_r^*)_R \leq (a_r^*)_R.$$

To prove the converse direction, we first suppose $f \in \mathcal{S}(\mathbb{R}^m)$ and $\operatorname{supp} \hat{f} \subseteq \{\xi : |\xi| \leq 3\}$. For every $P = I' \times J'$ with $l(P) = l(R)$, there exist some $\hat{R}_0 = \hat{I}_0 \times \hat{J}_0 \subseteq P$ with $l(\hat{I}_0) = 2^{-(j+\gamma)\wedge(k+\gamma)}$ and $l(\hat{J}_0) = 2^{-(j+\gamma)\wedge 2(k+\gamma)}$, and some $y_1 \in \hat{R}_0$ such that $a_P = f(y_1)$. Take $y_2 \in \hat{R}_0$ such that $\inf_{y\in\hat{R}_0} |f(y)| = f(y_2)$. Since $l(P) = l(R) = 1$, we get

$$|y_1 - y_2| \lesssim (2^{-2[(j+\gamma)\wedge(k+\gamma)]} + 2^{-2[(j+\gamma)\wedge 2(k+\gamma)]})^{1/2} \lesssim (2^{-2(j\wedge k)} + 2^{-2(j\wedge 2k)})^{1/2} 2^{-\gamma} \lesssim 2^{-\gamma}$$

by Lemma 2.5. So by the Mean-Value Theorem,

$$a_P - \inf_{y\in\hat{R}_0} |f(y)| = f(y_1) - f(y_2) \leq |y_1 - y_2| \sup_{y\in P} |\nabla f(y)| \lesssim 2^{-\gamma} \sup_{y\in P} |\nabla f(y)|,$$

which implies that

$$a_P \lesssim b_{P,\gamma} + 2^{-\gamma} \sup_{y\in P} |\nabla f(y)|.$$

Let $d_P = \sup_{y\in P} |\nabla f(y)|$ and $d = \{d_P\}_P$. Then

$$(a_r^*)_R \lesssim (b_r^*)_R + 2^{-\gamma} (d_r^*)_R. \tag{2.3}$$

Let $g \in \mathcal{S}$ with $\hat{g}(\xi) = 1$ if $|\xi| \leq 3$ and $\hat{g}(\xi) \subseteq \{\xi : |\xi| \leq \pi\}$. It is easy to see that $f = f * g = (\hat{f}\hat{g})^\vee$. Arguing as in the proof of [16, Lemma A.4] or [19, Theorem 1.3], we obtain

$$f(x) = \sum_{l(\tilde{I})=1} \sum_{l(\tilde{J})=1} f(x_{\tilde{I}}, x_{\tilde{J}}) g(x' - x_{\tilde{I}}, x_m - x_{\tilde{J}}),$$

where $x_{\tilde I}$ and $x_{\tilde J}$ are the left lower corners of $\tilde I$ and the left end points of $\tilde J$, respectively. Hence

$$\begin{aligned} d_P &\le \sup_{y\in P}\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}|f(x_{\tilde I},x_{\tilde J})|\nabla g(y'-x_{\tilde I},y_m-x_{\tilde J})\\ &\le \sup_{y\in P}\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}|f(x_{\tilde I},x_{\tilde J})|\nabla g(y'-x_{\tilde I},y_m-x_{\tilde J})\frac{[(1+|x_{\tilde I}-x_{I'}|)(1+|x_I-x_{I'}|)]^{\lambda/r}}{(1+|x_{\tilde I}-x_I|)^{\lambda/r}}\frac{[(1+|x_{\tilde J}-x_{J'}|)(1+|x_J-x_{J'}|)]^{\lambda/r}}{(1+|x_{\tilde J}-x_J|)^{\lambda/r}}, \end{aligned}$$

where $P=I'\times J'$ and where $x_{I'}$ and $x_{J'}$ are the left lower corners of I' and the left end points of J', respectively. Since $g\in\mathcal S$, we have

$$\sup_{y\in P}\nabla g(y'-x_{\tilde I},y_m-x_{\tilde J})\lesssim\frac{1}{(1+|x_{\tilde I}-x_{I'}|)^M}\frac{1}{(1+|x_{\tilde J}-x_{J'}|)^M}$$

for any larger $M>0$. Therefore,

$$\begin{aligned} (d_r^*)_R &= \Big(\sum_{\substack{P\\ l(P)=l(R)}}\frac{|d_P|^r}{(1+|x_{I'}-x_I|)^\lambda(1+|x_{J'}-x_J|)^\lambda}\Big)^{1/r}\\ &\lesssim\Big\{\sum_{\substack{P\\ l(P)=l(R)}}\Big(\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}\frac{|f(x_{\tilde I},x_{\tilde J})|}{(1+|x_{\tilde I}-x_I|)^{\lambda/r}(1+|x_{\tilde I}-x_{I'}|)^L}\frac{1}{(1+|x_{\tilde J}-x_J|)^{\lambda/r}(1+|x_{\tilde J}-x_{J'}|)^L}\Big)^r\Big\}^{1/r} \end{aligned}$$

for any sufficiently large L. If $r>1$, by Hölder's inequality,

$$\begin{aligned} (d_r^*)_R &\lesssim\Big\{\sum_{\substack{P\\ l(P)=l(R)}}\Big(\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}\frac{|f(x_{\tilde I},x_{\tilde J})|^r}{(1+|x_{\tilde I}-x_I|)^{\lambda}(1+|x_{\tilde I}-x_{I'}|)^{Lr/2}}\frac{1}{(1+|x_{\tilde J}-x_J|)^{\lambda}(1+|x_{\tilde J}-x_{J'}|)^{Lr/2}}\Big)\\ &\qquad\times\Big(\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}\frac{1}{(1+|x_{\tilde I}-x_{I'}|)^{Lr'/2}(1+|x_{\tilde J}-x_{J'}|)^{Lr'/2}}\Big)^{r/r'}\Big\}^{1/r}\\ &\lesssim\Big\{\sum_{\substack{P\\ l(P)=l(R)}}\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}\frac{|f(x_{\tilde I},x_{\tilde J})|^r}{(1+|x_{\tilde I}-x_I|)^{\lambda}(1+|x_{\tilde I}-x_{I'}|)^{Lr/2}}\frac{1}{(1+|x_{\tilde J}-x_J|)^{\lambda}(1+|x_{\tilde J}-x_{J'}|)^{Lr/2}}\Big\}^{1/r}\\ &\lesssim\Big(\sum_{l(\tilde I)=1}\sum_{l(\tilde J)=1}\frac{|f(x_{\tilde I},x_{\tilde J})|^r}{(1+|x_{\tilde I}-x_I|)^\lambda(1+|x_{\tilde J}-x_J|)^\lambda}\Big)^{1/r}\le(a_r^*)_R. \end{aligned}$$

Likewise, if $0<r\le1$, it suffices to use $(\sum a_i)^r\le\sum a_i^r$ to obtain the same estimate $(d_r^*)_R\lesssim(a_r^*)_R$. Therefore, by taking sufficiently large γ in (2.3) we have $(a_r^*)_R\lesssim(d_r^*)_R$.

