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Characterizations of second order Sobolev spaces ☆

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ABSTRACT

Second order Sobolev spaces are important in applications to partial differential equations and geometric analysis, in particular to equations such as the bi-Laplacian. The main purpose of this paper is to establish some new characterizations of the second order Sobolev spaces $W^{2,p}(\mathbb{R}^N)$ in Euclidean spaces. We will present here several types of characterizations: by second order differences, by the Taylor remainder of first order and by the differences of the first order gradient. Such characterizations are inspired by the works of Bourgain et al. (2001) and Nguyen (2006, 2008) on characterizations of first order Sobolev spaces in the Euclidean space.

1. Introduction

The classical definition of Sobolev space $W^{k,p}(\Omega)$ is as follows:

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ \forall |\alpha| \le k\}.$$

Here, α is a multi-index and $D^\alpha u$ is the derivative in the weak sense, Ω is an open set in $\mathbb{R}^N$ and $1 \le p \le \infty$. Moreover, in [28], the fractional Sobolev space is defined, here k is not a natural number. Since the theory of Sobolev spaces can be applied in many branches of modern mathematics, such as harmonic analysis, complex analysis, differential geometry and geometric analysis, partial differential equations, etc., there has been a substantial effort to characterize Sobolev spaces in different settings in various ways (see e.g., [16,14,12,11,15,18], etc.). However, even in the Euclidean spaces, the difficulties appear because the partial derivatives for the fractional Sobolev spaces are in a suitable weak sense. Gagliardo used the semi-norm in his paper [13]

$$|g|_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|g(x) - g(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}, \quad p > 1,$$

to characterize functions in $W^{s,p}$. However, when $s \to 1^-$, we have that $|g|_{W^{s,p}(\Omega)}$ does not converge to

$$|g|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla g(x)|^p \, dx \right)^{1/p}.$$

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In order to study this situation, Bourgain, Brezis and Mironescu established a new characterization of Sobolev spaces in [5]. Indeed, they proved that

Theorem A (*Bourgain, Brezis and Mironescu, [5]*). *Let $g \in L^p\left(\mathbb{R}^N\right)$, $1 < p < \infty$. Then $g \in W^{1,p}\left(\mathbb{R}^N\right)$ iff*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dxdy \le C, \quad \forall n \ge 1,$$

for some constant $C > 0$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dxdy = K_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx.$$

Here

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p\,d\sigma$$

for any $e \in \mathbb{S}^{N-1}$ and $d\sigma$ is the surface measure on $\mathbb{S}^{N-1}$. Here $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative radial mollifiers satisfying

$$\lim_{n\to\infty}\int_\tau^\infty \rho_n(r)\,r^{N-1}dr = 0, \quad \forall \tau > 0,$$

$$\lim_{n\to\infty}\int_0^\infty \rho_n(r)\,r^{N-1}dr = 1.$$

Theorem A has been extended to high order case by Bojarski, Ihnatsyeva and Kinnunen [3] using the high order Taylor remainder and by Borghol [4] using high order differences. We also mention related characterization of Sobolev spaces in [2,6,7,17,23,26,27].

We note here that as a consequence of Theorem A, we can characterize the Sobolev space $W^{1,p}(\mathbb{R}^N)$ as follows: Let $g \in L^p\left(\mathbb{R}^N\right)$, $1 < p < \infty$. Then $g \in W^{1,p}\left(\mathbb{R}^N\right)$ iff

$$\sup_{0<\delta<1}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\|x-y|<\delta}} \frac{|g(x)-g(y)|^p}{\delta^{N+p}}dxdy < \infty. \tag{1.1}$$

Recently, Nguyen [24] established some new characterizations of the Sobolev space $W^{1,p}(\mathbb{R}^N)$ which are closely related to Theorem A. More precisely, he used the dual form of (1.1) and proved the following results:

Theorem B (*H.M. Nguyen, [24]*). *Let $1 < p < \infty$. Then the following hold:*

(a) *Let $g \in W^{1,p}(\mathbb{R}^N)$. Then there exists a positive constant $C_{N,p}$ depending only on N and p such that*

$$\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\|g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx, \quad \forall\delta > 0,\ \forall g \in W^{1,p}\left(\mathbb{R}^N\right).$$

(b) *If $g \in L^p\left(\mathbb{R}^N\right)$ satisfies*

$$\sup_{0<\delta<1}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\|g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}dxdy < \infty,$$

then $g \in W^{1,p}\left(\mathbb{R}^N\right)$.

(c) *Moreover, for any $g \in W^{1,p}(\mathbb{R}^N)$,*

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\|g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}dxdy = \frac{1}{p}K_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx.$$

The works of Bourgain, Brezis and Mironescu [5] and H.M. Nguyen [24,25] on characterizations of first order Sobolev spaces in the Euclidean space were also investigated on the Heisenberg groups and Carnot groups by Barbieri [1] and the authors [10,9].

Motivated by Theorem B, it is natural to ask if the characterizations of type of Theorem B of H.M. Nguyen can be given for higher order Sobolev spaces. This is exactly the main purpose of this paper.

Inspired by the above two theorems (Theorems A and B), we will first establish in this paper characterizations of the second order Sobolev spaces in Euclidean spaces in the spirit of the work by H.M. Nguyen [24] using the method of first order differences. Here, we choose two different approaches to characterize the second order Sobolev spaces $W^{2,p}(\mathbb{R}^N)$: by the second order differences and by the Taylor remainder of first order. Our methods and results are in the spirit of the work of [24], namely using the mean value theorem, Hardy–Littlewood maximal functions, rotations in the Euclidean spaces, etc. Nevertheless, the situation in second order case is more complicated than in the first order case. Therefore, additional care is needed to handle our second order case.

We mention in passing that other type of characterizations of high order Sobolev spaces have been given using high order Poincaré inequalities on Euclidean spaces and Carnot (stratified) groups by Liu, Lu and Wheeden [18]. Such high order Poincaré inequalities have been extensively studied on stratified groups by the third author with his collaborators [8,19–22]. Nevertheless, those characterizations are in quite different nature than what we offer here.

The first purpose of this paper is to prove the following estimates for functions in the Sobolev spaces $W^{2,p}(\mathbb{R}^N)$.

Theorem 1.1. *Let $g \in W^{2,p}\left(\mathbb{R}^N\right),\ 1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that*

(1)

$$\underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N} |\Delta g|^p\, dx, \quad \forall \delta > 0.$$

(2)

$$\lim_{\delta\to 0} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} \left|D^2 g\,(x)\,(\sigma,\sigma)\right|^p dxd\sigma.$$

(3)

$$\sup_{0<\varepsilon<1} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+2p}}dxdy$$
$$\le C_{N,p}\int_{\mathbb{R}^N} |\Delta g|^p\, dx.$$

(4)

$$\lim_{\varepsilon\to 0} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} \left|D^2 g\,(x)\,(\sigma,\sigma)\right|^p dxd\sigma.$$

Here we have used the notation

$$\left|D^2 g\,(x)\,(\sigma,\sigma)\right| = \sum_{1\le i_1,i_2\le N} \sigma_{i_1}\sigma_{i_2}\frac{\partial^2 g}{\partial x_{i_1}\partial x_{i_2}}(x)\,.$$

We will use this notation frequently throughout this paper.

