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Liouville Type Theorems for PDE and IE Systems Involving Fractional Laplacian on a Half Space

Wei Dai[1] **· Zhao Liu**[2] **· Guozhen Lu**[3]

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Abstract In this paper, let α be any real number between 0 and 2, we study the Dirichlet problem for semi-linear elliptic system involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n_+, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n_+, \\ u(x) = v(x) = 0, & x \notin \mathbb{R}^n_+. \end{cases} \tag{1}$$

We will first establish the equivalence between PDE problem (1) and the corresponding integral equation (IE) system (Lemma 2). Then we use the moving planes method in integral forms to establish our main theorem, a Liouville type theorem for the integral system (Theorem 3). Then we conclude the Liouville type theorem for the above differential system involving the fractional Laplacian (Corollary 4).

Keywords Liouville type theorem · Dirichlet problem · Half space · Method of moving planes in integral forms · Nonexistence · Rotational symmetry · The fractional Laplacian

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✉ Guozhen Lu
guozhen.lu@uconn.edu

Wei Dai
weidai@buaa.edu.cn

Zhao Liu
liuzhao2008tj@sina.com

[1] School of Mathematics and Systems Science, Beihang University (BUAA), Beijing 100191, People's Republic of China

[2] School of Mathematics and Computer Science, Jiangxi Science and Technology Normal University, Nanchang 330038, People's Republic of China

[3] Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA

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1 Introduction

In this paper, we study the following Dirichlet problem for semi-linear elliptic system involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n_+, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n_+, \\ u(x) = v(x) = 0, & x \notin \mathbb{R}^n_+, \end{cases} \tag{2}$$

where $0 < \alpha < 2$ and the nonlocal operator $(-\Delta)^{\alpha/2}$ is defined by

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy, \tag{3}$$

here $P.V.$ means in the Cauchy principal value sense. Equivalently, we can also use the Fourier transform to define the fractional Laplacian:

$$\mathcal{F}((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}u(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform. The fractional Laplacian is well defined in the Schwartz space $\mathcal{S}$, the space of all rapidly decreasing C^∞ functions in $\mathbb{R}^n$. Therefore, the definition of the fractional Laplacian can also be extended further to distributions in the space

$$\mathcal{L}_{\alpha/2} = \left\{ u \in L^1_{loc} : \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha}}dx < \infty \right\}$$

by

$$\langle (-\Delta)^{\alpha/2}u, \varphi \rangle = \int_{\mathbb{R}^n} u(x)(-\Delta)^{\alpha/2}\varphi(x)dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n_+).$$

Throughout this paper, we consider the solution in the following distributional sense.

Definition 1 For any $f \in L^1_{loc}(\mathbb{R}^n_+)$, $u \in \mathcal{L}_{\alpha/2}$ satisfies

$$(-\Delta)^{\alpha/2}u(x) = f(x), \quad x \in \mathbb{R}^n_+,$$

if and only if

$$\int_{\mathbb{R}^n} u(x)(-\Delta)^{\alpha/2}\varphi(x)dx = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \ \forall \varphi \in C_0^\infty(\mathbb{R}^n_+).$$

In recent years, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars. However, the non-locality of the fractional Laplacian makes it difficult to study. L. Caffarelli and L. Silvestre [4] first introduced an extension method to overcome this difficulty, which reduced this nonlocal problem into a local one in higher dimensions. More precisely, for a function $u : \mathbb{R}^n \to \mathbb{R}$, consider the extension $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ that satisfies

$$\begin{cases} div(y^{1-\alpha}\nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x), \end{cases}$$

then

$$(-\Delta)^{\alpha/2}u := -C_{n,\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}.$$

In the present paper, we will prove a nonexistence result for locally bounded nonnegative solutions of problem (2), where $(-\Delta)^{\alpha/2}$ is given by the nonlocal definition (3). Instead of using the extension method of Caffarelli and Silvestre [4], our proof is based on the ideas similar to those from [5, 13] of deriving an integral representation of the solution using the Green Function in the half space, and then applying the method of moving planes in integral forms to the corresponding integral system.

In order to use the method of integral equations to study the semi-linear elliptic system involving the fractional Laplacian, we first establish the equivalence between Eq. 2 and the integral system in $\mathbb{R}^n_+$:

$$\begin{cases} u(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y) v^q(y) dy, \\ v(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y) u^p(y) dy, \end{cases} \tag{4}$$

where

$$G^+_\infty(x,y) = \frac{A_{n,\alpha}}{s^{\frac{n-\alpha}{2}}} \left(1 - \frac{1}{\int_0^\infty \frac{1}{\mu^{\alpha/2}(1+\mu)} d\mu} \int_0^{\frac{s}{t}} \frac{\left(\frac{s-t\mu}{s+t}\right)^{\frac{n-2}{2}}}{\mu^{\alpha/2}(1+\mu)} d\mu \right) \tag{5}$$

is the Green's function (see [5, 19]) for fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ $(0 < \alpha < 2)$ in $\mathbb{R}^n_+$ with Dirichlet boundary conditions (where we use the variables $t = 4x_n y_n$ and $s = |x-y|^2$).

We first establish the following equivalence between Eqs. 2 and 4:

Lemma 2 *Assume that* (u, v) *is a pair of locally bounded nonnegative solutions of problem* (2). *Then* (u, v) *is also a pair of solutions of integral system* (4), *and vice versa.*

It is well known that a Liouville type theorem (nonexistence of solutions in the whole space or on a half space) is very important in establishing a priori estimates for the solutions to a family of equations with the same boundary conditions on either bounded domains in Euclidean spaces or on Riemannian manifolds with boundaries. Using the rescaling method (also called the "blow-up" method) in [17], an equation in a bounded domain will blow up to become another equation in the whole Euclidean space or a half space. With the aid of the corresponding Liouville-type theorem in the Euclidean space $\mathbb{R}^n$ and half space $\mathbb{R}^n_+$ and a contradiction argument, the a priori bound could be derived. The Liouville type theorem for various PDE and IE systems have been extensively studied by many authors (see [1, 2, 5, 7, 11–14, 17, 21, 22, 26, 27] and the references therein). There are also large amounts of literature devoted to investigating the quantitative and qualitative properties of solutions to the IE systems of type (4) and its related PDE systems (see [18, 21, 24, 25] and the references therein).

In [9], the authors developed the method of moving planes of [15, 16] in integral forms and proved under global integrability conditions that in the critical cases every positive entire solutions (u, v) to a IE system involving fractional Laplacian are radially symmetric and monotonic decreasing about some point in the whole space $\mathbb{R}^n$. The first two authors of the current paper proved in [21] the non-existence of nontrivial nonnegative solutions for poly-harmonic systems in the half space $\mathbb{R}^n_+$ with Dirichlet boundary conditions. In [22], P. Wang, J. Zhu and the third author established some Liouville type theorems without the boundedness assumptions of nonnegative solutions to certain classes of poly-harmonic elliptic equations and systems. By deriving an equivalent relationship between the PDE and IE problems, W. Chen, Y. Fang and R. Yang proved in [5] the non-existence of positive solutions to the Dirichlet problem for $(-\Delta)^{\frac{\alpha}{2}} u = u^p$ in $\mathbb{R}^n_+$ in both the critical and subcritical cases. By applying the method of moving spheres to a reformulated problem generated from the Caffarelli-Silvestre extension, M. M. Fall and T. Weth [12] derived the nonexistence of positive continuous solutions to Dirichlet problem for $(-\Delta)^s u = f(x, u)$ on both star-shaped domains in supercritical cases and the half space $\mathbb{R}^n_+$ in subcritical and critical cases. In Fall and Weth [13], under mild assumptions on the nonlinearity f, they also proved the monotonicity and nonexistence of positive bounded solutions to Dirichlet problem for $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n_+$. A. Quaas and A. Xia [26] investigated both the single equation and the PDE system (2) in $\mathbb{R}^n_+$, they derived the nonexistence of positive viscosity bounded solutions to system (2) provided $p, q \in [\frac{n-1}{n-1-\alpha}, \frac{n-1+\alpha}{n-1-\alpha}]$ and satisfy more additional assumptions.

Comparing the results in [12, 13, 26] with our main results on system (2) (Corollary 4), one should observe two essential differences. First, we only assume the local boundedness in Corollary 4. Second, we must restrict the exponents p, q to subcritical and critical cases in Corollary 4, nevertheless, p, q are allowed to go beyond the lower bound $\frac{n-1}{n-1-\alpha}$ in [26].