For the general case, one can apply a standard regularization argument to remove the assumption $f\in\mathcal S(\mathbb R^m)$ (see [16, Lemma A.4]). □

For any $R\in\Pi_{j,k}$ and positive integer γ, denote

$$A_R^\gamma=\{\tilde R=\tilde I\times\tilde J\in\mathcal D: l(\tilde I)=2^{-(j+\gamma)\wedge(k+\gamma)},\ l(\tilde J)=2^{-(j+\gamma)\wedge2(k+\gamma)},\ \tilde R\subseteq R\}.$$

Define the sequence $\inf_\gamma(f)=\{\inf_{R,\gamma}(f)\}_R$ by $\inf_{R,\gamma}(f)=|R|^{1/2}\max\{\inf_{y\in\tilde R}|\phi_{j,k}*f(y)|:\tilde R\in A_R^\gamma\}$. At last, for convenience, we denote

$$\|f\|_{\dot{\mathcal F}_p^{\alpha,q}}=\Big\|\Big(\sum_{j,k\in\mathbb Z}2^{-[(j\wedge k)\alpha_1+(j\wedge2k)\alpha_2]q}|\psi_{j,k}*f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb R^m)}$$

for $f\in\mathcal S_0'$. Then we have the following lemma.

Lemma 2.8. *Suppose $\alpha=(\alpha_1,\alpha_2)\in\mathbb R^2$ and $0<p,q<\infty$. For any $\gamma\ge0$, one has*

$$\|\inf_\gamma(f)\|_{\dot f_p^{\alpha,q}}\le C\|f\|_{\dot{\mathcal F}_p^{\alpha,q}},$$

where C is independent of $f\in\mathcal S_0'$.

Proof. First define a sequence $s = \{s_P\}$ by

$$s_P = |P|^{1/2} \inf_{y\in P} |\phi_{\mu-\gamma,\nu-\gamma} * f(y)|$$

for $P = I' \times J' \in \mathcal{D}$ with $l(I') = 2^{-\mu\wedge\nu}$ and $l(J') = 2^{-\mu\wedge 2\nu}$, $\mu, \nu \in \mathbb{Z}$. For $R \in \Pi_{j,k}$, with $j + \gamma = \mu, k + \gamma = \nu$, we may assume $\inf_{R,\gamma}(f)$ takes its maximum at $\widetilde{R}_0 = \widetilde{I}_0 \times \widetilde{J}_0 \in A_R^\gamma$, that is,

$$\inf_{R,\gamma}(f) = |R|^{1/2} \inf_{y\in\widetilde{R}_0} |\phi_{j,k} * f(y)|.$$

For any $P \in A_R^\gamma$, since $l(I') = 2^{-(j+\gamma)\wedge(k+\gamma)}$ and $l(J') = 2^{-(j+\gamma)\wedge 2(k+\gamma)}$, by Lemma 2.5 together with $|x_{I'} - x_{\widetilde{I}_0}| \le 2^{-j\wedge k}$ and $|x_{J'} - x_{\widetilde{J}_0}| \le 2^{-j\wedge 2k}$, one has

$$\begin{aligned}
(s_r^*)_P &= \Bigg(\sum_{\substack{\widetilde{R}\\ l(\widetilde{R})=l(P)}} \frac{|s_{\widetilde{R}}|^r}{(1 + l(I')^{-1}|x_{I'} - x_{\widetilde{I}}|)^\lambda (1 + l(J')^{-1}|x_{J'} - x_{\widetilde{J}}|)^\lambda} \Bigg)^{1/r} \\
&\ge \frac{|s_{\widetilde{R}_0}|}{(1 + l(I')^{-1}|x_{I'} - x_{\widetilde{I}_0}|)^{\lambda/r} (1 + l(J')^{-1}|x_{J'} - x_{\widetilde{J}_0}|)^{\lambda/r}} \\
&\gtrsim |s_{\widetilde{R}_0}| 2^{-\gamma\lambda/r} 2^{-2\gamma\lambda/r}.
\end{aligned}$$

Therefore,

$$\inf_{R,\gamma}(f)\widetilde{\chi}_R \lesssim \sum_{P\in A_R^\gamma} 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} (s_r^*)_P \widetilde{\chi}_P.$$

Applying Lemma 2.5 and Lemma 2.7, when $\alpha_2 \ge 0$, one has

$$\begin{aligned}
\|\inf_\gamma(f)\|_{\dot{f}_p^{\alpha,q}} &= \Bigg\| \Bigg(\sum_{R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)} |J|^{\alpha_2} | \inf_{R,\gamma}(f) | \widetilde{\chi}_R(x))^q \Bigg)^{1/q} \Bigg\|_{L^p} \\
&\lesssim 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{\gamma\alpha_1 + 2\gamma\alpha_2} \|s_r^*\|_{\dot{f}_p^{\alpha,q}} \\
&\lesssim 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{\gamma\alpha_1 + 2\gamma\alpha_2} \|s\|_{\dot{f}_p^{\alpha,q}} \\
&= 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{\gamma(\alpha_1+\alpha_2)} \Bigg\| \Bigg(\sum_{P\in\mathcal{D}} (|I'|^{\alpha_1/(m-1)} |J'|^{\alpha_2} |s_R| \widetilde{\chi}_R(x))^q \Bigg)^{1/q} \Bigg\|_{L^p} \\
&\le 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{\gamma\alpha_1 + 2\gamma\alpha_2} \Bigg\| \Bigg(\sum_{\mu,\nu\in\mathbb{Z}} (2^{-(\mu\wedge\nu)\alpha_1} 2^{-(\mu\wedge 2\nu)\alpha_2} |\phi_{\mu-\gamma,\nu-\gamma} * f|)^q \Bigg)^{1/q} \Bigg\|_{L^p} \\
&= 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{\gamma\alpha_1 + 2\gamma\alpha_2} \Bigg\| \Bigg(\sum_{j,k\in\mathbb{Z}} (2^{-[(j+\gamma)\wedge(k+\gamma)]\alpha_1} 2^{-[(j+\gamma)\wedge 2(k+\gamma)]\alpha_2} |\phi_{j,k} * f|)^q \Bigg)^{1/q} \Bigg\|_{L^p} \\
&\le 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{\gamma\alpha_2} \Bigg\| \Bigg(\sum_{j,k\in\mathbb{Z}} (2^{-(j\wedge k)\alpha_1} 2^{-(j\wedge 2k)\alpha_2} |\phi_{j,k} * f|)^q \Bigg)^{1/q} \Bigg\|_{L^p}.
\end{aligned}$$

Similarly, we obtain

$$\|\inf_\gamma(f)\|_{\dot{f}_p^{\alpha,q}} \lesssim 2^{\gamma\lambda/r} 2^{2\gamma\lambda/r} 2^{-\gamma\alpha_2} \|f\|_{\dot{\mathcal{F}}_p^{\alpha,q}}$$

when $\alpha_2 < 0$. Then we complete the proof. □

Now we define the sequence $\sup(f) = \{\sup_R(f)\}_R$ by setting $\sup_R(f) = |R|^{1/2} \sup_{y\in R} |\phi_{j,k} * f(y)|$.

