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L^p ESTIMATES FOR BILINEAR AND MULTIPARAMETER HILBERT TRANSFORMS

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Muscalu, Pipher, Tao and Thiele proved that the standard bilinear and biparameter Hilbert transform does not satisfy any L^p estimates. They also raised a question asking if a bilinear and biparameter multiplier operator defined by

$$T_m(f_1, f_2)(x) := \int_{\mathbb{R}^4} m(\xi, \eta) \hat{f}_1(\xi_1, \eta_1) \hat{f}_2(\xi_2, \eta_2) e^{2\pi i x \cdot ((\xi_1,\eta_1)+(\xi_2,\eta_2))} \, d\xi \, d\eta$$

satisfies any L^p estimates, where the symbol m satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim \frac{1}{\operatorname{dist}(\xi, \Gamma_1)^{|\alpha|}} \cdot \frac{1}{\operatorname{dist}(\eta, \Gamma_2)^{|\beta|}}$$

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, Γ_i $(i = 1, 2)$ are subspaces in $\mathbb{R}^2$ and $\dim \Gamma_1 = 0$, $\dim \Gamma_2 = 1$. Silva partially answered this question and proved that T_m maps $L^{p_1} \times L^{p_2} \to L^p$ boundedly when $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ with $p_1, p_2 > 1$, $\frac{1}{p_1} + \frac{2}{p_2} < 2$ and $\frac{1}{p_2} + \frac{2}{p_1} < 2$. One notes that the admissible range here for these tuples (p_1, p_2, p) is a proper subset of the admissible range of the bilinear Hilbert transform (BHT) derived by Lacey and Thiele.

We establish the same L^p estimates as BHT in the full range for the bilinear and d-parameter ($d \geq 2$) Hilbert transforms with arbitrary symbols satisfying appropriate decay assumptions and having singularity sets $\Gamma_1, \ldots, \Gamma_d$ with $\dim \Gamma_i = 0$ for $i = 1, \ldots, d-1$ and $\dim \Gamma_d = 1$. Moreover, we establish the same L^p estimates as BHT for bilinear and biparameter Fourier multipliers of symbols with $\dim \Gamma_1 = \dim \Gamma_2 = 1$ and satisfying some appropriate decay estimates. In particular, our results include the L^p estimates as BHT in the full range for certain modified bilinear and biparameter Hilbert transforms of tensor-product type with $\dim \Gamma_1 = \dim \Gamma_2 = 1$ but with a slightly better logarithmic decay than that of the bilinear and biparameter Hilbert transform BHT $\otimes$ BHT.

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1. Introduction

The bilinear Hilbert transform is defined by

$$\mathrm{BHT}(f_1, f_2)(x) := \mathrm{p.v.} \int_{\mathbb{R}} f_1(x-t) f_2(x+t) \frac{dt}{t}; \tag{1-1}$$

or, equivalently, it can be written as the bilinear multiplier operator

$$\mathrm{BHT} : (f_1, f_2) \mapsto \int_{\xi<\eta} \hat{f}_1(\xi) \hat{f}_2(\eta) e^{2\pi i x(\xi+\eta)} \, d\xi \, d\eta, \tag{1-2}$$

where f_1 and f_2 are Schwartz functions on $\mathbb{R}$. M. Lacey and C. Thiele proved the following celebrated L^p estimates for the bilinear Hilbert transform:

Theorem 1.1 [Lacey and Thiele 1997; 1999]. *The bilinear operator* BHT *maps* $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ *into* $L^r(\mathbb{R})$ *boundedly for any* $1 < p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *and* $\frac{2}{3} < r < \infty$.

There are lots of works related to bilinear operators of BHT type. J. Gilbert and A. Nahmod [2001] and F. Bernicot [2008] proved that the same L^p estimates as BHT are valid for bilinear operators with more general symbols. Uniform estimates were obtained by Thiele [2002], L. Grafakos and X. Li [2004] and Li [2006]. A maximal variant of Theorem 1.1 was proved by Lacey [2000]. C. Muscalu, Thiele and T. Tao [Muscalu et al. 2004b] and J. Jung [≥ 2015] investigated various trilinear variants of the bilinear Hilbert transform. For more related results involving estimates for multilinear singular multiplier operators, we refer to, for example, [Christ and Journé 1987; Coifman and Meyer 1978; 1997; Fefferman and Stein 1982; Grafakos and Torres 2002a; 2002b; Journé 1985; Kenig and Stein 1999; Muscalu and Schlag 2013; Muscalu et al. 2002; Thiele 2006] and the references therein.

Since Lacey and Thiele [1997; 1999] established the L^p estimates for $\frac{2}{3} < p < \infty$, whether the bilinear operators of BHT type satisfy L^p estimates all the way down to $\frac{1}{2}$ has remained an open problem. Though we do not have a counterexample yet for the L^p estimates for the bilinear Hilbert transform in the range of $\frac{1}{2} < p < \frac{2}{3}$, we have established in [Dai and Lu ≥ 2015b] a counterexample for a modified version of bilinear operators of BHT type. To describe this result, we denote by $\mathscr{F}L^p(\mathbb{R})$ the space consisting of all functions f whose Fourier transform $\hat{f}$ satisfies $\hat{f} \in L^p(\mathbb{R})$. The Hausdorff–Young inequality indicates that $\|\hat{f}\|_{L^{p'}(\mathbb{R})} \lesssim_p \|f\|_{L^p(\mathbb{R})}$ for $1 \leq p \leq 2$. Then, by Theorem 1.1, it implies that the bilinear Hilbert transform maps $\mathscr{F}L^{p_1'} \times L^{p_2} \to L^p$ for $p_1 \geq 2$ and maps $L^{p_1} \times \mathscr{F}L^{p_2'} \to L^p$ for $p_2 \geq 2$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Thus it will be interesting to know whether the bilinear operators of BHT type map $\mathscr{F}L^{p_1'} \times L^{p_2} \to L^p$ for $p_1 < 2$ or $L^{p_1} \times \mathscr{F}L^{p_2'} \to L^p$ for $p_2 < 2$ boundedly with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Our work in [Dai and Lu ≥ 2015b] gives a negative answer to the boundedness of $\mathscr{F}L^{p_1'} \times L^{p_2} \to L^p$ for $p_1 < 2$ and $L^{p_1} \times \mathscr{F}L^{p_2'} \to L^p$ for $p_2 < 2$.

To date, we are still not aware of any uniform L^p estimates for bilinear Fourier multiplier operator of BHT type in the range $p \in \left(\frac{1}{2}, \frac{2}{3}\right)$. By decomposing the bilinear multiplier operator T_m into a summation of infinitely many bilinear paraproducts without modulation invariance, we have proved in [Dai and Lu ≥ 2015b] that there exists a class of symbols m (with one-dimensional singularity sets), which also satisfy the symbol estimates of BHT type operators investigated in [Gilbert and Nahmod 2001] and are arbitrarily close to the symbols of BHT type operators, such that the corresponding bilinear multiplier operators T_m associated with symbols m satisfy L^p estimates all the way down to $\frac{1}{2}$.

In multiparameter cases, there are also large amounts of literature devoted to studying the estimates of multiparameter and multilinear operators (see [Chen and Lu 2014; Dai and Lu $\geq$ 2015a; Demeter and Thiele 2010; Hong and Lu 2014; Kesler $\geq$ 2015; Luthy 2013; Muscalu and Schlag 2013; Muscalu et al. 2004a; 2006; Silva 2014] and the references therein). In the bilinear and biparameter cases, let Γ_i ($i = 1, 2$) be subspaces in $\mathbb{R}^2$, we consider operators T_m defined by

$$T_m(f_1, f_2)(x) := \int_{\mathbb{R}^4} m(\xi, \eta) \hat{f}_1(\xi_1, \eta_1) \hat{f}_2(\xi_2, \eta_2) e^{2\pi i x \cdot ((\xi_1, \eta_1) + (\xi_2, \eta_2))} \, d\xi \, d\eta, \tag{1-3}$$

where the symbol m satisfies[1]

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim \frac{1}{\operatorname{dist}(\xi, \Gamma_1)^{|\alpha|}} \cdot \frac{1}{\operatorname{dist}(\eta, \Gamma_2)^{|\beta|}} \tag{1-4}$$

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. If $\dim \Gamma_1 = \dim \Gamma_2 = 0$, Muscalu, J. Pipher, Tao and Thiele proved in [Muscalu et al. 2004a; 2006] that Hölder-type L^p estimates are available for T_m; however, if $\dim \Gamma_1 = \dim \Gamma_2 = 1$, let T_m be the double bilinear Hilbert transform on polydisks $\mathrm{BHT} \otimes \mathrm{BHT}$ defined by

$$\mathrm{BHT} \otimes \mathrm{BHT}(f_1, f_2)(x, y) := \mathrm{p.v.} \int_{\mathbb{R}^2} f_1(x - s, y - t) f_2(x + s, y + t) \, \frac{ds}{s} \, \frac{dt}{t}; \tag{1-5}$$

they also proved in [Muscalu et al. 2004a] that the operator $\mathrm{BHT} \otimes \mathrm{BHT}$ does not satisfy any L^p estimates of Hölder type by constructing a counterexample. In fact, consider bounded functions $f_1(x, y) = f_2(x, y) = e^{ixy}$; one has formally

$$\mathrm{BHT} \otimes \mathrm{BHT}(f_1, f_2)(x, y) = (f_1 \cdot f_2)(x, y) \int_{\mathbb{R}^2} \frac{e^{2ist}}{st} \, ds \, dt = i\pi (f_1 \cdot f_2)(x, y) \int_{\mathbb{R}} \frac{\operatorname{sgn}(s)}{s} \, ds,$$

then localize functions f_1, f_2 and let $f_1^N(x, y) = f_2^N(x, y) = e^{ixy} \chi_{[-N,N]}(x) \chi_{[-N,N]}(y)$. One can verify the pointwise estimate

$$|\mathrm{BHT} \otimes \mathrm{BHT}(f_1^N, f_2^N)(x, y)| \geq \left| \int_{-\frac{N}{10}}^{\frac{N}{10}} \int_{-\frac{N}{10}}^{\frac{N}{10}} \frac{e^{2ist}}{st} \, ds \, dt \right| + O(1) \geq C \log N + O(1) \tag{1-6}$$

for every $x, y \in \left[-\frac{1}{100} N, \frac{1}{100} N\right]$ and sufficiently large $N \in \mathbb{Z}^+$, which indicates that no Hölder-type L^p estimates are available for the bilinear operator $\mathrm{BHT} \otimes \mathrm{BHT}$. When $\dim \Gamma_1 = 0$ and $\dim \Gamma_2 = 1$, there is the following problem:

Question 1.2 [Muscalu et al. 2004a, Question 8.2]. Let $\dim \Gamma_1 = 0$ and $\dim \Gamma_2 = 1$ with Γ_2 nondegenerate in the sense of [Muscalu et al. 2002]. If m is a multiplier satisfying (1-4), does the corresponding operator T_m defined by (1-3) satisfy any L^p estimates?

[1] Throughout this paper, $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$. If necessary, we use explicitly $A \lesssim_{\star,\dots,\star} B$ to indicate that there exists a positive constant $C_{\star,\dots,\star}$, continuously depending only on the quantities appearing in the subscript, such that $A \leq C_{\star,\dots,\star} B$.

P. Silva [2014] answered this question partially and proved that T_m defined by (1-3), (1-4) with $\dim\Gamma_1 = 0$ and $\dim\Gamma_2 = 1$ maps $L^p \times L^q \to L^r$ boundedly when $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ with $p, q > 1$, $\frac{1}{p}+\frac{2}{q} < 2$ and $\frac{1}{q}+\frac{2}{p} < 2$. One should observe that the admissible range for these tuples (p, q, r) is a proper subset of the region $p, q > 1$ and $\frac{3}{4} < r < \infty$, which is also properly contained in the admissible range of BHT (see Theorem 1.1).

Naturally, we may wonder whether the biparameter bilinear operator T_m given by (1-3), (1-4) (with appropriate decay assumptions on the symbol m and singularity sets Γ_1, Γ_2 satisfying $\dim\Gamma_1 = 0$ or 1, $\dim\Gamma_2 = 1$) satisfies the same L^p estimates as BHT.

To study this problem, we must find the implicit decay assumptions on symbol m to preclude the existence of those kinds of counterexamples constructed in (1-6) for BHT $\otimes$ BHT. To this end, let us consider first the bilinear operator $T_m \otimes \mathrm{BHT}$ of tensor product type that is defined by

$$T_m \otimes \mathrm{BHT}(f_1, f_2)(x, y) := \mathrm{p.v.} \int_{\mathbb{R}^2} f_1(x-s, y-t) f_2(x+s, y+t) \frac{K(s)}{t}\, ds\, dt, \tag{1-7}$$

where the symbol $m(\xi_1^1, \xi_2^1) = m(\zeta) := \hat{K}(\zeta)$ with $\zeta := \xi_1^1 - \xi_2^1$ has one-dimensional nondegenerate singularity set Γ_1. Let $f_1(x, y) = f_2(x, y) = e^{ixy}$; one can easily derive that

$$T_m \otimes \mathrm{BHT}(f_1, f_2)(x, y) = (f_1 \cdot f_2)(x, y) \int_{\mathbb{R}^2} K(s) \frac{e^{2ist}}{t}\, ds\, dt. \tag{1-8}$$

From (1-8) and the above counterexample constructed in (1-6) for the operator BHT $\otimes$ BHT, we observe that one sufficient condition for precluding the existence of these kinds of counterexamples is $K \in L^1$ or, equivalently, $m = \hat{K} \in \mathcal{F}(L^1)$. From the Riemann–Lebesgue theorem, we know that a necessary condition for $m \in \mathcal{F}(L^1)$ is $m(\zeta) \to 0$ as $|\zeta| \to \infty$. Moreover, if $K \in L^1(\mathbb{R})$ is odd, one can even derive that $\left|\int_{\mathbb{R}} m(\zeta)/\zeta \, d\zeta\right| \lesssim \|K\|_{L^1}$ (this indicates that many uniformly continuous functions with logarithmic decay rate do not belong to $\mathcal{F}(L^1)$). Therefore, in order to guarantee that the same L^p estimates as the bilinear Hilbert transform are available for the bilinear operators $T_m \otimes \mathrm{BHT}$ and BHT $\otimes$ BHT, we need some appropriate decay assumptions on the symbol.

The purpose of this paper is to prove the same L^p estimates as BHT for modified bilinear operators $T_m \otimes \mathrm{BHT}$ with arbitrary nonsmooth symbols which decay faster than the logarithmic rate.

For $d \geq 2$, any two generic vectors $\xi_1 = (\xi_1^i)_{i=1}^d$, $\xi_2 = (\xi_2^i)_{i=1}^d$ in $\mathbb{R}^d$ generate naturally the following collection of d vectors in $\mathbb{R}^2$:

$$\bar{\xi}_1 = (\xi_1^1, \xi_2^1), \quad \bar{\xi}_2 = (\xi_1^2, \xi_2^2), \quad \ldots, \quad \bar{\xi}_d = (\xi_1^d, \xi_2^d). \tag{1-9}$$

Let $m = m(\xi) = m(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{2d})$ that is smooth away from the subspaces $\Gamma_1 \cup \cdots \cup \Gamma_{d-1} \cup \Gamma_d$ and satisfies

$$\mathrm{dist}(\bar{\xi}_d, \Gamma_d)^{|\alpha_d|} \cdot \int_{\mathbb{R}^{2(d-1)}} \frac{\left|\partial_{\bar{\xi}_1}^{\alpha_1} \cdots \partial_{\bar{\xi}_d}^{\alpha_d} m(\bar{\xi})\right|}{\prod_{i=1}^{d-1} \mathrm{dist}(\bar{\xi}_i, \Gamma_i)^{2-|\alpha_i|}}\, d\bar{\xi}_1 \cdots d\bar{\xi}_{d-1} \leq B < +\infty \tag{1-10}$$

for sufficiently many multi-indices $\alpha_1, \dots, \alpha_d$, where $\dim \Gamma_i = 0$ for $i = 1, \dots, d-1$ and $\Gamma_d := \{(\xi_1^d, \xi_2^d) \in \mathbb{R}^2 : \xi_1^d = \xi_2^d\}$. Denote by $T_m^{(d)}$ the bilinear multiplier operator defined by

$$T_m^{(d)}(f_1, f_2)(x) := \int_{\mathbb{R}^{2d}} m(\xi) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)}\, d\xi. \tag{1-11}$$

Our result for bilinear operators $T_m^{(d)}$ satisfying (1-10) and (1-11) is the following:

Theorem 1.3. *For any $d \geq 2$, the bilinear, d-parameter multiplier operator $T_m^{(d)}$ maps $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ boundedly for any $1 < p_1, p_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. The implicit constants in the bounds depend only on p_1, p_2, p, d and B.*

Remark 1.4. For arbitrarily small $\varepsilon > 0$, let $m^\varepsilon = m^\varepsilon(\xi) = m^\varepsilon(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{2d})$ that is smooth away from the subspaces $\Gamma_1 \cup \dots \cup \Gamma_{d-1} \cup \Gamma_d$ defined as in Theorem 1.3 and satisfying differential estimates

$$|\partial_{\bar{\xi}_1}^{\alpha_1} \cdots \partial_{\bar{\xi}_d}^{\alpha_d} m^\varepsilon(\bar{\xi})| \lesssim \prod_{i=1}^{d-1} \left(\frac{1}{\operatorname{dist}(\bar{\xi}_i, \Gamma_i)^{|\alpha_i|}} \cdot \langle \log_2 \operatorname{dist}(\bar{\xi}_i, \Gamma_i) \rangle^{-(1+\varepsilon)} \right) \cdot \frac{1}{\operatorname{dist}(\bar{\xi}_d, \Gamma_d)^{|\alpha_d|}} \tag{1-12}$$

for sufficiently many multi-indices $\alpha_1, \dots, \alpha_d$; then m^ε satisfies conditions (1-10).

As shown in [Muscalu et al. 2004a], the bilinear and biparameter Hilbert transform does not satisfy any L^p estimates. This is the case when the singularity sets Γ_1 and Γ_2 satisfy $\dim \Gamma_1 = \dim \Gamma_2 = 1$. Thus, it is natural to ask if the L^p estimates will break down for any bilinear and biparameter Fourier multiplier operator with $\dim \Gamma_1 = \dim \Gamma_2 = 1$. In other words, will a nonsmooth symbol with the same dimensional singularity sets but with a slightly better decay than that for the bilinear and biparameter Hilbert transform assure the L^p estimates? Our next two theorems will address this issue.

For $d = 2$ and arbitrarily small $\varepsilon > 0$, let $\tilde{m}^\varepsilon = \tilde{m}^\varepsilon(\xi) = \tilde{m}^\varepsilon(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^4)$ that is smooth away from the subspaces $\Gamma_1 \cup \Gamma_2$ and satisfies

$$|\partial_{\bar{\xi}_1}^{\alpha_1} \partial_{\bar{\xi}_2}^{\alpha_2} \tilde{m}^\varepsilon(\bar{\xi})| \lesssim \prod_{i=1}^{2} \frac{1}{\operatorname{dist}(\bar{\xi}_i, \Gamma_i)^{|\alpha_i|}} \cdot \langle \log_2 \operatorname{dist}(\bar{\xi}_1, \Gamma_1) \rangle^{-(1+\varepsilon)} \tag{1-13}$$

for sufficiently many multi-indices α_1, α_2, where $\langle x \rangle := \sqrt{1 + x^2}$ and $\Gamma_i := \{(\xi_1^i, \xi_2^i) \in \mathbb{R}^2 : \xi_1^i = \xi_2^i\}$ for $i = 1, 2$. Denote by $T_{\tilde{m}^\varepsilon}^{(2)}$ the bilinear multiplier operator defined by

$$T_{\tilde{m}^\varepsilon}^{(2)}(f_1, f_2)(x) := \int_{\mathbb{R}^4} \tilde{m}^\varepsilon(\xi) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)}\, d\xi. \tag{1-14}$$

Our result for bilinear operators $T_{\tilde{m}^\varepsilon}^{(2)}$ satisfying (1-13) and (1-14) is the following:

Theorem 1.5. *For $d = 2$ and any $\varepsilon > 0$, the bilinear and biparameter multiplier operator $T_{\tilde{m}^\varepsilon}^{(2)}$ maps $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$ boundedly for any $1 < p_1, p_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. The implicit constants in the bounds depend only on p_1, p_2, p, ε and tend to infinity as $\varepsilon \to 0$.*

Our result for modified bilinear and biparameter Hilbert transform of tensor product type with a slightly better decay than that of $\mathrm{BHT} \otimes \mathrm{BHT}$ is the following:

Theorem 1.6. *For any $\varepsilon > 0$, let the bilinear and biparameter operator* $\mathrm{BHT}^\varepsilon \otimes \mathrm{BHT}$ *be defined by*

$$\mathrm{BHT}^\varepsilon \otimes \mathrm{BHT}(f_1, f_2)(x_1, x_2) = p.v. \int_{\mathbb{R}^2} f_1(x-s) f_2(x+s) \frac{\Psi^\varepsilon(s_1)}{s_2} \, ds_1 \, ds_2$$

with the function Ψ^ε *satisfying*

$$|\partial_{\xi_1}^{\alpha_1} \hat{\Psi}^\varepsilon(\xi_1^1 - \xi_2^1)| \lesssim |\xi_1^1 - \xi_2^1|^{-|\alpha_1|} \cdot \langle \log_2 |\xi_1^1 - \xi_2^1| \rangle^{-(1+\varepsilon)} \tag{1-15}$$

for sufficiently many multi-indices α_1; then it satisfies the same L^p estimates as $T^{(2)}_{\tilde{m}^\varepsilon}$.