In this paper, we will study the non-existence of nontrivial nonnegative locally integrable solutions of the IE system (4) by using the method of moving planes [15, 16] (see also [3, 6, 20, 23]) in integral forms initially used in [8, 9]. Our main result is the following theorem.

Theorem 3 *For* $0 < \alpha < 2$ *and* $\frac{n}{n-\alpha} < p, q \leq \frac{n+\alpha}{n-\alpha}$, *assume that* $(u, v) \in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+) \times L_{loc}^{\frac{n(q-1)}{\alpha}}(\mathbb{R}^n_+)$. *If* (u, v) *is a pair of nonnegative solutions of* Eq. 4, *then* $(u, v) = (0, 0)$.

Combining Lemma 2 and Theorem 3, we conclude the following corollary immediately.

Corollary 4 *For* $0 < \alpha < 2$ *and* $\frac{n}{n-\alpha} < p, q \leq \frac{n+\alpha}{n-\alpha}$, *if* (u, v) *is a pair of locally bounded nonnegative solutions of problem* (2), *then* $(u, v) = (0, 0)$.

In order to prove our main result (Theorem 3) under the local integrability assumptions, we have to exploit the Kelvin transform properly. To this end, for $z^0 \in \partial\mathbb{R}^n_+$, we define the Kelvin transform of u and v centered at point z^0 by

$$\bar{u}(x) = \frac{1}{|x - z^0|^{n-\alpha}} u\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right), \quad \bar{v}(x) = \frac{1}{|x - z^0|^{n-\alpha}} v\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right).$$

We only need to discuss two different possibilities. First, if there exists some point $z_0 \in \partial\mathbb{R}^n_+$ such that both $\bar{u}$ and $\bar{v}$ are bounded near z_0, then by applying the method of moving plane in integral forms along the x_n-axis, we can show that u and v are globally integrable and strictly monotone increasing with respect to the variable x_n (see Proposition 9), which yields a contradiction. As to the second case, that is, at least one of $\bar{u}$ and $\bar{v}$ is singular for any $z_0 \in \partial\mathbb{R}^n_+$, we can prove that both u and v only depend on the variable x_n (see Proposition 10), which will also yield a contradiction by making use of the lower bound estimates of the Green's function G^+_∞ (see Eq. 8). In both of these two cases, we can derive the Liouville type theorem (Theorem 3) for IE system (4).

We end this introduction with the following remarks. Different from [5], in the proof of our main result Theorem 3 (see Section 3.4), we derive a more precise estimate (49) by using the lower bound estimates (8) of the Green's function G^+_∞, which will imply the existence of a infinitesimal sequence $\{v^q(x_n^j)(x_n^j)^{\frac{\alpha}{2}+1-\frac{1}{k}}\}_{j=1}^\infty$ (see Eq. 50) that has a positive lower bound (53) (a contradiction). Therefore, we can avoid applying the iteration technique used in [5] and simplify the proof to a large extent. Indeed, our approach also gives a simpler proof in the single equation case considered in [5].

The rest of this paper is arranged as follows. In Section 2, we establish the equivalence between problem (2) and integral system (4). In Section 3, we will first give some preliminary lemmas, then we use the method of moving planes in integral forms and Kelvin transforms to prove Theorem 3, and thus obtain the Liouville type theorem for the IE system (4).

2 The Equivalence

In this section, inspired by the ideas from [5, 13], we will establish the equivalence between PDE system (2) and IE system (4) (Lemma 2).

In order to prove Lemma 2, we need the following two lemmas established in L. Silvestre [28] and W. Chen, C. Li, L. Zhang and T. Cheng [10] respectively.

Lemma 5 ([28]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, if $u \in \mathcal{L}_{\alpha/2}$ and $(-\Delta)^{\frac{\alpha}{2}} u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u \geq 0$ in $\mathbb{R}^n$.*

Lemma 6 ([10]) *For $0 < \alpha < 2$, $u \in \mathcal{L}_{\alpha/2}$. Assume that u satisfies the following equation in the sense of distribution,*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = 0, \quad u(x) \geq 0, & x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, & x \notin \mathbb{R}^n_+. \end{cases}$$

Then either $u \equiv 0$ or

$$\begin{cases} u(x) = C {x_n}^{\alpha/2}, & x \in \mathbb{R}^n_+, \\ u(x) = 0, & x \notin \mathbb{R}^n_+, \end{cases}$$

for some positive condition C.

First, assume (u, v) is a pair of locally bounded nonnegative solutions to Eq. 2, our goal is to show that (u, v) also satisfies (4). Let $P_R = (0, \cdots, 0, R)$ and

$$\begin{cases} \tilde{u}_R(x) = \int_{B_R(P_R)} G_R^+(x, y) v^q(y) dy, \\ \tilde{v}_R(x) = \int_{B_R(P_R)} G_R^+(x, y) u^p(y) dy, \end{cases}$$

where $G_R^+(x, y)$ is the Green's function for $(-\Delta)^{\frac{\alpha}{2}}$ on $B_R(P_R)$.

Then, we can derive

$$\begin{cases} (-\Delta)^{\alpha/2} \tilde{u}_R(x) = v^q(x), & x \in B_R(P_R), \\ (-\Delta)^{\alpha/2} \tilde{v}_R(x) = u^p(x), & x \in B_R(P_R), \\ \tilde{u}_R(x) = \tilde{v}_R(x) = 0, & x \notin B_R(P_R). \end{cases}$$

Let $U_R(x) = u(x) - \tilde{u}_R(x)$ and $V_R(x) = v(x) - \tilde{v}_R(x)$, by Eq. 2, we have

$$\begin{cases} (-\Delta)^{\alpha/2} U_R(x) = 0, & x \in B_R(P_R), \\ (-\Delta)^{\alpha/2} V_R(x) = 0, & x \in B_R(P_R), \\ U_R(x) \geq 0, \quad V_R(x) \geq 0, & x \notin B_R(P_R). \end{cases}$$

By Lemma 5, for any $x \in B_R(P_R)$, we deduce

$$U_R(x) \geq 0, \quad V_R(x) \geq 0.$$

Letting $R \to \infty$, we have

$$\begin{cases} u(x) \geq \int_{\mathbb{R}^n_+} G^+_\infty(x, y) v^q(y) dy, \\ v(x) \geq \int_{\mathbb{R}^n_+} G^+_\infty(x, y) u^p(y) dy. \end{cases}$$

Define

$$\begin{cases} \tilde{u}(x) := \int_{\mathbb{R}^n_+} G^+_\infty(x, y) v^q(y) dy, \\ \tilde{v}(x) := \int_{\mathbb{R}^n_+} G^+_\infty(x, y) u^p(y) dy. \end{cases}$$

Then $(\tilde{u}, \tilde{v})$ is a pair of solution of

$$\begin{cases} (-\Delta)^{\alpha/2} \tilde{u}(x) = v^q(x), x \in \mathbb{R}^n_+, \\ (-\Delta)^{\alpha/2} \tilde{v}(x) = u^p(x), x \in \mathbb{R}^n_+, \\ \tilde{u}(x) = \tilde{v}(x) = 0, \qquad x \notin \mathbb{R}^n_+. \end{cases}$$

Define $U = u - \tilde{u}$ and $V = v - \tilde{v}$, we have

$$\begin{cases} (-\Delta)^{\alpha/2} U(x) = 0, \;\; U(x) \geq 0, \;\; x \in \mathbb{R}^n_+, \\ (-\Delta)^{\alpha/2} V(x) = 0, \;\; V(x) \geq 0, \;\; x \in \mathbb{R}^n_+, \\ U(x) = V(x) = 0, \qquad\qquad\quad x \notin \mathbb{R}^n_+. \end{cases}$$

From Lemma 6, we can deduce that either

$$U(x) = V(x) = 0, \;\; \forall x \in \mathbb{R}^n, \tag{6}$$

or there exist two positive C_1, C_2 such that

$$\begin{cases} U(x) = C_1 {x_n}^{\alpha/2}, \qquad x \in \mathbb{R}^n_+, \\ V(x) = C_2 {x_n}^{\alpha/2}, \qquad x \in \mathbb{R}^n_+, \\ U(x) = V(x) = 0, \qquad x \notin \mathbb{R}^n_+. \end{cases} \tag{7}$$