Theorem 1.2. *Let $g \in W^{2,p}(\mathbb{R}^N),\ 1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that*

(1)

$$\underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N} |\Delta g|^p dx, \quad \forall \delta > 0.$$

(2)

$$\lim_{\delta\to 0} \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dxd\sigma.$$

(3)

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1} \frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{1}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx.$$

(4)

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1} \frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.$$

Using Theorems 1.1 and 1.2, we can set up the new characterizations of the Sobolev space $W^{2,p}(\mathbb{R}^N)$ using the method of second order differences and the Taylor remainder of first order which are our main aims of this paper. Indeed, we prove the following two theorems:

Theorem 1.3. *Let $g \in A^p(\mathbb{R}^N)$, $1<p<\infty$ where $A^p(\mathbb{R}^N)$ is the set of all $g \in L^p\left(\mathbb{R}^N\right)$ such that $\exists\{g_n\}$ and $A(g)>0$: $\|g_n\|_p \le A(g)$; $\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right| \le A(g)$; $\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right| \le A(g)\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|$ a.e. $x,y\in\mathbb{R}^N$ and $g_n \to g$ a.e. Then the following are equivalent:*

(1) $g \in W^{2,p}\left(\mathbb{R}^N\right)$.

(2)

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy < \infty.$$

(3)

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1} \frac{1}{|x-y|^{N+2p}}dxdy < \infty.$$

Theorem 1.4. *Let $g\in B^p(\mathbb{R}^N)$, $1<p<\infty$ where $B^p(\mathbb{R}^N)$ is the set of all $g\in L^p\left(\mathbb{R}^N\right)$ such that $\exists\{g_n\}$ and $B(g)>0$: $\|g_n\|_p \le B(g)$; $|g_n(x)-g_n(y)-\nabla g_n(y)(x-y)| \le B(g)$; $|g_n(x)-g_n(y)-\nabla g_n(y)(x-y)| \le B(g)\,|g(x)-g(y)-\nabla g(y)(x-y)|$ a.e. $x,y\in\mathbb{R}^N$ and $g_n\to g$ a.e. $\mathbb{R}^N$. Then the following are equivalent:*

(1) $g\in W^{2,p}(\mathbb{R}^N)$

(2)

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy < \infty$$

(3)

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1} \frac{1}{|x-y|^{N+2p}}dxdy$$
$$<\infty.$$

Next, we will also study the characterizations of $W^{2,p}\left(\mathbb{R}^N\right)$ by the differences of the first order gradient in the spirit of Bourgain, Brezis and Mironescu [5] and H.M. Nguyen [24,25]. More precisely, we will prove that

Theorem 1.5. *Let $g\in W^{1,p}\left(\mathbb{R}^N\right)$, $1<p<\infty$. Then $g\in W^{2,p}\left(\mathbb{R}^N\right)$ iff*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \le C, \quad \forall n\ge 1,$$

for some constant $C>0$. Moreover,

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbf{S}^{N-1}}\left|D^2 g(x)(\sigma,\sigma)\right|^p d\sigma dx.$$

Here $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative radial mollifiers satisfying

$$\lim_{n\to\infty}\int_\tau^\infty \rho_n(r)\, r^{N-1}dr = 0, \quad \forall \tau > 0,$$

$$\lim_{n\to\infty}\int_0^\infty \rho_n(r)\, r^{N-1}dr = 1.$$

Theorem 1.6. *Let $g\in C^p(\mathbb{R}^N)$, $1<p<\infty$ where $C^p(\mathbb{R}^N)$ is the set of all $g\in L^p\left(\mathbb{R}^N\right)$ such that $\exists\,\{g_n\}$ and $C(g)>0$: $\|g_n\|_p\le C(g)$; $|(\nabla g_n(x)-\nabla g_n(y))\cdot(x-y)| \le C(g)$; $|(\nabla g_n(x)-\nabla g_n(y))\cdot(x-y)| \le C(g)\,|(\nabla g(x)-\nabla g(y))\cdot(x-y)|$ a.e. $x,y\in\mathbb{R}^N$ and $g_n\to g$ a.e. $\mathbb{R}^N$. Then the following are equivalent:*

(1) $g\in W^{2,p}(\mathbb{R}^N)$

(2)

$$\sup_{0<\delta<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}\frac{\delta^p}{|x-y|^{N+2p}}dxdy<\infty$$

(3)

$$\sup_{0<\varepsilon<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1}\frac{\varepsilon|\,(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1}\frac{1}{|x-y|^{N+2p}}dxdy<\infty.$$

It is worthy noting that if we use the term $|\nabla g(x)-\nabla g(y)|$ instead of $|(\nabla g(x)-\nabla g(y))\cdot(x-y)|$, then Theorem 1.5 is just a easy consequence of Theorem A. Indeed, if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))|^p}{|x-y|^p}\rho_n(|x-y|)\,dxdy\le C,\quad \forall n\ge 1,$$

then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\left(\frac{\partial g}{\partial x_i}(x)-\frac{\partial g}{\partial x_i}(y)\right)\right|^p}{|x-y|^p}\rho_n(|x-y|)\,dxdy\le C,\quad \forall n\ge 1,\ \forall i=1,\dots,N.$$

Hence, by Theorem A, $\frac{\partial g}{\partial x_i}\in W^{1,p}\left(\mathbb{R}^N\right)$ $\forall i=1,\dots,N$ which means that $g\in W^{2,p}\left(\mathbb{R}^N\right)$. However, in our case, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))|^p}{|x-y|^p}\rho_n(|x-y|)\,dxdy.$$

The plan of this paper is as follows: In Section 2, we will study some helpful lemmas and use them to prove Theorems 1.1 and 1.3 which will give characterizations of second order Sobolev spaces by the second order difference. In Section 3, we establish Theorems 1.2 and 1.4 which will give characterizations of the second order Sobolev spaces using the Taylor reminder of first order. In Section 4, we characterize the second order Sobolev spaces using the first order differences of the first order gradients of the functions.

2. Characterizations using second order differences

In this section, we will investigate the characterizations of second order Sobolev spaces $W^{2,p}\left(\mathbb{R}^N\right)$ in terms of the second order differences, namely Theorems 1.1 and 1.3.

In order to prove the above two theorems, we will study the following useful lemmas. First of all, we will need to use the following basic lemma from Fourier analysis.

Lemma 2.1. *Let $1<p<\infty$. Then there exists a constant $C_{N,p}>0$ such that for every $1\le i\le N$ we have for every $g\in L^p(\mathbb{R}^N)$*

$$\left\|\frac{\partial^2}{\partial^2 x_i}g\right\|_{L^p(\mathbb{R}^N)}\le C_{N,p}\|\Delta g\|_{L^p(\mathbb{R}^N)}.$$

Proof. It suffices to prove that the operator $T=\frac{\partial^2}{\partial^2 x_i}\cdot\Delta^{-1}$ is bounded on $L^p(\mathbb{R}^N)$. It is easy to see that the operator T is a multiplier operator with the symbol $\frac{\xi_i^2}{|\xi|^2}$ which is a Marcinkiewicz multiplier which is known to be bounded on $L^p(\mathbb{R}^N)$. The operator T can also be viewed as a composition of two Riesz transforms and is known to be bounded on $L^p(\mathbb{R}^N)$. We refer to Stein's book [28]. $\square$

Lemma 2.2. *There exists a constant $C_{N,p} > 0$ such that for all $\delta > 0$, all $g \in W^{2,p}(\mathbb{R}^N)$:*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx. \tag{2.1}$$

