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# New characterizations of Sobolev spaces on the Heisenberg group ☆

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ABSTRACT

The main aim of this paper is to present some new characterizations of Sobolev spaces on the Heisenberg group $\mathbb{H}$. First, among several results (Theorems 1.1 and 1.2), we prove that if $f \in L^p(\mathbb{H})$, $p > 1$, then $f \in W^{1,p}(\mathbb{H})$ if and only if

$$\sup_{0<\delta<1} \iint\limits_{|f(u)-f(v)|>\delta} \frac{\delta^p}{|u^{-1}v|^{Q+p}} dudv < \infty.$$

This characterizations is in the spirit of the work by Bourgain, Brezis and Mironescu [5], in particular, the work by Hoai-Minh Nguyen [29] in the Euclidean spaces. Our work extends that of Nguyen to Sobolev spaces $W^{1,p}(\mathbb{H})$ for $p > 1$ in the setting of Heisenberg group. Second, corresponding to the case $p = 1$, we give a characterizations of BV functions on the Heisenberg group (Theorems 4.1 and 4.2). Third, we give some more generalized characterizations of Sobolev spaces on the Heisenberg groups (Theorems 5.1 and 5.2).

It is worth to note that the underlying geometry of the Euclidean spaces, such as that any two points in $\mathbb{R}^N$ can be connected by a line-segment, plays an important role in the proof of the main theorems in [29]. Thus, one of the main techniques in [29] is to use the uniformity in every directions of the unit sphere in the Euclidean spaces. More precisely, to

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deal with the general case $\sigma \in \mathbb{S}^{N-1}$, it is often assumed that $\sigma = e_N = (0, \dots, 0, 1)$ and, hence, one just needs to work on the one-dimensional case. This can be done by using the rotation in the Euclidean spaces. Due to the non-commutative nature of the Heisenberg group, the absence of this uniformity on the Heisenberg group creates extra difficulties for us to handle. Hence, we need to find a different approach to establish this characterization.



## 1. Introduction

The theory of Sobolev spaces plays a crucial role in the study of many sides of partial differential equations and calculus of variations. Moreover, the range of its applications is much larger, such as problems in algebraic topology, complex analysis, differential geometry, probability theory, etc.

The classical definition of Sobolev space is as follows: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the set of all functions $u \in L^p(\Omega)$ such that for every multi-index $\alpha$ with $|\alpha| \le k$, the weak partial derivative $D^\alpha u$ belongs to $L^p(\Omega)$, i.e.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \ \forall |\alpha| \le k\}.$$

Here, $\Omega$ is an open set in $\mathbb{R}^n$ and $1 \le p \le +\infty$. The natural number $k$ is called the order of the Sobolev space $W^{k,p}(\Omega)$. This definition can be extended easily to other settings such as Riemannian manifolds, since the gradient is well-defined there [18]. Moreover, we can also define the fractional Sobolev space, where the order $k$ is not a natural number, via Bessel potentials [33].

Sobolev spaces on Riemannian manifolds or with metric measure spaces as targeted spaces have been studied by, e.g., Korevaar and Schoen [19], Hebey [18], etc. There have been characterizations of Sobolev spaces in doubling metric measure spaces. For instance, various characterizations of first order Sobolev spaces in metric measure spaces have been given using a Lipschitz type (pointwise) estimate by Hajlasz [17], then using Poincaré type inequalities by Franchi, Lu and Wheeden [16] for the first order Sobolev spaces (see also Franchi, Hajlasz and Koskela [15]), and subsequently by Liu, Lu and Wheeden [21] for high order Sobolev spaces, etc. The Heisenberg group (and more generally, stratified groups) is a special case of metric measure spaces with doubling measures. The characterizations given in [17,16] and [21] also give alternative definitions of non-isotropic Sobolev spaces on the Heisenberg group. Indeed, it was shown in [21] that the definition of non-isotropic Folland–Stein spaces [14] is equivalent to the Sobolev spaces on stratified groups using the higher order Poincaré inequalities (see also [23,24,26,27,10]).

Nevertheless, the main purpose of our paper focuses on those types of characterizations of Sobolev spaces on the Heisenberg group in the spirit of characterizations given by

Bourgain, Brezis and Mironescu [5] and Hoai-Minh Nguyen [29] in the Euclidean spaces. To this end, we will first recall those results of [5] and [29].