We can obtain a contradiction in the second case by deriving a lower bound estimates of the Green's function G^+_∞. In fact, by Eq. 5, for sufficiently large s, we derive

$$\begin{aligned} G^+_\infty(x, y) &= \frac{A_{n,\alpha}}{s^{\frac{n-\alpha}{2}}} \left(1 - \frac{1}{\int_0^\infty \frac{1}{\mu^{\alpha/2}(1+\mu)} d\mu} \int_0^{\frac{s}{t}} \frac{(\frac{s-t\mu}{s+t})^{\frac{n-2}{2}}}{\mu^{\alpha/2}(1+\mu)} d\mu \right) \\ &\geq \tilde{A}_{n,\alpha} \frac{1}{s^{\frac{n-\alpha}{2}}} \left(\int_0^\infty \frac{1}{\mu^{\alpha/2}(1+\mu)} d\mu - \int_0^{\frac{s}{t}} \frac{(\frac{s-t\mu}{s+t})^{\frac{n-2}{2}}}{\mu^{\alpha/2}(1+\mu)} d\mu \right) \\ &\geq \tilde{A}_{n,\alpha} \frac{1}{s^{\frac{n-\alpha}{2}}} \int_{\frac{s}{t}}^\infty \frac{1}{\mu^{\alpha/2}(1+\mu)} d\mu \\ &\geq C_{n,\alpha} \frac{t^{\alpha/2}}{s^{n/2}}. \end{aligned} \tag{8}$$

Then, for each fixed x and large enough R, we have

$$\begin{aligned} u(x) \geq \tilde{u}(x) &= \int_{\mathbb{R}^n_+} G^+_\infty(x,y)v^q(y)dy \\ &\geq C_1 \int_{\mathbb{R}^n_+} G^+_\infty(x,y)(y_n^{\alpha/2})^q dy \\ &\geq C_1 \int_{\mathbb{R}^n_+ \setminus B_R(0)} \frac{y_n^{\frac{\alpha(q+1)}{2}}}{|x-y|^n} dy \\ &\geq C_1 \int_R^\infty y_n^{\frac{\alpha(q+1)}{2}} \int_R^\infty \frac{r^{n-2}}{(r^2+|x_n-y_n|^2)^{\frac{n}{2}}} dr dy_n \\ &\geq C_1 \int_R^\infty y_n^{\frac{\alpha(q+1)}{2}-1} dy_n \\ &= \infty. \end{aligned}$$

This contradicts the local boundedness assumption on u, which implies that the second case (7) can not happen. Therefore, we can derive from Eq. 6 that

$$u(x) = \tilde{u}(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y)v^q(y)dy.$$

Similarly, we also have

$$v(x) = \tilde{v}(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y)u^p(y)dy,$$

that is, (u,v) satisfies IE system (4).

On the other hand, if (u,v) is a pair of solutions of integral system (4). Then for any $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^n_+)$, we have

$$\begin{aligned} \langle(-\Delta)^{\alpha/2}u, \phi_1\rangle &= \langle(-\Delta)^{\alpha/2}u, \phi_1\rangle \\ &= \int_{\mathbb{R}^n_+}\left(\int_{\mathbb{R}^n_+} G^+_\infty(x,y)v^q(y)dy\right)(-\Delta)^{\alpha/2}\phi_1(x)dx \\ &= \int_{\mathbb{R}^n_+}\left(\int_{\mathbb{R}^n_+} G^+_\infty(x,y)(-\Delta)^{\alpha/2}\phi_1(x)dx\right)v^q(y)dy \\ &= \int_{\mathbb{R}^n_+}\left(\int_{\mathbb{R}^n_+} \delta(x-y)\phi_1(x)dx\right)v^q(y)dy \\ &= \langle v^q, \phi_1\rangle. \end{aligned}$$

We can also derive

$$\langle(-\Delta)^{\alpha/2}v, \phi_2\rangle = \langle u^p, \phi_2\rangle.$$

Therefore, (u,v) also satisfies (2).

This completes the proof of Lemma 2.

3 Liouville Type Theorem

In this section, we will use the method of moving planes in integral forms and Kelvin transforms to prove Theorem 3, a Liouville type theorem for the IE system (4), which will imply Corollary 4 for the PDE system (2).

Let λ be a positive real number and let the moving plane be

$$T_\lambda = \{x \in \mathbb{R}^n_+ : x_n = \lambda\}.$$

We denote

$$\Sigma_\lambda = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : 0 < x_n < \lambda\}$$

and let

$$x^\lambda = (x_1, x_2, \cdots, 2\lambda - x_n)$$

be the reflection of the point $x = (x_1, x_2, \cdots, x_n)$ about the plane T_λ, and

$$\Sigma^c_\lambda = \mathbb{R}^n_+ \setminus \Sigma_\lambda, \quad \tilde{\Sigma}_\lambda = \{x^\lambda : x \in \Sigma_\lambda\},$$

$$u_\lambda(x) = u(x^\lambda), \quad v_\lambda(x) = v(x^\lambda).$$

To prove the proof of Theorem 3, we need the following lemma proved in [5].

Lemma 7 *(i) For any $x, y \in \Sigma_\lambda$, $x \neq y$, we have*

$$G^+_\infty(x^\lambda, y^\lambda) > \max\{G^+_\infty(x^\lambda, y), G^+_\infty(x, y^\lambda)\}$$

and

$$G^+_\infty(x^\lambda, y^\lambda) - G^+_\infty(x, y) > |G^+_\infty(x^\lambda, y) - G^+_\infty(x, y^\lambda)|.$$

(ii) For any $x \in \Sigma_\lambda$, $y \in \Sigma^c_\lambda$, it holds

$$G^+_\infty(x^\lambda, y) > G^+_\infty(x, y).$$

Next, we need to show the following

Lemma 8 *For any $x \in \Sigma_\lambda$, it holds*

$$u(x) - u_\lambda(x) \leq \int_{\Sigma_\lambda} [G^+_\infty(x^\lambda, y^\lambda) - G^+_\infty(x, y^\lambda)][v^q(y) - v^q_\lambda(y)]dy,$$

$$v(x) - v_\lambda(x) \leq \int_{\Sigma_\lambda} [G^+_\infty(x^\lambda, y^\lambda) - G^+_\infty(x, y^\lambda)][u^p(y) - u^p_\lambda(y)]dy.$$

Proof Since

$$u(x) = \int_{\Sigma_\lambda} G^+_\infty(x, y)v^q(y)dy + \int_{\Sigma_\lambda} G^+_\infty(x, y^\lambda)v^q_\lambda(y)dy + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} G^+_\infty(x, y)v^q(y)dy,$$

$$u(x^\lambda) = \int_{\Sigma_\lambda} G^+_\infty(x^\lambda, y)v^q(y)dy + \int_{\Sigma_\lambda} G^+_\infty(x^\lambda, y^\lambda)v^q_\lambda(y)dy + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} G^+_\infty(x^\lambda, y)v^q(y)dy.$$

By Lemma 7, we have

$$
\begin{aligned}
u(x)-u(x^\lambda) &= \int_{\Sigma_\lambda}[G^+_\infty(x,y)-G^+_\infty(x^\lambda,y)]v^q(y)dy \\
&\quad+\int_{\Sigma_\lambda}[G^+_\infty(x,y^\lambda)-G^+_\infty(x^\lambda,y^\lambda)]v^q_\lambda(y)dy \\
&\quad+\int_{\Sigma^c_\lambda\setminus\tilde{\Sigma}_\lambda}[G^+_\infty(x,y)-G^+_\infty(x^\lambda,y)]v^q(y)dy \\
&\le \int_{\Sigma_\lambda}[G^+_\infty(x,y)-G^+_\infty(x^\lambda,y)]v^q(y)dy \\
&\quad+\int_{\Sigma_\lambda}[G^+_\infty(x,y^\lambda)-G^+_\infty(x^\lambda,y^\lambda)]v^q_\lambda(y)dy \\
&\le \int_{\Sigma_\lambda}[G^+_\infty(x^\lambda,y^\lambda)-G^+_\infty(x,y^\lambda)]v^q(y)dy \\
&\quad-\int_{\Sigma_\lambda}[G^+_\infty(x^\lambda,y^\lambda)-G^+_\infty(x,y^\lambda)]v^q_\lambda(y)dy \\
&= \int_{\Sigma_\lambda}[G^+_\infty(x^\lambda,y^\lambda)-G^+_\infty(x,y^\lambda)][v^q(y)-v^q_\lambda(y)]dy.
\end{aligned}
$$

Similarly, we obtain

$$
v(x)-v_\lambda(x)\le\int_{\Sigma_\lambda}[G^+_\infty(x^\lambda,y^\lambda)-G^+_\infty(x,y^\lambda)][u^p(y)-u^p_\lambda(y)]dy.
$$

□

For clarity of the presentation, we divide this Section into several Sub-sections.