Lemma 2.9. *Suppose $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $0 < p, q \le \infty$, and $\gamma \ge 0$ is sufficiently large. Then for $f \in \mathcal{S}'(\mathbb{R}^m)$,*

$$\|f\|_{\dot{\mathcal{F}}_p^{s,q}} \approx \|\inf_\gamma(f)\|_{\dot{f}_p^{\alpha,q}} \approx \|\sup(f)\|_{\dot{f}_p^{\alpha,q}}.$$

Proof. Firstly, one has the following relationship:

$$\begin{aligned}
\|f\|_{\dot{\mathcal{F}}_p^{\alpha,q}} &= \Bigg(\int \Bigg(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} |\psi_{j,k} * f(x)|^q \Bigg)^{p/q} dx \Bigg)^{1/p} \\
&= \Bigg(\int \Bigg(\sum_{j,k\in\mathbb{Z}} \sum_{R\in\Pi_{j,k}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} |\psi_{j,k} * f(x)|^q \Bigg)^{p/q} \chi_R(x)\, dx \Bigg)^{1/p} \\
&\le \|\sup(f)\|_{\dot{f}_p^{\alpha,q}}.
\end{aligned}$$

Now it is easy to see that, for $R = I \times J$ with $l(I) = 2^{-j\wedge k}$ and $l(J) = 2^{-j\wedge 2k}$,

$$\sup_{x\in R} |\psi_{j,k} * f(x', x_m)| = \sup_{y\in Q} |\psi_{j,k} * f(2^{-j\wedge k}y', 2^{-j\wedge 2k}y_m)|,$$

where Q is a cube in $\mathbb{R}^m$ with $l(Q) = 1$. Note that

$$(\psi_{j,k} * f(2^{-j\wedge k}\cdot, 2^{-j\wedge 2k}\cdot))^\wedge(\xi', \xi_m) = 2^{(j\wedge k)(m-1)}2^{j\wedge 2k}\widehat{\psi^1}(2^{-j+j\wedge k}\xi', 2^{-j+j\wedge 2k}\xi_m)\widehat{\psi^2}(2^{-k+j\wedge k}\xi', 2^{-2k+j\wedge 2k}\xi_m)\hat{f}(\xi', \xi_m).$$

From the compact conditions (1.5), (1.7), ξ' must satisfy $|2^{-j+j\wedge k}\xi'| \le 2$ and $|2^{-k+j\wedge k}\xi'| \le 2$, which give $|\xi'| \le 2$ by comparing j and k. Similarly, by comparing j and $2k$, we obtain $|\xi_m| \le 2$, which is the aim to restrict the support of $\widehat{\psi^{(2)}}$ in the set $\{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \le |\xi|_h \le 2^{1/2}\}$, smaller than the range in references [7, 19]. Therefore applying Lemma 2.7 to each of the functions $\psi_{j,k} * f(2^{-j\wedge k}y', 2^{-j\wedge 2k}y_m)$ we obtain

$$(\sup(f)_r^*)_R \lesssim (\inf_\gamma(f)_r^*)_R.$$

Let $r = \min\{p, q\}$. Then Lemma 2.3 gives

$$\|\sup(f)\|_{\dot{f}_p^{\alpha,q}} \lesssim \|\inf_\gamma(f)\|_{\dot{f}_p^{\alpha,q}}.$$

Combining the above with Lemma 2.8, we complete the proof. □

Proof of Theorem 1.5. Similar to the proof of Lemma 3.4 in this paper, we can obtain $\|f\|_{\dot{\mathcal{F}}_p^{\alpha,q}} \lesssim \|f\|_{\dot{F}_p^{\alpha,q}}$. We omit the details. By Lemma 2.9 together with the obvious fact $\|f\|_{\dot{F}_p^{\alpha,q}} \le \|\sup(f)\|_{\dot{f}_p^{\alpha,q}}$, we complete the proof. □

3 Restriction estimates

Before we give the atomic decomposition, we need some lemmas. Firstly we need an almost orthogonality estimates proved in [19].

Lemma 3.1. *Let $\psi^i, \phi^i \in \mathcal{S}_L(\mathbb{R}^m)$, $i = 1, 2$, $\psi_{j,k}$ and $\phi_{j',k'}$ be defined as before and let L be a positive integer. Then for any given integer M, there exists a constant $C = C(L, M) > 0$ such that*

$$|\psi_{j,k} * \phi_{j',k'}(x', x_m)| \le C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}}\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1 + 2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)^{(M+1)}}.$$

The following lemma is a variant of [19, Lemma 3.2]. The main difference between them is that we restrict the sum in a collection.

Lemma 3.2. *For $j, j', k, k' \in \mathbb{Z}$, let $R = I \times J \in \Pi_{j,k}$ be fixed. Suppose that for a positive integer N, Π is a collection of $P = I' \times J'$ where I' are dyadic cubes in $\mathbb{R}^{m-1}$, J' are dyadic cubes in $\mathbb{R}$, with $l(I') = 2^{-(j'\wedge k')-N}$ and $l(J') = 2^{-(j'\wedge 2k')-N}$, and the left lower corners of I' are $x_{I'}$, the left end points of J' are $x_{J'}$. Then for any $u', v' \in I$, $u_m, v_m \in J$, and any $\frac{m-1}{M+m-1} < \delta \le 1$,*

$$\begin{aligned}\sum_{I'\times J'\in\Pi} &\frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')}\,2^{j\wedge j'\wedge 2k\wedge 2k'}\,2^{-(m-1)[(j'\wedge k')+N]}\,2^{-(j'\wedge 2k')-N}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|u' - x_{I'}|)^{(M+m-1)}}\frac{|(\phi_{j',k'} * f)(x_{I'}, x_{J'})|}{(1 + 2^{j\wedge j'\wedge 2k\wedge 2k'}|u_m - x_{J'}|)^{(M+1)}}\\ &\le C_1\Big\{\mathcal{M}_s\Big(\sum_{I'\times J'\in\Pi}|(\phi_{j',k'} * f)(x_{I'}, x_{J'})|\chi_{I'}\chi_{J'}\Big)^\delta(v', v_m)\Big\}^{1/\delta},\end{aligned}$$

where $C_1 = C2^{-mN(1-1/\delta)}2^{(m-1)(1/\delta-1)(j'\wedge k'-j\wedge k)_+}\,2^{(1/\delta-1)(j'\wedge 2k'-j\wedge 2k)_+}$; here $(a - b)_+ = \max\{a - b, 0\}$ and $\mathcal{M}_s$ is the strong maximal function.