Remark 1.7. For simplicity, we will only consider the biparameter case $d = 2$ and $\Gamma_i = \{(0, 0)\}$ $(i = 1, \ldots, d-1)$ in the proof of Theorem 1.3. It will be clear from the proof (see Section 4) that we can extend the argument to the general d-parameter and $\dim \Gamma_i = 0$ $(i = 1, \ldots, d-1)$ cases straightforwardly. We will only prove Theorem 1.5 in Section 5 and omit the proof of Theorem 1.6, since one can observe from the discretization procedure in Section 2 that the bilinear and biparameter operator $\mathrm{BHT}^\varepsilon \otimes \mathrm{BHT}$ can be reduced to the same bilinear model operators $\tilde{\Pi}^\varepsilon_{\vec{\mathbb{P}}}$ as $T^{(2)}_{\tilde{m}^\varepsilon}$.

It's well known that a standard approach to prove L^p estimates for one-parameter n-linear operators with singular symbols (e.g., Coifman–Meyer multiplier, BHT and one-parameter paraproducts) is by the generic estimates of the corresponding $(n+1)$-linear forms consisting of estimates for different sizes and energies (see [Jung $\geq$ 2015; Muscalu and Schlag 2013; Muscalu et al. 2002; 2004b]), which relies on the one-dimensional BMO theory, or, more precisely, the John–Nirenberg-type inequalities to get good control over the relevant sizes. Unfortunately, there is no routine generalization of such approach to multiparameter settings, for instance, we don't have analogues of the John–Nirenberg inequalities for dyadic rectangular BMO spaces in the two-parameter case (see [Muscalu and Schlag 2013]). To overcome these difficulties, Muscalu et al. [2004a] developed a completely new approach to prove L^p estimates for biparameter paraproducts; their essential idea is to apply the stopping-time decompositions based on hybrid square and maximal operators MM, MS, SM and SS, the one-dimensional BMO theory and Journé's lemma, and hence could not be extended to solve the general d-parameter $(d \geq 3)$ cases. As to the general d-parameter $(d \geq 3)$ cases, by proving a generic decomposition (see Lemma 4.1), Muscalu et al. [2006] simplified the arguments they introduced in [Muscalu et al. 2004a], and this simplification works equally well in all d-parameter settings. Recently, a pseudodifferential variant of the theorems in [Muscalu et al. 2004a; 2006] has been established in [Dai and Lu $\geq$ 2015a]. Moreover, J. Chen and G. Lu [Chen and Lu 2014] offer a different proof than those in [Muscalu et al. 2004a; 2006] to establish a Hörmander-type theorem of L^p estimates (and weighted estimates as well) for multilinear and multiparameter Fourier multiplier operators with limited smoothness in multiparameter Sobolev spaces.

However, in this paper, in order to prove our main results, Theorems 1.3 and 1.5 in biparameter settings, we have at least two different difficulties from [Muscalu et al. 2004a; 2006]. First, observe that if one restricts the sum of tritiles $P'' \in \mathbb{P}''$ in the definitions of discrete model operators (see Section 2) to a tree then one essentially gets a tensor product of two discrete paraproducts on x_1 and x_2, respectively, which can be estimated by the MM, MS, SM and SS functions, but, due to the extra degree of freedom in frequency in the x_2 direction, there are infinitely many such tensor products of paraproducts in the summation, so it's difficult for us to carry out the stopping-time decompositions by using the hybrid square and maximal operators as in [Muscalu et al. 2004a; 2006]. Second, in the proof of Theorem 1.5, note that there are infinitely many tritiles $P' \in \mathbb{P}'$ with the property that $I_{P'} = I_0$ for a certain fixed dyadic interval I_0 of the same length as $I_{P'}$, so we can't estimate $\sum_{P'} |I_{P'}| \lesssim |\tilde{I}|$ for all dyadic intervals $I_{P'} \subseteq \tilde{I}$ with comparable lengths, and hence we can't apply Journé's lemma as in [Muscalu et al. 2004a] either. By making use of the L^2 sizes and L^2 energies estimates of the trilinear forms, the almost orthogonality of wave packets associated with different tiles of distinct trees and the decay assumptions on the symbols, we are able to overcome these difficulties in the proof of Theorems 1.3 and 1.5 in biparameter settings.

Nevertheless, in the proof of Theorem 1.5 in general d-parameter settings ($d \geq 3$), one easily observes that the generic decomposition will destroy the perfect orthogonality of wave packets associated with distinct tiles which have disjoint frequency intervals in both the x_1 and x_2 directions, thus we can't apply the generic decomposition to extend the results of Theorem 1.5 to higher parameters $d \geq 3$ as in [Muscalu et al. 2006]. For the proof of Theorem 1.3, we are able to apply the generic decomposition lemma (Lemma 4.1) to the $d-1$ variables $x_1, \ldots, x_{d-1}$. Although one can't obtain that supp $\Phi^{3,\ell}_{\tilde{P}'_3} \otimes \Phi^3_{P''_3}$ is entirely contained in the exceptional set U as in [Muscalu et al. 2006], one can observe that the support set is contained in U in all the variables $x_1, \ldots, x_{d-1}$, but not the last, x_d. Therefore, we only need to consider the distance from the support set to the set E'_3 in the x_d direction and obtain enough decay factors for summation; the extension of the proof to the general d-parameter ($d \geq 3$) cases is straightforward.

The rest of this paper is organized as follows. In Section 2 we reduce the proof of Theorem 1.3 and Theorem 1.5 to proving restricted weak type estimates of discrete bilinear model operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ (Proposition 2.17). Section 3 is devoted to giving a review of the definitions and useful properties about trees, L^2 sizes and L^2 energies introduced in [Muscalu et al. 2004b]. In Sections 4 and 5 we carry out the proof of Proposition 2.17, which completes the proof of our main theorems, Theorem 1.3 and Theorem 1.5, respectively.

2. Reduction to restricted weak type estimates of discrete bilinear model operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$

2A. *Discretization.* As we can see from the study of multiparameter and multilinear Coifman–Meyer multiplier operators (see, e.g., [Muscalu et al. 2002; 2004a; 2004b; 2006]), a standard approach to obtain L^p estimates of bilinear operators $T^{(d)}_m$ and $T^{(2)}_{\tilde{m}^\varepsilon}$ is to reduce them into discrete sums of inner products with wave packets (see [Thiele 2006]).

2A1. *Discretization for bilinear, biparameter operators $T^{(2)}_m$ with $\Gamma_1 = \{(0,0)\}$.* We will use the following discretization procedure. First, we need to decompose the symbol $m(\xi)$ in a natural way. To this end, for the first spatial variable x_1, we decompose the region $\{\bar{\xi}_1 = (\xi^1_1, \xi^1_2) \in \mathbb{R}^2 \setminus \{(0,0)\}\}$ by using *Whitney squares* with respect to the singularity point $\{\xi^1_1 = \xi^1_2 = 0\}$, while, for the last spatial variable x_2, we decompose the region $\{\bar{\xi}_2 = (\xi^2_1, \xi^2_2) \in \mathbb{R}^2 : \xi^2_1 \neq \xi^2_2\}$ by using Whitney squares with respect to the singularity line $\Gamma_2 = \{\xi^2_1 = \xi^2_2\}$. In order to describe our discretization procedure clearly, let us first recall some standard notation and definitions in [Muscalu et al. 2004b].

An interval I on the real line $\mathbb{R}$ is called dyadic if it is of the form $I = 2^{-k}[n, n+1]$ for some $k, n \in \mathbb{Z}$. An interval is said to be a *shifted dyadic interval* if it is of the form $2^{-k}[j+\alpha, j+1+\alpha]$ for some $k, j \in \mathbb{Z}$ and $\alpha \in \{0, \frac{1}{3}, -\frac{1}{3}\}$. A *shifted dyadic cube* is a set of the form $Q = Q_1 \times Q_2 \times Q_3$, where each Q_j is a shifted dyadic interval and they all have the same length. A *shifted dyadic quasicube* is a set $Q = Q_1 \times Q_2 \times Q_3$, where Q_j $(j = 1, 2, 3)$ are shifted dyadic intervals satisfying the less restrictive condition $|Q_1| \simeq |Q_2| \simeq |Q_3|$. One easily observes that, for every cube $Q \subseteq \mathbb{R}^3$, there exists a shifted dyadic cube $\tilde{Q}$ such that $Q \subset \frac{7}{10}\tilde{Q}$ (the cube having the same center as $\tilde{Q}$ but with side length $\frac{7}{10}$ that of $\tilde{Q}$) and $\operatorname{diam}(Q) \simeq \operatorname{diam}(\tilde{Q})$.

The same terminology will also be used in the plane $\mathbb{R}^2$. The only difference is that the previous cubes become squares. For any cube or square Q, we will denote the side length of Q by $\ell(Q)$ and denote the reflection of Q with respect to the origin by $-Q$ hereafter.

Definition 2.1 [Muscalu and Schlag 2013; Muscalu et al. 2006]. For $J \subseteq \mathbb{R}$ an arbitrary interval, we say that a smooth function Φ_J is a bump adapted to J if and only if the following inequalities hold:

$$|\Phi_J^{(l)}(x)| \lesssim_{l,\alpha} \frac{1}{|J|^l} \cdot \frac{1}{(1+\operatorname{dist}(x, J)/|J|)^\alpha} \tag{2-1}$$

for every integer $\alpha \in \mathbb{N}$ and for sufficiently many derivatives $l \in \mathbb{N}$. If Φ_J is a bump adapted to J, we say that $|J|^{-\frac{1}{2}}\Phi_J$ is an L^2-normalized bump adapted to J.

Now let $\varphi \in \mathcal{S}(\mathbb{R})$ be an even Schwartz function such that $\operatorname{supp}\hat{\varphi} \subseteq \left[-\frac{3}{16}, \frac{3}{16}\right]$ and $\hat{\varphi}(\xi) = 1$ on $\left[-\frac{1}{6}, \frac{1}{6}\right]$, and define $\psi \in \mathcal{S}(\mathbb{R})$ to be the Schwartz function whose Fourier transform satisfies $\hat{\psi}(\xi) := \hat{\varphi}(\xi/4) - \hat{\varphi}(\xi/2)$ and $\operatorname{supp}\hat{\psi} \subseteq \left[-\frac{3}{4}, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, \frac{3}{4}\right]$, such that $0 \le \hat{\varphi}(\xi), \hat{\psi}(\xi) \le 1$. Then, for every integer $k \in \mathbb{Z}$, we define $\hat{\varphi}_k, \hat{\psi}_k \in \mathcal{S}(\mathbb{R})$ by

$$\hat{\varphi}_k(\xi) := \hat{\varphi}\left(\frac{\xi}{2^k}\right), \quad \hat{\psi}_k(\xi) := \hat{\psi}\left(\frac{\xi}{2^k}\right) = \hat{\varphi}_{k+2}(\xi) - \hat{\varphi}_{k+1}(\xi) \tag{2-2}$$

and observe that

$$\operatorname{supp}\hat{\varphi}_k \subseteq \left[-\tfrac{3}{16}\cdot 2^k, \tfrac{3}{16}\cdot 2^k\right], \quad \operatorname{supp}\hat{\psi}_k \subseteq \left[-\tfrac{3}{4}\cdot 2^k, -\tfrac{1}{3}\cdot 2^k\right] \cup \left[\tfrac{1}{3}\cdot 2^k, \tfrac{3}{4}\cdot 2^k\right],$$

and $\operatorname{supp}\hat{\psi}_k \cap \operatorname{supp}\hat{\psi}_{k'} = \varnothing$ for any integers $k, k' \in \mathbb{Z}$ such that $|k-k'| \ge 2$, and $\operatorname{supp}\hat{\varphi} \cap \operatorname{supp}\hat{\psi}_k = \varnothing$ for any integer $k \ge 0$. One easily obtains the homogeneous Littlewood–Paley dyadic decomposition

$$1 = \sum_{k\in\mathbb{Z}} \hat{\psi}_k(\xi) \quad \text{for all } \xi \in \mathbb{R}\setminus\{0\} \tag{2-3}$$

and inhomogeneous Littlewood–Paley dyadic decomposition

$$1 = \hat{\varphi}(\xi) + \sum_{k\ge -1} \hat{\psi}_k(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{2-4}$$

As a consequence, we get a decomposition for the product $1(\xi_1^1, \xi_2^1) = 1(\xi_1^1)\cdot 1(\xi_2^1)$ as follows:

$$1(\xi_1^1, \xi_2^1) = \sum_{k'\in\mathbb{Z}} \hat{\varphi}_{k'}(\xi_1^1)\hat{\psi}_{k'}(\xi_2^1) + \sum_{k'\in\mathbb{Z}} \hat{\psi}_{k'}(\xi_1^1)\hat{\tilde{\psi}}_{k'}(\xi_2^1) + \sum_{k'\in\mathbb{Z}} \hat{\psi}_{k'}(\xi_1^1)\hat{\varphi}_{k'}(\xi_2^1) \tag{2-5}$$

for every $(\xi_1^1, \xi_2^1) \neq (0,0)$, where

$$\hat{\hat{\psi}}_{k'} := \sum_{|k-k'|\leq 1,\, k\in\mathbb{Z}} \hat{\psi}_k \quad \text{for all } k' \in \mathbb{Z}.$$

By breaking the characteristic function of the plane (ξ_1^1, ξ_2^1) into finite sums of smoothed versions of characteristic functions of cones as in (2-5), we can decompose the operator $T_m^{(2)}$ into a finite sum of several parts in the x_1 direction. Since all the operators obtained in this decomposition can be treated in the same way, we will only discuss one of them in detail. More precisely, let

$$\tilde{\mathbb{Q}}' := \left\{\tilde{Q}' = \tilde{Q}_1' \times \tilde{Q}_2' \subseteq \mathbb{R}^2 : \tilde{Q}_1' := 2^{k'}\left[-\tfrac{1}{2}, \tfrac{1}{2}\right],\ \tilde{Q}_2' := 2^{k'}\left[\tfrac{1}{24}, \tfrac{25}{24}\right] \text{ for all } k' \in \mathbb{Z}\right\}. \tag{2-6}$$

For each square $\tilde{Q}' \in \tilde{\mathbb{Q}}'$, we define bump functions $\phi_{\tilde{Q}_i',i}$ $(i = 1, 2)$ adapted to intervals $\tilde{Q}_i'$ and satisfying $\operatorname{supp} \phi_{\tilde{Q}_i',i} \subseteq \frac{9}{10}\tilde{Q}_i'$ by

$$\phi_{\tilde{Q}_1',1}(\xi) := \hat{\varphi}\left(\frac{\xi}{\ell(\tilde{Q}')}\right) = \hat{\varphi}_{k'}(\xi) \tag{2-7}$$

and

$$\phi_{\tilde{Q}_2',2}(\xi) := \hat{\psi}\left(\frac{\xi}{\ell(\tilde{Q}')}\right) \cdot \chi_{\{\xi>0\}} = \hat{\psi}_{k'}(\xi) \cdot \chi_{\{\xi>0\}}, \tag{2-8}$$

respectively, and finally define smooth bump functions $\phi_{\tilde{Q}'}$ adapted to $\tilde{Q}'$ and satisfying $\operatorname{supp} \phi_{\tilde{Q}'} \subseteq \frac{9}{10}\tilde{Q}'$ by

$$\phi_{\tilde{Q}'}(\xi_1^1, \xi_2^1) := \phi_{\tilde{Q}_1',1}(\xi_1^1) \cdot \phi_{\tilde{Q}_2',2}(\xi_2^1). \tag{2-9}$$

Without loss of generality, we will only consider the smoothed characteristic function of the cone $\{(\xi_1^1, \xi_2^1) \in \mathbb{R}^2 : |\xi_1^1| \lesssim |\xi_2^1|,\ \xi_2^1 > 0\}$ in the decomposition (2-5) from now on, which is defined by

$$\sum_{\tilde{Q}'\in\tilde{\mathbb{Q}}'} \phi_{\tilde{Q}'}(\xi_1^1, \xi_2^1). \tag{2-10}$$

As to the x_2 direction, we consider the collection $\mathbb{Q}''$ of all shifted dyadic squares $Q'' = Q_1'' \times Q_2''$ satisfying

$$Q'' \subseteq \{(\xi_1^2, \xi_2^2) \in \mathbb{R}^2 : \xi_1^2 \neq \xi_2^2\}, \quad \operatorname{dist}(Q'', \Gamma_2) \simeq 10^4 \operatorname{diam}(Q''). \tag{2-11}$$

We can split the collection $\mathbb{Q}''$ into two disjoint subcollections, that is, define

$$\mathbb{Q}_{\mathrm{I}}'' := \{Q'' \in \mathbb{Q}'' : Q'' \subseteq \{\xi_1^2 < \xi_2^2\}\}, \quad \mathbb{Q}_{\mathrm{II}}'' := \{Q'' \in \mathbb{Q}'' : Q'' \subseteq \{\xi_1^2 > \xi_2^2\}\}. \tag{2-12}$$

Since the set of squares $\left\{\frac{7}{10}Q'' : Q'' \in \mathbb{Q}''\right\}$ also forms a finitely overlapping cover of the region $\{\xi_1^2 \neq \xi_2^2\}$, we can apply a standard partition of unity and write the symbol $\chi_{\{\xi_1^2\neq\xi_2^2\}}$ as

$$\chi_{\{\xi_1^2\neq\xi_2^2\}} = \sum_{Q''\in\mathbb{Q}''} \phi_{Q''}(\xi_1^2, \xi_2^2) = \left(\sum_{Q''\in\mathbb{Q}_{\mathrm{I}}''} + \sum_{Q''\in\mathbb{Q}_{\mathrm{II}}''}\right)\phi_{Q''}(\xi_1^2, \xi_2^2) = \chi_{\{\xi_1^2<\xi_2^2\}} + \chi_{\{\xi_1^2>\xi_2^2\}}, \tag{2-13}$$

where each $\phi_{Q''}$ is a smooth bump function adapted to Q'' and supported in $\frac{8}{10}Q''$.

One can easily observe that we only need to discuss in detail one term in the decomposition (2-13), since the other term can be treated in the same way. Without loss of generality, we will only consider the first term in (2-13), that is, the characteristic function $\chi_{\{\xi_1^2<\xi_2^2\}}$ of the upper half plane with respect to the singularity line Γ_2, which can be written as

$$\chi_{\{\xi_1^2<\xi_2^2\}}=\sum_{Q''\in\mathbb{Q}''_{\mathbb{I}}}\phi_{Q''}(\xi_1^2,\xi_2^2). \tag{2-14}$$

In a word, we only need to consider the bilinear operator $T^{(2)}_{m,(lh,\mathbb{I})}$ given by

$$T^{(2)}_{m,(lh,\mathbb{I})}(f_1,f_2)(x):=\sum_{\substack{\tilde{Q}'\in\tilde{\mathbb{Q}}'\\ Q''\in\mathbb{Q}''_{\mathbb{I}}}}\int_{\mathbb{R}^4}m(\xi)\phi_{\tilde{Q}'}(\bar{\xi}_1)\phi_{Q''}(\bar{\xi}_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi ix\cdot(\xi_1+\xi_2)}\,d\xi \tag{2-15}$$

from now on, and the proof of Theorem 1.3 can be reduced to proving the L^p estimates

$$\|T^{(2)}_{m,(lh,\mathbb{I})}(f_1,f_2)\|_{L^p(\mathbb{R}^2)}\lesssim_{p,p_1,p_2,B}\|f_1\|_{L^{p_1}(\mathbb{R}^2)}\cdot\|f_2\|_{L^{p_2}(\mathbb{R}^2)} \tag{2-16}$$

as long as $1<p_1,p_2\le\infty$ and $0<\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}<\frac{3}{2}$.

On one hand, since $\xi_1^1\in\operatorname{supp}\phi_{\tilde{Q}'_1,1}\subseteq\ell(\tilde{Q}')\big[-\frac{3}{16},\frac{3}{16}\big]$ and $\xi_2^1\in\operatorname{supp}\phi_{\tilde{Q}'_2,2}\subseteq\ell(\tilde{Q}')\big[\frac{1}{3},\frac{3}{4}\big]$, it follows that $-\xi_1^1-\xi_2^1\in\ell(\tilde{Q}')\big[-\frac{15}{16},-\frac{7}{48}\big]$, and, as a consequence, there exists an interval $\tilde{Q}'_3:=\ell(\tilde{Q}')\big[-\frac{25}{24},-\frac{1}{24}\big]$ and a bump function $\phi_{\tilde{Q}'_3,3}$ adapted to $\tilde{Q}'_3$ such that $\operatorname{supp}\phi_{\tilde{Q}'_3,3}\subseteq\ell(\tilde{Q}')\big[-\frac{23}{24},-\frac{1}{8}\big]\subseteq\frac{9}{10}\tilde{Q}'_3$ and $\phi_{\tilde{Q}'_3,3}\equiv 1$ on $\ell(\tilde{Q}')\big[-\frac{15}{16},-\frac{7}{48}\big]$.

On the other hand, observe that there exist bump functions $\phi_{Q''_i,i}$ $(i=1,2)$ adapted to the shifted dyadic interval Q''_i such that $\operatorname{supp}\phi_{Q''_i,i}\subseteq\frac{9}{10}Q''_i$ and $\phi_{Q''_i,i}\equiv 1$ on $\frac{8}{10}Q''_i$ $(i=1,2)$, respectively, and $\operatorname{supp}\phi_{Q''}\subseteq\frac{8}{10}Q''$, thus one has $\phi_{Q''_1,1}\cdot\phi_{Q''_2,2}\equiv 1$ on $\operatorname{supp}\phi_{Q''}$. Since $\xi_1^2\in\operatorname{supp}\phi_{Q''_1,1}\subseteq\frac{9}{10}Q''_1$ and $\xi_2^2\in\operatorname{supp}\phi_{Q''_2,2}\subseteq\frac{9}{10}Q''_2$, it follows that $-\xi_1^2-\xi_2^2\in-\frac{9}{10}Q''_1-\frac{9}{10}Q''_2$, and, as a consequence, one can find a shifted dyadic interval Q''_3 with the property that $-\frac{9}{10}Q''_1-\frac{9}{10}Q''_2\subseteq\frac{7}{10}Q''_3$ and also satisfying $|Q''_1|=|Q''_2|\simeq|Q''_3|$. In particular, there exists a bump function $\phi_{Q''_3,3}$ adapted to Q''_3 and supported in $\frac{9}{10}Q''_3$ such that $\phi_{Q''_3,3}\equiv 1$ on $-\frac{9}{10}Q''_1-\frac{9}{10}Q''_2$.