3.1 Kelvin Transform

In virtue of Kelvin transforms, we only need to assume that u and v are locally integrable. We place the centers at boundary $\partial\mathbb{R}^n_+$ to ensure that the half space $\mathbb{R}^n_+$ is invariant under the inversion. For $z^0\in\partial\mathbb{R}^n_+$, let

$$
\bar{u}(x)=\frac{1}{|x-z^0|^{n-\alpha}}u\left(\frac{x-z^0}{|x-z^0|^2}+z^0\right),\quad \bar{v}(x)=\frac{1}{|x-z^0|^{n-\alpha}}v\left(\frac{x-z^0}{|x-z^0|^2}+z^0\right)
$$

be the Kelvin transform of u and v centered at point z^0.

3.2 If Both $\bar{u}(x)$ and $\bar{v}(x)$ are not Singular at Some Point $z_0\in\partial\mathbb{R}^n_+$

If there is a $z^0=(z^0_1,\cdots,z^0_{n-1},0)\in\partial\mathbb{R}^n_+$ such that $\bar{u}(x)$ and $\bar{v}(x)$ are not singular at z^0, then we can deduce

$$
u(x)=O\left(\frac{1}{|x|^{n-\alpha}}\right),\quad v(x)=O\left(\frac{1}{|x|^{n-\alpha}}\right)\quad\text{for } |x| \text{ large.} \tag{9}
$$

Since $u\in L^{\frac{n(p-1)}{\alpha}}_{loc}(\mathbb{R}^n_+)$ and $v\in L^{\frac{n(q-1)}{\alpha}}_{loc}(\mathbb{R}^n_+)$, we have

$$
\int_{\mathbb{R}^n_+}u^{\frac{n(p-1)}{\alpha}}(y)dy<\infty,\quad \int_{\mathbb{R}^n_+}v^{\frac{n(q-1)}{\alpha}}(y)dy<\infty. \tag{10}
$$

In this case, u and v are globally integrable, and we will move the planes in the direction of x_n-axis to show that u and v are monotone increasing in x_n, which will yield a contradiction. The proof consists of two steps.

Step 1 Define

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda : u_\lambda(x) - u(x) < 0\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda : v_\lambda(x) - v(x) < 0\}.$$

For positive λ sufficiently small, we will show that the measure of Σ_λ^u and Σ_λ^v must be zero. In fact, for any $x \in \Sigma_\lambda^u$, by the Mean Value Theorem and Lemma 8, we obtain

$$\begin{aligned}
u(x) - u_\lambda(x) &\leq \int_{\Sigma_\lambda} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][v^q(y) - v_\lambda^q(y)]dyc \\
&\leq \int_{\Sigma_\lambda^v} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][v^q(y) - v_\lambda^q(y)]dy \\
&\quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^v} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][v^q(y) - v_\lambda^q(y)]dy \\
&\leq \int_{\Sigma_\lambda^v} [G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda)][v^q(y) - v_\lambda^q(y)]dy \\
&\leq \int_{\Sigma_\lambda^v} G_\infty^+(x^\lambda, y^\lambda)[v^q(y) - v_\lambda^q(y)]dy \\
&\leq q \int_{\Sigma_\lambda^v} G_\infty^+(x^\lambda, y^\lambda)\varphi_\lambda^{q-1}(y)[v(y) - v_\lambda(y)]dy \\
&\leq q \int_{\Sigma_\lambda^v} G_\infty^+(x^\lambda, y^\lambda)v^{q-1}(y)[v(y) - v_\lambda(y)]dy,
\end{aligned} \tag{11}$$

where $\varphi_\lambda(y)$ is valued between $v_\lambda(y)$ and $v(y)$, therefore on Σ_λ^v, we have

$$0 \leq v_\lambda(y) \leq \varphi_\lambda(y) \leq v(y).$$

Similarly, we have

$$v(x) - v_\lambda(x) \leq p \int_{\Sigma_\lambda^u} G_\infty^+(x^\lambda, y^\lambda)u^{p-1}(y)[u(y) - u_\lambda(y)]dy. \tag{12}$$

By the expression of $G_\infty^+(x, y)$, we have

$$\begin{aligned}
G_\infty^+(x, y) &= \frac{A_{n,\alpha}}{s^{\frac{n-\alpha}{2}}} \left(1 - \frac{1}{\int_0^\infty \frac{1}{\mu^{\alpha/2}(1+\mu)}d\mu} \int_0^{\frac{s}{t}} \frac{(\frac{s-t\mu}{s+t})^{\frac{n-2}{2}}}{\mu^{\alpha/2}(1+\mu)}d\mu\right) \\
&\leq \frac{A_{n,\alpha}}{|x - y|^{n-\alpha}}.
\end{aligned}$$

By Eq. 18, we obtain

$$0 < u(x) - u_\lambda(x) \leq \int_{\Sigma_\lambda^v} \frac{C}{|x - y|^{n-\alpha}} v^{q-1}(y)[v(y) - v_\lambda(y)]dy. \tag{13}$$

Now, we apply Hardy-Littlewood-Sobolev and Hölder inequalities to Eq. 16 and obtain, for some $r > \frac{n}{n-\alpha}$,

$$\begin{aligned} \|u-u_\lambda\|_{L^r(\Sigma_\lambda^u)} &\leq C\|v^{q-1}(v-v_\lambda)\|_{L^{\frac{nr}{n+\alpha r}}(\Sigma_\lambda^v)} \\ &\leq C\|v^{q-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^v)}\|v-v_\lambda\|_{L^r(\Sigma_\lambda^v)}. \end{aligned} \tag{14}$$

Similarly, we can also derive

$$\begin{aligned} \|v-v_\lambda\|_{L^r(\Sigma_\lambda^u)} &\leq C\|u^{p-1}(u-u_\lambda)\|_{L^{\frac{nr}{n+\alpha r}}(\Sigma_\lambda^u)} \\ &\leq C\|u^{p-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^u)}\|u-u_\lambda\|_{L^r(\Sigma_\lambda^u)}. \end{aligned} \tag{15}$$

Combining Eqs. 14 and 15, we have

$$\|u-u_\lambda\|_{L^r(\Sigma_\lambda^u)} \leq C\|v^{q-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^v)}\|u^{p-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^u)}\|u-u_\lambda\|_{L^r(\Sigma_\lambda^u)}. \tag{16}$$

Since $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)$ and $v \in L^{\frac{n(q-1)}{\alpha}}(\mathbb{R}_+^n)$, we can choose sufficiently small positive λ such that

$$C\|v^{q-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^v)}\|u^{p-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^u)} \leq \frac{1}{2}. \tag{17}$$

By inequalities (16) and (17), we obtain

$$\|u-u_\lambda\|_{L^r(\Sigma_\lambda^u)} = 0, \quad \|v-v_\lambda\|_{L^r(\Sigma_\lambda^v)} = 0,$$

and therefore Σ_λ^u and Σ_λ^v must be measure zero. So, for positive λ sufficiently small, we must have

$$u_\lambda(x) \geq u(x), \quad v_\lambda(x) \geq v(x), \quad a.e.\ x \in \Sigma_\lambda. \tag{18}$$

Furthermore, we can deduce from Eqs. 11 and 12 that Eq. 18 also holds for arbitrary $x \in \Sigma_\lambda$.

Step 2 Inequality (18) provides a starting point to move the plane $T_\lambda = \{x \in \mathbb{R}_+^n : x_n = \lambda\}$. Now we start from the neighborhood of $x_n = 0$ and move the plane up as long as Eq. 18 holds.

Define

$$\lambda_0 = \sup\{\lambda : u_\lambda(x) \geq u(x),\ v_\lambda(x) \geq v(x),\ \mu \leq \lambda,\ \forall\ x \in \Sigma_\mu\}.$$

We will prove

$$\lambda_0 = +\infty. \tag{19}$$

Suppose on the contrary that $\lambda_0 < \infty$, we will show that $u(x)$ and $v(x)$ are symmetric about the plane T_{λ_0}, that is

$$u_{\lambda_0}(x) \equiv u(x), \quad v_{\lambda_0}(x) \equiv v(x), \quad \forall\ x \in \Sigma_{\lambda_0}. \tag{20}$$

Otherwise, on Σ_{λ_0},

$$u_{\lambda_0}(x) \geq u(x), \quad v_{\lambda_0}(x) \geq v(x), \text{ but } u_{\lambda_0}(x) \not\equiv u(x) \text{ and } v_{\lambda_0}(x) \not\equiv v(x).$$

We show that the plane can be moved upward further. More precisely, there exists an $\varepsilon > 0$ small enough such that for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$,

$$u_\lambda(x) \geq u(x), \quad v_\lambda(x) \geq v(x), \quad \forall x \in \Sigma_\lambda.$$

By the integrability conditions, we can choose ε sufficiently small so that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$,

$$C\|v^{q-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^v)}\|u^{p-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^u)} \leq \frac{1}{2}. \tag{21}$$

For the continuity of our work, let us postpone the proof of Eq. 21. Now together with Eqs. 14 and 15, we arrive at

$$\|u - u_\lambda\|_{L^r(\Sigma_\lambda^u)} = 0, \quad \|v - v_\lambda\|_{L^r(\Sigma_\lambda^v)} = 0,$$

and therefore Σ_λ^u and Σ_λ^v must be measure zero. Hence, for $\lambda > \lambda_0$ and sufficiently close to λ_0, we can deduce from deduce from Eqs. 11 and 12 that

$$u_\lambda(x) \geq u(x), \;\; v_\lambda(x) \geq v(x), \quad \forall x \in \Sigma_\lambda.$$

This contradicts with the definition of λ_0, therefore Eq. 20 must hold.