Proof. Let

$$A_0 = \Big\{I' : \frac{|u' - x_{I'}|}{2^{-(j\wedge j'\wedge k\wedge k')}} \le 1\Big\}, \quad B_0 = \Big\{J' : I' \times J', \frac{|u_m - x_{J'}|}{2^{-(j\wedge j'\wedge 2k\wedge 2k')}} \le 1\Big\},$$

and for $r \ge 1$, $s \ge 1$,

$$A_r = \Big\{I' : 2^{r-1} < \frac{|u' - x_{I'}|}{2^{-(j\wedge j'\wedge k\wedge k')}} \le 2^r\Big\}, \quad B_s = \Big\{J' : I' \times J', 2^{s-1} < \frac{|u_m - x_{J'}|}{2^{-(j\wedge j'\wedge 2k\wedge 2k')}} \le 2^s\Big\}.$$

For any fixed $r, s \geq 0$, denote

$$E = \{(w', w_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : |w' - u'| \leq 2^{r-(j\wedge j'\wedge k\wedge k')} + (m-1)^{1/2} 2^{-(j\wedge k)},\ |w_m - u_m| \leq 2^{r-(j\wedge j'\wedge 2k\wedge 2k')} + 2^{-(j\wedge 2k)}\}.$$

Then $A_r \times B_s \subset E$ and $I \times J \subseteq E$. Obviously,

$$|E| \leq C 2^{(m-1)[r-(j\wedge j'\wedge k\wedge k')]}\, 2^{[s-(j\wedge j'\wedge 2k\wedge 2k')]}.$$

Thus for $\frac{m-1}{M+m-1} < \delta \leq 1$,

$$\begin{aligned}
&\sum_{I'\times J'\in\Pi} \frac{2^{(m-1)(j\wedge j'\wedge k\wedge k')} 2^{j\wedge j'\wedge 2k\wedge 2k'}\, 2^{-(m-1)[(j'\wedge k')+N]} 2^{-(j'\wedge 2k')-N}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|u' - x_{I'}|)^{(M+m-1)}} \frac{|(\phi_{j',k'} * f)(x_{I'}, x_{J'})|}{(1 + 2^{j\wedge j'\wedge 2k\wedge 2k'}|u_m - x_{J'}|)^{(M+1)}} \\
&\leq C \sum_{r,s\geq 0} 2^{-r(M+m-1)} 2^{-s(M+1)} 2^{(m-1)(j\wedge j'\wedge k\wedge k')} 2^{j\wedge j'\wedge 2k\wedge 2k'} 2^{-(m-1)[(j'\wedge k')+N]} 2^{-(j'\wedge 2k')-N} \\
&\qquad \times \Big(\sum_{I'\times J'\in(A_r\times B_s)\cap\Pi} |(\phi_{j',k'} * f)(x_{I'}, x_{J'})|^{\delta} \Big)^{1/\delta} \\
&= C \sum_{r,s\geq 0} 2^{-r(M+m-1)} 2^{-s(M+1)} 2^{(m-1)(j\wedge j'\wedge k\wedge k')} 2^{j\wedge j'\wedge 2k\wedge 2k'} |I'|^{1-1/\delta} |J'|^{1-1/\delta} |E|^{1/\delta} \\
&\qquad \times \Big\{ \frac{1}{|E|} \int_E \sum_{I'\times J'\in(A_r\times B_s)\cap\Pi} |(\phi_{j',k'} * f)(x_{I'}, x_{J'})|^{\delta} \chi_{I'}\chi_{J'}\, dx \Big\}^{1/\delta} \\
&\leq C_1 \Big\{ \mathcal{M}_s \Big(\sum_{I'\times J'\in\Pi} |(\phi_{j',k'} * f)(x_{I'}, x_{J'})| \chi_{I'}\chi_{J'} \Big)^{\delta} (v', v_m) \Big\}^{1/\delta},
\end{aligned}$$

where in the last step we use the following deduction:

$$\begin{aligned}
\sum_{r,s\geq 0} 2^{-r(M+m-1)} 2^{-s(M+1)} 2^{(m-1)(j\wedge j'\wedge k\wedge k')} 2^{j\wedge j'\wedge 2k\wedge 2k'} |I'|^{1-1/\delta} |J'|^{1-1/\delta} |E|^{1/\delta} &= C_1 \sum_{r,s\geq 0} 2^{-r(M+m-1)} 2^{-s(M+1)} 2^{(m-1)r/\delta} 2^{s/\delta} \\
&\leq CC_1
\end{aligned}$$

because of $\delta > \frac{m-1}{M+m-1}$. □

In order to obtain the compact support of the atoms, we need a discrete Calderón-type identity on the space $L^2(\mathbb{R}^m) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ which is dense in $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ (see [7]). To do this, let $\phi^{(1)} \in \mathcal{S}_L(\mathbb{R}^m)$ with $\operatorname{supp} \phi^{(1)} \subseteq B(0,1)$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\phi^{(1)}}(2^{-j}\xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^m \setminus \{0\}, \tag{3.1}$$

where $L \geq 10M$ will be specified in application. We also let $\phi^{(2)} \in \mathcal{S}_L(\mathbb{R}^m)$ with $\operatorname{supp} \phi^{(2)} \subseteq B(0,1)$,

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \quad \text{for all } (\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \setminus \{(0,0)\}. \tag{3.2}$$

Set

$$\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)},$$

where $\phi_j^{(1)}(x) = 2^{jm}\phi^{(1)}(2^j x)$ and $\phi_k^{(2)}(x', x_m) = 2^{k(m+1)}\phi^{(2)}(2^k x', 2^{2k} x_m)$. The discrete Calderón-type identity is then given by the following [7].