We denote by $\tilde{\boldsymbol{Q}}'$ the collection of all cubes $\tilde{Q}':=\tilde{Q}'_1\times\tilde{Q}'_2\times\tilde{Q}'_3$ with $\tilde{Q}'_1\times\tilde{Q}'_2\in\tilde{\mathbb{Q}}'$ and $\tilde{Q}'_3$ defined as above, and denote by $\boldsymbol{Q}''$ the collection of all shifted dyadic quasicubes $Q'':=Q''_1\times Q''_2\times Q''_3$ with $Q''_1\times Q''_2\in\mathbb{Q}''_{\mathbb{I}}$ and Q''_3 defined as above.

Definition 2.2 [Muscalu et al. 2004b]. We say that a collection of shifted dyadic quasicubes (cubes) is *sparse* if and only if, for every $j=1,2,3$:

(i) If Q and $\tilde{Q}$ belong to this collection and $|Q_j|<|\tilde{Q}_j|$, then $10^8|Q_j|\le|\tilde{Q}_j|$.

(ii) If Q and $\tilde{Q}$ belong to this collection and $|Q_j|=|\tilde{Q}_j|$, then $10^8Q_j\cap 10^8\tilde{Q}_j=\varnothing$.

In fact, it is not difficult to see that the collection $\boldsymbol{Q}''$ can be split into a sum of finitely many sparse collection of shifted dyadic quasicubes. Therefore, we can assume from now on that the collection $\boldsymbol{Q}''$ is sparse.

Assuming this we then observe that, for any Q'' in such a sparse collection $\boldsymbol{Q}''$, there exists a unique shifted dyadic cube $\tilde{Q}''$ in $\mathbb{R}^3$ such that $Q''\subseteq\frac{7}{10}\tilde{Q}''$ and with the property that $\operatorname{diam}(Q'')\simeq\operatorname{diam}(\tilde{Q}'')$.

This allows us in particular to assume further that $\boldsymbol{Q}''$ is a sparse collection of shifted dyadic cubes (that is, $|Q''_1| = |Q''_2| = |Q''_3| = \ell(Q'')$).

Now consider the trilinear form $\Lambda^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2, f_3)$ associated to $T^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2)$, which can be written as

$$\begin{aligned}\Lambda^{(2)}_{m,(lh,\mathbb{I})}&(f_1, f_2, f_3)\\ &:= \int_{\mathbb{R}^2} T^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2)(x) f_3(x)\,dx\\ &= \sum_{\substack{\tilde{Q}'\in\tilde{\boldsymbol{Q}}'\\ Q''\in\boldsymbol{Q}''}} \int_{\xi_1+\xi_2+\xi_3=0} m_{\tilde{Q}',Q''}(\xi_1,\xi_2,\xi_3) \prod_{i=1}^{3} (f_i * (\check{\phi}_{\tilde{Q}'_i,i} \otimes \check{\phi}_{Q''_i,i}))^{\wedge}(\xi_i)\,d\xi_1\,d\xi_2\,d\xi_3, \end{aligned} \tag{2-17}$$

where $\xi_i = (\xi_i^1, \xi_i^2)$ for $i = 1, 2, 3$, while

$$m_{\tilde{Q}',Q''}(\xi_1,\xi_2,\xi_3) := m(\xi_1,\xi_2)\cdot(\tilde{\phi}_{\tilde{Q}'} \otimes (\phi_{Q''_1\times Q''_2}\cdot\tilde{\phi}_{Q''_3,3}))(\xi_1,\xi_2,\xi_3), \tag{2-18}$$

where $\tilde{\phi}_{\tilde{Q}'}$ is an appropriate smooth function of $(\xi_1^1, \xi_2^1, \xi_3^1)$ which is supported on a slightly larger cube (with a constant magnification independent of $\ell(\tilde{Q}')$) than $\operatorname{supp}(\phi_{\tilde{Q}'_1,1}(\xi_1^1)\phi_{\tilde{Q}'_2,2}(\xi_2^1)\phi_{\tilde{Q}'_3,3}(\xi_3^1))$ and equals 1 on $\operatorname{supp}(\phi_{\tilde{Q}'_1,1}(\xi_1^1)\phi_{\tilde{Q}'_2,2}(\xi_2^1)\phi_{\tilde{Q}'_3,3}(\xi_3^1))$, the function $\phi_{Q''_1\times Q''_2}(\xi_1^2, \xi_2^2)$ is one term of the partition of unity defined in (2-14), and $\tilde{\phi}_{Q''_3,3}$ is an appropriate smooth function of ξ_3^2 supported on a slightly larger interval (with a constant magnification independent of $\ell(Q'')$) than $\operatorname{supp}\phi_{Q''_3,3}$ which equals 1 on $\operatorname{supp}\phi_{Q''_3,3}$. We can decompose $m_{\tilde{Q}',Q''}(\xi_1,\xi_2,\xi_3)$ as a Fourier series,

$$m_{\tilde{Q}',Q''}(\xi_1,\xi_2,\xi_3) = \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3\in\mathbb{Z}^2} C^{\tilde{Q}',Q''}_{\vec{n}_1,\vec{n}_2,\vec{n}_3} e^{2\pi i(n'_1,n'_2,n'_3)\cdot(\xi_1^1,\xi_2^1,\xi_3^1)/\ell(\tilde{Q}')} e^{2\pi i(n''_1,n''_2,n''_3)\cdot(\xi_1^2,\xi_2^2,\xi_3^2)/\ell(Q'')}, \tag{2-19}$$

where the Fourier coefficients $C^{\tilde{Q}',Q''}_{\vec{n}_1,\vec{n}_2,\vec{n}_3}$ are given by

$$\begin{aligned}C^{\tilde{Q}',Q''}_{\vec{n}_1,\vec{n}_2,\vec{n}_3} = \int_{\mathbb{R}^6} m_{\tilde{Q}',Q''}\big((\ell(\tilde{Q}')\xi_1^1, \ell(Q'')\xi_1^2), (\ell(\tilde{Q}')\xi_2^1, \ell(Q'')\xi_2^2), (\ell(\tilde{Q}')\xi_3^1, \ell(Q'')\xi_3^2)\big)&\\ \times e^{-2\pi i(\vec{n}_1\cdot\xi_1+\vec{n}_2\cdot\xi_2+\vec{n}_3\cdot\xi_3)}\,d\xi_1\,d\xi_2\,d\xi_3.&\end{aligned} \tag{2-20}$$

Then, by a straightforward calculation, we can rewrite (2-17) as

$$\begin{aligned}\Lambda^{(2)}_{m,(lh,\mathbb{I})}&(f_1, f_2, f_3)\\ &= \sum_{\substack{\tilde{Q}'\in\tilde{\boldsymbol{Q}}'\\ Q''\in\boldsymbol{Q}''}} \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3\in\mathbb{Z}^2} C^{\tilde{Q}',Q''}_{\vec{n}_1,\vec{n}_2,\vec{n}_3} \int_{\mathbb{R}^2} \prod_{j=1}^{3} (f_j * (\check{\phi}_{\tilde{Q}'_j,j} \otimes \check{\phi}_{Q''_j,j}))\left(x - \left(\frac{n'_j}{\ell(\tilde{Q}')}, \frac{n''_j}{\ell(Q'')}\right)\right)dx.\end{aligned} \tag{2-21}$$

Definition 2.3 [Muscalu et al. 2004b; Thiele 2006]. An arbitrary dyadic rectangle of area 1 in the phase-space plane is called a *Heisenberg box* or *tile*. Let $P := I_P \times \omega_P$ be a tile. An L^2-normalized wave packet on P is a function Φ_P which has Fourier support $\operatorname{supp}\hat{\Psi}_P \subseteq \frac{9}{10}\omega_P$ and obeys the estimates

$$|\Phi_P(x)| \lesssim |I_P|^{-\frac{1}{2}}\left(1 + \frac{\operatorname{dist}(x, I_P)}{|I_P|}\right)^{-M}$$

for all $M > 0$, where the implicit constant depends on M.

Now we define $\phi^{n'_i}_{\tilde{Q}'_i,i} := e^{2\pi i n'_i \xi^1_i/\ell(\tilde{Q}')} \cdot \phi_{\tilde{Q}'_i,i}$ and $\phi^{n''_i}_{Q''_i,i} := e^{2\pi i n''_i \xi^2_i/\ell(Q'')} \cdot \phi_{Q''_i,i}$ for $i = 1, 2, 3$. Since any $\tilde{Q}' \in \tilde{\boldsymbol{Q}}'$ and $Q'' \in \boldsymbol{Q}''$ are both shifted dyadic cubes, there exist integers $k', k'' \in \mathbb{Z}$ such that $\ell(\tilde{Q}') = |\tilde{Q}'_1| = |\tilde{Q}'_2| = |\tilde{Q}'_3| = 2^{k'}$ and $\ell(Q'') = |Q''_1| = |Q''_2| = |Q''_3| = 2^{k''}$, respectively. By splitting the integral region $\mathbb{R}^2$ into the union of unit squares, using the L^2-normalization procedure and simple calculations, we can rewrite (2-21) as

$$\Lambda^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2, f_3)$$
$$= \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3\in\mathbb{Z}^2} \sum_{\substack{\tilde{Q}'\in\tilde{\boldsymbol{Q}}'\\ Q''\in\boldsymbol{Q}''}} \int_0^1\int_0^1 \sum_{\substack{\tilde{I}'\text{ dyadic},\\ |\tilde{I}'|=2^{-k'}}} \sum_{\substack{I''\text{ dyadic},\\ |I''|=2^{-k''}}} \frac{C^{\tilde{Q}',Q''}_{\vec{n}_1,\vec{n}_2,\vec{n}_3}}{|\tilde{I}'|^{\frac12}\times|I''|^{\frac12}} \prod_{j=1}^3 \langle f_j, \check{\phi}^{n'_j,\nu'}_{\tilde{I}',\tilde{Q}'_j,j} \otimes \check{\phi}^{n''_j,\nu''}_{I'',Q''_j,j}\rangle\, d\nu'\, d\nu''$$
$$=: \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3\in\mathbb{Z}^2} \int_0^1\int_0^1 \sum_{\vec{P}:=\tilde{P}'\otimes P''\in\vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}}{|I_{\vec{P}}|^{\frac12}} \prod_{j=1}^3 \langle f_j, \Phi^{j,\vec{n}_j,\nu}_{\vec{P}_j}\rangle\, d\nu, \tag{2-22}$$

where $\langle\cdot,\cdot\rangle$ denotes the complex scalar L^2 inner product, and we have:

- Fourier coefficients $C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3} := C^{\tilde{Q}',Q''}_{\vec{n}_1,\vec{n}_2,\vec{n}_3}$;
- tritiles $\tilde{P}' := (\tilde{P}'_1, \tilde{P}'_2, \tilde{P}'_3)$ and $P'' := (P''_1, P''_2, P''_3)$;
- tiles $\tilde{P}'_i := I_{\tilde{P}'_i}\times\omega_{\tilde{P}'_i}$, where $I_{\tilde{P}'_i} := \tilde{I}' = 2^{-k'}[l', l'+1] =: I_{\tilde{P}'}$ and the frequency intervals are $\omega_{\tilde{P}'_i} := \tilde{Q}'_i$ for $i = 1, 2, 3$;
- tiles $P''_j := I_{P''_j}\times\omega_{P''_j}$, where $I_{P''_j} := I'' = 2^{-k''}[l'', l''+1] =: I_{P''}$ and the frequency intervals are $\omega_{P''_j} := Q''_j$ for $j = 1, 2, 3$;
- frequency cubes $Q_{\tilde{P}'} := \omega_{\tilde{P}'_1}\times\omega_{\tilde{P}'_2}\times\omega_{\tilde{P}'_3}$ and $Q_{P''} := \omega_{P''_1}\times\omega_{P''_2}\times\omega_{P''_3}$;
- $\tilde{\mathbb{P}}'$ denotes a collection of such tritiles $\tilde{P}'$ and $\mathbb{P}''$ denotes a collection of such tritiles P'';
- bitiles $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}_3$ defined by
$$\vec{P}_1 := (\tilde{P}'_1, P''_1) = \big(2^{-k'}[l', l'+1]\times 2^{k'}\big[-\tfrac12, \tfrac12\big], 2^{-k''}[l'', l''+1]\times Q''_1\big),$$
$$\vec{P}_2 := (\tilde{P}'_2, P''_2) = \big(2^{-k'}[l', l'+1]\times 2^{k'}\big[\tfrac{1}{24}, \tfrac{25}{24}\big], 2^{-k''}[l'', l''+1]\times Q''_2\big),$$
$$\vec{P}_3 := (\tilde{P}'_3, P''_3) = \big(2^{-k'}[l', l'+1]\times 2^{k'}\big[-\tfrac{25}{24}, -\tfrac{1}{24}\big], 2^{-k''}[l'', l''+1]\times Q''_3\big);$$
- the biparameter tritile $\vec{P} := \tilde{P}'\otimes P'' = (\vec{P}_1, \vec{P}_2, \vec{P}_3)$;
- rectangles $I_{\vec{P}_i} := I_{\tilde{P}'_i}\times I_{P''_i} = I_{\tilde{P}'}\times I_{P''} =: I_{\vec{P}}$ for $i = 1, 2, 3$, and hence $|I_{\vec{P}}| = |I_{\tilde{P}'}\times I_{P''}| = |I_{\vec{P}_1}| = |I_{\vec{P}_2}| = |I_{\vec{P}_3}| = 2^{-k'}\cdot 2^{-k''}$;
- the double frequency cube $Q_{\vec{P}} := (Q_{\tilde{P}'}, Q_{P''}) = (\omega_{\tilde{P}'_1}\times\omega_{\tilde{P}'_2}\times\omega_{\tilde{P}'_3}, \omega_{P''_1}\times\omega_{P''_2}\times\omega_{P''_3})$;
- $\vec{\mathbb{P}} := \tilde{\mathbb{P}}'\times\mathbb{P}''$ denotes a collection of such biparameter tritiles $\vec{P}$;
- L^2-normalized wave packets $\Phi^{i,n'_i,\nu'}_{\tilde{P}'_i}$ associated with the Heisenberg boxes $\tilde{P}'_i$ defined by
$$\Phi^{i,n'_i,\nu'}_{\tilde{P}'_i}(x_1) := \check{\phi}^{n'_i,\nu'}_{\tilde{I}',\tilde{Q}'_i,i}(x_1) := 2^{-k'/2}\overline{\check{\phi}^{n'_i}_{\tilde{Q}'_i,i}(2^{-k'}(l'+\nu')-x_1)} \quad \text{for } i = 1, 2, 3;$$

- L^2-normalized wave packets $\Phi^{i,n_i'',\nu''}_{P_i''}$ associated with the Heisenberg boxes P_i'' defined by

$$\Phi^{i,n_i'',\nu''}_{P_i''}(x_2) := \check{\phi}^{n_i'',\nu''}_{I'',Q_i'',i}(x_2) := 2^{-k''/2}\overline{\check{\phi}^{n_i''}_{Q_i'',i}(2^{-k''}(l''+\nu'')-x_2)} \quad \text{for } i=1,2,3;$$

- smooth bump functions $\Phi^{i,\vec{n}_i,\nu}_{\vec{P}_i} := \Phi^{i,n_i',\nu'}_{\tilde{P}_i'} \otimes \Phi^{i,n_i'',\nu''}_{P_i''}$ for $i=1,2,3$.

We have the following rapid decay estimates of the Fourier coefficients $C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}$ with respect to the parameters $\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2$:

Lemma 2.4. *The Fourier coefficients $C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}$ satisfy estimates*

$$|C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}| \lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{n}_j|)^M} \cdot C_{|I_{\tilde{P}'}|} \tag{2-23}$$

for any biparameter tritile $\vec{P} \in \vec{\mathbb{P}}$, where M is sufficiently large and the sequence $C_{k'} := C_{|I_{\tilde{P}'}|}$ for $|I_{\tilde{P}'}| = 2^{-k'}$ ($k' \in \mathbb{Z}$) satisfies

$$\sum_{k' \in \mathbb{Z}} C_{k'} \le B < +\infty. \tag{2-24}$$

Proof. Let $\ell(Q_{\tilde{P}'}) = 2^{k'}$ and $\ell(Q_{P''}) = 2^{k''}$ for $k', k'' \in \mathbb{Z}$. For any $\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2$ and $\vec{P} \in \vec{\mathbb{P}}$, we deduce from (2-18) and (2-20) that

$$\begin{aligned} &C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3} \\ &= \int_{\mathbb{R}^6} m_{Q_{\tilde{P}'},Q_{P''}}((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2)) e^{-2\pi i(\vec{n}_1\cdot\xi_1+\vec{n}_2\cdot\xi_2+\vec{n}_3\cdot\xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3, \end{aligned} \tag{2-25}$$

where

$$\begin{aligned} &m_{Q_{\tilde{P}'},Q_{P''}}((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2)) \\ &\qquad := m(2^{k'}\bar{\xi}_1, 2^{k''}\bar{\xi}_2)\tilde{\phi}_{Q_{\tilde{P}'}}(2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1)\phi_{\omega_{P_1''}\times\omega_{P_2''}}(2^{k''}\bar{\xi}_2)\tilde{\phi}_{\omega_{P_3''},3}(2^{k''}\xi_3^2). \end{aligned} \tag{2-26}$$

Since $\operatorname{supp}(\tilde{\phi}_{Q_{\tilde{P}'}}(\xi_1^1, \xi_2^1, \xi_3^1)\phi_{\omega_{P_1''}\times\omega_{P_2''}}(\bar{\xi}_2)\tilde{\phi}_{\omega_{P_3''},3}(\xi_3^2)) \subseteq Q_{\tilde{P}'} \times Q_{P''}$, we have that

$$\operatorname{supp}(\tilde{\phi}_{Q_{\tilde{P}'}}(2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1)\phi_{\omega_{P_1''}\times\omega_{P_2''}}(2^{k''}\bar{\xi}_2)\tilde{\phi}_{\omega_{P_3''},3}(2^{k''}\xi_3^2)) \subseteq Q^0_{\tilde{P}'} \times Q^0_{P''},$$

where the cubes $Q^0_{\tilde{P}'}$ and $Q^0_{P''}$ are defined by

$$Q^0_{\tilde{P}'} = \omega^0_{\tilde{P}_1'} \times \omega^0_{\tilde{P}_2'} \times \omega^0_{\tilde{P}_3'} := \{(\xi_1^1, \xi_2^1, \xi_3^1) \in \mathbb{R}^3 : (2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1) \in Q_{\tilde{P}'}\}, \tag{2-27}$$

$$Q^0_{P''} = \omega^0_{P_1''} \times \omega^0_{P_2''} \times \omega^0_{P_3''} := \{(\xi_1^2, \xi_2^2, \xi_3^2) \in \mathbb{R}^3 : (2^{k''}\xi_1^2, 2^{k''}\xi_2^2, 2^{k''}\xi_3^2) \in Q_{P''}\} \tag{2-28}$$

and satisfy $|Q^0_{\tilde{P}'}| \simeq |Q^0_{P''}| \simeq 1$. From the properties of the Whitney squares we constructed above, one obtains that $\operatorname{dist}(2^{k'}\bar{\xi}_1, \Gamma_1) \simeq 2^{k'}$ for any $\bar{\xi}_1 \in \omega^0_{\tilde{P}_1'} \times \omega^0_{\tilde{P}_2'}$ and $\operatorname{dist}(2^{k''}\bar{\xi}_2, \Gamma_2) \simeq 2^{k''}$ for any $\bar{\xi}_2 \in \omega^0_{P_1''} \times \omega^0_{P_2''}$.

One can deduce from (2-25), (2-26) and integrating by parts sufficiently many times that

$$
\begin{aligned}
&|C_{Q_{\tilde{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}| \\
&\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{n}_j|)^M} \\
&\qquad \times \int_{Q^0_{\tilde{P}'}\times Q^0_{P''}} \big|\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2}\partial^{\alpha_3}_{\xi_3}\big[m_{Q_{\tilde{P}'},Q_{P''}}((2^{k'}\xi^1_1,2^{k''}\xi^2_1),(2^{k'}\xi^1_2,2^{k''}\xi^2_2),(2^{k'}\xi^1_3,2^{k''}\xi^2_3))\big]\big|\,d\xi_1\,d\xi_2\,d\xi_3 \\
&\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{n}_j|)^M} \int_{\omega^0_{P''_1}\times\omega^0_{P''_2}} \operatorname{dist}(2^{k''}\bar{\xi}_2,\Gamma_2)^{|\alpha''|} \int_{\omega^0_{\tilde{P}'_1}\times\omega^0_{\tilde{P}'_2}} \operatorname{dist}(2^{k'}\bar{\xi}_1,\Gamma_1)^{|\alpha'|}|\partial^{\alpha'}_{\bar{\xi}_1}\partial^{\alpha''}_{\bar{\xi}_2} m(2^{k'}\bar{\xi}_1,2^{k''}\bar{\xi}_2)|\,d\bar{\xi}_1\,d\bar{\xi}_2 \\
&\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{n}_j|)^M}\cdot\frac{1}{\ell(Q_{P''})^2} \int_{\omega_{P''_1}\times\omega_{P''_2}} \operatorname{dist}(\bar{\xi}_2,\Gamma_2)^{|\alpha''|} \int_{\omega_{\tilde{P}'_1}\times\omega_{\tilde{P}'_2}} \operatorname{dist}(\bar{\xi}_1,\Gamma_1)^{|\alpha'|-2}|\partial^{\alpha'}_{\bar{\xi}_1}\partial^{\alpha''}_{\bar{\xi}_2} m(\bar{\xi}_1,\bar{\xi}_2)|\,d\bar{\xi}_1\,d\bar{\xi}_2 \\
&=: \prod_{j=1}^{3} \frac{1}{(1+|\vec{n}_j|)^M}\cdot C_{|I_{\tilde{P}'}|},
\end{aligned}
$$

where the multi-indices $\alpha_i := (\alpha^1_i, \alpha^2_i)$ for $i = 1, 2, 3$ and $|\alpha_1| = |\alpha_2| = |\alpha_3| = M$ are sufficiently large, the multi-indices $\alpha' := (\alpha'_1, \alpha'_2, \alpha'_3)$, $\alpha'' := (\alpha''_1, \alpha''_2, \alpha''_3)$ with $\alpha'_i \le \alpha^1_i$ and $\alpha''_j \le \alpha^2_j$ for $i, j = 1, 2, 3$. This proves the estimates (2-23).