Proof. First, using the polar coordinates, we get

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ \left|g(x)+g(x+h\sigma)-2g\left(x+\frac{h}{2}\sigma\right)\right|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}}dhdxd\sigma.$$

Hence, to prove (2.1), it is enough to prove that for every $\sigma \in \mathbb{S}^{N-1}$, we can obtain

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ \left|g(x)+g(x+h\sigma)-2g\left(x+\frac{h}{2}\sigma\right)\right|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}}dhdx \le C_{N,p}\int\limits_{\mathbb{R}^N}|\Delta g|^p\,dx. \tag{2.2}$$

Because of the rotation, we now can assume without loss of generality that $\sigma = e_N = (0, \dots, 0, 1)$. By the mean value theorem, one has

$$\begin{aligned}
\left|g(x)+g(x+he_N)-2g\left(x+\frac{h}{2}e_N\right)\right| &= \left|\left[g(x+he_N)-g\left(x+\frac{h}{2}e_N\right)\right]-\left[g\left(x+\frac{h}{2}e_N\right)-g(x)\right]\right| \\
&= \left|\int_{x_N+\frac{h}{2}}^{x_N+h}\frac{\partial g}{\partial x_N}\left(x',s\right)ds-\int_{x_N}^{x_N+\frac{h}{2}}\frac{\partial g}{\partial x_N}\left(x',s\right)ds\right| \\
&= \left|\int_{x_N}^{x_N+\frac{h}{2}}\left[\frac{\partial g}{\partial x_N}\left(x',s+\frac{h}{2}\right)-\frac{\partial g}{\partial x_N}\left(x',s\right)\right]ds\right| \\
&= \left|\int_{x_N}^{x_N+\frac{h}{2}}\left(\int_{s}^{s+\frac{h}{2}}\frac{\partial^2 g}{\partial x_N^2}\left(x',t\right)dt\right)ds\right| \\
&\le \int_{x_N}^{x_N+h}\int_{x_N}^{x_N+h}\left|\frac{\partial^2 g}{\partial x_N^2}\left(x',t\right)\right|dtds \\
&\le \int_{x_N}^{x_N+h} hM_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\,ds \\
&\le h^2M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ \left|g(x)+g(x+h\sigma)-2g\left(x+\frac{h}{2}\sigma\right)\right|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}}dhdx &\le \int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ h^2M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}}dhdx \\
&= \int\limits_{\mathbb{R}^N}\int_{\sqrt{\frac{\delta}{M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)}}}^{\infty}\frac{\delta^p}{h^{2p+1}}dhdx
\end{aligned}$$

$$= \frac{1}{2p} \int\limits_{\mathbb{R}^N} \left| M_N \left(\frac{\partial^2 g}{\partial x_N^2} \right)(x) \right|^p dx$$

$$\le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p \, dx.$$

The proof of Lemma 2.2 is now completed. □

Lemma 2.3. *There holds*

$$\lim_{\delta \to 0} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N}_{\left| g(x)+g(y)-2g\left(\frac{x+y}{2}\right) \right| > \delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \frac{1}{2^{2p+1}p} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} \left| D^2 g(x)(\sigma, \sigma) \right|^p dxd\sigma$$

for all $g \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$.

Proof. Again, by changing of variables, we obtain

$$\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N}_{\left| g(x)+g(y)-2g\left(\frac{x+y}{2}\right) \right| > \delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} \int_0^\infty{}_{\left| \frac{g(x)+g(x+\sqrt{\delta} h\sigma)-2g\left(x+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta} \right| h^2 > 1} \frac{1}{h^{2p+1}} dhdxd\sigma.$$

Define $g_\delta : \mathbb{S}^{N-1} \to \mathbb{R}$ by

$$g_\delta(\sigma) = \int\limits_{\mathbb{R}^N} \int_0^\infty{}_{\left| \frac{g(x)+g(x+\sqrt{\delta} h\sigma)-2g\left(x+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta} \right| h^2 > 1} \frac{1}{h^{2p+1}} dhdx.$$

We first prove that for all $\sigma \in \mathbb{S}^{N-1}$, $\forall \delta > 0$:

$$g_\delta(\sigma) \le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p \, dx. \tag{2.3}$$

Indeed, again, without loss of generality, we assume that $\sigma = e_N = (0, \ldots, 0, 1)$. Hence, we need to verify that

$$\int\limits_{\mathbb{R}^N} \int_0^\infty{}_{\left| \frac{g(x',x_N)+g(x',x_N+\sqrt{\delta} h\sigma)-2g\left(x',x_N+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta} \right| h^2 > 1} \frac{1}{h^{2p+1}} dhdx \le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p \, dx. \tag{2.4}$$

Similar to what is done in Lemma 2.2, we have

$$\left| \frac{g(x', x_N) + g(x', x_N + \sqrt{\delta} h\sigma) - 2g\left(x', x_N + \frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta} \right| \le M_N \left(\frac{\partial^2 g}{\partial x_N^2} \right)(x).$$

Thus,

$$\int\limits_{\mathbb{R}^N} \int_0^\infty{}_{\left| \frac{g(x',x_N)+g(x',x_N+\sqrt{\delta} h\sigma)-2g\left(x',x_N+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta} \right| h^2 > 1} \frac{1}{h^{2p+1}} dhdx \le \int\limits_{\mathbb{R}^N} \int_0^\infty{}_{h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x) > 1} \frac{1}{h^{2p+1}} dhdx$$

$$\le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p \, dx.$$

Next, we will show that

$$g_\delta(\sigma) \to \frac{1}{2^{2p+1}p}\int_{\mathbb{R}^N} \left|D^2 g(x)(\sigma,\sigma)\right|^p dx \quad \text{as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1} \tag{2.5}$$

where

$$\left|D^2 g(x)(\sigma,\sigma)\right| = \sum_{1\le i_1,i_2\le N} \sigma_{i_1}\sigma_{i_2}\frac{\partial^2 g}{\partial x_{i_1}\partial x_{i_2}}(x).$$

Again, without loss of generality, we suppose that $\sigma = e_N = (0,\ldots,0,1)$. We write

$$g_\delta(\sigma) = \int_{\mathbb{R}^N}\int_0^\infty G_\delta(x,h)\,dhdx$$

where

$$G_\delta(x,h) = \frac{1}{h^{2p+1}}\chi_{\left\{\left|\frac{g(x)+g(x+\sqrt{\delta}h\sigma)-2g\left(x+\frac{h}{2}\sqrt{\delta}\sigma\right)}{h^2\delta}\right|h^2>1\right\}}(x,h).$$

Noting that for all $\sigma \in \mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{2p+1}}\chi_{\{|D^2g(x)(\sigma,\sigma)|h^2>4\}}(x,h) \quad \text{as } \delta \to 0 \text{ for a.e. } (x,h) \in \mathbb{R}^N \times [0,\infty),$$

and

$$G_\delta(x,h) \le \frac{1}{h^{2p+1}}\chi_{\left\{h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>1\right\}}(x,h) \in L^1\left(\mathbb{R}^N\times[0,\infty)\right).$$

Hence, by Lebesgue's dominated convergence theorem, we get (2.5).

Using (2.3) and (2.5) and the Lebesgue dominated convergence theorem again, we can conclude that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma. \qquad \square$$

The following elementary lemma was proved and used in [24]. For the sake of completeness, we include a proof.