Theorem B. *Let $\phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions* (3.1) *and* (3.2) *respectively. Then for any $f \in L^2(\mathbb{R}^m) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, there exists some $h \in L^2(\mathbb{R}^m) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ such that for a sufficiently large $N \in \mathbb{N}$,*

$$f(x', x_m) = \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |I||J| \phi_{j,k}(x' - 2^{-(j\wedge k)-N}\ell', x_m - 2^{-(j\wedge 2k)-N}\ell_m)(\phi_{j,k} * h)(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m),$$

where the series converge in L^2 and in $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with side-length $l(I) = 2^{-(j\wedge k)-N}$ and $l(J) = 2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)-N}\ell'$ and $2^{-(j\wedge 2k)-N}\ell_m$, respectively. Moreover,

$$\|f\|_{L^2(\mathbb{R}^m)} \approx \|h\|_{L^2(\mathbb{R}^m)}$$

and

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \approx \|h\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}. \tag{3.3}$$

Remark 3.3. The above theorem gives an equivalent norm of $\dot F_p^{\alpha,q}(\mathbb{R}^m)$, that is, if $f \in L^2 \cap \dot F_p^{\alpha,q}(\mathbb{R}^m)$, then

$$\begin{aligned}C_2^{-1}\|f\|_{\dot F_p^{\alpha,q}(\mathbb{R}^m)} &\le \bigg\|\bigg(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |\phi_{j,k} * h(2^{-(j\wedge k)-N}\ell', 2^{-(j\wedge 2k)-N}\ell_m)|^q \chi_I(x')\chi_J(x_m)\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)}\\ &\le C_2\|f\|_{\dot F_p^{\alpha,q}(\mathbb{R}^m)}.\end{aligned}$$

for some positive constant C_2. Note that the above relationship is different from (3.3). One can prove it by applying a similar proof as the following Lemma 3.4 combining Theorem A with Theorem B. See [7] for details.

In the pure product Hardy space case [24], to get atomic decompositions, one needs to establish duality of corresponding space. Our approach is different from this scheme by using the following lemma.

Lemma 3.4. *Let $0 < p, q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions* (3.1) *and* (3.2) *respectively, with*

$$L > (m+1)\Big(\frac{1}{\delta} - 1\Big) + |\alpha_1| + 2|\alpha_2|.$$

For $(j,k) \in \mathbb{Z}^2$, let $\widetilde\Pi_{j,k}$ be a collection of rectangles $R = I \times J$ where I are dyadic cubes in $\mathbb{R}^{m-1}$, J are dyadic intervals in $\mathbb{R}$, with the side length $l(I) = 2^{-(j\wedge k)-N}$ and $l(J) = 2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $x_I = 2^{-(j\wedge k)-N}\ell'$ and $x_J = 2^{-(j\wedge 2k)-N}\ell_m$, respectively, for some $(\ell', \ell_m) \in \mathbb{Z}^m$. If $0 < \delta < \min\{p, q, 1\}$, then

$$\begin{aligned}&\bigg\|\sum_{j,k}\sum_{R\in\widetilde\Pi_{j,k}} |R|\phi_{j,k}(x'-x_I, x_m-x_J)\times(\phi_{j,k}*h)(x_I,x_J)\bigg\|_{\dot F_p^{\alpha,q}(\mathbb{R}^m)}\\ &\qquad\le C_2\bigg\|\bigg(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}\sum_{R\in\widetilde\Pi_{j,k}} |\phi_{j,k}*h(x_I,x_J)|^q\chi_I(x')\chi_J(x_m)\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^m)}\end{aligned}$$

for the positive constance C_2 in Remark 3.3.

Remark 3.5. From this lemma, one can see that the order L in the cancellation condition of $\phi^{(1)}$ and $\phi^{(2)}$ should be bigger than $(m+1)(\frac{1}{\min\{p,q,1\}} - 1) + |\alpha_1| + 2|\alpha_2|$ which is reflected in (iv′) of Definition 1.6.

Proof. Let $\psi_{j',k'}$ be the same as in Theorem A. By Lemma 3.1, for $(\ell'', \ell'_m) \in \mathbb{Z}^{m-1}\times\mathbb{Z}$,

$$\begin{aligned}&|\psi_{j',k'} * \phi_{j,k}(2^{-(j'\wedge k')}\ell'' - x_I, 2^{-(j'\wedge 2k')}\ell'_m - x_J)|\\ &\quad\le C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|2^{-(j'\wedge k')}\ell''-x_I|)^{(M+m-1)}}\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|2^{-(j'\wedge 2k')}\ell'_m - x_J|)^{(M+1)}}.\end{aligned}$$

Hence, with Lemma 3.2, for $\frac{m-1}{M+m-1} < \delta \le 1$ and any $v'' \in I'$, $v'_m \in J'$, where I' are dyadic cubes in $\mathbb{R}^{m-1}$ and J' are dyadic intervals in $\mathbb{R}$ with side-length $l(I') = 2^{-(j'\wedge k')}$ and $l(J') = 2^{-(j'\wedge 2k')}$, and the left lower corners of I' and the left end points of J' are $x_{I'} = 2^{-(j'\wedge k')}\ell''$ and $x_{J'} = 2^{-(j'\wedge 2k')}\ell'_m$, respectively, we have

$$\begin{aligned}&\bigg|\sum_{j,k}\sum_{R\in\widetilde\Pi_{j,k}} |R|(\phi_{j,k}(\cdot - x_I, \cdot - x_J)\times(\phi_{j,k}*h)(x_I,x_J)) * \psi_{j',k'}(2^{-(j'\wedge k')}\ell'', 2^{-(j'\wedge 2k')}\ell'_m)\bigg|\\ &\quad\le C\sum_{j,k}2^{-|j-j'|L}2^{-|k-k'|L}\sum_{R\in\widetilde\Pi_{j,k}} 2^{-[(j\wedge k)+N](m-1)}2^{-(j\wedge 2k)-N}\frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge k\wedge k'}|2^{-(j'\wedge k')}\ell''-x_I|)^{(M+m-1)}}\\ &\qquad\qquad\times\frac{(\phi_{j,k}*h)(x_I,x_J)}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|2^{-(j'\wedge 2k')}\ell'_m - x_J|)^{(M+1)}}\\ &\quad\le C\sum_{j,k}2^{-|j-j'|L}2^{-|k-k'|L}\bigg\{\mathcal{M}_s\bigg(\sum_{R\in\widetilde\Pi_{j,k}}|(\phi_{j,k}*h)(x_I,x_J)|\chi_I\chi_J\bigg)^{\delta}(v'', v'_m)\bigg\}^{1/\delta}.\end{aligned}$$

Set

$$\widetilde h = \sum_{j,k}\sum_{R\in\widetilde\Pi_{j,k}} |R|\phi_{j,k}(x'-x_I, x_m-x_J)\times(\phi_{j,k}*h)(x_I,x_J).$$

Summing over j', k' and (ℓ'', ℓ'_m), for any $v'' \in I', v'_m \in J'$, we have

$$\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q} \sum_{(\ell'',\ell'_m)} |\psi_{j',k'} * \widetilde{h}(x_{I'}, x_{J'})|^q \chi_{I'}(x'')\chi_{J'}(x'_m)\Big)^{1/q}$$
$$\le C\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q}\Big[\sum_{j,k} 2^{-|j-j'|L}2^{-|k-k'|L}C_1\Big\{\mathcal{M}_s\Big(\sum_{R\in\widetilde{\Pi}_{j,k}} |(\phi_{j,k} * h)(x_I, x_J)|\chi_I\chi_J\Big)^{\delta}(v'', v'_m)\Big\}^{1/\delta}\Big]^q\Big)^{1/q}.$$