Moreover, for $|I_{\tilde{P}'}| = 2^{-k'}$, we define the sequence $C_{k'} := C_{|I_{\tilde{P}'}|}$ ($k' \in \mathbb{Z}$). From the estimates (1-10) for symbol $m(\bar{\xi}_1, \bar{\xi}_2)$, we get that

$$
\operatorname{dist}(\bar{\xi}_2,\Gamma_2)^{|\alpha''|}\cdot\int_{\mathbb{R}^2} \operatorname{dist}(\bar{\xi}_1,\Gamma_1)^{|\alpha'|-2}|\partial^{\alpha'}_{\bar{\xi}_1}\partial^{\alpha''}_{\bar{\xi}_2} m(\bar{\xi})|\,d\bar{\xi}_1 \le B < +\infty, \tag{2-29}
$$

and hence we can deduce the following summable property for the sequence $\{C_{k'}\}_{k'\in\mathbb{Z}}$:

$$
\begin{aligned}
\sum_{k'\in\mathbb{Z}} C_{k'} &\le \frac{1}{\ell(Q_{P''})^2}\int_{\omega_{P''_1}\times\omega_{P''_2}} \operatorname{dist}(\bar{\xi}_2,\Gamma_2)^{|\alpha''|}\int_{\bigcup_{\tilde{P}'\in\tilde{\mathbb{P}}'}(\omega_{\tilde{P}'_1}\times\omega_{\tilde{P}'_2})_{\tilde{P}'}} \operatorname{dist}(\bar{\xi}_1,\Gamma_1)^{|\alpha'|-2}|\partial^{\alpha'}_{\bar{\xi}_1}\partial^{\alpha''}_{\bar{\xi}_2} m(\bar{\xi}_1,\bar{\xi}_2)|\,d\bar{\xi}_1\,d\bar{\xi}_2 \\
&\le \frac{1}{\ell(Q_{P''})^2}\int_{\omega_{P''_1}\times\omega_{P''_2}} B\,d\bar{\xi}_2 \le B < +\infty.
\end{aligned} \tag{2-30}
$$

This ends the proof of the summable estimate (2-24). □

Observe that the rapid decay with respect to the parameters $\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2$ in (2-23) is acceptable for summation, all the functions $\Phi^{i,n'_i,\nu'}_{\tilde{P}'_i}$ ($i = 1, 2, 3$) are L^2-normalized and are wave packets associated with the Heisenberg boxes $\tilde{P}'_i$ uniformly with respect to the parameters n'_i, and all the functions $\Phi^{j,n''_j,\nu''}_{P''_j}$ ($j = 1, 2, 3$) are L^2-normalized and are wave packets associated with the Heisenberg boxes P''_j uniformly with respect to the parameters n''_j; therefore we only need to consider from now on the part of the trilinear form $\Lambda^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2, f_3)$ defined in (2-22) corresponding to $\vec{n}_1 = \vec{n}_2 = \vec{n}_3 = \vec{0}$,

$$\dot{\Lambda}^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2, f_3) := \int_0^1 \int_0^1 \sum_{\vec{P}\in\vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi^{1,\nu}_{\vec{P}_1}\rangle \langle f_2, \Phi^{2,\nu}_{\vec{P}_2}\rangle \langle f_3, \Phi^{3,\nu}_{\vec{P}_3}\rangle \, d\nu, \tag{2-31}$$

where $C_{Q_{\vec{P}}} := C_{Q_{\vec{P}},\vec{0},\vec{0},\vec{0}}$, we have parameters $\nu = (\nu', \nu'')$ and $\Phi^{i,\nu}_{\vec{P}_i} := \Phi^{i,\vec{0},\nu}_{\vec{P}_i}$ for $i = 1, 2, 3$.

Remark 2.5. We should point out two important properties of the tritiles in $\mathbb{P}''$ (see [Muscalu and Schlag 2013; Muscalu et al. 2004b]). First, if one knows the position of P_1'', P_2'' or P_3'', then one knows precisely the positions of the other two as well. Second, if one assumes for instance that all the frequency intervals $\omega_{P_1''}$ of the P_1'' tiles intersect each other (say, they are nonlacunary about a fixed frequency ξ_0), then the frequency intervals $\omega_{P_2''}$ of the corresponding P_2'' tiles are disjoint and lacunary around ξ_0 (that is, $\operatorname{dist}(\xi_0, \omega_{P_2''}) \simeq |\omega_{P_2''}|$ for all $P'' \in \mathbb{P}''$). A similar conclusion can also be drawn for the P_3'' tiles modulo certain translations. This observation motivates the introduction of trees in Definition 3.1.

We review the following definitions from [Muscalu et al. 2004b].

Definition 2.6. A collection $\mathbb{P}$ of tritiles is called sparse if all tritiles in $\mathbb{P}$ have the same shift and the sets $\{Q_P : P \in \mathbb{P}\}$ and $\{I_P : P \in \mathbb{P}\}$ are sparse.

Definition 2.7. Let P and P' be tiles. Then we write:

(i) $P' < P$ if $I_{P'} \subsetneq I_P$ and $\omega_P \subseteq 3\omega_{P'}$;

(ii) $P' \le P$ if $P' < P$ or $P' = P$;

(iii) $P' \lesssim P$ if $I_{P'} \subseteq I_P$ and $\omega_P \subseteq 10^6 \omega_{P'}$;

(iv) $P' \lesssim' P$ if $P' \lesssim P$ but $P' \not\le P$.

Definition 2.8. A collection $\mathbb{P}$ of tritiles is said to have *rank* 1 if the following properties are satisfied for all $P, P' \in \mathbb{P}$:

(i) If $P \neq P'$, then $P_j \neq P_j'$ for $1 \le j \le 3$.

(ii) If $\omega_{P_j} = \omega_{P_j'}$ for some j, then $\omega_{P_j} = \omega_{P_j'}$ for all $1 \le j \le 3$.

(iii) If $P_j' \le P_j$ for some j, then $P_j' \lesssim P_j$ for all $1 \le j \le 3$.

(iv) If in addition to $P_j' \le P_j$ one also assumes that $10^8 |I_{P'}| \le |I_P|$, then one has $P_i' \lesssim' P_i$ for every $i \neq j$.

It is not difficult to see that the collection of tritiles $\mathbb{P}''$ can be written as a finite union of sparse collections of rank 1; thus we may assume further that $\mathbb{P}''$ is a sparse collection of rank 1 from now on.

The bilinear operator corresponding to the trilinear form $\dot{\Lambda}^{(2)}_{m,(lh,\mathbb{I})}(f_1, f_2, f_3)$ can be written as

$$\dot{\Pi}_{\vec{\mathbb{P}}}(f_1, f_2)(x) = \int_0^1 \int_0^1 \sum_{\vec{P}\in\vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi^{1,\nu}_{\vec{P}_1}\rangle \langle f_2, \Phi^{2,\nu}_{\vec{P}_2}\rangle \Phi^{3,\nu}_{\vec{P}_3}(x) \, d\nu. \tag{2-32}$$

Since $\dot{\Pi}_{\vec{\mathbb{P}}}(f_1, f_2)$ is an average of some discrete bilinear model operators depending on the parameters $\nu = (\nu_1, \nu_2) \in [0, 1]^2$, it is enough to prove the Hölder-type L^p estimates for each of them, uniformly with respect to parameters $\nu = (\nu_1, \nu_2)$. From now on, we will do this in the particular case when the parameters $\nu=(\nu_1,\nu_2)=(0,0)$, but the same argument works in general. By Fatou's lemma, we can also replace the summation in the definition (2-32) of $\dot{\Pi}_{\vec{\mathbb{P}}}(f_1,f_2)$ on the collection $\vec{\mathbb{P}}=\tilde{\mathbb{P}}'\times\mathbb{P}''$ by arbitrary finite collections $\tilde{\mathbb{P}}'$ and $\mathbb{P}''$ of tritiles, and prove the estimates are uniform with respect to different choices of the set $\vec{\mathbb{P}}$.

Therefore, one can reduce the bilinear operator $\dot{\Pi}_{\vec{\mathbb{P}}}$ further to the discrete bilinear model operator $\Pi_{\vec{\mathbb{P}}}$ defined by

$$\Pi_{\vec{\mathbb{P}}}(f_1,f_2)(x):=\sum_{\vec{P}\in\vec{\mathbb{P}}}\frac{C_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}}\langle f_1,\Phi^1_{\vec{P}_1}\rangle\langle f_2,\Phi^2_{\vec{P}_2}\rangle\Phi^3_{\vec{P}_3}(x), \tag{2-33}$$

where $\Phi^j_{\vec{P}_j}:=\Phi^{j,(0,0)}_{\vec{P}_j}$ for $j=1,2,3$, respectively, $\vec{\mathbb{P}}=\tilde{\mathbb{P}}'\times\mathbb{P}''$ with an arbitrary finite collection $\tilde{\mathbb{P}}'$ of tritiles and an arbitrary finite sparse collection $\mathbb{P}''$ of rank 1. As discussed above, we now reach a conclusion that the proof of Theorem 1.3 can be reduced to proving the following L^p estimates for discrete bilinear model operators $\Pi_{\vec{\mathbb{P}}}$:

Proposition 2.9. *If the finite set $\vec{\mathbb{P}}$ is chosen arbitrarily, as above, then the operator $\Pi_{\vec{\mathbb{P}}}$ given by (2-33) maps $L^{p_1}(\mathbb{R}^2)\times L^{p_2}(\mathbb{R}^2)\to L^p(\mathbb{R}^2)$ boundedly for any $1<p_1,p_2\le\infty$ satisfying $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{2}{3}<p<\infty$. Moreover, the implicit constants in the bounds depend only on p_1, p_2, p, B and are independent of the particular choice of the finite collection $\vec{\mathbb{P}}$.*

2A2. *Discretization for bilinear, biparameter operators $T^{(2)}_{\tilde{m}^\varepsilon}$.* We will use the discretization procedure as follows. First, we need to decompose the symbol $\tilde{m}^\varepsilon(\xi)$ in a natural way. To this end, for both the spatial variables x_i $(i=1,2)$, we decompose the regions $\{\bar{\xi}_i=(\xi^i_1,\xi^i_2)\in\mathbb{R}^2:\xi^i_1\neq\xi^i_2\}$ by using Whitney squares with respect to the singularity lines $\Gamma_i=\{\xi^i_1=\xi^i_2\}$ $(i=1,2)$ respectively. Since the Whitney dyadic square decomposition for the x_2 direction has already been described in (2-11), (2-12), (2-13) and (2-14) in Section 2A1, we only need to discuss the Whitney decomposition with respect to the singularity line Γ_1 in the x_1 direction.

To be specific, we consider the collection $\mathbb{Q}'$ of all shifted dyadic squares $Q'=Q'_1\times Q'_2$ satisfying

$$Q'\subseteq\{(\xi^1_1,\xi^1_2)\in\mathbb{R}^2:\xi^1_1\neq\xi^1_2\},\quad \operatorname{dist}(Q',\Gamma_1)\simeq 10^4\operatorname{diam}(Q'). \tag{2-34}$$

We can split the collection $\mathbb{Q}'$ into two disjoint subcollections, that is, define

$$\mathbb{Q}'_{\mathrm{I}}:=\{Q'\in\mathbb{Q}':Q'\subseteq\{\xi^1_1<\xi^1_2\}\},\quad \mathbb{Q}'_{\mathrm{II}}:=\{Q'\in\mathbb{Q}':Q'\subseteq\{\xi^1_1>\xi^1_2\}\}. \tag{2-35}$$

Since the set of squares $\{\frac{7}{10}Q':Q'\in\mathbb{Q}'\}$ also forms a finitely overlapping cover of the region $\{\xi^1_1\neq\xi^1_2\}$, we can apply a standard partition of unity and write the symbol $\chi_{\{\xi^1_1\neq\xi^1_2\}}$ as

$$\chi_{\{\xi^1_1\neq\xi^1_2\}}=\sum_{Q'\in\mathbb{Q}'}\phi_{Q'}(\xi^1_1,\xi^1_2)=\Bigg(\sum_{Q'\in\mathbb{Q}'_{\mathrm{I}}}+\sum_{Q'\in\mathbb{Q}'_{\mathrm{II}}}\Bigg)\phi_{Q'}(\xi^1_1,\xi^1_2)=\chi_{\{\xi^1_1<\xi^1_2\}}+\chi_{\{\xi^1_1>\xi^1_2\}}, \tag{2-36}$$

where each $\phi_{Q'}$ is a smooth bump function adapted to Q' and supported in $\frac{8}{10}Q'$.

Notice that, by splitting the symbol $\tilde{m}^\varepsilon(\xi)$, we can decompose the operator $T^{(2)}_{\tilde{m}^\varepsilon}$ correspondingly into a finite sum of several parts, and we only need to discuss one of them in detail. From the decompositions

(2-13) and (2-36), we obtain that

$$\begin{aligned}\tilde{m}^{\varepsilon}(\bar{\xi}_1,\bar{\xi}_2) &= \Bigg(\sum_{\substack{Q'\in\mathbb{Q}'_{\mathbb{I}}\\ Q''\in\mathbb{Q}''_{\mathbb{I}}}}+\sum_{\substack{Q'\in\mathbb{Q}'_{\mathbb{I}}\\ Q''\in\mathbb{Q}''_{\mathbb{II}}}}+\sum_{\substack{Q'\in\mathbb{Q}'_{\mathbb{II}}\\ Q''\in\mathbb{Q}''_{\mathbb{I}}}}+\sum_{\substack{Q'\in\mathbb{Q}'_{\mathbb{II}}\\ Q''\in\mathbb{Q}''_{\mathbb{II}}}}\Bigg)\phi_{Q'}(\xi_1^1,\xi_2^1)\phi_{Q''}(\xi_1^2,\xi_2^2)\cdot\tilde{m}^{\varepsilon}(\bar{\xi}_1,\bar{\xi}_2)\\ &=: \tilde{m}^{\varepsilon}_{\mathbb{I},\mathbb{I}}(\bar{\xi}_1,\bar{\xi}_2)+\tilde{m}^{\varepsilon}_{\mathbb{I},\mathbb{II}}(\bar{\xi}_1,\bar{\xi}_2)+\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{I}}(\bar{\xi}_1,\bar{\xi}_2)+\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{II}}(\bar{\xi}_1,\bar{\xi}_2).\end{aligned}\tag{2-37}$$

One can easily see that we only need to discuss in detail one term in the decomposition (2-37), since the other terms can be treated in the same way. Without loss of generality, we will only consider the third term in (2-37), which can be written as

$$\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{I}}(\bar{\xi}_1,\bar{\xi}_2) := \sum_{\substack{Q'\in\mathbb{Q}'_{\mathbb{II}}\\ Q''\in\mathbb{Q}''_{\mathbb{I}}}} \tilde{m}^{\varepsilon}(\bar{\xi}_1,\bar{\xi}_2)\phi_{Q'}(\xi_1^1,\xi_2^1)\phi_{Q''}(\xi_1^2,\xi_2^2).\tag{2-38}$$

In other words, we only need to consider the bilinear operator $T^{(2)}_{\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{I}}}$ given by

$$T^{(2)}_{\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{I}}}(f_1,f_2)(x) := \sum_{\substack{Q'\in\mathbb{Q}'_{\mathbb{II}}\\ Q''\in\mathbb{Q}''_{\mathbb{I}}}} \int_{\mathbb{R}^4} \tilde{m}^{\varepsilon}(\xi)\phi_{Q'}(\bar{\xi}_1)\phi_{Q''}(\bar{\xi}_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}\,d\xi \tag{2-39}$$

from now on, and the proof of Theorem 1.5 can be reduced to proving the following L^p estimates for $T^{(2)}_{\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{I}}}$:

$$\|T^{(2)}_{\tilde{m}^{\varepsilon}_{\mathbb{II},\mathbb{I}}}(f_1,f_2)\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,p,p_1,p_2} \|f_1\|_{L^{p_1}(\mathbb{R}^2)}\cdot\|f_2\|_{L^{p_2}(\mathbb{R}^2)} \tag{2-40}$$

as long as $1<p_1,\ p_2\le\infty$ and $0<\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}<\frac{3}{2}$.

Observe that there exist bump functions $\phi_{Q'_i,i}$ $(i=1,2)$ adapted to the shifted dyadic interval Q'_i such that $\operatorname{supp}\phi_{Q'_i,i}\subseteq\frac{9}{10}Q'_i$ and $\phi_{Q'_i,i}\equiv 1$ on $\frac{8}{10}Q'_i$ $(i=1,2)$ respectively, and $\operatorname{supp}\phi_{Q'}\subseteq\frac{8}{10}Q'$, so one has $\phi_{Q'_1,1}\cdot\phi_{Q'_2,2}\equiv 1$ on $\operatorname{supp}\phi_{Q'}$. Since $\xi_1^1\in\operatorname{supp}\phi_{Q'_1,1}\subseteq\frac{9}{10}Q'_1$ and $\xi_2^1\in\operatorname{supp}\phi_{Q'_2,2}\subseteq\frac{9}{10}Q'_2$, it follows that $-\xi_1^1-\xi_2^1\in-\frac{9}{10}Q'_1-\frac{9}{10}Q'_2$, and, as a consequence, one can find a shifted dyadic interval Q'_3 with the property that $-\frac{9}{10}Q'_1-\frac{9}{10}Q'_2\subseteq\frac{7}{10}Q'_3$ and also satisfying $|Q'_1|=|Q'_2|\simeq|Q'_3|$. In particular, there exists a bump function $\phi_{Q'_3,3}$ adapted to Q'_3 and supported in $\frac{9}{10}Q'_3$ such that $\phi_{Q'_3,3}\equiv 1$ on $-\frac{9}{10}Q'_1-\frac{9}{10}Q'_2$. Recall that the smooth functions $\phi_{Q''_j,j}$ $(j=1,2,3)$ and shifted dyadic intervals Q''_3 have already been defined in Section 2A1.

We denote by $\boldsymbol{Q}'$ the collection of all shifted dyadic quasicubes $Q':=Q'_1\times Q'_2\times Q'_3$ with $Q'_1\times Q'_2\in\mathbb{Q}'_{\mathbb{II}}$ and Q'_3 defined as above, and denote by $\boldsymbol{Q}''$ the collection of all shifted dyadic quasicubes $Q'':=Q''_1\times Q''_2\times Q''_3$ with $Q''_1\times Q''_2\in\mathbb{Q}''_{\mathbb{I}}$ and Q''_3 defined in Section 2A1.

In fact, it is not difficult to see that the collections $\boldsymbol{Q}'$ and $\boldsymbol{Q}''$ can be split into a sum of finitely many sparse collection of shifted dyadic quasicubes. Therefore, we can assume from now on that the collections $\boldsymbol{Q}'$ and $\boldsymbol{Q}''$ are sparse.

Assuming this, we then observe that, for any Q' in such a sparse collection $\boldsymbol{Q}'$, there exists a unique shifted dyadic cube $\tilde{Q}'$ in $\mathbb{R}^3$ such that $Q'\subseteq\frac{7}{10}\tilde{Q}'$ and with the property that $\operatorname{diam}(Q')\simeq\operatorname{diam}(\tilde{Q}')$. This allows us in particular to assume further that $\boldsymbol{Q}'$ is a sparse collection of shifted dyadic cubes (that is,

$|Q_1'| = |Q_2'| = |Q_3'| = \ell(Q')$). Similarly, we can also assume that $\boldsymbol{Q}''$ is a sparse collection of shifted dyadic cubes.