By Eq. 20, we derive that the plane $x_n = 2\lambda_0$ is the symmetric image of the boundary $\partial\mathbb{R}^n_+$ with respect to the plane T_{λ_0}, and hence $(u(x), v(x)) = 0$ when x is on the plane $x_n = 2\lambda_0$. This contradicts with our assumption $u(x) > 0$ and $v(x) > 0$. Therefore Eq. 19 must hold.

Now, we prove inequality (21). For any small $\delta > 0$, we can choose R large enough so that

$$\left(\int_{\mathbb{R}^n_+\setminus B_R(0)} u^{\frac{n(p-1)}{\alpha}}(y)dy\right)^{\frac{\alpha}{n}} < \delta, \quad \left(\int_{\mathbb{R}^n_+\setminus B_R(0)} v^{\frac{n(q-1)}{\alpha}}(y)dy\right)^{\frac{\alpha}{n}} < \delta. \tag{22}$$

We fix R and then show that the measure of Σ_λ^u and Σ_λ^v are sufficiently small as λ close to λ_0. First, for any $x \in \Sigma_{\lambda_0}$, we have

$$u_{\lambda_0}(x) - u(x) > 0, \;\; v_{\lambda_0}(x) - v(x) > 0. \tag{23}$$

In fact, from the proof of Lemmas 7 and 8, we have

$$\begin{aligned} u_{\lambda_0}(x) - u(x) \;&\geq \textstyle\int_{\Sigma_{\lambda_0}} [G_\infty^+(x^{\lambda_0}, y^{\lambda_0}) - G_\infty^+(x, y^{\lambda_0})][v_{\lambda_0}^q(y) - v^q(y)]dy \\ &\quad + \textstyle\int_{\Sigma_{\lambda_0}^c\setminus\tilde{\Sigma}_{\lambda_0}} [G_\infty^+(x^{\lambda_0}, y^{\lambda_0}) - G_\infty^+(x, y^{\lambda_0})]v^q(y)dy \\ &\geq \textstyle\int_{\Sigma_{\lambda_0}^c\setminus\tilde{\Sigma}_{\lambda_0}} [G_\infty^+(x^{\lambda_0}, y^{\lambda_0}) - G_\infty^+(x, y^{\lambda_0})]v^q(y)dy. \end{aligned} \tag{24}$$

Similarly,

$$v_{\lambda_0}(x) - v(x) \geq \int_{\Sigma_{\lambda_0}^c\setminus\tilde{\Sigma}_{\lambda_0}} [G_\infty^+(x^{\lambda_0}, y^{\lambda_0}) - G_\infty^+(x, y^{\lambda_0})]u^p(y)dy. \tag{25}$$

If the inequalities in Eq. 23 are wrong, then there exists some point $x^0 \in \Sigma_{\lambda_0}$ such that

$$u_{\lambda_0}(x^0) = u(x^0) \quad \text{or} \quad v_{\lambda_0}(x^0) = v(x^0).$$

Combining this with Eqs. 24 and 25, for any $y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}$, we have

$$u^p(y) = 0 \quad \text{or} \quad v^p(y) = 0.$$

Therefore, we obtain

$$u(y) = 0 \quad \text{or} \quad v(y) = 0, \quad \forall y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}.$$

This is a contraction with our assumption that $u > 0$ and $v > 0$. Therefore Eq. 23 must hold.

For any $\eta > 0$, define

$$E_\eta = \{x \in \Sigma_{\lambda_0} \cap B_R(0) : u_{\lambda_0}(x) - u(x) > \eta\}$$

and

$$F_\eta = \{\Sigma_{\lambda_0} \cap B_R(0)\} \setminus E_\eta.$$

Obviously,

$$\lim_{\eta\to 0} \mu(F_\eta) = 0.$$

For $\lambda > \lambda_0$, let

$$D_\lambda = (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_R(0).$$

Then it is easy to prove that

$$\{\Sigma_\lambda^u \cap B_R(0)\} \subset (\Sigma_\lambda^u \cap E_\eta) \cup F_\eta \cup D_\eta. \tag{26}$$

Apparently, the measure of D_λ is small for λ close to λ_0. We will show that the measure of $\Sigma_\lambda^u \cap E_\eta$ can be sufficiently small as λ close to λ_0. Actually, for any $x \in \Sigma_\lambda^u \cap E_\eta$, we have

$$u_\lambda(x) - u(x) = u_\lambda(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u(x) < 0.$$

Therefore,

$$u_{\lambda_0}(x) - u_\lambda(x) > u_{\lambda_0}(x) - u(x) > \eta.$$

So, we obtain

$$(\Sigma_\lambda^u \cap E_\eta) \subset G_\eta \equiv \{x \in B_R(0) : u_{\lambda_0}(x) - u_\lambda(x) > \eta\}. \tag{27}$$

By the well-known Chebyshev inequality, we have

$$\mu(G_\eta) \le \frac{1}{\eta^{p+1}} \int_{G_\eta} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} dx \le \frac{1}{\eta^{p+1}} \int_{B_R(0)} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} dx. \tag{28}$$

For each fixed η, the right hand side of inequality (28) can be sufficiently small as λ close to λ_0. Therefore, by Eqs. 26 and 27, the measure of $\Sigma_\lambda^u \cap E_\eta$ can be made sufficiently small. Similarly, the measure of $\Sigma_\lambda^u \cap E_\eta$ can be made sufficiently small. Combining this with Eq. 22, we derive Eq. 21.

Now, by Eq. 19, $u(x)$ and $v(x)$ are monotone increasing with respect to x_n.

Proposition 9 *For $\frac{n}{n-\alpha} < p, q \le \frac{n+\alpha}{n-\alpha}$, Assume that both $\bar{u}(x)$ and $\bar{v}(x)$ are not singular at z_0. If $(u, v) \in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n) \times L_{loc}^{\frac{n(q-1)}{\alpha}}(\mathbb{R}_+^n)$ is a pair of nonnegative solutions of* Eq. 4, *then u and v are strictly monotone increasing with respect to the variable x_n.*

3.3 If at Least one of $\bar{u}(x)$ and $\bar{v}(x)$ is Singular for any $z^0 \in \partial\mathbb{R}_+^n$

Without loss of generality, we may assume that both $\bar{u}(x)$ and $\bar{v}(x)$ are singular at z^0. We will prove that $(\bar{u}(x), \bar{u}(x))$ is rotationally symmetric about the line passing through z^0 and parallel to the x_n-axis. For $\forall\varepsilon > 0$, $x \in \mathbb{R}^n_+ \backslash B_\varepsilon(z^0)$, we have

$$
\begin{aligned}
\bar{u}(x) &= \frac{1}{|x-z^0|^{n-2m}} u\left(\frac{x-z^0}{|x-z^0|^2}+z^0\right)\\
&= \frac{1}{|x-z^0|^{n-2m}} \int_{\mathbb{R}^n_+} G^+_\infty\left(\frac{x-z^0}{|x-z^0|^2}+z^0, y\right) v^q(y)dy\\
&= \frac{1}{|x-z^0|^{n-2m}} \int_{\mathbb{R}^n_+} \frac{G^+_\infty(\frac{x-z^0}{|x-z^0|^2}+z^0, \frac{\tilde{y}-z^0}{|\tilde{y}-z^0|^2}+z^0)}{|\tilde{y}-z^0|^{2n}} v^q\left(\frac{\tilde{y}-z^0}{|\tilde{y}-z^0|^2}+z^0\right)d\tilde{y}\\
&= \int_{\mathbb{R}^n_+} \frac{G^+_\infty(\frac{x-z^0}{|x-z^0|^2}+z^0, \frac{\tilde{y}-z^0}{|\tilde{y}-z^0|^2}+z^0)}{|x-z^0|^{n-2m}|\tilde{y}-z^0|^{n-2m}} \left[\frac{1}{|\tilde{y}-z^0|^{n-2m}} v\left(\frac{\tilde{y}-z^0}{|\tilde{y}-z^0|^2}+z^0\right)\right]^q\\
&\quad \cdot \frac{1}{|\tilde{y}-z^0|^{n-\alpha}} d\tilde{y}\\
&= \int_{\mathbb{R}^n_+} G^+_\infty(x,\tilde{y}) \frac{\bar{v}^q(\tilde{y})}{|\tilde{y}-z^0|^{\beta_1}} d\tilde{y}, \qquad (29)
\end{aligned}
$$

similarly,

$$
\bar{v}(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y)(y) \frac{\bar{u}^p(y)}{|y-z^0|^{\beta_2}} dy, \qquad (30)
$$

where $\frac{n}{n-\alpha} < p, q \le \frac{n+\alpha}{n-\alpha}$, $\beta_1 = n+\alpha-q(n-\alpha) \ge 0$ and $\beta_2 = n+\alpha-p(n-\alpha) \ge 0$.