When $0 < q \le 1$, using the inequality $(\sum_l a_l)^q \le \sum_l a_l^q$,

$$\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q} \sum_{(\ell'',\ell'_m)} |\psi_{j',k'} * \widetilde{h}(x_{I'}, x_{J'})|^q \chi_{I'}(x'')\chi_{J'}(x'_m)\Big)^{1/q}$$
$$\le C\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q}\sum_{j,k} 2^{-|j-j'|Lq}2^{-|k-k'|Lq}C_1^q\Big\{\mathcal{M}_s\Big(\sum_{R\in\widetilde{\Pi}_{j,k}} |(\phi_{j,k} * h)(x_I, x_J)|\chi_I\chi_J\Big)^{\delta}(v'', v'_m)\Big\}^{q/\delta}\Big)^{1/q}$$
$$\le C\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}\Big\{\mathcal{M}_s\Big(\sum_{R\in\widetilde{\Pi}_{j,k}} |(\phi_{j,k} * h)(x_I, x_J)|\chi_I\chi_J\Big)^{\delta}(v'', v'_m)\Big\}^{q/\delta}\Big)^{1/q}$$

where in the last inequality we use the facts that

$$(j'\wedge k' - j\wedge k)_+ \le |j-j'| + |k-k'|, \quad (j'\wedge 2k' - j\wedge 2k)_+ \le |j-j'| + 2|k-k'|$$

and

$$L > (m+1)\Big(\frac{1}{\delta} - 1\Big) + |\alpha_1| + 2|\alpha_2|,$$

then

$$\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k'-j\wedge k)\alpha_1+(j'\wedge 2k'-j\wedge 2k)\alpha_2]q}2^{-|j-j'|Lq}2^{-|k-k'|Lq}C_1^q \le C.$$

When $q > 1$, by Cauchy's inequality with exponents q, q', $\frac{1}{q} + \frac{1}{q'} = 1$, for all $0 < \varepsilon < 1$,

$$\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q} \sum_{(\ell'',\ell'_m)} |\psi_{j',k'} * \widetilde{h}(x_{I'}, x_{J'})|^q \chi_{I'}(x'')\chi_{J'}(x'_m)\Big)^{1/q}$$
$$\le C\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1+(j'\wedge 2k')\alpha_2]q}\Big[\sum_{j,k} 2^{-|j-j'|Lq'\varepsilon}2^{-|k-k'|Lq'\varepsilon}\Big]^{q/q'}$$
$$\times\Big[\sum_{j,k} 2^{-|j-j'|Lq(1-\varepsilon)}2^{-|k-k'|Lq(1-\varepsilon)}C_1^q\Big\{\mathcal{M}_s\Big(\sum_{R\in\widetilde{\Pi}_{j,k}} |(\phi_{j,k} * h)(x_I, x_J)|\chi_I\chi_J\Big)^{\delta}(v'', v'_m)\Big\}^{q/\delta}\Big]\Big)^{1/q}$$
$$\le C\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}\Big\{\mathcal{M}_s\Big(\sum_{R\in\widetilde{\Pi}_{j,k}} |(\phi_{j,k} * h)(x_I, x_J)|\chi_I\chi_J\Big)^{\delta}(v'', v'_m)\Big\}^{q/\delta}\Big)^{1/q},$$

where in the last inequality we use the similar estimates as in the case of $0 < q \le 1$, since

$$\sum_{j,k} 2^{-|j-j'|Lq'\varepsilon}2^{-|k-k'|Lq'\varepsilon} \le C$$

and

$$\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k'-j\wedge k)\alpha_1+(j'\wedge 2k'-j\wedge 2k)\alpha_2]q}2^{-|j-j'|Lq(1-\varepsilon)}2^{-|k-k'|Lq(1-\varepsilon)}C_1^q \le C$$

when ε is close to 0, since $L > (m+1)(\frac{1}{\delta} - 1) + |\alpha_1| + 2|\alpha_2|$. Applying Fefferman–Stein's vector-valued strong maximal inequality on $L^{p/\delta}(\ell^{q/\delta})$ provided $\delta < \min\{p, q, 1\}$, we complete the proof. □

Remark 3.6. All conditions on δ can be satisfied if we choose M large enough.

4 Atomic decomposition

We are ready to give the atomic decompositions. For convenience, in this section, the function $h(x)$, the positive integers N are always from the discrete Calderón-type identity in Theorem B and $\alpha = (\alpha_1, \alpha_2)$. For fixed $j, k \in \mathbb{Z}$, denote by $\Pi^N_{j,k}$ the set of all $R = I \times J$ such that I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side lengths $l(I) = 2^{-(j\wedge k)-N}$ and $l(J) = 2^{-(j\wedge 2k)-N}$, and the left lower corners of I and the left end points of J are $x_I = 2^{-(j\wedge k)-N}\ell'$ and $x_J = 2^{-(j\wedge 2k)-N}\ell_m$, respectively, $(\ell', \ell_m) \in \mathbb{Z}^m$.

Proof of Theorem 1.7. Consider

$$S^\alpha_q(f)(x) = \Big\{ \sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \sum_{R\in\Pi^N_{j,k}} |\phi_{j,k} * h(x_I, x_J)|^q \chi_I(x')\chi_J(x_m)\Big\}^{1/q}.$$

Then by Remark 3.3,

$$\|f\|_{\dot F^{\alpha,q}_p(\mathbb{R}^m)} \approx \|S^\alpha_q(f)\|_{L^p(\mathbb{R}^m)}.$$

For any $f \in L^2(\mathbb{R}^m) \cap \dot F^{\alpha,q}_p(\mathbb{R}^m)$, set

$$\Omega_i = \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : S^\alpha_q(f)(x) > 2^i\},$$

and

$$\mathcal{B}_i = \Big\{(j, k, I, J) : |(I \times J) \cap \Omega_i| > \frac{1}{2}|I \times J|,\ |(I \times J) \cap \Omega_{i+1}| \le \frac{1}{2}|I \times J|\Big\}.$$

Obviously, $\Omega_{i+1} \subseteq \Omega_i$, and if $\Omega_{i+1} = \Omega_i$, then $\mathcal{B}_i = \emptyset$, hence $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ if $i \ne j$. We say a rectangle $R \in \mathcal{B}_i$ means $R = I \times J$ with $l(I) = 2^{-(j\wedge k)-N}$, $l(J) = 2^{-(j\wedge 2k)-N}$, and $(j, k, I, J) \in \mathcal{B}_i$.