Lemma 2.4. *Let Ω be a measurable set in $\mathbb{R}^m$, Φ and Ψ be two measurable nonnegative functions on Ω, and $\alpha > -1$. Then*

$$\int_0^1\int_{\Phi(x)>\delta}\delta^\alpha\Psi(x)\,dxd\delta = \int_{\Phi(x)\le 1}\frac{1}{\alpha+1}\Phi^{\alpha+1}(x)\Psi(x)\,dx + \int_{\Phi(x)>1}\frac{1}{\alpha+1}\Psi(x)\,dx.$$

Proof. Using Fubini's theorem, we get

$$\begin{aligned}\int_0^1\int_{\Phi(x)>\delta}\delta^\alpha\Psi(x)\,dxd\delta &= \int_{\Phi(x)>1}\int_0^1\delta^\alpha\Psi(x)\,d\delta dx + \int_{\Phi(x)\le 1}\int_0^{\Phi(x)}\delta^\alpha\Psi(x)\,d\delta dx \\ &= \int_{\Phi(x)>1}\frac{1}{\alpha+1}\Psi(x)\,dx + \int_{\Phi(x)\le 1}\frac{1}{\alpha+1}\Phi^{\alpha+1}(x)\Psi(x)\,dx. \qquad \square\end{aligned}$$

Proof of Theorem 1.1. (1) and (2) are consequences of Lemmas 2.2 and 2.3.

Now we will prove (3). By (1), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p dx, \quad \forall\delta > 0. \tag{2.6}$$

In particular,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1} \frac{1}{|x-y|^{N+2p}}dxdy \leq C_{N,p}\int\limits_{\mathbb{R}^N} |\Delta g|^p\, dx. \tag{2.7}$$

Now, from (2.6), one has

$$\int_0^1\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta} \frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta \leq C_{N,p}\int\limits_{\mathbb{R}^N} |\Delta g|^p\, dx.$$

Using Lemma 2.4, we deduce

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\leq 1} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1} \frac{\varepsilon}{|x-y|^{N+2p}}dxdy\\ &\leq C_{N,p}\int\limits_{\mathbb{R}^N} |\Delta g|^p\, dx.\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we get the assertion (3).

Now, set

$$G(\delta) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}dxdy.$$

So by the previous results, we have

$$G(\delta) \leq C_{N,p}\int\limits_{\mathbb{R}^N} |\Delta g|^p\, dx, \quad \forall\delta > 0$$

and

$$\lim_{\delta\to 0} G(\delta) = \frac{1}{2^{2p+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N} \left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Now, we claim that

$$\lim_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}G(\delta)\,d\delta = \frac{1}{2^{2p+1}}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N} \left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma. \tag{2.9}$$

Indeed, for every $\epsilon > 0$, we can find a number $X(\epsilon) \in (0, 1)$ such that

$$\left|G(\delta) - \frac{1}{2^{2p+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N} \left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right| < \epsilon \quad \text{for all } \delta \in (0, X(\epsilon)).$$

Now, we have:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left|\int_{X(\epsilon)}^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[G(\delta) - \frac{1}{2^{2p+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N} \left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta\right|\\ &\leq \lim_{\varepsilon\to 0}\int_{X(\epsilon)}^1 (p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left|G(\delta) - \frac{1}{2^{2p+1}p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N} \left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right|d\delta\end{aligned}$$

$$\leq \lim_{\varepsilon\to 0}\int_{X(\epsilon)}^{1}(p+\varepsilon)\,\varepsilon X(\epsilon)^{\varepsilon-1}\left[C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx\right]d\delta$$
$$=0.$$

Moreover,

$$\lim_{\varepsilon\to 0}\left|\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta\right|$$
$$\leq \lim_{\varepsilon\to 0}\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left|G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right|d\delta$$
$$\leq \lim_{\varepsilon\to 0}\int_0^{X(\epsilon)}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\epsilon\, d\delta$$
$$\leq p\epsilon.$$

Thus,

$$\lim_{\varepsilon\to 0}\left|\int_0^{1}(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[G(\delta)-\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta\right|\leq p\epsilon,\quad \forall\epsilon>0.$$

Hence we can get

$$\lim_{\varepsilon\to 0}\int_0^1(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}G(\delta)\,d\delta=\lim_{\varepsilon\to 0}\int_0^1(p+\varepsilon)\,\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{2^{2p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma\right]d\delta$$
$$=\frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\int_0^1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>\delta}\frac{(p+\varepsilon)\,\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta=\frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Now, using Lemma 2.4 with $\alpha=p+\varepsilon-1$, $\Phi(x,y)=\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|$, $\Psi(x,y)=\frac{1}{|x-y|^{N+2p}}$, we obtain

$$\lim_{\varepsilon\to 0}\left[\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\leq 1}\frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy+\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy\right]=\frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

Noting that

$$\lim_{\varepsilon\to 0}\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy=0,$$

we have

$$\lim_{\varepsilon\to 0}\iint\limits_{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\leq 1}\frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy=\frac{1}{2^{2p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g(x)(\sigma,\sigma)\right|^p dxd\sigma.$$

We have the statement (4).

Proof of Theorem 1.3. First, it is clear that statements (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are consequences of Theorem 1.1. Now, we will prove (3) $\Longrightarrow$ (1):

First, we assume further that $\left|g(x)+g(y)-2g(\frac{x+y}{2})\right|$ is bounded by $M(g)$ on $\mathbb{R}^N \times \mathbb{R}^N$. Then since

$$\sup_{0<\varepsilon<1} \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \underset{\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+2p}}dxdy < \infty$$

we get

$$\sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\gamma}}{|x-y|^{N+2p}}dxdy = C\left(g, M\left(g\right)\right) < \infty.$$

Let η_ε be any sequence of smooth mollifiers and set $g^\varepsilon = g * \eta_\varepsilon$. Then we can get $g^\varepsilon \in L^p(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N) \subseteq W^{2,p}(\mathbb{R}^N)$.

Using (4) of Theorem 1.1, we can have

$$\begin{aligned}
C_{N,p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g^\varepsilon(x)(\sigma,\sigma)|^p dxd\sigma &\le \sup_{0<\gamma<1}\underset{\left|g^\varepsilon(x)+g^\varepsilon(y)-2g^\varepsilon\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\gamma\left|g^\varepsilon(x)+g^\varepsilon(y)-2g^\varepsilon\left(\frac{x+y}{2}\right)\right|^{p+\gamma}}{|x-y|^{N+2p}}dxdy\\
&\le \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma\left|g^\varepsilon(x)+g^\varepsilon(y)-2g^\varepsilon\left(\frac{x+y}{2}\right)\right|^{p+\gamma}}{|x-y|^{N+2p}}dxdy\\
&= \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma\left|\int_{\mathbb{R}^N}\left\{g(x-z)+g(y-z)-2g\left(\frac{x+y}{2}-z\right)\right\}\eta^\varepsilon(z)dz\right|^{p+\gamma}}{|x-y|^{N+2p}}dxdy.
\end{aligned}$$

Since the function $x^{p+\varepsilon}$ is convex on $[0,\infty)$, by Jensen's inequality, we can deduce

$$\begin{aligned}
&\sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma\left|\int_{\mathbb{R}^N}\left\{g(x-z)+g(y-z)-2g\left(\frac{x+y}{2}-z\right)\right\}\eta^\varepsilon(z)dz\right|^{p+\gamma}}{|x-y|^{N+2p}}dxdy\\
&\le \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma\left|g(x-z)+g(y-z)-2g\left(\frac{x+y}{2}-z\right)\right|^{p+\gamma}\int_{\mathbb{R}^N}\eta^\varepsilon(z)dz}{|x-y|^{N+2p}}dxdy\\
&= \sup_{0<\gamma<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\gamma\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\gamma}}{|x-y|^{N+2p}}dxdy \le C\left(g, M\left(g\right)\right).
\end{aligned}$$

So $\|g^\varepsilon\|_{W^{2,p}(\mathbb{R}^N)}$ is bounded. Since $g^\varepsilon \to g$ a.e., we get $g \in W^{2,p}(\mathbb{R}^N)$.