We will discuss the following two different cases separately.

(i) Critical case: $p = q = \frac{n+\alpha}{n-\alpha}$. If $(u(x), v(x))$ is a pair of solutions of

$$
\begin{cases} u(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y) v^{\frac{n+\alpha}{n-\alpha}}(y)dy,\\ v(x) = \int_{\mathbb{R}^n_+} G^+_\infty(x,y) u^{\frac{n+\alpha}{n-\alpha}}(y)dy, \end{cases} \qquad (31)
$$

then $(\bar{u}(x), \bar{v}(x))$ is also a pair of solutions of Eq. 31. Since $u \in L^{\frac{2n}{n-\alpha}}_{loc}(\mathbb{R}^n_+)$ and $v \in L^{\frac{2n}{n-\alpha}}_{loc}(\mathbb{R}^n_+)$, for any domain Ω that is a positive distance away from z_0, we have

$$
\int_\Omega \bar{u}^{\frac{n+\alpha}{n-\alpha}}(x)dx < \infty, \quad \int_\Omega \bar{v}^{\frac{n+\alpha}{n-\alpha}}(x)dx < \infty. \qquad (32)
$$

Now, we apply method of moving planes to $(\bar{u}(x), \bar{v}(x))$.

In this case, for a given real number λ, we redefine

$$
\hat{\Sigma}_\lambda = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ : x_1 < \lambda\}, \quad \hat{T}_\lambda = \{x \in \mathbb{R}^n_+ : x_1 = \lambda\},
$$

and let

$$
x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n).
$$

For $x, y \in \hat{\Sigma}_\lambda$, $x \ne y$, we have

$$
G^+_\infty(x,y) = G^+_\infty(x^\lambda, y^\lambda) > G^+_\infty(x, y^\lambda) = G^+_\infty(x^\lambda, y). \qquad (33)
$$

Obviously, we have

$$
\bar{u}(x) = \int_{\hat{\Sigma}_\lambda} G^+_\infty(x,y)\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)dy + \int_{\hat{\Sigma}_\lambda} G^+_\infty(x,y^\lambda)\bar{v}_\lambda^{\frac{n+\alpha}{n-\alpha}}(y)dy, \qquad (34)
$$

$$
\bar{u}(x^\lambda) = \int_{\hat{\Sigma}_\lambda} G^+_\infty(x^\lambda,y)\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)dy + \int_{\hat{\Sigma}_\lambda} G^+_\infty(x^\lambda,y^\lambda)\bar{v}_\lambda^{\frac{n+\alpha}{n-\alpha}}(y)dy. \qquad (35)
$$

By Eqs. 33, 34 and 35, it is easy to see

$$\begin{aligned}\bar{u}(x)-\bar{u}(x^{\lambda}) &= \int_{\hat{\Sigma}_{\lambda}}[G_{\infty}^{+}(x,y)-G_{\infty}^{+}(x^{\lambda},y)]\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)dy \\ &\quad+\int_{\hat{\Sigma}_{\lambda}}[G_{\infty}^{+}(x,y^{\lambda})-G_{\infty}^{+}(x^{\lambda},y^{\lambda})]\bar{v}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)dy \\ &= \int_{\hat{\Sigma}_{\lambda}}[G_{\infty}^{+}(x,y)-G_{\infty}^{+}(x^{\lambda},y)][\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)-\bar{v}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)]dy. \end{aligned} \tag{36}$$

Similarly,

$$\bar{v}(x)-\bar{v}(x^{\lambda}) = \int_{\hat{\Sigma}_{\lambda}}[G_{\infty}^{+}(x,y)-G_{\infty}^{+}(x^{\lambda},y)][\bar{u}^{\frac{n+\alpha}{n-\alpha}}(y)-\bar{u}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)]dy. \tag{37}$$

In the sequel, the proof will consist of two steps. We will move the plane $\hat{T}_{\lambda}$ along the direction of x_1-axis until $\lambda = z_1^0$, x_1-axis can be chosen any direction, we can show that the solutions $\bar{u}(x)$ and $\bar{v}(x)$ are rotationally symmetric about the line passing through z^0 and parallel to x_n-axis.

Step 1 In this step, we will show that for λ sufficiently negative, and $\varepsilon > 0$ small enough,

$$\bar{u}_{\lambda}(x)-\bar{u}(x)\geq 0,\quad \bar{v}_{\lambda}(x)-\bar{v}(x)\geq 0,\quad \text{a.e. } x\in\hat{\Sigma}_{\lambda}\setminus B_{\varepsilon}((z^0)^{\lambda}). \tag{38}$$

Define

$$\hat{\Sigma}_{\lambda}^{u}=\{x\in\hat{\Sigma}_{\lambda}\setminus B_{\varepsilon}((z^0)^{\lambda}) : \bar{u}_{\lambda}(x)-\bar{u}(x)<0\},\quad \hat{\Sigma}_{\lambda}^{v}=\{x\in\hat{\Sigma}_{\lambda}\setminus B_{\varepsilon}((z^0)^{\lambda}) : \bar{v}_{\lambda}(x)-\bar{v}(x)<0\}.$$

In fact, for $x\in\Sigma_{\lambda}^{u}$, by Eq. 36 and the Mean Value Theorem, we obtain

$$\begin{aligned}\bar{u}(x)-\bar{u}_{\lambda}(x) &= \int_{\hat{\Sigma}_{\lambda}^{v}}[G_{\infty}^{+}(x,y)-G_{\infty}^{+}(x^{\lambda},y)][\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)-\bar{v}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)]dy \\ &\quad+\int_{\hat{\Sigma}_{\lambda}\setminus\hat{\Sigma}_{\lambda}^{v}}[G_{\infty}^{+}(x,y)-G_{\infty}^{+}(x^{\lambda},y)][\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)-\bar{v}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)]dy \\ &\leq \int_{\hat{\Sigma}_{\lambda}^{v}}[G_{\infty}^{+}(x,y)-G_{\infty}^{+}(x^{\lambda},y)][\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)-\bar{v}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)]dy \\ &\leq \int_{\hat{\Sigma}_{\lambda}^{v}}G_{\infty}^{+}(x,y)[\bar{v}^{\frac{n+\alpha}{n-\alpha}}(y)-\bar{v}_{\lambda}^{\frac{n+\alpha}{n-\alpha}}(y)]dy \\ &\leq \int_{\hat{\Sigma}_{\lambda}^{v}}\frac{C}{|x-y|^{n-\alpha}}\bar{v}^{\frac{2\alpha}{n-\alpha}}(y)(\bar{v}(y)-\bar{v}_{\lambda}(y))dy. \end{aligned} \tag{39}$$

Similarly,

$$\bar{v}(x)-\bar{v}_{\lambda}(x)\leq\int_{\hat{\Sigma}_{\lambda}^{u}}\frac{C}{|x-y|^{n-\alpha}}\bar{u}^{\frac{2\alpha}{n-\alpha}}(y)(\bar{u}(y)-\bar{u}_{\lambda}(y))dy. \tag{40}$$