**Theorem A.** *(See Bourgain, Brezis and Mironescu [5].) Let $g \in L^p(\mathbb{R}^N)$, $1 < p < \infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ iff*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy \le C, \quad \forall n \ge 1,$$

*for some constant $C > 0$. Moreover,*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p dx, \quad \forall g \in L^p\big(\mathbb{R}^N\big).$$

*Here*

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p dx$$

*for any $e \in \mathbb{S}^{N-1}$. Here $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative radial mollifiers satisfying*

$$\lim_{n\to\infty}\int_\tau^\infty \rho_n(r)r^{N-1}dr = 0, \quad \forall \tau > 0,$$

$$\lim_{n\to\infty}\int_0^\infty \rho_n(r)r^{N-1}dr = 1.$$

This result is studied further and extended in [2–4,7,20,28,31].

Recently, Hoai-Minh Nguyen [29] established some new characterizations of the Sobolev space $W^{1,p}(\mathbb{R}^N)$ that are closely related to Theorem A. More precisely, it was conjectured by Brezis and confirmed in [29] that

**Theorem B.** *(See H.M. Nguyen [29].) Let $1 < p < \infty$. Then*

(a) *There exists a positive constant $C_{N,p}$ depending only on $N$ and $p$ such that*

$$\int\limits_{\substack{\mathbb{R}^N\mathbb{R}^N \\ |g(x)-g(y)|>\delta}}\int \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p dx, \quad \forall \delta > 0,\ \forall g \in W^{1,p}\big(\mathbb{R}^N\big).$$

(b) *If $g \in L^p(\mathbb{R}^N)$ satisfies*

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} dxdy < \infty,$$

*then* $g \in W^{1,p}(\mathbb{R}^N)$.

(c) *Moreover, for any* $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\delta\to 0} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} dxdy = \frac{1}{p} K_{N,p} \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx.$$

This result is considered further in [6,30].

In this paper, we will establish results of the type similar to Theorem B in the setting of Heisenberg groups. Let $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ be the $n$-dimensional Heisenberg group whose group structure is given by

$$(z,t)\cdot(z',t') = \left(z+z', t+t'+2\,\mathrm{Im}\left(z\cdot\overline{z'}\right)\right),$$

for any two points $(z,t)$ and $(z',t')$ in $\mathbb{H}$. The Lie algebra of $\mathbb{H}$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \qquad X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}$$

for $i = 1,\ldots,n$. These generators satisfy the non-commutative relationship

$$[X_i, Y_j] = -4\delta_{ij}T.$$

Moreover, all the commutators of length greater than two vanish, and thus this is a nilpotent, graded, and stratified group of step two.

For each real number $r \in \mathbb{R}$, there is a dilation naturally associated with Heisenberg group structure which is usually denoted as

$$\delta_r(z,t) = \left(rz, r^2 t\right), \quad \forall (z,t) \in \mathbb{H}.$$

However, for simplicity we will write $ru$ to denote $\delta_r u$. The Jacobian determinant of $\delta_r$ is $r^Q$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}$.

We use $\xi = (z,t)$ to denote any point $(z,t) \in \mathbb{H}$ and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ to denote the homogeneous norm of $\xi \in \mathbb{H}$. With this norm, we can define a Heisenberg ball centered at $\xi = (z,t)$ with radius $R$: $B(\xi,R) = \{v \in \mathbb{H} : \rho(\xi^{-1}\cdot v) < R\}$. The volume of such a ball is $\sigma_Q = C_Q R^Q$ for some constant $C_Q$ depending only on $Q$. We also define $\varSigma$ the unit sphere in the Heisenberg group $\mathbb{H}$:

$$\varSigma = \left\{\xi \in \mathbb{H} : \rho(\xi) = 1\right\}.$$

We use $\nabla_{\mathbb{H}} f$ to express the horizontal subgradient of the function $f : \mathbb{H} \to \mathbb{R}$:

$$\nabla_{\mathbb{H}} f = \sum_{j=1}^{n} \big((X_j f)X_j + (Y_j f)Y_j\big).$$

Let $\Omega$ be an open set in $\mathbb{H}$. We use $W_0^{1,p}(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ under the norm $\|f\|_{W_0^{1,p}(\Omega)} = (\int_\Omega (|\nabla_{\mathbb{H}} f|^p + |f|^p)du)^{1/p}$.