In the general case, since $g \in A^p\left(\mathbb{R}^N\right)$, we can find a sequence $\{g_n\}$ and $A(g) > 0$ such that $\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|$ is bounded by $A(g)$ and $\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right| \le A(g)\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|$ a.e. $x, y \in \mathbb{R}^N$. Then it is clear that $g_n \in W^{2,p}(\mathbb{R}^N)$ since

$$\begin{aligned}
&\sup_{0<\varepsilon<1}\underset{\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy\\
&\quad+\underset{\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{1}{|x-y|^{N+2p}}dxdy\\
&= \sup_{0<\varepsilon<1}\underset{\substack{\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|\le 1\\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon\left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy
\end{aligned}$$

$$+ \int\limits_{\substack{\mathbb{R}^N \\ \left|\frac{g_n}{A(g)}(x)+\frac{g_n}{A(g)}(y)-2\frac{g_n}{A(g)}\left(\frac{x+y}{2}\right)\right|\le 1 \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}} \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N+2p}} dxdy$$

$$\le \sup_{0<\varepsilon<1} \int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}} \int\limits_{\mathbb{R}^N} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy + 2\int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|>1}} \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N+2p}} dxdy$$

$$< \infty.$$

Moreover, by (4) in Theorem 1.1, we have

$$\frac{1}{2^{2p+1}} \int\limits_{\mathbb{S}^{N-1}} \int\limits_{\mathbb{R}^N} \left|D^2 g_n(x)(\sigma,\sigma)\right|^p dx d\sigma = \lim_{\varepsilon\to 0} \int\limits_{\substack{\mathbb{R}^N \\ \left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|\le 1}} \int\limits_{\mathbb{R}^N} \frac{\varepsilon\left|g_n(x)+g_n(y)-2g_n\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$\le \lim_{\varepsilon\to 0} C(A(g)) \int\limits_{\substack{\mathbb{R}^N \\ \left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|\le 1}} \int\limits_{\mathbb{R}^N} \frac{\varepsilon\left|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$

$$\le C(g, A(g)).$$

Hence, $\|g_n\|_{W^{2,p}(\mathbb{R}^N)}$ is bounded. Since $g_n \to g$ a.e. $\mathbb{R}^N$, we get $g \in W^{2,p}(\mathbb{R}^N)$.

3. Characterizations of the second type: the Taylor remainder

The man purpose of this section is to establish Theorems 1.2 and 1.4, namely, characterizing the second order Sobolev spaces $W^{2,p}(\mathbb{R}^N)$ using the method of the Taylor remainder of first order.

In order to prove these two theorems, we will need to adapt the following useful lemmas:

Lemma 3.1. *There exists a constant $C_{N,p} > 0$ such that for all $\delta > 0$, all $g \in W^{2,p}\left(\mathbb{R}^N\right)$:*

$$\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)-\nabla g(y)(x-y)|>\delta}} \int\limits_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+2p}} dxdy \le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p dx. \tag{3.1}$$

Proof. Again, using the polar coordinates, we get

$$\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)-\nabla g(y)(x-y)|>\delta}} \int\limits_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \int\limits_{\substack{\mathbb{S}^{N-1} \\ |g(x+h\sigma)-g(x)-h\nabla g(x)\sigma|>\delta}} \int\limits_{\mathbb{R}^N} \int_0^\infty \frac{\delta^p}{h^{2p+1}} dhdxd\sigma.$$

Thus, again, to prove (3.1), it is enough to prove that for every $\sigma \in \mathbb{S}^{N-1}$, we get

$$\int\limits_{\substack{\mathbb{R}^N \\ |g(x+h\sigma)-g(x)-h\nabla g(x)\sigma|>\delta}} \int_0^\infty \frac{\delta^p}{h^{2p+1}} dhdx \le C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p dx. \tag{3.2}$$

Because of the rotation, we assume without loss of generality that $\sigma = e_N = (0, \ldots, 0, 1)$.

Now, by the mean value theorem, one has

$$|g(x+he_N) - g(x) - h\nabla g(x)e_N| = \left| \int_0^1 \frac{\partial g}{\partial x_N}\left(x', x_N + sh\right) h ds - h\frac{\partial g}{\partial x_N}(x) \right|$$

$$= \left| h\int_0^1 \left[\frac{\partial g}{\partial x_N}\left(x', x_N + sh\right) hds - \frac{\partial g}{\partial x_N}(x', x_N) \right] ds \right|$$

$$= \left| h \int_0^1 \int_{x_N}^{x_N+sh} \frac{\partial^2 g}{\partial x_N^2}\left(x', t\right) dt ds \right|$$
$$\leq h \int_0^1 sh M_N \left(\frac{\partial^2 g}{\partial x_N^2} \right)(x)\, ds$$
$$\leq \frac{1}{2} h^2 M_N \left(\frac{\partial^2 g}{\partial x_N^2} \right)(x)\,.$$

Thus,

$$\int\limits_{\mathbb{R}^N} \int\limits_{\substack{0 \\ |g(x+h\sigma)-g(x)-h\nabla g(x)\sigma|>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}} dh dx \leq \int\limits_{\mathbb{R}^N} \int\limits_{\substack{0 \\ h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>\delta}}^{\infty} \frac{\delta^p}{h^{2p+1}} dh dx$$
$$= \int\limits_{\mathbb{R}^N} \int\limits_{\sqrt{\frac{\delta}{M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)}}}^{\infty} \frac{\delta^p}{h^{2p+1}} dh dx$$
$$= \frac{1}{2p} \int\limits_{\mathbb{R}^N} \left| M_N \left(\frac{\partial^2 g}{\partial x_N^2} \right)(x) \right|^p dx$$
$$\leq C_{N,p} \int\limits_{\mathbb{R}^N} \left| \frac{\partial^2 g}{\partial x_N^2} \right|^p dx$$
$$\leq C_{N,p} \int\limits_{\mathbb{R}^N} |\Delta g|^p\, dx. \quad \square$$

Lemma 3.2. *There holds*

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dx dy = \frac{1}{2^{p+1} p} \int_{\mathbb{S}^N} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma$$

for all $g \in W^{2,p}(\mathbb{R}^N),\ 1 < p < \infty$.