We apply the Hardy-Littlewood-Sobolev inequality and Hölder inequality to Eqs. 39 and 40 and obtain, for any $r>\frac{n}{n-\alpha}$,

$$\begin{aligned}\|\bar{u}-\bar{u}_{\lambda}\|_{L^{r}(\hat{\Sigma}_{\lambda}^{u})} &\leq C\|\bar{v}^{\frac{2\alpha}{n-\alpha}}(\bar{v}-\bar{v}_{\lambda})\|_{L^{\frac{nr}{n+\alpha r}}(\hat{\Sigma}_{\lambda}^{v})} \\ &\leq C\|\bar{v}^{\frac{2\alpha}{n-\alpha}}\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}_{\lambda}^{v})}\|\bar{v}-\bar{v}_{\lambda}\|_{L^{r}(\hat{\Sigma}_{\lambda}^{v})} \end{aligned} \tag{41}$$

and

$$
\begin{aligned}
\|\bar{v}-\bar{v}_\lambda\|_{L^r(\hat{\Sigma}^v_\lambda)} &\le C\|\bar{u}^{\frac{2\alpha}{n-\alpha}}(\bar{u}-\bar{u}_\lambda)\|_{L^{\frac{nr}{n+\alpha r}}(\hat{\Sigma}^u_\lambda)} \\
&\le C\|\bar{u}^{\frac{2\alpha}{n-\alpha}}\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}^u_\lambda)}\|\bar{u}-\bar{u}_\lambda\|_{L^r(\hat{\Sigma}^u_\lambda)}. \qquad (42)
\end{aligned}
$$

By Eq. 32, we can choose N sufficiently large such that for $\lambda \le -N$,

$$
C\|\bar{u}^{\frac{2\alpha}{n-\alpha}}\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}^u_\lambda)} < \frac{1}{2}, \quad C\|\bar{v}^{\frac{2\alpha}{n-\alpha}}\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}^v_\lambda)} < \frac{1}{2}. \qquad (43)
$$

Now inequalities (41), (42) and (43) imply

$$
\|\bar{u}-\bar{u}_\lambda\|_{L^r(\hat{\Sigma}^u_\lambda)} = 0, \quad \|\bar{v}-\bar{v}_\lambda\|_{L^r(\hat{\Sigma}^v_\lambda)} = 0,
$$

and therefore $\hat{\Sigma}^u_\lambda$ and $\hat{\Sigma}^v_\lambda$ must be measure zero. Thus we derive Eq. 38, moreover, Eqs. 39 and 40 will yield that Eq. 38 also holds for any $x \in \hat{\Sigma}_\lambda \setminus B_\varepsilon((z^0)^\lambda)$.

Step 2 We now move the plane $\hat{T}_\lambda$ continuously toward the right as long as inequality (38) holds to its limiting position. Define

$$
\lambda_0 = \sup\{\lambda \le z^0_1 : \bar{u}_\lambda(x) \ge \bar{u}(x),\ \bar{v}_\lambda(x) \ge \bar{v}(x),\ \mu \le \lambda,\ \forall x \in \hat{\Sigma}_\mu\}.
$$

We will prove that $\lambda_0 = z^0_1$. On the contrary, suppose that $\lambda_0 < z^0_1$. We will show that $\bar{u}(x)$ and $\bar{v}(x)$) are rotationally symmetric about $\hat{T}_{\lambda_0}$, that is

$$
\bar{u}_{\lambda_0}(x) \equiv \bar{u}(x), \quad \bar{v}_{\lambda_0}(x) \equiv \bar{v}(x), \quad \forall x \in \hat{\Sigma}_{\lambda_0}\setminus B_\varepsilon((z^0)^{\lambda_0}). \qquad (44)
$$

Otherwise, on $\hat{\Sigma}_{\lambda_0}\setminus B_\varepsilon((z^0)^{\lambda_0})$,

$$
\bar{u}_{\lambda_0}(x) \ge \bar{u}(x), \quad \bar{v}_{\lambda_0}(x) \ge \bar{v}(x), \text{ but } \bar{u}_{\lambda_0}(x) \not\equiv \bar{u}(x) \text{ and } \bar{v}_{\lambda_0}(x) \not\equiv \bar{v}(x).
$$

We show that the plane can be moved further to the right. More precisely, there exists an $\varepsilon > 0$ such that for any λ in $[\lambda_0, \lambda_0+\varepsilon)$,

$$
\bar{u}_\lambda(x) \ge \bar{u}(x), \quad \bar{v}_\lambda(x) \ge \bar{v}(x), \quad \forall x \in \hat{\Sigma}_\lambda\setminus B_\varepsilon((z^0)^\lambda). \qquad (45)
$$

The proof is similar to Step 2 in Case 1. We only need to use $\hat{\Sigma}_\lambda\setminus B_\varepsilon((z^0)^\lambda)$ instead of Σ_λ and $\hat{\Sigma}_{\lambda_0}\setminus B_\varepsilon((z^0)^{\lambda_0})$ instead of Σ_{λ_0}. Thus, Eq. 45 contradicts with the definition of λ_0. Therefore, Eq. 44 must hold. That is, if $\lambda_0 < z^0_1$, for any $\varepsilon > 0$,

$$
\bar{u}_{\lambda_0}(x) \equiv \bar{u}(x), \quad \forall x \in \hat{\Sigma}_{\lambda_0}\setminus B_\varepsilon((z^0)^{\lambda_0}).
$$

Since $\bar{u}$ is singular at z^0, $\bar{u}$ must be singular at $(z^0)^\lambda$. This is impossible. So it is easy to see $\lambda_0 = z^0_1$.

In this situation, for any two points X^1 and X^2, with $X^i = (x^i, x_n) \in \mathbb{R}^{n-1}\times[0,\infty)$, $i = 1, 2$. Let z^0 be the projection of the midpoint $X^0 = \frac{X^1+X^2}{2}$ on $\partial\mathbb{R}^n_+$. Set $Y^i = \frac{X^i-z^0}{|X^i-z^0|^2} + z^0$, $i = 1, 2$. From the above discussions, it is also easy to see $\bar{u}(Y^1) = \bar{u}(Y^2)$, and hence $u(X^1) = u(X^2)$. This implies that $u(x)$ only depends on the x_n-variable. Similarly, we can also deduce that $v(x)$ only depends on the x_n-variable.

(ii) Subcritical cases: $\frac{n}{n-\alpha} < p, q \le \frac{n+\alpha}{n-\alpha}$ and at least one of p, q is not equal to $\frac{n+\alpha}{n-\alpha}$. Since $u \in L^{\frac{n(p-1)}{\alpha}}_{loc}(\mathbb{R}^n_+)$ and $v \in L^{\frac{n(q-1)}{\alpha}}_{loc}(\mathbb{R}^n_+)$, for any domain Ω that is positive distance away from z^0, we have

$$\int_{\Omega}\left(\frac{\bar{u}^{p-1}(y)}{|y-z^0|^{\beta_2}}\right)^{\frac{n}{\alpha}}dy<\infty,\quad \int_{\Omega}\left(\frac{\bar{v}^{q-1}(y)}{|y-z^0|^{\beta_1}}\right)^{\frac{n}{\alpha}}dy<\infty, \tag{46}$$

where $\beta_1=n+\alpha-q(n-\alpha)\geq 0$, $\beta_2=n+\alpha-p(n-\alpha)\geq 0$.

By Eq. 29, we have

$$\bar{u}(x)=\int_{\hat{\Sigma}_\lambda}G_\infty^+(x,y)\frac{\bar{v}^q(y)}{|y-z^0|^{\beta_1}}dy+\int_{\hat{\Sigma}_\lambda}G_\infty^+(x,y^\lambda)\frac{\bar{v}_\lambda^q(y)}{|y^\lambda-z^0|^{\beta_1}}dy,$$

then,

$$\bar{u}(x^\lambda)=\int_{\hat{\Sigma}_\lambda}G_\infty^+(x^\lambda,y)\frac{\bar{v}^q(y)}{|y-z^0|^{\beta_1}}dy+\int_{\hat{\Sigma}_\lambda}G_\infty^+(x^\lambda,y^\lambda)\frac{\bar{v}_\lambda^q(y)}{|y^\lambda-z^0|^{\beta_1}}dy.$$

As a consequence, we obtain

$$\begin{aligned}\bar{u}(x)-\bar{u}(x^\lambda)&=\int_{\hat{\Sigma}_\lambda}(G_\infty^+(x,y)-G_\infty^+(x^\lambda,y))\frac{\bar{v}^q(y)}{|y-z^0|^{\beta_1}}dy\\&\quad+\int_{\hat{\Sigma}_\lambda}(G_\infty^+(x,y^\lambda)-G_\infty^+(x^\lambda,y^\lambda))\frac{\bar{v}_\lambda^q(y)}{|y^\lambda-z^0|^{\beta_1}}dy\\&=\int_{\hat{\Sigma}_\lambda}(G_\infty^+(x,y)-G_\infty^+(x^\lambda,y))\left(\frac{\bar{v}^q(y)}{|y-z^0|^{\beta_1}}-\frac{\bar{v}_\lambda^q(y)}{|y^\lambda-z^0|^{\beta_1}}\right)dy.\end{aligned} \tag{47}$$

Similar to the critical case $p=q=\frac{n+\alpha}{n-\alpha}$, the proof for the subcritical cases will also be divided into two steps. We can also prove that $\bar{u}$ and $\bar{v}$ are rotationally symmetric about the line passing through z^0 and parallel to the x_n-axis, as a consequence, $u(x)$ and $v(x)$ only depend on the x_n-variable.