The first aim of this paper is to prove the following estimates for functions in the Sobolev space $W^{1,p}(\mathbb{H})$:

**Theorem 1.1.** *Let $1 < p < \infty$ and $f \in W^{1,p}(\mathbb{H})$. Then*

(a) *There exists a positive constant $C_{Q,p}$ depending only on $Q, p$ such that*

$$\sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon |f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \le C_{Q,p} \int\limits_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du.$$

(b) *There holds*

$$\lim_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon |f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv = K_{Q,p} \int\limits_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du,$$

*where $K_{Q,p}$ is a constant defined as follows*

$$K_{Q,p} = \int\limits_{\Sigma} \big|\langle e, \sigma'\rangle\big|^p d\sigma = \int\limits_{\Sigma} |\langle (e,0), \sigma\rangle|^p d\sigma$$

*for any $(e, 0) \in \Sigma$.*

(c) *There exists a positive constant $C_{Q,p}$ such that*

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \le C_{Q,p} \int\limits_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du, \quad \forall \delta > 0.$$

(d) *Moreover,*

$$\liminf_{\delta\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv = \frac{1}{p} K_{Q,p} \int\limits_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du.$$

Our Theorem 1.1 extends Theorem 3 of H.M. Nguyen [29] in the Euclidean spaces. Using Theorem 1.1, we set up new characterizations of the Sobolev space $W^{1,p}(\mathbb{H})$ which is one of the main purposes of this paper. More precisely, we prove that

**Theorem 1.2.** *Let $1 < p < \infty$ and $f \in L^p(\mathbb{H})$. Then the following are equivalent:*

(1) $f \in W^{1,p}(\mathbb{H})$.

(2)

$$\sup_{0<\varepsilon<1} \iint_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv + \iint_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < \infty.$$

(3)

$$\sup_{0<\delta<1} \iint_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < \infty.$$

(4) *There exists a nonnegative function $F \in L^p(\mathbb{H})$ such that*

$$\bigl|f(u)-f(v)\bigr| \le \rho\bigl(u^{-1}\cdot v\bigr)\bigl(F(u)+F(v)\bigr) \quad \textit{for a.e. } u, v \in \mathbb{H}.$$

(5) *The $L^1$ to $L^p$ Poincaré inequalities hold for every metric ball $B$ in $\mathbb{H}$. Namely, there exists a function $g \in L^p_{loc}(\mathbb{H})$ and an absolute constant $C > 0$ independent of the ball $B$ such that*

$$\frac{1}{|B|}\int_B \bigl|f(u)-f_B\bigr|du \le Cr(B)\biggl(\frac{1}{|B|}\int_B |g|^q du\biggr)^{\frac{1}{q}}$$

*for some $1 \le q < p$, where $r(B)$ is the radius of the ball $B$.*

The equivalence of (1), (2) and (3) in the above Theorem 1.2 in the Euclidean spaces was given in Theorem 3 of [29].

The following remarks are in order. First, the proofs of the main theorems (e.g., Theorem B) in [29] rely on the underlying geometry of the Euclidean spaces, such as that any two points can be connected by a line-segment. Second, it is worth to note that one of the main techniques in the proof of Theorem B is to use the uniformity in every directions of the unit sphere in the Euclidean spaces. More precisely, to deal with the general case $\sigma \in \mathbb{S}^{N-1}$, it is often to be assumed that $\sigma = e_N = (0,\dots,0,1)$ and, hence, one just needs to work on 1-dimensional case. This can be done by using the rotation in the Euclidean spaces. In the case of Heisenberg groups, this type of property is not available because of the structure of the Heisenberg groups, in particular, the dilation. Hence, we need to find a different approach to this characterization. In fact, we will use the representation formula on the Heisenberg group proved in [22] to obtain estimate (2.1). This estimate will allow us to establish a useful lemma (Lemma 2.2 in Section 2). Third, as we have shown in [21], (1), (4) and (5) are all equivalent. Therefore, the new ingredient here is that (1), (2) and (3) are equivalent. Fourth, results of this paper together with characterizations of second order Sobolev spaces in Euclidean spaces established in [12] have been presented in [11]. We have also extended results in this paper to general stratified groups in [13].

The plan of this paper is as follows: In Section 2, we will study some helpful lemmas and use them to prove Theorem 1.1 which gives properties of Sobolev functions in $W^{1,p}(\mathbb{H})$ for $1 < p < \infty$. Theorem 1.2 gives the characterizations of Sobolev functions in $W^{1,p}(\mathbb{H})$ for $1 < p < \infty$ and will be considered in Section 3. The borderline case $p = 1$ (i.e., for BV functions) will be investigated in Section 4. We also study some generalizations and variants in Section 5 which extends Theorems 1.1 and 1.2.

## 2. Proof of Theorem 1.1

### 2.1. Some preliminary lemmas

We first recall an elementary lemma from [29] and include a proof.