Proof. Again, by changing of variables, we obtain

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dx dy = \int_{\mathbb{S}^N} \int_{\mathbb{R}^N} \int_0^\infty\limits_{\left|\frac{g(x+\sqrt{\delta}h\sigma)-g(x)-\sqrt{\delta}h\nabla g(x)\sigma}{h^2\delta}\right| h^2>1} \frac{1}{h^{2p+1}} dh dx d\sigma.$$

Define $g_\delta : \mathbb{S}^{N-1} \to \mathbb{R}$ by

$$g_\delta(\sigma) = \int_{\mathbb{R}^N} \int_0^\infty\limits_{\left|\frac{g(x+\sqrt{\delta}h\sigma)-g(x)-\sqrt{\delta}h\nabla g(x)\sigma}{h^2\delta}\right| h^2>1} \frac{1}{h^{2p+1}} dh dx.$$

We first prove that for all $\sigma \in \mathbb{S}^{N-1},\ \forall \delta > 0$:

$$g_\delta(\sigma) \leq C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx. \tag{3.3}$$

Indeed, again, without loss of generality, we assume that $\sigma = e_N = (0, \ldots, 0, 1)$. Hence, we need to verify that

$$\int_{\mathbb{R}^N} \int_0^\infty\limits_{\left|\frac{g(x',x_N+\sqrt{\delta}h)-g(x',x_N)-\nabla g(x',x_N)\sqrt{\delta}he_N}{h^2\delta}\right| h^2>1} \frac{1}{h^{2p+1}} dh dx \leq C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx. \tag{3.4}$$

Similar to the proof of Lemma 2.2, we have

$$\left|\frac{g(x', x_N+\sqrt{\delta}h)-g(x', x_N)-\nabla g(x', x_N)\sqrt{\delta}he_N}{h^2\delta}\right| \le \frac{1}{2}M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x).$$

Thus,

$$\int_{\mathbb{R}^N}\int_0^\infty\limits_{\left|\frac{g(x', x_N+\sqrt{\delta}h)-g(x', x_N)-\nabla g(x', x_N)\sqrt{\delta}he_N}{h^2\delta}\right|h^2>1} \frac{1}{h^{2p+1}}dhdx \le \int_{\mathbb{R}^N}\int_0^\infty\limits_{h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>2} \frac{1}{h^{2p+1}}dhdx.$$

So we can get

$$\int_{\mathbb{R}^N}\int_0^\infty\limits_{\left|\frac{g(x', x_N+\sqrt{\delta}h)-g(x', x_N)-\nabla g(x', x_N)\sqrt{\delta}he_N}{h^2\delta}\right|h^2>1} \frac{1}{h^{2p+1}}dhdx \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx.$$

Next we will show that

$$g_\delta(\sigma) \to \frac{1}{2^{p+1}p}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dx \quad \text{as } \delta\to 0 \text{ for every } \sigma\in\mathbb{S}^{N-1} \tag{3.5}$$

where

$$|D^2g(x)(\sigma,\sigma)| = \sum_{1\le i_1,i_2\le N}\sigma_{i_1}\sigma_{i_2}\frac{\partial^2 g}{\partial x_{i_1}\partial x_{i_2}(x)}.$$

Again, with loss of generality, we suppose that $\sigma = e_N = (0,\ldots,0,1)$. We write

$$g_\delta(\sigma) = \int_{\mathbb{R}^N}\int_0^\infty G_\delta(x,h)dhdx$$

where

$$G_\delta(x,h) = \frac{1}{h^{2p+1}}\chi_{\left\{\left|\frac{g(x+\sqrt{\delta}h\sigma)-g(x)-\nabla g(x)\sqrt{\delta}h\sigma}{h^2\delta}\right|h^2>1\right\}}(x,h).$$

Noting that for all $\sigma\in\mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{2p+1}}\chi_{\{|D^2g(x)(\sigma,\sigma)|h^2>2\}} \quad \text{as } \delta\to 0 \text{ for a.e. } (x,h)\in\mathbb{R}^N\times[0,\infty),$$

and

$$G_\delta(x,h) \le \frac{1}{h^{2p+1}}\chi_{\left\{h^2M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>2\right\}}(x,h)\in L^1(\mathbb{R}^N\times[0,\infty)).$$

Hence, by the Lebesgue dominated convergence theorem, we get (3.5). Using (3.3), (3.5) and the Lebesgue dominated convergence theorem again, we can conclude that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}\frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{p+1}p}\int_{\mathbb{S}^N}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma. \quad\square$$

Proof of Theorem 1.2. (1) and (2) are consequences of Lemmas 3.1 and 3.2. Now we will prove (3). By (1), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}\frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx, \quad \forall\delta>0. \tag{3.6}$$

In particular,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)-\nabla g(y)(x-y)|>1}\frac{1}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx, \quad \forall\delta>0. \tag{3.7}$$

Now, from (3.6), one has

$$\int_0^1 \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx. \tag{3.8}$$

Using Lemma 2.4, we deduce

$$\underset{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+2p}}dxdy$$
$$\le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx.$$

Hence, we get the assertion (3). Now set

$$H(\delta) = \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}.$$

So from what we have proved, we have

$$H(\delta) \le C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx, \quad \forall\delta > 0$$

and

$$\lim_{\delta\to 0} H(\delta) = \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma. \tag{3.9}$$

Indeed, for every $\theta > 0$, we can find a number $X(\theta) \in (0, 1)$ such that

$$\left|H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right| < \theta \quad \text{for all } \delta \in (0, X(\theta)).$$

Now, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left|\int_{X(\theta)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right]d\delta\right| \\
&\le \lim_{\varepsilon\to 0}\int_{X(\theta)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left|H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right|d\delta \\
&\le \lim_{\varepsilon\to 0}\int_{X(\theta)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx\right]d\delta \\
&= \lim_{\varepsilon\to 0}(p+\varepsilon)(1-X(\theta)^{\varepsilon})\left[C_{N,p}\int_{\mathbb{R}^N}|\triangle g|^p dx\right] \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left|\int_0^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right]d\delta\right| \\
&\le \lim_{\varepsilon\to 0}\int_0^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left|H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right|d\delta \\
&\le \lim_{\varepsilon\to 0}\int_0^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\theta d\delta \\
&\le p\theta.
\end{aligned}$$

Thus, $\forall$ sufficiently small $\theta > 0$:

$$\lim_{\varepsilon\to 0}\left|\int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[H(\delta) - \frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dx d\sigma\right]d\delta\right| \le p\theta.$$

Hence we can get

$$\lim_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}H(\delta)d\delta = \lim_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\left[\frac{1}{2^{p+1}p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma\right]d\delta$$
$$= \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Consequently, we have

$$\lim_{\varepsilon\to 0}\int_0^1\underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon(p+\varepsilon)\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}}dxdyd\delta = \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Now, using Lemma 2.4 with

$$\alpha = p+\varepsilon-1, \qquad \Phi(x,y) = |g(x)-g(y)-\nabla g(y)(x-y)|, \qquad \Psi(x,y) = \frac{1}{|x-y|^{N+2p}},$$

we obtain

$$\lim_{\varepsilon\to 0}\left[\begin{array}{c}\displaystyle\iint_{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1}\frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy\\ \displaystyle+\iint_{|g(x)-g(y)-\nabla g(y)(x-y)|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy\end{array}\right] = \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\left|D^2g\,(x)\,(\sigma,\sigma)\right|^p dxd\sigma.$$

Noting that

$$\lim_{\varepsilon\to 0}\underset{|g(x)-g(y)-\nabla g(y)(x-y)|>1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy = 0,$$

we have

$$\lim_{\varepsilon\to 0}\underset{|g(x)-g(y)-\nabla g(y)(x-y)|\le 1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}}\frac{\varepsilon|g(x)-g(y)-\nabla g(y)(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy = \frac{1}{2^{p+1}}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}|D^2g(x)(\sigma,\sigma)|^p dxd\sigma.$$

Proof of Theorem 1.4. is similar to the proof of Theorem 1.3 and will be omitted.