By Theorem B, we write

$$f(x', x_m) = \sum_i \sum_{(j,k,I,J)\in\mathcal{B}_i} |R|\phi_{j,k}(x' - x_I, x_m - x_J) \times (\phi_{j,k} * h)(x_I, x_J),$$

where the series converges in $L^2(\mathbb{R}^m)$ and in $\dot F^{\alpha,q}_p(\mathbb{R}^m)$. To obtain the atomic decomposition, rewrite

$$f(x', x_m) = \sum_i \lambda_i a_i(x', x_m),$$

where

$$a_i(x', x_m) = \frac{1}{\lambda_i} \sum_{(j,k,I,J)\in\mathcal{B}_i} |R|\phi_{j,k}(x' - x_I, x_m - x_J) \times (\phi_{j,k} * h)(x_I, x_J),$$

and

$$\lambda_i = C_2|\widetilde\Omega_i|^{\frac{1}{p}-\frac{1}{q}}\Big(\int \sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I, x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx\Big)^{1/q},$$

where

$$\widetilde\Omega_i = \Big\{(x', x_m) : M_s(\chi_{\Omega_i})(x', x_m) > \frac{1}{10^{Nm}}\Big\}.$$

Note that functions $\phi^{(1)}$ and $\phi^{(2)}$ are supported in the unit ball. Then for a fixed $R \in \mathcal{B}_i$, $\phi_{j,k}(x' - x_I, x_m - x_J)$ is supported in $2^{N+3}R$, which implies that a_i is supported in $\widetilde\Omega_i$. To see that the a_i satisfy (ii) in Definition 1.6, we write

$$a_i(x', x_m) = \frac{1}{\lambda_i} b_i(x', x_m),$$

where

$$\begin{aligned} b_i(x', x_m) &= \sum_{(j,k,I,J)\in\mathcal{B}_i} |R|\ \phi_{j,k}(x' - x_I, x_m - x_J) \times (\phi_{j,k} * h)(x_I, x_J) \\ &= \sum_{j,k} \sum_{I\times J\in\widetilde\Pi_{j,k}} |R|\ \phi_{j,k}(x' - x_I, x_m - x_J) \times (\phi_{j,k} * h)(x_I, x_J), \end{aligned}$$

and

$$\widetilde\Pi_{j,k} = \{R = I \times J : (j, k, I, J) \in \mathcal{B}_i\}.$$

By Lemma 3.4

$$\|b_i\|_{\dot F^{\alpha,q}_q} \le C_2\Big(\int \sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I, x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx\Big)^{1/q},$$

which implies that

$$\|a_i\|_{\dot{F}_q^{\alpha,q}(\mathbb{R}^m)} \le |\widetilde{\Omega}_i|^{\frac{1}{q}-\frac{1}{p}}.$$

Again by Lemma 3.4, we have

$$\begin{aligned}\|b_i\|_{\dot{F}_p^{\alpha,q}} &\le C_2\Big(\int\Big(\sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\Big)^{p/q}dx\Big)^{1/p}\\ &\le C_2\Big(\int\sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx\Big)^{1/q}|\widetilde{\Omega}_i|^{\frac{1}{p}-\frac{1}{q}}\end{aligned}$$

by Hölder's inequality, which implies that

$$\|a_i\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \le 1.$$

Now, we check (iii). Firstly, we consider the case $p < q$. It is easy to have that, if $x = (x', x_m) \in R \in \mathcal{B}_i$ such that $M_s(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x) > \frac{1}{2}$, then

$$\chi_R(x) = \chi_R^{q/p}(x) \le 2^{q/p} M_s^{q/p}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x).$$

Thus, by the Fefferman–Stein vector-valued strong maximal inequality, for $0 < p < q < \infty$,

$$\begin{aligned}\lambda_i^p &= C^p|\widetilde{\Omega}_i|^{1-\frac{p}{q}}\Big(\int\sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx\Big)^{p/q}\\ &\le C^p|\widetilde{\Omega}_i|^{1-\frac{p}{q}}\Big(\int\sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}M_s^{q/p}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x)\,dx\Big)^{p/q}\\ &\lesssim |\widetilde{\Omega}_i|^{1-\frac{p}{q}}\Big(\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx\Big)^{p/q}\\ &\lesssim |\widetilde{\Omega}_i|^{1-\frac{p}{q}}(2^{iq}|\widetilde{\Omega}_i\setminus\Omega_{i+1}|)^{\frac{p}{q}}\\ &\le 2^{ip}|\widetilde{\Omega}_i| \lesssim 2^{ip}|\Omega_i|.\end{aligned}$$

Hence

$$\sum_i \lambda_i^p \lesssim \|S_q^\alpha(f)\|^p_{L^p(\mathbb{R}^m)} \approx \|f\|^p_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}.$$

If $p = q$, then (iii) easily follows from the definition of λ_i and Remark 3.3.

To verify conditions (iv), note that $a_i = \sum_{R\in m(\widetilde{\Omega}_i)} a_{i,R}$, where

$$a_{i,R}(x',x_m) = \frac{1}{\lambda_i}\sum_{Q\in\mathcal{B}_i,\,Q\subset R}|Q|\phi_{j,k}(x'-x_I, x_m-x_J)\times(\phi_{j,k} * h)(x_I,x_J).$$

Since for each fixed $Q \in \mathcal{B}_i$, $\operatorname{supp}\phi_{j,k}(x'-x_I, x_m-x_J) \subseteq 2^{N+3}Q$, it follows that $\operatorname{supp} a_{i,R} \subseteq 2^{N+3}R$. Generally, for any $Q \in \mathcal{B}_i$, there may exist more than one $R \in m(\widetilde{\Omega}_i)$ such that $Q \subset R$. In order to ensure non-intersection, once an $R_0 \in m(\widetilde{\Omega}_i)$ has been selected such that $Q \subset R_0$, then any other $R \in m(\widetilde{\Omega}_i)$ will not be allowed to do so. Obviously, the $a_{i,R}$ satisfy the cancellation condition (iv′) since $\phi_{j,k} \in \mathcal{S}_L(\mathbb{R}^m)$.

With the same proof as in the estimation of $\|b_i\|_{\dot{F}_q^{s,q}(\mathbb{R}^m)}$, one has

$$\|a_{i,R}\|_{\dot{F}_q^{\alpha,q}(\mathbb{R}^m)} \le \frac{1}{\lambda_i}\Big(\int\sum_{Q\in\mathcal{B}_i,\,2^{N+3}Q\subset R} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx\Big)^{1/q}$$

which shows that

$$\sum_{R\in m(\widetilde{\Omega}_i)}\|a_{i,R}\|^q_{\dot{F}_q^{\alpha,q}(\mathbb{R}^m)} \le \frac{1}{\lambda_i^q}\int\sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * h(x_I,x_J)|^q 2^{-(j\alpha_1+k\alpha_2)q}\chi_R(x)\,dx.$$

Hence

$$\Big(\sum_{R\in m(\widetilde{\Omega}_i)}\|a_{i,R}\|^q_{\dot{F}_q^{\alpha,q}(\mathbb{R}^m)}\Big)^{1/q} \le |\widetilde{\Omega}_i|^{\frac{1}{q}-\frac{1}{p}}.$$ □

Proof of Theorem 1.9. From Definition 1.6, for any (p, q, α)-atom a, we have $\|a\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \le C_0$. So when T is a bounded operator on $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, we have the one direction.