In a word, we have reached the following conclusions.

Proposition 10 *For $\frac{n}{n-\alpha}<p,q\leq\frac{n+\alpha}{n-\alpha}$, assume that at least one of $\bar{u}(x)$ and $\bar{v}(x)$ is singular at any $z^0=(z_1^0,\cdots,z_{n-1}^0,0)\in\partial\mathbb{R}_+^n$. If $(u,v)\in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)\times L_{loc}^{\frac{n(q-1)}{\alpha}}(\mathbb{R}_+^n)$ is a pair of nonnegative solutions of* Eq. 4*, then u and v depend only on the variable x_n.*

3.4 The Proof of Theorem 3

In this subsection, we will prove that the nonnegative solutions $(u,v)\equiv(0,0)$. If we assume that both $\bar{u}(x)$ and $\bar{v}(x)$ are not singular at some point $z_0\in\partial\mathbb{R}_+^n$, then Proposition 9 will yield a contradiction with the global integrability estimates (9). In what follows, we only need to consider the second possibility. In this situation, we can deduce from Proposition 10 that u and v depend only on the variable x_n. For $x=(x',x_n)$, $y=(y',y_n)\in\mathbb{R}^{n-1}\times[0,+\infty)$, we assume $(u(x),v(x))=(u(x_n),v(x_n))$ is a pair of solutions of

$$\begin{cases}u(x)=\int_{\mathbb{R}_+^n}G_\infty^+(x,y)v^q(y)dy,\\ v(x)=\int_{\mathbb{R}_+^n}G_\infty^+(x,y)u^p(y)dy,\end{cases} \tag{48}$$

where

$$G_{\infty}^{+}(x,y)=\frac{A_{n,\alpha}}{s^{\frac{n-\alpha}{2}}}\left(1-\frac{1}{\int_0^{\infty}\frac{1}{\mu^{\alpha/2}(1+\mu)}d\mu}\int_0^{\frac{s}{t}}\frac{(\frac{s-t\mu}{s+t})^{\frac{n-2}{2}}}{\mu^{\alpha/2}(1+\mu)}d\mu\right)$$

is the Green's function for fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ in $\mathbb{R}_+^n$ with Dirichlet boundary conditions (the variables $t=4x_ny_n$ and $s=|x-y|^2$).

For each fixed $x\in\mathbb{R}_+^n$, choosing R large enough and letting $k=\frac{2}{2-\alpha}\left[\frac{2+\alpha}{2-\alpha}-\frac{n+\alpha}{n-\alpha}\right]^{-1}$, by Eqs. 8 and 48, we can derive that

$$\begin{aligned}
+\infty &> u(x)=u(x_n)\\
&=\int_0^{\infty}v^q(y_n)\int_{\mathbb{R}^{n-1}}G_{\infty}^{+}(x,y)dy'dy_n\\
&\geq\int_{R^k}^{\infty}v^q(y_n)\int_{\mathbb{R}^{n-1}\setminus B_R(0)}G_{\infty}^{+}(x,y)dy'dy_n\\
&\geq C_{n,\alpha}x_n^{\alpha/2}\int_{R^k}^{\infty}v^q(y_n)y_n^{\alpha/2}\int_{\mathbb{R}^{n-1}\setminus B_R(0)}\frac{1}{|x-y|^n}dy'dy_n\\
&\geq C_{n,\alpha}x_n^{\alpha/2}\int_{R^k}^{\infty}v^q(y_n)y_n^{\alpha/2}\int_R^{\infty}\frac{r^{n-2}}{(r^2+|x_n-y_n|^2)^{\frac{n}{2}}}drdy_n\\
&\geq C_{n,\alpha}x_n^{\alpha/2}\int_{R^k}^{\infty}v^q(y_n)y_n^{\alpha/2}\frac{1}{|x_n-y_n|}\int_{R/|x_n-y_n|}^{\infty}\frac{\tilde{r}^{n-2}}{(\tilde{r}^2+1)^{\frac{n}{2}}}d\tilde{r}dy_n\\
&\geq C_{n,\alpha}x_n^{\alpha/2}\int_{R^k}^{\infty}v^q(y_n)y_n^{\frac{\alpha}{2}}\frac{1}{R}dy_n\geq C_{n,\alpha}x_n^{\alpha/2}\int_{R^k}^{\infty}v^q(y_n)y_n^{\frac{\alpha}{2}-\frac{1}{k}}dy_n,
\end{aligned}\tag{49}$$

where $C_{n,\alpha}$ is a positive constant depending on n,α that may change from line to line. It follows from Eq. 49 that there exists a sequence $\{x_n^j\}$ such that

$$v^q(x_n^j)(x_n^j)^{\frac{\alpha}{2}+1-\frac{1}{k}}\to 0,\ \text{ as } x_n^j\to\infty.\tag{50}$$

Similar to Eq. 49, we can also deduce the following estimate:

$$\begin{aligned}
v(x_n^j) &= \int_0^{\infty}u^p(y_n)\int_{\mathbb{R}^{n-1}}G_{\infty}^{+}(x^j,y)dy'dy_n\\
&\geq C_{n,\alpha}(x_n^j)^{\alpha/2}\int_0^{\infty}u^p(y_n)y_n^{\alpha/2}\int_{\mathbb{R}^{n-1}}\frac{1}{|x^j-y|^n}dy'dy_n\\
&\geq C_{n,\alpha}(x_n^j)^{\alpha/2}\int_0^{\infty}u^p(y_n)y_n^{\alpha/2}\int_0^{\infty}\frac{r^{n-2}}{(r^2+|x_n^j-y_n|^2)^{\frac{n}{2}}}drdy_n\\
&\geq C_{n,\alpha}(x_n^j)^{\alpha/2}\int_0^{\infty}u^p(y_n)y_n^{\alpha/2}\frac{1}{|x_n^j-y_n|}\int_0^{\infty}\frac{\tilde{r}^{n-2}}{(\tilde{r}^2+1)^{\frac{n}{2}}}d\tilde{r}dy_n\\
&\geq C_{n,\alpha}\int_0^{\infty}u^p(y_n)\frac{(x_n^j)^{\alpha/2}y_n^{\alpha/2}}{|x_n^j-y_n|}dy_n.
\end{aligned}\tag{51}$$

Therefore, by Eq. 51, for $j \geq J_0$ large enough, we have

$$\begin{aligned} v(x_n^j) &\geq C_{n,\alpha} \int_0^1 u^p(y_n) \frac{(x_n^j)^{\alpha/2} y_n^{\alpha/2}}{|x_n^j - y_n|} dy_n \\ &\geq C_{n,\alpha} (x_n^j)^{\frac{\alpha}{2}-1} \int_0^1 u^p(y_n) y_n^{\alpha/2} dy_n \\ &\geq C_{n,\alpha,p} (x_n^j)^{\frac{\alpha}{2}-1}. \end{aligned} \tag{52}$$

Recall that $k = \frac{2}{2-\alpha}\left[\frac{2+\alpha}{2-\alpha} - \frac{n+\alpha}{n-\alpha}\right]^{-1}$, one can verify that $\frac{\alpha}{2}+1-\frac{1}{k}+\max\{p,q\}\left(\frac{\alpha}{2}-1\right) \geq 0$ since $p, q \leq \frac{n+\alpha}{n-\alpha}$, thus we can deduce from Eq. 52 that, for $j \geq J_0$ large enough,

$$v^q(x_n^j)(x_n^j)^{\frac{\alpha}{2}+1-\frac{1}{k}} \geq C_{n,p,q,\alpha}(x_n^j)^{\frac{\alpha}{2}+1-\frac{1}{k}+q(\frac{\alpha}{2}-1)} \geq C_{n,p,q,\alpha} > 0, \tag{53}$$

which contradicts with Eq. 50. So there is no nontrivial nonnegative solution of Eq. 48.

This completes the proof of Theorem 3.

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