4. More characterizations of second order Sobolev spaces: combinations of first order difference and Taylor remainder

In this section, we will study some other characterizations of the second order Sobolev spaces. Namely, we will give characterizations motivated by the observation that $g \in W^{2,p}(\mathbb{R}^N)$ is essentially equivalent to $\nabla g \in W^{1,p}(\mathbb{R}^N)$.

4.1. Characterization of Bourgain–Brezis–Mironescu type

Lemma 4.1. *Let $g \in W^{2,p}\left(\mathbb{R}^N\right)$. Then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n\left(|x-y|\right)dxdy \le C, \quad \forall n\ge 1, \tag{4.1}$$

for some constant $C > 0$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n\left(|x-y|\right)dxdy = \int_{\mathbb{R}^N}\int_{\mathbf{S}^{N-1}}\left|D^2g\,(x)\,(\sigma,\sigma)\right|^p d\sigma dx. \tag{4.2}$$

Proof. Since $g \in W^{2,p}\left(\mathbb{R}^N\right)$, $\frac{\partial g}{\partial x_i} \in W^{1,p}\left(\mathbb{R}^N\right)$. Using Theorem A (noting Theorem A still holds when $\Omega = \mathbb{R}^N$), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\frac{\partial g}{\partial x_i}(x)-\frac{\partial g}{\partial x_i}(y)\right|^p}{|x-y|^p}\rho_n\left(|x-y|\right)dxdy \le C.$$

Hence,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))|^p}{|x-y|^{p}}\rho_n(|x-y|)\,dxdy \le C$$

for some constant $C > 0$.

Now, suppose that $g \in C_0^\infty\left(\mathbb{R}^N\right)$. Then by Taylor's formula, we have

$$\begin{aligned}|(\nabla g(x+h)-\nabla g(x))\cdot h| &= \left|\sum\left(\frac{\partial g}{\partial x_i}(x+h)-\frac{\partial g}{\partial x_i}(x)\right)h_i\right| \\ &\le \left|D^2 g\,(x)\,(h,h)\right| + c\,|h|^3 .\end{aligned}$$

Hence, for every $\theta > 0$:

$$|(\nabla g(x+h)-\nabla g(x))\cdot h|^p \le (1+\theta)\left|D^2 g\,(x)\,(h,h)\right|^p + c_\theta\,|h|^{3p} .$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x+h)-\nabla g(x))\cdot h|^p}{|h|^{2p}}\rho_n(|h|)\,dhdx \\ &\quad\le (1+\theta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\rho_n(|h|)}{|h|^{2p}}\left|D^2 g\,(x)\,(h,h)\right|^p dhdx + c_\theta\,|\mathrm{supp}\,(g)|\int_{\mathbb{R}^N}\rho_n(|h|)\,|h|^p\,dh.\end{aligned}$$

Also,

$$\int_{\mathbb{R}^N}\frac{\rho_n(|h|)}{|h|^{2p}}\left|D^2 g\,(x)\,(h,h)\right|^p dh = \int_0^\infty \rho_n(r)\,r^{N-1}dr\int_{\mathbf{S}^{N-1}}\left|D^2 g\,(x)\,(\sigma,\sigma)\right|^p d\sigma .$$

Now, let $n \to \infty$ and then $\theta \to 0$, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \le \int_{\mathbb{R}^N}\int_{\mathbf{S}^{N-1}}\left|D^2 g\,(x)\,(\sigma,\sigma)\right|^p d\sigma dx. \tag{4.3}$$

Now, for any $A > 0$, then again by Taylor's formula:

$$\left|(\nabla g(x+h)-\nabla g(x))\cdot h - D^2 g\,(x)\,(h,h)\right| \le C_A\,|h|^3 \quad \text{a.e. } x\in B\,(0,A)\,;\ h\in B\,(0,1)\,.$$

Then

$$\left|D^2 g\,(x)\,(h,h)\right|^p \le (1+\theta)\,|(\nabla g(x+h)-\nabla g(x))\cdot h|^p + C_{A,\theta}\,|h|^{3p} \quad \text{a.e. } x\in B\,(0,A)\,;\ h\in B\,(0,1)\,.$$

Hence

$$\begin{aligned}&\int_{B(0,A)}\int_{B(0,1)}\frac{\rho_n(|h|)}{|h|^{2p}}\left|D^2 g\,(x)\,(h,h)\right|^p dhdx \\ &\quad\le (1+\theta)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x+h)-\nabla g(x))\cdot h|^p}{|h|^{2p}}\rho_n(|h|)\,dhdx + C_{A,\theta}\,|B\,(0,A)|\int_{B(0,1)}\rho_n(|h|)\,|h|^p\,dh.\end{aligned}$$

Let $n \to \infty$ and $\theta \to 0$:

$$\begin{aligned}\int_{B(0,A)}\int_{\mathbf{S}^{N-1}}\left|D^2 g\,(x)\,(\sigma,\sigma)\right|^p d\sigma dx &\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x+h)-\nabla g(x))\cdot h|^p}{|h|^{2p}}\rho_n(|h|)\,dhdx \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy.\end{aligned}$$

Since $A > 0$ is arbitrary, we get (4.2) for $C_0^\infty\left(\mathbb{R}^N\right)$-functions.

By density argument, we also have (4.2) for $W^{2,p}\left(\mathbb{R}^N\right)$-functions. □

Proof of Theorem 1.5. Assume that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy \le C(g), \quad \forall n\ge 1.$$

Let $g_k = g * \eta_k$ where η_k is a sequence of smooth mollifiers. Noting that $D^\alpha(g_k) = D^\alpha(g) * \eta_k$, we have from the convexity of the function t^p on $[0,\infty)$ that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g_k(x)-\nabla g_k(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^p}{|x-y|^{2p}}\rho_n(|x-y|)\,dxdy\\ &\le C(g).\end{aligned}$$

Moreover, since g_k is smooth, by Lemma 4.1, we get

$$\int_{\mathbb{R}^N}\int_{S^{N-1}} \left|D^2 g_k(x)(\sigma,\sigma)\right|^p d\sigma dx \le C(g)$$

which means that

$$\int_{\mathbb{R}^N} |\Delta g_k(x)|^p\,dx \le C_1(g).$$

Since $g_k\to g$ a.e. $\mathbb{R}^N$, we can conclude that $g\in W^{2,p}\left(\mathbb{R}^N\right)$.

4.2. Characterization of H.-M. Nguyen type

Lemma 4.2. *There exists a constant $C_{N,p}>0$ such that for all $\delta>0$, all $g\in W^{2,p}\left(\mathbb{R}^N\right)$:*

$$\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy \le C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx.$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy &\le \underset{|g(x)-g(y)-\nabla g(y)(x-y)|>\frac{\delta}{2}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} + \underset{|g(y)-g(x)-\nabla g(x)\cdot(y-x)|>\frac{\delta}{2}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy\\ &\le C_{N,p}\int_{\mathbb{R}^N}|\Delta g|^p\,dx. \quad\square\end{aligned}$$

Lemma 4.3. *There holds*

$$\lim_{\delta\to 0}\underset{|(\nabla g(x)-\nabla g(y))(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \frac{1}{2p}\int_{\mathbb{S}^N}\int_{\mathbb{R}^N}|D^2 g(x)(\sigma,\sigma)|^p dxd\sigma$$

for all $g\in W^{2,p}(\mathbb{R}^N)$, $1<p<\infty$.