On the other hand, for an $f \in L^2(\mathbb{R}^m) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^m)$, by Corollary 1.8, we have

$$f = \sum_i \lambda_i a_i$$

in $L^2(\mathbb{R}^m)$ and $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ with $\{a_i\}$ being (p, q, α)- atoms of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$. Since T is a bounded operator on $L^2(\mathbb{R}^m)$, $T(f)(x) = \sum_i \lambda_i T(a_i)(x)$, using that $f \to \|f\|^p_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}$ is subadditive when $0 < p \le q < \infty$ and $p \le 1$, one has

$$\|T(f)\|^p_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \le \sum_i \lambda_i^p \|T(a_i)\|^p_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \lesssim \sum_i \lambda_i^p \lesssim \|f\|^p_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}$$

which can be extended to the entire $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ by density. □

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References

[1] S.-Y. A. Chang and R. Fefferman, A continuous version of duality of H^1 with BMO on the bidisc, *Ann. of Math. (2)* **112** (1980), 179–201.
[2] S.-Y. A. Chang and R. Fefferman, The Calderón–Zygmund decomposition on product domains, *Amer. J. Math.* **104** (1982) 455–468.
[3] S.-Y. A. Chang and R. Fefferman, Some recent developments in Fourier analysis and H^p theory on product domains, *Bull. Amer. Math. Soc.* **12** (1985), 1–43.
[4] L. Carleson, A counterexample for measures bounded on H^p for the bidisc, Mittag-Leffler Rep. 7, Institut Mittag-Leffler, Djursholm, 1974.
[5] R. R. Coifman, A real variable characterization of H_p, *Stud. Math.* **51** (1974), 269–274.
[6] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
[7] W. Ding, The boundedness of composition operators on Triebel–Lizorkin and Besov spaces with different homogeneities, *Acta Math. Sin. (Engl. Ser.)* **30** (2014), no. 6, 933–948.
[8] W. Ding and G. Lu, Duality of multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular integral operators, submitted.
[9] Y. Ding, G. Lu and B. Ma, Multi-parameter Triebel–Lizorkin and Besov spaces associated with flag singular integrals, *Acta Math. Sin. (Engl. Ser.)* **26** (2010), no. 4, 603–620.
[10] R. Fefferman, Calderón–Zygmund theory for product domains: H^p spaces, *Proc. Natl. Acad. Sci. USA* **83** (1986), 840–843.
[11] R. Fefferman, Harmonic analysis on product spaces, *Ann. of Math. (2)* **126** (1987), 109–130.
[12] R. Fefferman and E. M. Stein, Singular integrals on product spaces, *Adv. Math.* **45** (1982), 117–143.
[13] S. Ferguson and M. Lacey, A characterization of product BMO by commutators, *Acta Math.* **189** (2002), no. 2, 143–160.
[14] G. B. Folland and E. M. Stein, Estimates for the ∂_b-complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974), 429–522.
[15] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton University Press, Princeton, 1982.
[16] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.* **93** (1990), 34–170.
[17] R. Gundy and E. M. Stein, H^p theory for the polydisk, *Proc. Natl. Acad. Sci. USA* **76** (1979), 1026–1029.
[18] Y. Han, J. Li and G. Lu, Multiparameter Hardy space theory on Carnot–Carathéodory spaces and product spaces of homogeneous type, *Trans. Amer. Math. Soc.* **365** (2013), no. 1, 319–360.
[19] Y. Han, C. Lin, G. Lu, T. Ruan and E. Sawyer, Hardy spaces associated with different homogeneities and boundedness of composition operators, *Rev. Mat. Iberoam.* **29** (2013), no. 4, 1127–1157.
[20] Y. Han and G. Lu, Discrete Littlewood–Paley–Stein theory and multi-parameter Hardy spaces associated with flag singular integrals, preprint (2008), http://arxiv.org/abs/0801.1701.

[21] Y. Han, G. Lu and Z. Ruan, Boundedness criterion of Journé’s class of singular integrals on multiparameter Hardy spaces, *J. Funct. Anal.* **264** (2013), no. 5, 1238–1268.
[22] Y. Han, G. Lu and Z. Ruan, Boundedness of singular integrals in Journe’s class on weighted multiparameter Hardy spaces, *J. Geom. Anal.* (2014), DOI 10.1007/s12220-013-9421-x.
[23] Y. Han, G. Lu and E. Sawyer, Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group, preprint (2012), http://arxiv.org/abs/1208.2484; to appear in *Anal. PDE*.
[24] Y. Han, G. Lu and K. Zhao, Discrete Calderón’s identity, atomic decomposition and boundedness criterion of operators on multiparameter Hardy spaces, *J. Geom. Anal.* **20** (2010), 670–689.
[25] J. L. Journe, Calderón–Zygmund operators on product spaces, *Rev. Mat. Iberoam.* **1** (1985), 55–92.
[26] J. L. Journe, Two problems of Calderón–Zygmund theory on product spaces, *Ann. Inst. Fourier (Grenoble)* **38** (1988), 111–132
[27] R. H. Latter, A decomposition of $H_p(R_n)$ in terms of atoms, *Stud. Math.* **62** (1978), 92–101.
[28] G. Lu and Z. Ruan, Duality theory of weighted Hardy spaces with arbitrary number of parameters, *Forum Math.* (2012), DOI 10.1515/forum-2012-0018.
[29] G. Lu and Y. Zhu, Singular integrals and weighted Triebel–Lizorkin and Besov spaces of arbitrary number of parameters, *Acta Math. Sin. (Engl. Ser.)* **29** (2013), no. 1, 39–52.
[30] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992.
[31] Y. Meyer and R. R. Coifman, *Wavelets, Calderón–Zygmund and Multilinear Operators*, Cambridge University Press, Cambridge, 1997.
[32] D. Phong and E. Stein, Some further classes of pseudo-differential and singular integral operators arising in boundary valve problems. I: Composition of operators, *Amer. J. Math.* **104** (1982), 141–172.
[33] J. Pipher, Journer’s covering lemma and its extension to higher dimensions, *Duke Math. J.* **53** (1986), 683–690.
[34] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.
[35] S. Wainger and G. Weiss, *Harmonic Analysis in Euclidean Spaces* (Williamstown 1978), Proc. Sympos. Pure Math. 35, American Mathematical Society, Providence, 1979.

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