Now consider the trilinear form $\Lambda^{(2)}_{\tilde{m}^{\varepsilon}_{\square\square,\square}}(f_1, f_2, f_3)$ associated to $T^{(2)}_{\tilde{m}^{\varepsilon}_{\square\square,\square}}(f_1, f_2)$, which can be written as

$$\begin{aligned}
\Lambda^{(2)}_{\tilde{m}^{\varepsilon}_{\square\square,\square}}&(f_1, f_2, f_3) \\
&:= \int_{\mathbb{R}^2} T^{(2)}_{\tilde{m}^{\varepsilon}_{\square\square,\square}}(f_1, f_2)(x) f_3(x)\, dx \\
&= \sum_{\substack{Q' \in \boldsymbol{Q}' \\ Q'' \in \boldsymbol{Q}''}} \int_{\xi_1+\xi_2+\xi_3=0} \tilde{m}^{\varepsilon}_{Q',Q''}(\xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} (f_j * (\check{\phi}_{Q_j',j} \otimes \check{\phi}_{Q_j'',j}))^{\wedge}(\xi_j)\, d\xi_1\, d\xi_2\, d\xi_3, \qquad \text{(2-41)}
\end{aligned}$$

where $\xi_i = (\xi_i^1, \xi_i^2)$ for $i = 1, 2, 3$, while

$$\tilde{m}^{\varepsilon}_{Q',Q''}(\xi_1, \xi_2, \xi_3) := \tilde{m}^{\varepsilon}(\xi_1, \xi_2) \cdot ((\phi_{Q_1' \times Q_2'} \cdot \tilde{\phi}_{Q_3',3}) \otimes (\phi_{Q_1'' \times Q_2''} \cdot \tilde{\phi}_{Q_3'',3}))(\xi_1, \xi_2, \xi_3), \qquad \text{(2-42)}$$

where $\tilde{\phi}_{Q_3',3}$ is an appropriate smooth function of ξ_3^1 which equals 1 on $\operatorname{supp} \phi_{Q_3',3}$ and is supported on a slightly larger interval (with a constant magnification independent of $\ell(Q')$) than $\operatorname{supp} \phi_{Q_3',3}$, and $\tilde{\phi}_{Q_3'',3}$ is an appropriate smooth function of ξ_3^2 which equals 1 on $\operatorname{supp} \phi_{Q_3'',3}$ and is supported on a slightly larger interval (with a constant magnification independent of $\ell(Q'')$) than $\operatorname{supp} \phi_{Q_3'',3}$. We can decompose $\tilde{m}^{\varepsilon}_{Q',Q''}(\xi_1, \xi_2, \xi_3)$ as a Fourier series,

$$\tilde{m}^{\varepsilon}_{Q',Q''}(\xi_1, \xi_2, \xi_3) = \sum_{\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2} \tilde{C}^{\varepsilon, Q', Q''}_{\vec{l}_1, \vec{l}_2, \vec{l}_3} e^{2\pi i (l_1', l_2', l_3') \cdot (\xi_1^1, \xi_2^1, \xi_3^1)/\ell(Q')} e^{2\pi i (l_1'', l_2'', l_3'') \cdot (\xi_1^2, \xi_2^2, \xi_3^2)/\ell(Q'')}, \qquad \text{(2-43)}$$

where the Fourier coefficients $C^{\varepsilon, Q', Q''}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}$ are given by

$$\begin{aligned}
\tilde{C}^{\varepsilon, Q', Q''}_{\vec{l}_1, \vec{l}_2, \vec{l}_3} = \int_{\mathbb{R}^6} \tilde{m}^{\varepsilon}_{Q',Q''}\big((\ell(Q')\xi_1^1, \ell(Q'')\xi_1^2), (\ell(Q')\xi_2^1, \ell(Q'')\xi_2^2), (\ell(Q')\xi_3^1, \ell(Q'')\xi_3^2)\big) \\
\times e^{-2\pi i (\vec{l}_1 \cdot \xi_1 + \vec{l}_2 \cdot \xi_2 + \vec{l}_3 \cdot \xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3. \qquad \text{(2-44)}
\end{aligned}$$

Then, by a straightforward calculation, we can rewrite (2-41) as

$$\begin{aligned}
\Lambda^{(2)}_{\tilde{m}^{\varepsilon}_{\square\square,\square}}&(f_1, f_2, f_3) \\
&= \sum_{\substack{Q' \in \boldsymbol{Q}' \\ Q'' \in \boldsymbol{Q}''}} \sum_{\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2} \tilde{C}^{\varepsilon, Q', Q''}_{\vec{l}_1, \vec{l}_2, \vec{l}_3} \int_{\mathbb{R}^2} \prod_{i=1}^{3} (f_i * (\check{\phi}_{Q_i',i} \otimes \check{\phi}_{Q_i'',i}))\left(x - \left(\frac{l_i'}{\ell(Q')}, \frac{l_i''}{\ell(Q'')}\right)\right) dx. \qquad \text{(2-45)}
\end{aligned}$$

Now we define $\phi^{l_i'}_{Q_i',i} := e^{2\pi i l_i' \xi_i^1/\ell(Q')} \cdot \phi_{Q_i',i}$ and $\phi^{l_i''}_{Q_i'',i} := e^{2\pi i l_i'' \xi_i^2/\ell(Q'')} \cdot \phi_{Q_i'',i}$ for $i = 1, 2, 3$. Since any $Q' \in \boldsymbol{Q}'$ and $Q'' \in \boldsymbol{Q}''$ are shifted dyadic cubes, there exist integers $k', k'' \in \mathbb{Z}$ such that $\ell(Q') = |Q_1'| = |Q_2'| = |Q_3'| = 2^{k'}$ and $\ell(Q'') = |Q_1''| = |Q_2''| = |Q_3''| = 2^{k''}$, respectively. By splitting the integral region $\mathbb{R}^2$ into the union of unit squares, the L^2-normalization procedure and simple calculations, we can rewrite (2-45) as

$$\Lambda^{(2)}_{\tilde{m}^{\varepsilon}_{\mathbb{I}\mathbb{I},\mathbb{I}}}(f_1, f_2, f_3)$$

$$= \sum_{\vec{l}_1,\vec{l}_2,\vec{l}_3\in\mathbb{Z}^2} \sum_{\substack{Q'\in\boldsymbol{Q}'\\ Q''\in\boldsymbol{Q}''}} \int_0^1\int_0^1 \sum_{\substack{I' \text{ dyadic}\\ |I'|=2^{-k'}}} \sum_{\substack{I'' \text{ dyadic}\\ |I''|=2^{-k''}}} \frac{\tilde{C}^{\varepsilon,Q',Q''}_{\vec{l}_1,\vec{l}_2,\vec{l}_3}}{|I'|^{\frac{1}{2}}\times|I''|^{\frac{1}{2}}} \prod_{i=1}^{3}\langle f_i, \check{\phi}^{l'_i,\lambda'}_{I',Q'_i,i}\otimes\check{\phi}^{l''_i,\lambda''}_{I'',Q''_i,i}\rangle\, d\lambda'\, d\lambda''$$

$$=: \sum_{\vec{l}_1,\vec{l}_2,\vec{l}_3\in\mathbb{Z}^2} \int_0^1\int_0^1 \sum_{\vec{P}:=P'\otimes P''\in\vec{\mathbb{P}}} \frac{\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3}}{|I_{\vec{P}}|^{\frac{1}{2}}} \prod_{i=1}^{3}\langle f_i, \Phi^{i,\vec{l}_i,\lambda}_{\vec{P}_i}\rangle\, d\lambda, \tag{2-46}$$

where we have:

- Fourier coefficients $\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3} := \tilde{C}^{\varepsilon,Q',Q''}_{\vec{l}_1,\vec{l}_2,\vec{l}_3}$;
- tritiles $P' := (P'_1, P'_2, P'_3)$ and $P'' := (P''_1, P''_2, P''_3)$;
- tiles $P'_i := I_{P'_i}\times\omega_{P'_i}$, where $I_{P'_i} := I' = 2^{-k'}[n', n'+1] =: I_{P'}$ and the frequency intervals are $\omega_{P'_i} := Q'_i$ for $i = 1, 2, 3$;
- tiles $P''_j := I_{P''_j}\times\omega_{P''_j}$, where $I_{P''_j} := I'' = 2^{-k''}[n'', n''+1] =: I_{P''}$ and the frequency intervals are $\omega_{P''_j} := Q''_j$ for $j = 1, 2, 3$;
- frequency cubes $Q_{P'} := \omega_{P'_1}\times\omega_{P'_2}\times\omega_{P'_3}$ and $Q_{P''} := \omega_{P''_1}\times\omega_{P''_2}\times\omega_{P''_3}$;
- $\mathbb{P}'$ denotes a collection of such tritiles P' and $\mathbb{P}''$ denotes a collection of such tritiles P'';
- bitiles $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}_3$ defined by
$$\vec{P}_i := (P'_i, P''_i) = (2^{-k'}[n', n'+1]\times Q'_i, 2^{-k''}[n'', n''+1]\times Q''_i) \quad \text{for } i = 1, 2, 3;$$
- the biparameter tritile $\vec{P} := P'\otimes P'' = (\vec{P}_1, \vec{P}_2, \vec{P}_3)$;
- rectangles $I_{\vec{P}_i} := I_{P'_i}\times I_{P''_i} = I_{P'}\times I_{P''} =: I_{\vec{P}}$ for $i = 1, 2, 3$, and hence $|I_{\vec{P}}| = |I_{P'}\times I_{P''}| = |I_{\vec{P}_1}| = |I_{\vec{P}_2}| = |I_{\vec{P}_3}| = 2^{-k'}\cdot 2^{-k''}$;
- the double frequency cube $Q_{\vec{P}} := (Q_{P'}, Q_{P''}) = (\omega_{P'_1}\times\omega_{P'_2}\times\omega_{P'_3}, \omega_{P''_1}\times\omega_{P''_2}\times\omega_{P''_3})$;
- $\vec{\mathbb{P}} := \mathbb{P}'\times\mathbb{P}''$ denotes a collection of such biparameter tritiles $\vec{P}$;
- L^2-normalized wave packets $\Phi^{i,l'_i,\lambda'}_{P'_i}$ associated with the Heisenberg boxes P'_i defined by
$$\Phi^{i,l'_i,\lambda'}_{P'_i}(x_1) := \check{\phi}^{l'_i,\lambda'}_{I',Q'_i,i}(x_1) := 2^{-k'/2}\overline{\check{\phi}^{l'_i}_{Q'_i,i}(2^{-k'}(n'+\lambda')-x_1)} \quad \text{for } i = 1, 2, 3,$$
- L^2-normalized wave packets $\Phi^{i,l''_i,\lambda''}_{P''_i}$ associated with the Heisenberg boxes P''_i defined by
$$\Phi^{i,l''_i,\lambda''}_{P''_i}(x_2) := \check{\phi}^{l''_i,\lambda''}_{I'',Q''_i,i}(x_2) := 2^{-k''/2}\overline{\check{\phi}^{l''_i}_{Q''_i,i}(2^{-k''}(n''+\lambda'')-x_2)} \quad \text{for } i = 1, 2, 3,$$
- smooth bump functions $\Phi^{i,\vec{l}_i,\lambda}_{\vec{P}_i} := \Phi^{i,l'_i,\lambda'}_{P'_i}\otimes\Phi^{i,l''_i,\lambda''}_{P''_i}$ for $i = 1, 2, 3$.

We have the following rapid decay estimates of the Fourier coefficients $\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3}$ with respect to the parameters $\vec{l}_1, \vec{l}_2, \vec{l}_3\in\mathbb{Z}^2$:

Lemma 2.10. *The Fourier coefficients* $\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3}$ *satisfy estimates*

$$|\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3}| \lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{l}_j|)^M} \cdot \langle \log_2 \ell(Q_{P'})\rangle^{-(1+\varepsilon)} \tag{2-47}$$

for any biparameter tritile $\vec{P} \in \vec{\mathbb{P}}$, *where* M *is sufficiently large.*

Proof. Let $\ell(Q_{P'}) = 2^{k'}$ and $\ell(Q_{P''}) = 2^{k''}$ for $k', k'' \in \mathbb{Z}$. For any $\varepsilon > 0$, $\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2$ and $\vec{P} \in \vec{\mathbb{P}}$, we deduce from (2-42) and (2-44) that

$$\begin{aligned}\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3} \\ = \int_{\mathbb{R}^6} \tilde{m}^{\varepsilon}_{Q_{P'},Q_{P''}}((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2)) e^{-2\pi i(\vec{l}_1\cdot\xi_1+\vec{l}_2\cdot\xi_2+\vec{l}_3\cdot\xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3,\end{aligned} \tag{2-48}$$

where

$$\begin{aligned}&\tilde{m}^{\varepsilon}_{Q_{P'},Q_{P''}}((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2)) \\ &\qquad := \tilde{m}^{\varepsilon}(2^{k'}\bar{\xi}_1, 2^{k''}\bar{\xi}_2)\phi_{\omega_{P'_1}\times\omega_{P'_2}}(2^{k'}\bar{\xi}_1)\tilde{\phi}_{\omega_{P'_3},3}(2^{k'}\xi_3^1)\phi_{\omega_{P''_1}\times\omega_{P''_2}}(2^{k''}\bar{\xi}_2)\tilde{\phi}_{\omega_{P''_3},3}(2^{k''}\xi_3^2).\end{aligned} \tag{2-49}$$

Since $\mathrm{supp}(\phi_{\omega_{P'_1}\times\omega_{P'_2}}(\bar{\xi}_1)\tilde{\phi}_{\omega_{P'_3},3}(\xi_3^1)\phi_{\omega_{P''_1}\times\omega_{P''_2}}(\bar{\xi}_2)\tilde{\phi}_{\omega_{P''_3},3}(\xi_3^2)) \subseteq Q_{P'} \times Q_{P''}$, we have that

$$\mathrm{supp}(\phi_{\omega_{P'_1}\times\omega_{P'_2}}(2^{k'}\bar{\xi}_1)\tilde{\phi}_{\omega_{P'_3},3}(2^{k'}\xi_3^1)\phi_{\omega_{P''_1}\times\omega_{P''_2}}(2^{k''}\bar{\xi}_2)\tilde{\phi}_{\omega_{P''_3},3}(2^{k''}\xi_3^2)) \subseteq Q^0_{P'} \times Q^0_{P''},$$

where the cubes $Q^0_{P'}$ and $Q^0_{P''}$ are defined by

$$Q^0_{P'} = \omega^0_{P'_1} \times \omega^0_{P'_2} \times \omega^0_{P'_3} := \{(\xi_1^1, \xi_2^1, \xi_3^1) \in \mathbb{R}^3 : (2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1) \in Q_{P'}\}, \tag{2-50}$$

$$Q^0_{P''} = \omega^0_{P''_1} \times \omega^0_{P''_2} \times \omega^0_{P''_3} := \{(\xi_1^2, \xi_2^2, \xi_3^2) \in \mathbb{R}^3 : (2^{k''}\xi_1^2, 2^{k''}\xi_2^2, 2^{k''}\xi_3^2) \in Q_{P''}\} \tag{2-51}$$

and satisfy $|Q^0_{P'}| \simeq |Q^0_{P''}| \simeq 1$. From the properties of the Whitney squares we constructed above, one obtains that $\mathrm{dist}(2^{k'}\bar{\xi}_1, \Gamma_1) \simeq 2^{k'}$ for any $\bar{\xi}_1 \in \omega^0_{P'_1} \times \omega^0_{P'_2}$ and $\mathrm{dist}(2^{k''}\bar{\xi}_2, \Gamma_2) \simeq 2^{k''}$ for any $\bar{\xi}_2 \in \omega^0_{P''_1} \times \omega^0_{P''_2}$.

By taking advantage of the estimates (1-13) for symbol $\tilde{m}^{\varepsilon}(\bar{\xi})$, one can deduce from (2-48), (2-49) and integrating by parts sufficiently many times that

$$\begin{aligned}&|\tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{l}_1,\vec{l}_2,\vec{l}_3}| \\ &\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{l}_j|)^M} \\ &\qquad \times \int_{Q^0_{P'}\times Q^0_{P''}} \left|\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2}\partial^{\alpha_3}_{\xi_3}\left[\tilde{m}^{\varepsilon}_{Q_{P'},Q_{P''}}((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2))\right]\right| d\xi_1\, d\xi_2\, d\xi_3 \\ &\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{l}_j|)^M} \int_{\omega^0_{P''_1}\times\omega^0_{P''_2}} \mathrm{dist}(2^{k''}\bar{\xi}_2, \Gamma_2)^{|\alpha''|} \int_{\omega^0_{P'_1}\times\omega^0_{P'_2}} \mathrm{dist}(2^{k'}\bar{\xi}_1, \Gamma_1)^{|\alpha'|} |\partial^{\alpha'}_{\bar{\xi}_1}\partial^{\alpha''}_{\bar{\xi}_2}\tilde{m}^{\varepsilon}(2^{k'}\bar{\xi}_1, 2^{k''}\bar{\xi}_2)|\, d\bar{\xi}_1\, d\bar{\xi}_2\end{aligned}$$

$$\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{l}_j|)^M} \cdot 2^{-2k'} 2^{-2k''} \int_{\omega_{P''_1} \times \omega_{P''_2}} \int_{\omega_{P'_1} \times \omega_{P'_2}} \operatorname{dist}(\bar{\xi}_2, \Gamma_2)^{|\alpha''|} \cdot \operatorname{dist}(\bar{\xi}_1, \Gamma_1)^{|\alpha'|} |\partial_{\bar{\xi}_1}^{\alpha'} \partial_{\bar{\xi}_2}^{\alpha''} \tilde{m}^{\varepsilon}(\bar{\xi}_1, \bar{\xi}_2)| \, d\bar{\xi}_1 \, d\bar{\xi}_2$$
$$\lesssim \prod_{j=1}^{3} \frac{1}{(1+|\vec{l}_j|)^M} \cdot \langle \log_2 \ell(Q_{P'}) \rangle^{-(1+\varepsilon)},$$

where the multi-indices $\alpha_i := (\alpha_i^1, \alpha_i^2)$ for $i = 1, 2, 3$ and $|\alpha_1| = |\alpha_2| = |\alpha_3| = M$ are sufficiently large, the multi-indices $\alpha' := (\alpha'_1, \alpha'_2, \alpha'_3)$, $\alpha'' := (\alpha''_1, \alpha''_2, \alpha''_3)$ with $\alpha'_i \leq \alpha_i^1$ and $\alpha''_j \leq \alpha_j^2$ for $i, j = 1, 2, 3$. This ends our proof of the estimates (2-47). □

Note that the rapid decay with respect to the parameters $\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2$ in (2-47) is acceptable for summation, all the functions $\Phi_{P'_i}^{i,l'_i,\lambda'}$ $(i = 1, 2, 3)$ are L^2-normalized and are wave packets associated with the Heisenberg boxes P'_i uniformly with respect to the parameters l'_i, and all the functions $\Phi_{P''_j}^{j,l''_j,\lambda''}$ $(j = 1, 2, 3)$ are L^2-normalized and are wave packets associated with the Heisenberg boxes P''_j uniformly with respect to the parameters l''_j, therefore we only need to consider from now on the part of the trilinear form $\Lambda_{\tilde{m}^{\varepsilon}_{\mathbb{0}\mathbb{0},\mathbb{0}}}^{(2)}(f_1, f_2, f_3)$ defined in (2-46) corresponding to $\vec{l}_1 = \vec{l}_2 = \vec{l}_3 = \vec{0}$,

$$\dot{\Lambda}_{\tilde{m}^{\varepsilon}_{\mathbb{0}\mathbb{0},\mathbb{0}}}^{(2)}(f_1, f_2, f_3) := \int_0^1 \int_0^1 \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1,\lambda} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2,\lambda} \rangle \langle f_3, \Phi_{\vec{P}_3}^{3,\lambda} \rangle \, d\lambda, \tag{2-52}$$

where $\tilde{C}^{\varepsilon}_{Q_{\vec{P}}} := \tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{0},\vec{0},\vec{0}}$, we have parameters $\lambda = (\lambda', \lambda'')$, and $\Phi_{\vec{P}_i}^{i,\lambda} := \Phi_{\vec{P}_i}^{i,\vec{0},\lambda}$ for $i = 1, 2, 3$.

The tritiles $P' = (P'_1, P'_2, P'_3)$ in the collection $\mathbb{P}'$ also satisfy the same properties (as $P'' \in \mathbb{P}''$) described in Remark 2.5. It is not difficult to see that both the collections of tritiles $\mathbb{P}'$ and $\mathbb{P}''$ can be written as a finite union of sparse collections of rank 1; thus we may assume further that $\mathbb{P}'$ and $\mathbb{P}''$ are sparse collections of rank 1 from now on.

The bilinear operator corresponding to the trilinear form $\dot{\Lambda}_{\tilde{m}^{\varepsilon}_{\mathbb{0}\mathbb{0},\mathbb{0}}}^{(2)}(f_1, f_2, f_3)$ can be written as

$$\dot{\tilde{\Pi}}_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2)(x) = \int_0^1 \int_0^1 \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1,\lambda} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2,\lambda} \rangle \Phi_{\vec{P}_3}^{3,\lambda}(x) \, d\lambda. \tag{2-53}$$

Since $\dot{\tilde{\Pi}}_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2)$ is an average of some discrete bilinear model operators depending on the parameters $\lambda = (\lambda_1, \lambda_2) \in [0, 1]^2$, it is enough to prove the Hölder-type L^p estimates for each of them, uniformly with respect to parameters $\lambda = (\lambda_1, \lambda_2)$. From now on, we will do this in the particular case when the parameters $\lambda = (\lambda_1, \lambda_2) = (0, 0)$, but the same argument works in general. By Fatou's lemma, we can also replace the summation in the definition (2-53) of $\dot{\tilde{\Pi}}_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2)$ on the collection $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$ by arbitrary finite collections $\mathbb{P}'$ and $\mathbb{P}''$ of tritiles, and prove the estimates are uniform with respect to different choices of the set $\vec{\mathbb{P}}$.

Definition 2.11. A finite collection $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$ of biparameter tritiles is said to be sparse and of rank 1 if both the finite collections $\mathbb{P}'$ and $\mathbb{P}''$ are sparse and of rank 1.