Proof. By changing of variables, we obtain

$$\underset{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+2p}}dxdy = \underset{\left|\frac{(\nabla g(x+\sqrt{\delta}h\sigma)-\nabla g(x))\cdot\sqrt{\delta}h\sigma}{h^2\delta}\right|h^2>1}{\int_{\mathbb{S}^N}\int_{\mathbb{R}^N}\int_0^\infty} \frac{1}{h^{2p+1}}dhdxd\sigma.$$

Define $g_\delta:\mathbb{S}^{N-1}\to\mathbb{R}$ by

$$g_\delta(\sigma) = \underset{\left|\frac{(\nabla g(x+\sqrt{\delta}h\sigma)-\nabla g(x))\cdot\sqrt{\delta}h\sigma}{h^2\delta}\right|h^2>1}{\int_{\mathbb{R}^N}\int_0^\infty} \frac{1}{h^{2p+1}}dhdx.$$

Then by the same argument as in Lemma 4.2 and by Lemma 3.2, we can prove that for all $\sigma \in \mathbb{S}^{N-1}$, $\forall \delta > 0$:

$$g_\delta(\sigma) \le C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx.$$

Now we will show that

$$g_\delta(\sigma) \to \frac{1}{2p} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx \quad \text{as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1}. \tag{4.4}$$

Indeed, again, with loss of generality, we suppose that $\sigma = e_N = (0, \dots, 0, 1)$. We write

$$g_\delta(\sigma) = \int_{\mathbb{R}^N} \int_0^\infty G_\delta(x,h) dh dx$$

where

$$G_\delta(x,h) = \frac{1}{h^{2p+1}} \chi_{\left\{\left|\frac{(\nabla g(x+\sqrt{\delta}h\sigma)-\nabla g(x))\cdot\sqrt{\delta}h\sigma}{h^2\delta}\right|h^2>1\right\}}(x,h).$$

Noting that for all $\sigma \in \mathbb{S}^{N-1}$:

$$G_\delta(x,h) \to \frac{1}{h^{2p+1}} \chi_{\{|D^2g(x)(\sigma,\sigma)|h^2>1\}} \quad \text{as } \delta \to 0 \text{ for a.e. } (x,h) \in \mathbb{R}^N \times [0,\infty),$$

and

$$G_\delta(x,h) \le \frac{1}{h^{2p+1}} \chi_{\left\{h^2 M_N\left(\frac{\partial^2 g}{\partial x_N^2}\right)(x)>1\right\}}(x,h) \in L^1(\mathbb{R}^N \times [0,\infty)).$$

Hence, by the Lebesgue dominated convergence theorem, we get (4.4). Using the Lebesgue dominated convergence theorem again, we can conclude that

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \frac{1}{2p}\int_{\mathbb{S}^N}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma. \quad \square$$

Theorem 4.1. *Let $g \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$. Then there exists a constant $C_{N,p}$ such that*

(1)

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy \le C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx, \quad \forall \delta > 0.$$

(2)

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}} dxdy = \frac{1}{2p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.$$

(3)

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|\le 1} \frac{\varepsilon|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy$$
$$+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1} \frac{1}{|x-y|^{N+2p}} dxdy \le C_{N,p}\int_{\mathbb{R}^N} |\triangle g|^p dx.$$

(4)

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|\le 1} \frac{\varepsilon|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}} dxdy = \frac{1}{2}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} |D^2 g(x)(\sigma,\sigma)|^p dx d\sigma.$$

Proof. (3) is just a consequence of (1), the Fatou lemma and Lemma 2.4. So we just have to prove (4). Set

$$H(\delta) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>\delta} \frac{\delta^p}{|x-y|^{N+2p}}.$$

So from what we have proved, we get

$$H(\delta) \le C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx, \quad \forall \delta > 0$$

and

$$\lim_{\delta \to 0} H(\delta) = \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma.$$

Thus, for every $\theta > 0$, we can find a number $X(\theta) \in (0, 1)$ such that

$$\left| H(\delta) - \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right| < \theta \quad \text{for all } \delta \in (0, X(\theta)).$$

Now, we have

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left| \int_{X(\theta)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left[H(\delta) - \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right] d\delta \right| \\
&\quad \le \lim_{\varepsilon \to 0} \int_{X(\theta)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left| H(\delta) - \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right| d\delta \\
&\quad \le \lim_{\varepsilon \to 0} \int_{X(\theta)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left[C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx \right] d\delta \\
&\quad = \lim_{\varepsilon \to 0} (p+\varepsilon)(1 - X(\theta)^{\varepsilon}) \left[C_{N,p} \int_{\mathbb{R}^N} |\triangle g|^p dx \right] \\
&\quad = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left| \int_0^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left[H(\delta) - \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right] d\delta \right| \\
&\quad \le \lim_{\varepsilon \to 0} \int_0^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left| H(\delta) - \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right| d\delta \\
&\quad \le \lim_{\varepsilon \to 0} \int_0^{X(\theta)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \theta d\delta \\
&\quad \le p\theta.
\end{aligned}$$

Thus,

$$\lim_{\varepsilon \to 0} \left| \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left[H(\delta) - \frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right] d\delta \right| \le p\theta, \quad \forall \text{ sufficiently small } \theta > 0.$$

Hence we can get

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} H(\delta) d\delta &= \lim_{\varepsilon \to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} \left[\frac{1}{2p} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma \right] d\delta \\
&= \frac{1}{2} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma.
\end{aligned}$$

Consequently, we have

$$\lim_{\varepsilon \to 0} \underset{|(\nabla g(x) - \nabla g(y))\cdot(x-y)| > \delta}{\int_0^1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\varepsilon(p+\varepsilon)\delta^{p+\varepsilon-1}}{|x-y|^{N+2p}} dx dy d\delta = \frac{1}{2} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |D^2 g(x)(\sigma, \sigma)|^p dx d\sigma.$$

Now, using Lemma 2.4 with

$$\alpha = p + \varepsilon - 1, \qquad \Phi(x, y) = |(\nabla g(x) - \nabla g(y)) \cdot (x - y)|, \qquad \Psi(x, y) = \frac{1}{|x-y|^{N+2p}},$$

we obtain

$$\lim_{\varepsilon\to 0}\left[\iint\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|\le 1}\frac{\varepsilon|(\nabla g(x)-\nabla g(y))\cdot(x-y)|^{p+\varepsilon}}{|x-y|^{N+2p}}dxdy + \iint\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy\right] = \frac{1}{2}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|D^2 g\,(x)\,(\sigma,\sigma)\right|^p dxd\sigma.$$

We have the statement (4) by noting that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|(\nabla g(x)-\nabla g(y))\cdot(x-y)|>1}\frac{\varepsilon}{|x-y|^{N+2p}}dxdy = 0. \quad \square$$

Proof of Theorem 1.6. is similar to the proof of Theorem 1.3; Theorem 1.4 and will be omitted.

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