Therefore, one can reduce the bilinear operator $\dot{\tilde{\Pi}}^{\varepsilon}_{\vec{\mathbb{P}}}$ further to the discrete bilinear model operator $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ defined by

$$\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2)(x) := \sum_{\vec{P}\in\vec{\mathbb{P}}} \frac{\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi^1_{\vec{P}_1}\rangle \langle f_2, \Phi^2_{\vec{P}_2}\rangle \Phi^3_{\vec{P}_3}(x), \tag{2-54}$$

where $\Phi^j_{\vec{P}_j} := \Phi^{j,(0,0)}_{\vec{P}_j}$ for $j = 1, 2, 3$, and the finite set $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$ is an arbitrary sparse collection (of biparameter tritiles) of rank 1. As discussed above, we now reach a conclusion that the proof of Theorem 1.5 can be reduced to proving the following L^p estimates for discrete bilinear model operators $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$:

Proposition 2.12. *If the finite set $\vec{\mathbb{P}}$ is an arbitrary sparse collection of rank 1, then the operator $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ given by (2-54) maps $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$ boundedly for any $1 < p_1, p_2 \le \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. Moreover, the implicit constants in the bounds depend only on ε, p_1, p_2, p and are independent of the particular finite sparse collection $\vec{\mathbb{P}}$ of rank 1.*

2B. *Multilinear interpolations.* First, let's review the following terminologies and definitions of multilinear interpolation arguments:

Definition 2.13 [Muscalu and Schlag 2013; Muscalu et al. 2002]. An n-tuple $\beta = (\beta_1, \dots, \beta_n)$ is said to be *admissible* if and only if $\beta_j < 1$ for every $1 \le j \le n$, $\sum_{j=1}^n \beta_j = 1$ and there is at most one index j for which $\beta_j < 0$. An index j is called *good* if $\beta_j \ge 0$ and *bad* if $\beta_j < 0$. A *good tuple* is an admissible tuple that contains only good indices; a *bad tuple* is an admissible tuple that contains precisely one bad index.

Definition 2.14 [Muscalu et al. 2002]. Let E, E' be sets of finite measure. We say that E' is a *major subset* of E if $E' \subseteq E$ and $|E'| \ge \frac{1}{2}|E|$.

Definition 2.15 [Muscalu and Schlag 2013; Muscalu et al. 2002]. If $\beta = (\beta_1, \dots, \beta_n)$ is an admissible tuple, we say that an n-linear form Λ is of *restricted weak type* β if and only if, for every sequence $E_1, \dots, E_n$ of measurable sets with positive and finite measure, there exists a major subset E'_j of E_j for the bad index j (if there is one) such that

$$|\Lambda(f_1, \dots, f_n)| \lesssim |E_1|^{\beta_1} \cdots |E_n|^{\beta_n} \tag{2-55}$$

for all measurable functions $|f_i| \le \chi_{E'_i}$ $(i = 1, \dots, n)$, where we adopt the convention $E'_i = E_i$ for good indices i. If β is bad with bad index j_0, and it happens that one can choose the major subset $E'_{j_0} \subseteq E_{j_0}$ in a way that depends only on the measurable sets $E_1, \dots, E_n$ and not on β, we say that Λ is of *uniformly restricted weak type*.

Definition 2.16 [Muscalu and Schlag 2013]. Let $1 < p_1, p_2 \le \infty$ and $0 < p < \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. An arbitrary bilinear operator T is said to be of the restricted weak type (p_1, p_2, p) if and only if, for all measurable sets E_1, E_2, E of finite measure, there exists $E' \subseteq E$ with $|E'| \simeq |E|$ such that

$$\left| \int_{\mathbb{R}^d} T(f_1, f_2)(x) f(x)\, dx \right| \lesssim |E_1|^{1/p_1} |E_2|^{1/p_2} |E'|^{1/p'} \tag{2-56}$$

for every $|f_1| \le \chi_{E_1}$, $|f_2| \le \chi_{E_2}$ and $|f| \le \chi_{E'}$.

By using multilinear interpolation (see [Grafakos and Tao 2003; Janson 1988; Muscalu and Schlag 2013; Muscalu et al. 2002]) and the symmetry of the operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$, we can reduce further the proof of Proposition 2.9 and Proposition 2.12 to proving the following restricted weak type estimates for the model operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$:

Proposition 2.17. *Let p_1 and p_2 be such that p_1 is strictly larger than 1 and arbitrarily close to 1 and p_2 is strictly smaller than 2 and arbitrarily close to 2 and such that, for $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$, one has $\frac{2}{3} < p < 1$. Then both the model operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ defined in (2-33) and (2-54) are of the restricted weak type (p_1, p_2, p). Moreover, the implicit constants in the bounds depend only on p_1, p_2, p, ε and B, and are independent of the particular choice of the finite collection $\vec{\mathbb{P}}$.*

Indeed, first we should note that, if p_1, p_2, p are as in Propositions 2.9 and 2.12, then the 3-tuple $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p'}\right)$ lies in the interior of the convex hull of the following six extremal points: $\beta^1 := \left(-\frac{1}{2}, \frac{1}{2}, 1\right)$, $\beta^2 := \left(-\frac{1}{2}, 1, \frac{1}{2}\right)$, $\beta^3 := \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$, $\beta^4 := \left(1, -\frac{1}{2}, \frac{1}{2}\right)$, $\beta^5 := \left(\frac{1}{2}, 1, -\frac{1}{2}\right)$ and $\beta^6 := \left(1, \frac{1}{2}, -\frac{1}{2}\right)$. Then, if we assume that Proposition 2.17 has been proved, from the symmetry of operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ and their adjoints we deduce that both the trilinear forms associated to bilinear operators $\Pi_{\vec{\mathbb{P}}}$ and $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ are of uniformly restricted weak type β for 3-tuples $\beta = (\beta_1, \beta_2, \beta_3)$ arbitrarily close to the six extremal points $\beta^1, \dots, \beta^6$ inside their convex hull and satisfying that, if β_j is close to $\frac{1}{2}$ for some $j = 1, 2, 3$, then β_j is strictly larger than $\frac{1}{2}$. By using the multilinear interpolation lemma, [Muscalu and Schlag 2013, Lemmas 9.4 and 9.6] or [Muscalu et al. 2002, Lemma 3.8], we first obtain restricted weak type estimates of Λ for good tuples inside the smaller convex hull of the three coordinate points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. After that, we use the interpolation lemma [Muscalu and Schlag 2013, Lemma 9.5] or [Muscalu et al. 2002, Lemma 3.10] to obtain restricted weak type estimates of Λ for bad tuples and finally conclude that restricted weak type estimates of Λ hold for all tuples β inside the convex hull of the six extremal points $\beta^1, \dots, \beta^6$.

It only remains to convert these restricted weak type estimates into strong type estimates. To do this, one just has to apply (exactly as in [Muscalu et al. 2002]) the multilinear Marcinkiewicz interpolation theorem in [Janson 1988] in the case of good tuples and the interpolation lemma [Muscalu et al. 2002, Lemma 3.11] in the case of bad tuples. This ends the proof of Propositions 2.9 and 2.12, and, as a consequence, completes the proof of our main results, Theorems 1.3 and 1.5. Therefore, we only have the task of proving Proposition 2.17 from now on.

3. Trees, L^2 sizes and L^2 energies

3A. ***Trees.*** We should recall that, for discrete bilinear paraproducts, the frequency intervals have already been organized with the lacunary properties (see [Muscalu and Schlag 2013; Muscalu et al. 2004a; 2006]), so we could use square function and maximal function estimates to handle the corresponding terms easily, at least in the Banach case. By the properties of the collection $\mathbb{P}''$ of tritiles we have explained in Remark 2.5, we can organize our collections of tritiles $\mathbb{P}'$, $\mathbb{P}''$ into trees as in [Grafakos and Li 2004], which satisfy lacunary properties about a certain frequency. We review the following standard definitions and properties for trees from [Muscalu et al. 2004b]:

Definition 3.1. Let $\mathbb{P}$ be a sparse rank-1 collection of tritiles and $j \in \{1, 2, 3\}$. A subcollection $T \subseteq \mathbb{P}$ is called a *j-tree* if and only if there exists a tritile P_T (called the top of the tree) such that

$$P_j \leq P_{T,j} \tag{3-1}$$

for every $P \in T$.

Remark 3.2. A tree does not necessarily have to contain the corresponding top P_T. From now on, we will write I_T and $\omega_{T,j}$ for I_{P_T} and $\omega_{P_T,j}$ for $j = 1, 2, 3$. Then, we simply say that T is a tree if it is a j-tree for some $j = 1, 2, 3$.

For every given dyadic interval I_0, there are potentially many tritiles P in $\mathbb{P}'$ and $\mathbb{P}''$ with the property that $I_P = I_0$. Due to this extra degree of freedom in frequency, we have infinitely many trees in our collections $\mathbb{P}'$ and $\mathbb{P}''$. We need to estimate each of these trees separately, and then add all these estimates together, by using the almost orthogonality conditions for distinct trees. This motivates the following definition:

Definition 3.3. Let $1 \leq i \leq 3$. A finite sequence of trees $T_1, \ldots, T_M$ is said to be a *chain of strongly i-disjoint trees* if and only if:

(i) $P_i \neq P'_i$ for every $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 \neq l_2$.

(ii) Whenever $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 \neq l_2$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \varnothing$, then if $|\omega_{P_i}| < |\omega_{P'_i}|$ one has $I_{P'} \cap I_{T_{l_1}} = \varnothing$ and if $|\omega_{P'_i}| < |\omega_{P_i}|$ one has $I_P \cap I_{T_{l_2}} = \varnothing$.

(iii) Whenever $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 < l_2$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \varnothing$, then if $|\omega_{P_i}| = |\omega_{P'_i}|$ one has $I_{P'} \cap I_{T_{l_1}} = \varnothing$.

3B. *L^2 sizes and L^2 energies.*

Following [Muscalu et al. 2004b], we give the definitions of standard norms on sequences of tiles:

Definition 3.4. Let $\mathbb{P}$ be a finite collection of tritiles, $j \in \{1, 2, 3\}$, and let f be an arbitrary function. We define the *size* of the sequence $(\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}$ by

$$\operatorname{size}_j\big((\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}\big) := \sup_{T\subseteq\mathbb{P}} \bigg(\frac{1}{|I_T|}\sum_{P\in T} |\langle f, \Phi^j_{P_j}\rangle|^2\bigg)^{\frac{1}{2}}, \tag{3-2}$$

where T ranges over all trees in $\mathbb{P}$ that are i-trees for some $i \neq j$. For $j = 1, 2, 3$, we define the *energy* of the sequence $(\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}$ by

$$\operatorname{energy}_j\big((\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}\big) := \sup_{n\in\mathbb{Z}} \sup_{\mathbb{T}} 2^n \bigg(\sum_{T\in\mathbb{T}} |I_T|\bigg)^{\frac{1}{2}}, \tag{3-3}$$

where now $\mathbb{T}$ ranges over all chains of strongly j-disjoint trees in $\mathbb{P}$ (which are i-trees for some $i \neq j$) having the property that

$$\bigg(\sum_{P\in T} |\langle f, \Phi^j_{P_j}\rangle|^2\bigg)^{\frac{1}{2}} \geq 2^n |I_T|^{\frac{1}{2}} \tag{3-4}$$

for all $T \in \mathbb{T}$ and such that

$$\Big(\sum_{P\in T'} |\langle f, \Phi^j_{P_j}\rangle|^2\Big)^{\frac{1}{2}} \le 2^{n+1}|I_{T'}|^{\frac{1}{2}} \tag{3-5}$$

for all subtrees $T' \subseteq T \in \mathbb{T}$.

The size measures the extent to which the sequences $(\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}$ $(j = 1, 2, 3)$ can concentrate on a single tree and should be thought of as a phase-space variant of the BMO norm. The energy is a phase-space variant of the L^2 norm. As the notation suggests, the number $\langle f, \Phi^j_{P_j}\rangle$ should be thought of as being associated with the tile P_j $(j = 1, 2, 3)$ rather than the full tritile P.

Let $\mathbb{P}$ be a finite collection of tritiles. Denote by $\Pi_{\mathbb{P}}$ the discrete bilinear operator given by

$$\Pi_{\mathbb{P}}(f_1, f_2)(x) = \sum_{P\in\mathbb{P}} \frac{1}{|I_P|^{\frac{1}{2}}} \langle f_1, \Phi^1_{P_1}\rangle\langle f_2, \Phi^2_{P_2}\rangle \Phi^3_{P_3}(x).$$

The following proposition provides a way of estimating the trilinear form associated with the bilinear operator $\Pi_{\mathbb{P}}(f_1, f_2)$. We define

$$\Lambda_{\mathbb{P}}(f_1, f_2, f_3) := \int_{\mathbb{R}} \Pi_{\mathbb{P}}(f_1, f_2)(x) f_3(x)\,dx.$$

Proposition 3.5 [Muscalu et al. 2004b]. *Let $\mathbb{P}$ be a finite collection of tritiles. Then*

$$|\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \lesssim \prod_{j=1}^{3} \big(\text{size}_j\big((\langle f_j, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}\big)\big)^{\theta_j} \big(\text{energy}_j\big((\langle f_j, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}\big)\big)^{1-\theta_j} \tag{3-6}$$

for any $0 \le \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1+\theta_2+\theta_3 = 1$; the implicit constants depend on the θ_j but are independent of the other parameters.

3C. *Estimates for sizes and energies.* In order to apply Proposition 3.5, we need to estimate further the sizes and energies appearing on the right-hand side of (3-6).

Lemma 3.6 [Muscalu and Schlag 2013; Muscalu et al. 2004b]. *Let $j \in \{1, 2, 3\}$ and $f \in L^2(\mathbb{R})$. Then one has*

$$\text{size}_j\big((\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}\big) \lesssim \sup_{P\in\mathbb{P}} \frac{1}{|I_P|} \int_{\mathbb{R}} |f| \tilde{\chi}^M_{I_P}\,dx \tag{3-7}$$

for every $M > 0$, where the approximate cutoff function $\tilde{\chi}^M_{I_P}(x)$ equals $(1+\operatorname{dist}(x, I_P)/|I_P|)^{-M}$ and the implicit constants depend on M.

Lemma 3.7 (Bessel-type estimates [Muscalu et al. 2004b]). *Let $j \in \{1, 2, 3\}$ and $f \in L^2(\mathbb{R})$. Then*

$$\text{energy}_j\big((\langle f, \Phi^j_{P_j}\rangle)_{P\in\mathbb{P}}\big) \lesssim \|f\|_{L^2}. \tag{3-8}$$

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by carrying out the proof of Proposition 2.17 for model operators $\Pi_{\vec{\mathbb{P}}}$ defined in (2-33) with $\vec{\mathbb{P}} = \tilde{\mathbb{P}}' \times \mathbb{P}''$.

Fix indices p_1, p_2, p as in the hypothesis of Proposition 2.17. Fix arbitrary measurable sets E_1, E_2, E_3 of finite measure (by using the scaling invariance of $\Pi_{\vec{\mathbb{P}}}$, we can assume further that $|E_3| = 1$). Our goal is to find $E_3' \subseteq E_3$ with $|E_3'| \simeq |E_3| = 1$ such that, when $|f_1| \le \chi_{E_1}$, $|f_2| \le \chi_{E_2}$ and $|f_3| \le \chi_{E_3'}$, the trilinear form $\Lambda_{\vec{\mathbb{P}}}(f_1, f_2, f_3)$ defined by

$$\Lambda_{\vec{\mathbb{P}}}(f_1, f_2, f_3) := \int_{\mathbb{R}^2} \Pi_{\vec{\mathbb{P}}}(f_1, f_2)(x) f_3(x)\, dx \tag{4-1}$$

satisfies the estimate

$$|\Lambda_{\vec{\mathbb{P}}}(f_1, f_2, f_3)| = \left| \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi^1_{\vec{P}_1}\rangle \langle f_2, \Phi^2_{\vec{P}_2}\rangle \langle f_3, \Phi^3_{\vec{P}_3}\rangle \right| \lesssim_{p,p_1,p_2,B} |E_1|^{1/p_1} |E_2|^{1/p_2}, \tag{4-2}$$

where p_1 is larger than but close to 1, while p_2 is smaller than but close to 2.

In order to prove our Theorem 1.3 in biparameter settings, one can easily observe that the main difficulty from [Muscalu et al. 2004a; 2006] is that, if we restrict the sum of tritiles $P'' \in \mathbb{P}''$ in the definition of discrete model operators $\Pi_{\vec{\mathbb{P}}}$ to a tree, then we essentially get a tensor product of two discrete paraproducts on x_1 and x_2 respectively, which can be estimated by the MM, MS, SM and SS functions, but, due to the extra degree of freedom in frequency in the x_2 direction, there are infinitely many such tensor products of paraproducts in the summation, so it's difficult for us to carry out the stopping-time decompositions by using the hybrid square and maximal operators as in [Muscalu et al. 2004a; 2006]. Instead, we will make use of the L^2 size and L^2 energy estimates of the trilinear forms, the almost orthogonality of wave packets associated with different tiles and the decay assumptions on the symbols. Furthermore, we can extend our proof of Theorem 1.3 to general d-parameter settings ($d \ge 3$) by applying the generic decomposition lemma (Lemma 4.1) to the $d-1$ variables $x_1, \dots, x_{d-1}$. Although one can't obtain that supp $\Phi^{3,\ell}_{\vec{P}_3'} \otimes \Phi^3_{P_3''}$ is entirely contained in the exceptional set U as in [Muscalu et al. 2006], one can show that this support set is contained in U in all the variables $x_1, \dots, x_{d-1}$, but not x_d. Therefore, we only need to consider the distance from this support set to the set E_3' in the x_d direction and obtain enough decay factors for summation; the extension of the proof from biparameter case to the general d-parameter ($d \ge 3$) cases is straightforward.

From [Muscalu et al. 2006], we can find the following generic decomposition lemma:

Lemma 4.1. *Let $J \subseteq \mathbb{R}$ be a fixed interval. Then every smooth bump function ϕ_J adapted to J can be naturally decomposed as*

$$\phi_J = \sum_{\ell \in \mathbb{N}} 2^{-100\ell} \phi_J^\ell,$$

where, for every $\ell \in \mathbb{N}$, ϕ_J^ℓ is also a bump function adapted to J but having the additional property that $\operatorname{supp}(\phi_J^\ell) \subseteq 2^\ell J$. *If in addition we assume that $\int_{\mathbb{R}} \phi_J(x)\, dx = 0$, then the functions ϕ_J^ℓ can be chosen so that $\int_{\mathbb{R}} \phi_J^\ell(x)\, dx = 0$ for every $\ell \in \mathbb{N}$.*

We use $2^\ell J$ to denote the interval having the same center as J but with length 2^ℓ times that of J.

By using Lemma 4.1, we can estimate the left-hand side of (4-2) by

$$|\Lambda_{\vec{\mathbb{P}}}(f_1, f_2, f_3)| \lesssim \sum_{\ell \in \mathbb{N}} 2^{-100\ell} \Lambda_{\vec{\mathbb{P}}}^{\ell}(f_1, f_2, f_3). \tag{4-3}$$

The trilinear forms $\Lambda_{\vec{\mathbb{P}}}^{\ell}(f_1, f_2, f_3)$ ($\ell \in \mathbb{N}$) are defined by

$$\Lambda_{\vec{\mathbb{P}}}^{\ell}(f_1, f_2, f_3) := \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{|C_{Q_{\vec{P}}}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi^1_{\vec{P}_1}\rangle| |\langle f_2, \Phi^2_{\vec{P}_2}\rangle| |\langle f_3, \Phi^{3,\ell}_{\vec{P}_3}\rangle|, \tag{4-4}$$

where the new biparameter wave packets are $\Phi^{3,\ell}_{\vec{P}_3} := \Phi^{3,\ell}_{\tilde{P}'_3} \otimes \Phi^3_{P''_3}$ with the additional property that $\operatorname{supp}(\Phi^{3,\ell}_{\tilde{P}'_3}) \subseteq 2^{\ell} I_{\tilde{P}'_3} = 2^{\ell} I_{\tilde{P}'}$.

For every $\ell \in \mathbb{N}$, we define the sets

$$\Omega_{-10\ell} := \bigcup_{j=1}^{2} \left\{ x \in \mathbb{R}^2 : \mathrm{MM}\left(\frac{\chi_{E_j}}{|E_j|}\right)(x) > C 2^{10\ell} \right\} \tag{4-5}$$

and

$$\tilde{\Omega}_{-10\ell} := \{x \in \mathbb{R}^2 : \mathrm{MM}(\chi_{\Omega_{-10\ell}})(x) > 2^{-\ell}\}, \tag{4-6}$$

where the double maximal operator MM is given by

$$\mathrm{MM}(h)(x, y) := \sup_{\substack{\text{dyadic rectangle } R\\ (x,y) \in R}} \frac{1}{|R|} \int_R |h(u, v)| \, du \, dv. \tag{4-7}$$

Finally, we define the exceptional set

$$U := \bigcup_{\ell \in \mathbb{N}} \tilde{\Omega}_{-10\ell}. \tag{4-8}$$

It is clear that $|U| < \frac{1}{10}$ if C is a large enough constant, which we fix from now on. Then, we define $E'_3 := E_3 \setminus U$ and note that $|E'_3| \simeq 1$.

Now fix $\ell \in \mathbb{N}$, and split the trilinear form $\Lambda_{\vec{\mathbb{P}}}^{\ell}(f_1, f_2, f_3)$ defined in (4-4) into two parts as follows:

$$\begin{aligned}
&\Lambda_{\vec{\mathbb{P}}}^{\ell}(f_1, f_2, f_3)\\
&= \sum_{\substack{\vec{P} \in \vec{\mathbb{P}}:\\ I_{\vec{P}} \cap \Omega^c_{-10\ell} \neq \varnothing}} \frac{|C_{Q_{\vec{P}}}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi^1_{\vec{P}_1}\rangle| |\langle f_2, \Phi^2_{\vec{P}_2}\rangle| |\langle f_3, \Phi^{3,\ell}_{\vec{P}_3}\rangle| + \sum_{\substack{\vec{P} \in \vec{\mathbb{P}}:\\ I_{\vec{P}} \cap \Omega^c_{-10\ell} = \varnothing}} \frac{|C_{Q_{\vec{P}}}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi^1_{\vec{P}_1}\rangle| |\langle f_2, \Phi^2_{\vec{P}_2}\rangle| |\langle f_3, \Phi^{3,\ell}_{\vec{P}_3}\rangle|\\
&=: \Lambda_{\vec{\mathbb{P}},I}^{\ell}(f_1, f_2, f_3) + \Lambda_{\vec{\mathbb{P}},II}^{\ell}(f_1, f_2, f_3),
\end{aligned} \tag{4-9}$$

where A^c denotes the complement of a set A.

4A. ***Estimates for trilinear form $\Lambda_{\vec{\mathbb{P}},I}^{\ell}(f_1, f_2, f_3)$.*** We can decompose the collection $\tilde{\mathbb{P}}'$ of tritiles into

$$\tilde{\mathbb{P}}' = \bigcup_{k' \in \mathbb{Z}} \tilde{\mathbb{P}}'_{k'}, \tag{4-10}$$

where

$$\tilde{\mathbb{P}}'_{k'} := \{\tilde{P}' \in \tilde{\mathbb{P}}' : |I_{\tilde{P}'}| = 2^{-k'}\}. \tag{4-11}$$

As a consequence, we can split the trilinear form $\Lambda^{\ell}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3)$ into

$$\Lambda^{\ell}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) = \sum_{k'\in\mathbb{Z}} \sum_{\substack{\vec{P}\in\tilde{\mathbb{P}}'_{k'}\times\mathbb{P}'': \\ I_{\vec{P}}\cap\Omega^c_{-10\ell}\neq\varnothing}} |C_{Q_{\vec{P}}}| \frac{|I_{\tilde{P}'}|}{|I_{P''}|^{\frac{1}{2}}} \prod_{j=1}^{2} \left| \left\langle \frac{\langle f_j, \Phi^j_{\tilde{P}'_j}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \right\rangle \right| \times \left| \left\langle \frac{\langle f_3, \Phi^{3,l}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \right\rangle \right|. \tag{4-12}$$

By Lemma 2.4, we can estimate the Fourier coefficients $C_{Q_{\vec{P}}} := C_{Q_{\vec{P}},\vec{0},\vec{0},\vec{0}}$ for each $\vec{P} \in \tilde{\mathbb{P}}'_{k'} \times \mathbb{P}''$ ($k' \in \mathbb{Z}$) by

$$|C_{Q_{\vec{P}}}| \lesssim C_{k'} \quad \text{with} \quad \sum_{k'\in\mathbb{Z}} C_{k'} \leq B < +\infty. \tag{4-13}$$

For each fixed $\tilde{P}' \in \tilde{\mathbb{P}}'$, we define the subcollection

$$\mathbb{P}''_{\tilde{P}'} := \{P'' \in \mathbb{P}'' : I_{\vec{P}} \cap \Omega^c_{-10\ell} \neq \varnothing\}.$$

Therefore, by using Proposition 3.5, we derive the estimates

$$\begin{aligned}
&\Lambda^{\ell}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) \\
&\lesssim \sum_{k'\in\mathbb{Z}} C_{k'} \sum_{\tilde{P}'\in\tilde{\mathbb{P}}'_{k'}} |I_{\tilde{P}'}| \\
&\qquad \times \left[\prod_{j=1}^{2} \left(\text{energy}_j \left(\left(\left\langle \frac{\langle f_j, \Phi^j_{\tilde{P}'_j}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \right)^{1-\theta_j} \left(\text{size}_j \left(\left(\left\langle \frac{\langle f_j, \Phi^j_{\tilde{P}'_j}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \right)^{\theta_j} \right] \\
&\qquad \times \left(\text{size}_3 \left(\left(\left\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \right)^{\theta_3} \left(\text{energy}_3 \left(\left(\left\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \right)^{1-\theta_3}
\end{aligned} \tag{4-14}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the right-hand side of (4-14), note that $I_{\vec{P}} \cap \Omega^c_{-10\ell} \neq \varnothing$ and $\operatorname{supp} f_3 \subseteq E'_3 \subseteq \mathbb{R}^2 \setminus U$; we apply the size estimates in Lemma 3.6 and get, for each $\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}$,

$$\text{size}_1 \left(\left(\left\langle \frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^1_{P''_1} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \lesssim \sup_{P''\in\mathbb{P}''_{\tilde{P}'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}} \right| \tilde{\chi}^M_{I_{P''}} \, dx \lesssim 2^{10\ell} |E_1|, \tag{4-15}$$

$$\text{size}_2 \left(\left(\left\langle \frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^2_{P''_2} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \lesssim \sup_{P''\in\mathbb{P}''_{\tilde{P}'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}} \right| \tilde{\chi}^M_{I_{P''}} \, dx \lesssim 2^{10\ell} |E_2|, \tag{4-16}$$

$$\text{size}_3 \left(\left(\left\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \right\rangle \right)_{P''\in\mathbb{P}''_{\tilde{P}'}} \right) \lesssim \sup_{P''\in\mathbb{P}''_{\tilde{P}'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}} \right| \tilde{\chi}^M_{I_{P''}} \, dx \lesssim 1, \tag{4-17}$$

where $M > 0$ is sufficiently large. By applying the energy estimates in Lemma 3.7 and Hölder estimates, we have, for each $\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}$,

$$\text{energy}_1\left(\left(\left\langle\frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^1_{P''_1}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}'}}\right) \lesssim \left\|\frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}\right\|_{L^2(\mathbb{R})} \lesssim \left(\int_{E_1} \frac{\tilde{\chi}^{100}_{I_{\tilde{P}'}}(x_1)}{|I_{\tilde{P}'}|}\, dx_1\, dx_2\right)^{\frac{1}{2}}, \tag{4-18}$$

$$\text{energy}_2\left(\left(\left\langle\frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^2_{P''_2}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}'}}\right) \lesssim \left\|\frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}\right\|_{L^2(\mathbb{R})} \lesssim \left(\int_{E_2} \frac{\tilde{\chi}^{100}_{I_{\tilde{P}'}}(x_1)}{|I_{\tilde{P}'}|}\, dx_1\, dx_2\right)^{\frac{1}{2}}, \tag{4-19}$$

$$\text{energy}_3\left(\left(\left\langle\frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}'}}\right) \lesssim \left\|\frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}\right\|_{L^2(\mathbb{R})} \lesssim \left(\int_{E'_3} \frac{\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}(x_1)}{|I_{\tilde{P}'}|}\, dx_1\, dx_2\right)^{\frac{1}{2}}, \tag{4-20}$$

where the approximate cutoff function $\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}(x_1)$ decays rapidly (of order 100) away from the interval $I_{\tilde{P}'}$ at scale $|I_{\tilde{P}'}|$ and satisfies the additional property that $\operatorname{supp} \tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}} \subseteq 2^\ell I_{\tilde{P}'}$.

Now we insert the size and energy estimates (4-15)–(4-20) into (4-14) and get

$$\begin{aligned}
&\Lambda^{\ell}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3)\\
&\quad\lesssim 2^{10\ell}|E_1|^{\theta_1}|E_2|^{\theta_2}\sum_{k'\in\mathbb{Z}} C_{k'}\sum_{\tilde{P}'\in\tilde{\mathbb{P}}'_{k'}}\left(\int_{E_1}\tilde{\chi}^{100}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_1}{2}}\left(\int_{E_2}\tilde{\chi}^{100}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_2}{2}}\left(\int_{E'_3}\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_3}{2}}.
\end{aligned}\tag{4-21}$$

Since $|I_{\tilde{P}'}| = 2^{-k'}$ for every $\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}$, all the dyadic intervals $I_{\tilde{P}'}$ are disjoint, thus, by using Hölder's inequality, we can estimate the inner sum in the right-hand side of (4-21) by

$$\prod_{j=1}^{2}\left(\sum_{\tilde{P}'\in\tilde{\mathbb{P}}'_{k'}}\int_{E_j}\tilde{\chi}^{100}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_j}{2}}\left(\sum_{\tilde{P}'\in\tilde{\mathbb{P}}'_{k'}}\int_{E'_3}\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_3}{2}} \lesssim |E_1|^{(1-\theta_1)/2}|E_2|^{(1-\theta_2)/2}. \tag{4-22}$$

Combining the estimates (4-13), (4-21) and (4-22), we arrive at

$$\begin{aligned}
\Lambda^{\ell}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) &\lesssim 2^{10\ell}|E_1|^{\theta_1}|E_2|^{\theta_2}|E_1|^{(1-\theta_1)/2}|E_2|^{(1-\theta_2)/2}\sum_{k'\in\mathbb{Z}} C_{k'}\\
&\lesssim_{\theta_1,\theta_2,\theta_3,B} 2^{10\ell}|E_1|^{(1+\theta_1)/2}|E_2|^{(1+\theta_2)/2}
\end{aligned}\tag{4-23}$$

for every $\ell \in \mathbb{N}$ and $0 \le \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponent $2/(1+\theta_1) = p_1$ strictly larger than 1 and close to 1, and $2/(1+\theta_2) = p_2$ strictly smaller than 2 and close to 2. We finally get the estimate

$$\Lambda^{\ell}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) \lesssim_{p,p_1,p_2,B} 2^{10\ell}|E_1|^{1/p_1}|E_2|^{1/p_2} \tag{4-24}$$

for every $\ell \in \mathbb{N}$ and p, p_1, p_2 satisfying the hypothesis of Proposition 2.17.

4B. *Estimates for the trilinear form $\Lambda^{\ell}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3)$.* If $I_{\vec{P}} \subseteq \Omega_{-10\ell}$, then $2^{\ell} I_{\tilde{P}'} \times I_{P''} \subseteq \tilde{\Omega}_{-10\ell}$. Therefore, for each fixed $\tilde{P}' \in \tilde{\mathbb{P}}'$, we define the corresponding subcollection of $\mathbb{P}''$ by

$$\mathbb{P}''_{\tilde{P}'} := \{P'' \in \mathbb{P}'' : I_{\vec{P}} \subseteq \Omega_{-10\ell}\},$$

then we can decompose the collection $\mathbb{P}''_{\tilde{P}'}$ further, as follows:

$$\mathbb{P}''_{\tilde{P}'} = \bigcup_{d'' \in \mathbb{N}} \mathbb{P}''_{\tilde{P}',d''}, \tag{4-25}$$

where

$$\mathbb{P}''_{\tilde{P}',d''} := \{P'' \in \mathbb{P}''_{\tilde{P}'} : 2^{\ell} I_{\tilde{P}'} \times 2^{d''} I_{P''} \subseteq \tilde{\Omega}_{-10\ell}\} \tag{4-26}$$

and d'' is maximal with this property.

Now we apply both the decompositions of $\tilde{\mathbb{P}}'$ and $\mathbb{P}''_{\tilde{P}'}$ defined in (4-10) and (4-25) at the same time, and split the trilinear form $\Lambda^{\ell}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3)$ into

$$\Lambda^{\ell}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3) = \sum_{k' \in \mathbb{Z}} \sum_{\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}} |C_{Q_{\vec{P}}}| |I_{\tilde{P}'}| \sum_{d'' \in \mathbb{N}} \sum_{P'' \in \mathbb{P}''_{\tilde{P}',d''}} \frac{1}{|I_{P''}|^{\frac{1}{2}}} \prod_{j=1}^{2} \left| \left\langle \frac{\langle f_j, \Phi^j_{\tilde{P}'_j} \rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \right\rangle \right| \times \left| \left\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3} \rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \right\rangle \right|. \tag{4-27}$$

In the inner sum of (4-27), since $2^{\ell} I_{\tilde{P}'} \times 2^{d''} I_{P''} \subseteq \tilde{\Omega}_{-10\ell}$,

$$\operatorname{supp}(\Phi^{3,\ell}_{\tilde{P}'_3}) \subseteq 2^{\ell} I_{\tilde{P}'} \quad \text{and} \quad \operatorname{supp} f_3 \subseteq E'_3 \subseteq \mathbb{R}^2 \setminus U,$$

we can assume hereafter in this subsection that

$$|f_3| \le \chi_{E'_3} \chi_{2^{\ell} I_{\tilde{P}'}} \chi_{(2^{d''} I_{P''})^c}. \tag{4-28}$$

By using Proposition 3.5 and (4-13), we derive from (4-27) the estimates

$$\begin{aligned}
&\Lambda^{\ell}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3) \\
&\lesssim \sum_{k' \in \mathbb{Z}} C_{k'} \sum_{\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}} |I_{\tilde{P}'}| \sum_{d'' \in \mathbb{N}} \Bigg[\prod_{j=1}^{2} \Bigg(\operatorname{energy}_j \Bigg(\Bigg(\Bigg\langle \frac{\langle f_j, \Phi^j_{\tilde{P}'_j} \rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \Bigg\rangle \Bigg)_{P'' \in \mathbb{P}''_{\tilde{P}',d''}} \Bigg) \Bigg)^{1-\theta_j} \\
&\qquad \times \Bigg(\operatorname{size}_j \Bigg(\Bigg(\Bigg\langle \frac{\langle f_j, \Phi^j_{\tilde{P}'_j} \rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \Bigg\rangle \Bigg)_{P'' \in \mathbb{P}''_{\tilde{P}',d''}} \Bigg) \Bigg)^{\theta_j} \Bigg] \\
&\times \Bigg(\operatorname{size}_3 \Bigg(\Bigg(\Bigg\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3} \rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \Bigg\rangle \Bigg)_{P'' \in \mathbb{P}''_{\tilde{P}',d''}} \Bigg) \Bigg)^{\theta_3} \Bigg(\operatorname{energy}_3 \Bigg(\Bigg(\Bigg\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3} \rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3} \Bigg\rangle \Bigg)_{P'' \in \mathbb{P}''_{\tilde{P}',d''}} \Bigg) \Bigg)^{1-\theta_3}
\end{aligned} \tag{4-29}$$

for any $0 \le \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the inner sum in the right-hand side of (4-29), note that $I_{\vec{P}} \subseteq \Omega_{-10\ell}$, $P'' \in \mathbb{P}''_{\tilde{P}',d''}$ and f_3 satisfies (4-28), so we apply the size estimates in Lemma 3.6 and get, for each $\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}$ and $d'' \in \mathbb{N}$,

$$\mathrm{size}_1\left(\left(\left\langle \frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^1_{P''_1}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}',d''}}\right) \lesssim \sup_{P''\in\mathbb{P}''_{\tilde{P}',d''}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}} \left|\frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}\right| \tilde{\chi}^M_{I_{P''}}\,dx \lesssim 2^{11\ell+d''}|E_1|, \tag{4-30}$$

$$\mathrm{size}_2\left(\left(\left\langle \frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^2_{P''_2}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}',d''}}\right) \lesssim \sup_{P''\in\mathbb{P}''_{\tilde{P}',d''}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}} \left|\frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}\right| \tilde{\chi}^M_{I_{P''}}\,dx \lesssim 2^{11\ell+d''}|E_2|, \tag{4-31}$$

$$\mathrm{size}_3\left(\left(\left\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}',d''}}\right) \lesssim \sup_{P''\in\mathbb{P}''_{\tilde{P}',d''}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}} \left|\frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}\right| \tilde{\chi}^M_{I_{P''}}\,dx \lesssim 2^{-(M-100)d''}, \tag{4-32}$$

where $M > 0$ is arbitrarily large. Similar to the energy estimates obtained in (4-18), (4-19) and (4-20), by applying the energy estimates in Lemma 3.7 and Hölder estimates we have, for each $\tilde{P}' \in \tilde{\mathbb{P}}'_{k'}$ and $d'' \in \mathbb{N}$,

$$\mathrm{energy}_1\left(\left(\left\langle \frac{\langle f_1, \Phi^1_{\tilde{P}'_1}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^1_{P''_1}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}',d''}}\right) \lesssim \left(\int_{E_1} \frac{\tilde{\chi}^{100}_{I_{\tilde{P}'}}(x_1)}{|I_{\tilde{P}'}|}\,dx_1\,dx_2\right)^{\frac{1}{2}}, \tag{4-33}$$

$$\mathrm{energy}_2\left(\left(\left\langle \frac{\langle f_2, \Phi^2_{\tilde{P}'_2}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^2_{P''_2}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}',d''}}\right) \lesssim \left(\int_{E_2} \frac{\tilde{\chi}^{100}_{I_{\tilde{P}'}}(x_1)}{|I_{\tilde{P}'}|}\,dx_1\,dx_2\right)^{\frac{1}{2}}, \tag{4-34}$$

$$\mathrm{energy}_3\left(\left(\left\langle \frac{\langle f_3, \Phi^{3,\ell}_{\tilde{P}'_3}\rangle}{|I_{\tilde{P}'}|^{\frac{1}{2}}}, \Phi^3_{P''_3}\right\rangle\right)_{P''\in\mathbb{P}''_{\tilde{P}',d''}}\right) \lesssim \left(\int_{E'_3} \frac{\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}(x_1)}{|I_{\tilde{P}'}|}\,dx_1\,dx_2\right)^{\frac{1}{2}}, \tag{4-35}$$

where the approximate cutoff function $\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}(x_1)$ decays rapidly (of order 100) away from the interval $I_{\tilde{P}'}$ at scale $|I_{\tilde{P}'}|$ and satisfies the additional property that $\operatorname{supp}\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}} \subseteq 2^\ell I_{\tilde{P}'}$.

Now we insert the size and energy estimates (4-30)–(4-35) into (4-29); by using the estimates (4-13), (4-22) and Hölder's inequality, we then get

$$\begin{aligned}
&\Lambda^{\ell}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3)\\
&\lesssim 2^{11\ell}|E_1|^{\theta_1}|E_2|^{\theta_2}\sum_{k'\in\mathbb{Z}} C_{k'}\sum_{d''\in\mathbb{N}} 2^{-(M\theta_3-100)d''}\prod_{j=1}^{2}\left(\sum_{\tilde{P}'\in\tilde{\mathbb{P}}'_{k'}}\int_{E_j}\tilde{\chi}^{100}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_j}{2}}\times\left(\sum_{\tilde{P}'\in\tilde{\mathbb{P}}'_{k'}}\int_{E'_3}\tilde{\chi}^{100,\ell}_{I_{\tilde{P}'}}\,dx\right)^{\frac{1-\theta_3}{2}}\\
&\lesssim_{\theta_1,\theta_2,\theta_3,B,M} 2^{11\ell}|E_1|^{(1+\theta_1)/2}|E_2|^{(1+\theta_2)/2}\sum_{d''\in\mathbb{N}} 2^{-(M\theta_3-100)d''}.
\end{aligned}\tag{4-36}$$

for every $\ell \in \mathbb{N}$ and $0 \le \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponent $2/(1+\theta_1) = p_1$ strictly larger than 1 and close to 1, and $2/(1+\theta_2) = p_2$ strictly smaller than 2 and close to 2. The series over $d'' \in \mathbb{N}$ in (4-36) is summable if we choose M large enough (say, $M \simeq 200\theta_3^{-1}$). We finally get the estimate

$$\Lambda^{\ell}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3) \lesssim_{p,p_1,p_2,B} 2^{11\ell}|E_1|^{1/p_1}|E_2|^{1/p_2} \tag{4-37}$$

for every $\ell \in \mathbb{N}$ and p, p_1, p_2 satisfying the hypothesis of Proposition 2.17.

4C. *Conclusions.* By inserting the estimates (4-9), (4-24) and (4-37) into (4-3), we finally get

$$|\Lambda_{\vec{\mathbb{P}}}(f_1, f_2, f_3)| \lesssim_{p,p_1,p_2,B} \sum_{\ell\in\mathbb{N}} 2^{-100\ell}2^{12\ell}|E_1|^{1/p_1}|E_2|^{1/p_2} \lesssim_{p,p_1,p_2,B} |E_1|^{1/p_1}|E_2|^{1/p_2}, \tag{4-38}$$

which completes the proof of Proposition 2.17 for the model operators $\Pi_{\vec{\mathbb{P}}}$.

This concludes the proof of Theorem 1.3.

5. Proof of Theorem 1.5

In this section, we prove Theorem 1.5 by carrying out the proof of Proposition 2.17 for the model operators $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$ defined in (2-54) with $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$.

Fix indices p_1, p_2, p as in the hypothesis of Proposition 2.17. Fix arbitrary measurable sets E_1, E_2, E_3 of finite measure (by using the scaling invariance of $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$, we can assume further that $|E_3| = 1$). Our goal is to find $E_3' \subseteq E_3$ with $|E_3'| \simeq |E_3| = 1$ such that, for any functions $|f_1| \le \chi_{E_1}$, $|f_2| \le \chi_{E_2}$ and $|f_3| \le \chi_{E_3'}$, one has the corresponding trilinear forms $\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2, f_3)$ defined by

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2, f_3) := \int_{\mathbb{R}^2} \tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2)(x) f_3(x)\, dx \tag{5-1}$$

satisfy estimates

$$|\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2, f_3)| = \left|\sum_{\vec{P}\in\vec{\mathbb{P}}} \frac{\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi^1_{\vec{P}_1}\rangle\langle f_2, \Phi^2_{\vec{P}_2}\rangle\langle f_3, \Phi^3_{\vec{P}_3}\rangle\right| \lesssim_{\varepsilon,p,p_1,p_2} |E_1|^{1/p_1}|E_2|^{1/p_2}, \tag{5-2}$$

where p_1 is larger than but close to 1, while p_2 is smaller than but close to 2.

In the proof of Theorem 1.5 in biparameter settings, besides the difficulty that one can't carry out the stopping-time decompositions by using the hybrid square and maximal operators as in [Muscalu et al. 2004a; 2006], we can't apply Journé's lemma as in [Muscalu et al. 2004a] either, since we can't get the estimate $\sum_{P'} |I_{P'}| \lesssim |\tilde{I}|$ for all dyadic intervals $I_{P'} \subseteq \tilde{I}$ with comparable lengths. Therefore, in order to prove Theorem 1.5, we will take advantage of the almost orthogonality of wave packets associated with different tiles of distinct trees and the decay assumptions on the symbols to overcome these difficulties.

We define the exceptional set

$$\Omega := \bigcup_{j=1}^{2}\left\{x \in \mathbb{R}^2 : \mathrm{MM}\left(\frac{\chi_{E_j}}{|E_j|}\right)(x) > C\right\} \tag{5-3}$$

It is clear that $|\Omega| < \frac{1}{10}$ if C is a large enough constant, which we fix from now on. Then, we define $E_3' := E_3 \setminus \Omega$ and observe that $|E_3'| \simeq 1$.

Now we estimate the trilinear form $\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2, f_3)$ defined in (5-1) by two terms as follows:

$$
\begin{aligned}
&|\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2, f_3)| \\
&\lesssim \sum_{\substack{\vec{P}\in\vec{\mathbb{P}}:\\ I_{\vec{P}}\cap\Omega^c\neq\varnothing}} \frac{|\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi^1_{\vec{P}_1}\rangle||\langle f_2, \Phi^2_{\vec{P}_2}\rangle||\langle f_3, \Phi^3_{\vec{P}_3}\rangle| + \sum_{\substack{\vec{P}\in\vec{\mathbb{P}}:\\ I_{\vec{P}}\cap\Omega^c=\varnothing}} \frac{|\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi^1_{\vec{P}_1}\rangle||\langle f_2, \Phi^2_{\vec{P}_2}\rangle||\langle f_3, \Phi^3_{\vec{P}_3}\rangle| \\
&=: \tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) + \tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3).
\end{aligned} \tag{5-4}
$$

5A. ***Estimates for trilinear form $\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3)$.*** We can decompose the collection $\tilde{\mathbb{P}}'$ of tritiles into

$$\mathbb{P}' = \bigcup_{k'\in\mathbb{Z}} \mathbb{P}'_{k'}, \tag{5-5}$$

where

$$\mathbb{P}'_{k'} := \{P' \in \mathbb{P}' : \ell(Q_{P'}) = 2^{k'}\}. \tag{5-6}$$

As a consequence, we can split the trilinear form $\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3)$ into

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) = \sum_{k'\in\mathbb{Z}} \sum_{\substack{\vec{P}\in\mathbb{P}'_{k'}\times\mathbb{P}'':\\ I_{\vec{P}}\cap\Omega^c\neq\varnothing}} |\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}| \frac{|I_{P'}|}{|I_{P''}|^{\frac{1}{2}}} \prod_{j=1}^{3} \left| \left\langle \frac{\langle f_j, \Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \right\rangle \right|. \tag{5-7}$$

By Lemma 2.10, we can estimate the Fourier coefficients $\tilde{C}^{\varepsilon}_{Q_{\vec{P}}} := \tilde{C}^{\varepsilon}_{Q_{\vec{P}},\vec{0},\vec{0},\vec{0}}$ for each $\vec{P} \in \mathbb{P}'_{k'} \times \mathbb{P}''$ ($k' \in \mathbb{Z}$) by

$$|\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}| \lesssim \tilde{C}^{\varepsilon}_{k'} := \langle k'\rangle^{-(1+\varepsilon)} = (1 + |k'|^2)^{-(1+\varepsilon)/2}. \tag{5-8}$$

For each fixed $P' \in \mathbb{P}'$, we define the subcollection $\mathbb{P}''_{P'}$ of $\mathbb{P}''$ by

$$\mathbb{P}''_{P'} := \{P'' \in \mathbb{P}'' : I_{\vec{P}} \cap \Omega^c \neq \varnothing\}.$$

Therefore, by using Proposition 3.5, we derive the estimates

$$
\begin{aligned}
&\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) \\
&\lesssim \sum_{k'\in\mathbb{Z}} \tilde{C}^{\varepsilon}_{k'} \sum_{P'\in\mathbb{P}'_{k'}} |I_{P'}| \prod_{j=1}^{3} \Bigg[\Bigg(\text{energy}_j \Bigg(\Bigg(\Bigg\langle \frac{\langle f_j, \Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \Bigg\rangle \Bigg)_{P''\in\mathbb{P}''_{P'}} \Bigg) \Bigg)^{1-\theta_j} \\
&\qquad \times \Bigg(\text{size}_j \Bigg(\Bigg(\Bigg\langle \frac{\langle f_j, \Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi^j_{P''_j} \Bigg\rangle \Bigg)_{P''\in\mathbb{P}''_{P'}} \Bigg) \Bigg)^{\theta_j} \Bigg]
\end{aligned} \tag{5-9}
$$

for any $0 \le \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the right-hand side of (5-9), note that $I_{\vec{P}} \cap \Omega^c \neq \varnothing$ and supp $f_3 \subseteq E_3'$, so we apply the size estimates in Lemma 3.6 and get, for each $P' \in \mathbb{P}_{k'}'$ and $j = 1, 2, 3$,

$$\mathrm{size}_j\left(\left(\left(\frac{\langle f_j, \Phi^j_{P_j'}\rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi^j_{P_j''}\right)\right)_{P''\in\mathbb{P}''_{P'}}\right) \lesssim \sup_{P''\in\mathbb{P}''_{P'}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}}\left|\frac{\langle f_j, \Phi^j_{P_j'}\rangle}{|I_{P'}|^{\frac{1}{2}}}\right| \tilde{\chi}^M_{I_{P''}}\,dx \lesssim |E_j|, \tag{5-10}$$

where $M > 0$ is sufficiently large. By applying the energy estimates in Lemma 3.7, we have, for each $P' \in \mathbb{P}_{k'}'$ and $j = 1, 2, 3$,

$$\mathrm{energy}_j\left(\left(\left(\frac{\langle f_j, \Phi^j_{P_j'}\rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi^j_{P_j''}\right)\right)_{P''\in\mathbb{P}''_{P'}}\right) \lesssim \frac{1}{|I_{P'}|^{\frac{1}{2}}}\left(\int_{\mathbb{R}} |\langle f_j, \Phi^j_{P_j'}\rangle|^2\,dx_2\right)^{\frac{1}{2}}. \tag{5-11}$$

Now we insert the size and energy estimates (5-10) and (5-11) into (5-9) and get

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) \lesssim |E_1|^{\theta_1}|E_2|^{\theta_2}\sum_{k'\in\mathbb{Z}}\tilde{C}^{\varepsilon}_{k'}\sum_{P'\in\mathbb{P}'_{k'}}\prod_{j=1}^{3}\left(\int_{\mathbb{R}}|\langle f_j, \Phi^j_{P_j'}\rangle|^2\,dx_2\right)^{\frac{1-\theta_j}{2}}. \tag{5-12}$$

Observe that, for any different tritiles $\bar{P}' \in \mathbb{P}'_{k'}$ and $\bar{\bar{P}}' \in \mathbb{P}'_{k'}$, one has $I_{\bar{P}'} \cap I_{\bar{\bar{P}}'} = \varnothing$, or otherwise one has $I_{\bar{P}'} = I_{\bar{\bar{P}}'}$ but $\omega_{\bar{P}'_j} \cap \omega_{\bar{\bar{P}}'_j} = \varnothing$ for every $j = 1, 2, 3$. By taking advantage of such orthogonality in L^2 of the wave packets $\Phi^j_{P_j'}$ corresponding to the tiles P_j' $(j = 1, 2, 3)$, one has that, for any function $F \in L^2(\mathbb{R})$ and $k' \in \mathbb{Z}$,

$$\begin{aligned}
\Big\|\sum_{P'\in\mathbb{P}'_{k'}}\langle F, \Phi^j_{P_j'}\rangle\Phi^j_{P_j'}\Big\|^2_{L^2} &\le \sum_{\substack{\bar{P}',\,\bar{\bar{P}}'\in\mathbb{P}'_{k'}:\\ \omega_{\bar{P}'_j}=\omega_{\bar{\bar{P}}'_j}\\ I_{\bar{P}'}\cap I_{\bar{\bar{P}}'}=\varnothing}} |\langle F, \Phi^j_{\bar{P}'_j}\rangle||\langle F, \Phi^j_{\bar{\bar{P}}'_j}\rangle||\langle \Phi^j_{\bar{P}'_j}, \Phi^j_{\bar{\bar{P}}'_j}\rangle| \\
&\lesssim 2^{k'}\sum_{\bar{P}'\in\mathbb{P}'_{k'}}|\langle F, \Phi^j_{\bar{P}'_j}\rangle|^2 \sum_{\substack{\bar{\bar{P}}'\in\mathbb{P}'_{k'}:\\ \omega_{\bar{P}'_j}=\omega_{\bar{\bar{P}}'_j}\\ I_{\bar{P}'}\cap I_{\bar{\bar{P}}'}=\varnothing}} |\langle\tilde{\chi}^{1000}_{I_{\bar{P}'}}, \tilde{\chi}^{1000}_{I_{\bar{\bar{P}}'}}\rangle| \\
&\lesssim \sum_{\bar{P}'\in\mathbb{P}'_{k'}}|\langle F, \Phi^j_{\bar{P}'_j}\rangle|^2 \sum_{\substack{\bar{\bar{P}}'\in\mathbb{P}'_{k'}:\\ \omega_{\bar{P}'_j}=\omega_{\bar{\bar{P}}'_j}\\ I_{\bar{P}'}\cap I_{\bar{\bar{P}}'}=\varnothing}} \left(1+\frac{\mathrm{dist}(I_{\bar{P}'}, I_{\bar{\bar{P}}'})}{|I_{\bar{P}'}|}\right)^{-100} \\
&\lesssim \sum_{P'\in\mathbb{P}'_{k'}}|\langle F, \Phi^j_{P_j'}\rangle|^2,
\end{aligned} \tag{5-13}$$

from which we deduce the Bessel-type inequality

$$\sum_{P'\in\mathbb{P}'_{k'}}|\langle F, \Phi^j_{P_j'}\rangle|^2 = \left|\left\langle\sum_{P'\in\mathbb{P}'_{k'}}\langle F, \Phi^j_{P_j'}\rangle\Phi^j_{P_j'}, F\right\rangle\right| \le \Big\|\sum_{P'\in\mathbb{P}'_{k'}}\langle F, \Phi^j_{P_j'}\rangle\Phi^j_{P_j'}\Big\|_{L^2}\cdot\|F\|_{L^2} \lesssim \|F\|^2_{L^2}, \tag{5-14}$$

where the implicit constants in the bounds are independent of $k' \in \mathbb{Z}$. Then, we can use the Bessel-type inequality (5-14) and Hölder's inequality to estimate the inner sum in the right-hand side of (5-12) by

$$\sum_{P'\in\mathbb{P}'_{k'}} \prod_{j=1}^{3}\left(\int_{\mathbb{R}} |\langle f_j, \Phi^j_{P'_j}\rangle|^2\,dx_2\right)^{\frac{1-\theta_j}{2}} \lesssim \prod_{j=1}^{3}\left(\int_{\mathbb{R}} \sum_{P'\in\mathbb{P}'_{k'}} |\langle f_j, \Phi^j_{P'_j}\rangle|^2\,dx_2\right)^{\frac{1-\theta_j}{2}}$$
$$\lesssim \prod_{j=1}^{3} \|f_j\|_{L^2(\mathbb{R}^2)}^{1-\theta_j} \lesssim |E_1|^{(1-\theta_1)/2}|E_2|^{(1-\theta_2)/2}. \tag{5-15}$$

Combining the estimates (5-8), (5-12) and (5-15), we arrive at

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) \lesssim |E_1|^{\theta_1}|E_2|^{\theta_2}|E_1|^{(1-\theta_1)/2}|E_2|^{(1-\theta_2)/2}\sum_{k'\in\mathbb{Z}}\tilde{C}^{\varepsilon}_{k'} \lesssim_{\varepsilon,\theta_1,\theta_2,\theta_3} |E_1|^{(1+\theta_1)/2}|E_2|^{(1+\theta_2)/2} \tag{5-16}$$

for any $0 \le \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1+\theta_2+\theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponent $2/(1+\theta_1) = p_1$ strictly larger than 1 and close to 1, and $2/(1+\theta_2) = p_2$ strictly smaller than 2 and close to 2. We finally get the estimate

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},I}(f_1, f_2, f_3) \lesssim_{\varepsilon,p,p_1,p_2} |E_1|^{1/p_1}|E_2|^{1/p_2} \tag{5-17}$$

for every $\varepsilon > 0$, and p, p_1, p_2 satisfy the hypothesis of Proposition 2.17.

5B. ***Estimates for the trilinear form $\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3)$.*** For each fixed $P' \in \mathbb{P}'$, we define the corresponding subcollection of $\mathbb{P}''$ by

$$\mathbb{P}''_{P'} := \{P'' \in \mathbb{P}'' : I_{\vec{P}} \subseteq \Omega\},$$

then we can decompose the collection $\mathbb{P}''_{P'}$ further, as follows:

$$\mathbb{P}''_{P'} = \bigcup_{\mu\in\mathbb{N}} \mathbb{P}''_{P',\mu}, \tag{5-18}$$

where

$$\mathbb{P}''_{P',\mu} := \{P'' \in \mathbb{P}''_{P'} : \mathrm{Dil}_{2^\mu}(I_{P'}\times I_{P''}) \subseteq \Omega\} \tag{5-19}$$

and μ is maximal with this property. By $\mathrm{Dil}_{2^\mu}(I_{\vec{P}})$ we mean the rectangle having the same center as the original $I_{\vec{P}}$ but whose side lengths are 2^μ times larger.

Now we apply both the decompositions of $\tilde{\mathbb{P}}'$ and $\mathbb{P}''_{P'}$ defined in (5-5) and (5-18) at the same time, and split the trilinear form $\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3)$ into

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3) = \sum_{k'\in\mathbb{Z}}\sum_{P'\in\mathbb{P}'_{k'}} |\tilde{C}^{\varepsilon}_{Q_{\vec{P}}}||I_{P'}|\sum_{\mu\in\mathbb{N}}\sum_{P''\in\mathbb{P}''_{P',\mu}} \frac{1}{|I_{P''}|^{\frac{1}{2}}}\prod_{j=1}^{3}\left|\left\langle \frac{\langle f_j, \Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi^j_{P''_j}\right\rangle\right| \tag{5-20}$$

In the inner sum of (5-20), since $\mathrm{Dil}_{2^\mu}(I_{P'}\times I_{P''}) \subseteq \Omega$ and $\mathrm{supp}\, f_3 \subseteq E'_3 \subseteq \mathbb{R}^2\setminus\Omega$, we get that

$$|f_3| \le \chi_{E'_3}\chi_{(\mathrm{Dil}_{2^\mu}(I_{P'}\times I_{P''}))^c} = \chi_{E'_3}\{\chi_{(2^\mu I_{P'})^c} + \chi_{(2^\mu I_{P''})^c} - \chi_{(2^\mu I_{P'})^c}\chi_{(2^\mu I_{P''})^c}\}, \tag{5-21}$$

and hence we can assume hereafter in this subsection that

$$|f_3| \le \chi_{E_3'}\chi_{(2^\mu I_{P'})^c}, \tag{5-22}$$

and the other two terms can be handled similarly.

By using Proposition 3.5 and (5-8), we derive from (5-20) the estimates

$$\begin{aligned}\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}&(f_1,f_2,f_3)\\ &\lesssim \sum_{k'\in\mathbb{Z}} \tilde{C}^{\varepsilon}_{k'} \sum_{P'\in\mathbb{P}'_{k'}} |I_{P'}| \sum_{\mu\in\mathbb{N}} \prod_{j=1}^{3}\Bigg[\Bigg(\mathrm{energy}_j\Bigg(\Bigg(\Bigg\langle \frac{\langle f_j,\Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac12}},\Phi^j_{P''_j}\Bigg\rangle\Bigg)_{P''\in\mathbb{P}''_{P',\mu}}\Bigg)\Bigg)^{1-\theta_j}\\ &\qquad\times\Bigg(\mathrm{size}_j\Bigg(\Bigg(\Bigg\langle \frac{\langle f_j,\Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac12}},\Phi^j_{P''_j}\Bigg\rangle\Bigg)_{P''\in\mathbb{P}''_{P',\mu}}\Bigg)\Bigg)^{\theta_j}\Bigg]\end{aligned} \tag{5-23}$$

for any $0 \le \theta_1,\theta_2,\theta_3 < 1$ with $\theta_1+\theta_2+\theta_3 = 1$.

To estimate the inner sum in the right-hand side of (5-23), note that $I_{\vec{P}} \subseteq \Omega$, $P'' \in \mathbb{P}''_{P',\mu}$ and f_3 satisfies (5-22), so we apply the size estimates in Lemma 3.6 and get, for each $P' \in \mathbb{P}'_{k'}$ and $\mu\in\mathbb{N}$,

$$\mathrm{size}_1\Bigg(\Bigg(\Bigg\langle \frac{\langle f_1,\Phi^1_{P'_1}\rangle}{|I_{P'}|^{\frac12}},\Phi^1_{P''_1}\Bigg\rangle\Bigg)_{P''\in\mathbb{P}''_{P',\mu}}\Bigg) \lesssim \sup_{P''\in\mathbb{P}''_{P',\mu}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}}\Bigg|\frac{\langle f_1,\Phi^1_{P'_1}\rangle}{|I_{P'}|^{\frac12}}\Bigg|\tilde{\chi}^M_{I_{P''}}\,dx \lesssim 2^{2\mu}|E_1|, \tag{5-24}$$

$$\mathrm{size}_2\Bigg(\Bigg(\Bigg\langle \frac{\langle f_2,\Phi^2_{P'_2}\rangle}{|I_{P'}|^{\frac12}},\Phi^2_{P''_2}\Bigg\rangle\Bigg)_{P''\in\mathbb{P}''_{P',\mu}}\Bigg) \lesssim \sup_{P''\in\mathbb{P}''_{P',\mu}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}}\Bigg|\frac{\langle f_2,\Phi^2_{P'_2}\rangle}{|I_{P'}|^{\frac12}}\Bigg|\tilde{\chi}^M_{I_{P''}}\,dx \lesssim 2^{2\mu}|E_2|, \tag{5-25}$$

$$\mathrm{size}_3\Bigg(\Bigg(\Bigg\langle \frac{\langle f_3,\Phi^3_{P'_3}\rangle}{|I_{P'}|^{\frac12}},\Phi^3_{P''_3}\Bigg\rangle\Bigg)_{P''\in\mathbb{P}''_{P',\mu}}\Bigg) \lesssim \sup_{P''\in\mathbb{P}''_{P',\mu}} \frac{1}{|I_{P''}|}\int_{\mathbb{R}}\Bigg|\frac{\langle f_3,\Phi^3_{P'_3}\rangle}{|I_{P'}|^{\frac12}}\Bigg|\tilde{\chi}^M_{I_{P''}}\,dx \lesssim 2^{-N\mu}, \tag{5-26}$$

where $M > 0$ and $N > 0$ are arbitrarily large. By applying the energy estimates in Lemma 3.7, we have, for each $P' \in \mathbb{P}'_{k'}$, $\mu\in\mathbb{N}$ and $j = 1, 2, 3$,

$$\mathrm{energy}_j\Bigg(\Bigg(\Bigg\langle \frac{\langle f_j,\Phi^j_{P'_j}\rangle}{|I_{P'}|^{\frac12}},\Phi^j_{P''_j}\Bigg\rangle\Bigg)_{P''\in\mathbb{P}''_{P',\mu}}\Bigg) \lesssim \frac{1}{|I_{P'}|^{\frac12}}\bigg(\int_{\mathbb{R}}|\langle f_j,\Phi^j_{P'_j}\rangle|^2\,dx_2\bigg)^{\frac12}. \tag{5-27}$$

Now we insert the size and energy estimates (5-24)–(5-27) into (5-23); by using the estimates (5-8) and (5-15), we derive that

$$\begin{aligned}\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}(f_1,f_2,f_3) &\lesssim |E_1|^{\theta_1}|E_2|^{\theta_2}\sum_{k'\in\mathbb{Z}}\tilde{C}^{\varepsilon}_{k'}\sum_{\mu\in\mathbb{N}}2^{-(N\theta_3-2)\mu}\sum_{P'\in\mathbb{P}'_{k'}}\prod_{j=1}^{3}\bigg(\int_{\mathbb{R}}|\langle f_j,\Phi^j_{P'_j}\rangle|^2\,dx_2\bigg)^{\frac{1-\theta_j}{2}}\\ &\lesssim_{\varepsilon,\theta_1,\theta_2,\theta_3,N} |E_1|^{(1+\theta_1)/2}|E_2|^{(1+\theta_2)/2}\sum_{\mu\in\mathbb{N}}2^{-(N\theta_3-2)\mu}.\end{aligned} \tag{5-28}$$

for every $0 \le \theta_1,\theta_2,\theta_3 < 1$ with $\theta_1+\theta_2+\theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponent $2/(1+\theta_1) = p_1$ strictly larger than 1 and close to 1, and $2/(1+\theta_2) = p_2$ strictly smaller than 2 and close to 2. The series over $\mu \in \mathbb{N}$ in (5-28) is summable if we choose N large enough (say, $N \simeq 4\theta_3^{-1}$). We finally get the estimate

$$\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}},II}(f_1, f_2, f_3) \lesssim_{\varepsilon,p,p_1,p_2} |E_1|^{1/p_1}|E_2|^{1/p_2} \tag{5-29}$$

for any $\varepsilon > 0$, and p, p_1, p_2 satisfy the hypothesis of Proposition 2.17.

5C. ***Conclusions.*** By inserting the estimates (5-17) and (5-29) into (5-4), we finally get

$$|\tilde{\Lambda}^{\varepsilon}_{\vec{\mathbb{P}}}(f_1, f_2, f_3)| \lesssim_{\varepsilon,p,p_1,p_2} |E_1|^{1/p_1}|E_2|^{1/p_2} \tag{5-30}$$

for any $\varepsilon > 0$, which completes the proof of Proposition 2.17 for the model operators $\tilde{\Pi}^{\varepsilon}_{\vec{\mathbb{P}}}$.

This concludes the proof of Theorem 1.5.

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