**Lemma 2.1.** *Let $\Omega$ be a measurable set in $\mathbb{R}^m$, $\Phi$ and $\Psi$ be two measurable nonnegative functions on $\Omega$, and $\alpha > -1$. Then*

$$\int_0^1 \int_{\Phi(x)>\delta} \delta^\alpha \Psi(x)\,dx\,d\delta = \int_{\Phi(x)\le 1} \frac{1}{\alpha+1}\Phi^{\alpha+1}(x)\Psi(x)\,dx + \int_{\Phi(x)>1} \frac{1}{\alpha+1}\Psi(x)\,dx.$$

**Proof.** Using Fubini's theorem, we get

$$\begin{aligned}
&\int_0^1 \int_{\Phi(x)>\delta} \delta^\alpha \Psi(x)\,dx\,d\delta \\
&= \int_{\Phi(x)>1} \int_0^1 \delta^\alpha \Psi(x)\,d\delta\,dx + \int_{\Phi(x)\le 1} \int_0^{\Phi(x)} \delta^\alpha \Psi(x)\,d\delta\,dx \\
&= \int_{\Phi(x)>1} \frac{1}{\alpha+1}\Psi(x)\,dx + \int_{\Phi(x)\le 1} \frac{1}{\alpha+1}\Phi^{\alpha+1}(x)\Psi(x)\,dx. \qquad \square
\end{aligned}$$

Next lemma is crucial in establishing our new characterizations of Sobolev spaces on the Heisenberg group $\mathbb{H}$. In the Euclidean spaces, H.M. Nguyen [29] used the property that every two points can be connected by a line-segment and then used the mean-value theorem to control the difference of $|f(x) - f(x+he)|$ (where $h \in \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$) by the Hardy–Littlewood maximal function of the partial derivative of $f$ in the direction of $e$. Such an argument does not work on the Heisenberg group. Therefore, we need to adapt a new argument by using the representation formula on the Heisenberg group $\mathbb{H}$ established in [22].

**Lemma 2.2.** *Let $f \in W^{1,p}(\mathbb{H})$, $1 < p < \infty$. Then we have*

$$\iint\limits_{|f(u)-f(v)|>\delta} \frac{\delta^p}{\rho(u^{-1} \cdot v)^{Q+p}} du dv \le C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du, \quad \forall \delta > 0$$

*where $C_{Q,p}$ is a positive constant depending only on $Q$ and $p$.*

**Proof.** First, we recall the following pointwise estimate on stratified groups proved in [22] (see Lemma 3.1 on page 382 there), for any metric ball $B$ in $\mathbb{H}$ and every $u \in B$, we have

$$\left|f(u) - f_B\right| \le C \int\limits_{cB} \frac{|\nabla_{\mathbb{H}} f(v)|}{|u^{-1}v|^{Q-1}} dv$$

where $f_B$ is the average of $f$ over $B$ and $c$ is a positive uniform constant bigger than or equal to 1.

Then we can show that

$$\begin{aligned} &\left|f(u) - f(v)\right| \le A_{Q,p}\rho\left(u^{-1} \cdot v\right)\left(M\left(|\nabla_{\mathbb{H}} f|\right)(u) + M\left(|\nabla_{\mathbb{H}} f|\right)(v)\right) \\ &\quad \text{for a.e. } u, v \in \mathbb{H} \end{aligned} \tag{2.1}$$

where $M$ denoted the Hardy–Littlewood maximal function

$$M(f)(u) = \sup_{r>0} \frac{1}{|B(u,r)|} \int\limits_{B(u,r)} f(v) dv$$

and $A_{Q,p}$ is the universal constant depending only on $Q$ and $p$.

Now noting that by (2.1):

$$\begin{aligned} &\left\{\left|f(u) - f(v)\right| > \delta\right\} \\ &\quad \subset \left\{A_{Q,p}\rho\left(u^{-1} \cdot v\right)\left(M\left(|\nabla_{\mathbb{H}} f|\right)(u) + M\left(|\nabla_{\mathbb{H}} f|\right)(v)\right) > \delta\right\} \\ &\quad \subset \left\{\rho\left(u^{-1} \cdot v\right) M\left(|\nabla_{\mathbb{H}} f|\right)(u) > \frac{\delta}{2A_{Q,p}}\right\} \cup \left\{\rho\left(u^{-1} \cdot v\right) M\left(|\nabla_{\mathbb{H}} f|\right)(v) > \frac{\delta}{2A_{Q,p}}\right\}, \end{aligned}$$

we get

$$\begin{aligned}
&\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>\delta}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \\
&\leq \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ \rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}}f|)(u)>\frac{\delta}{2A_{Q,p}}}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \\
&\quad + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ \rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}}f|)(v)>\frac{\delta}{2A_{Q,p}}}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}dudv.
\end{aligned}$$

Denote

$$\begin{aligned}
I_1 &:= \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ \rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}}f|)(u)>\frac{\delta}{2A_{Q,p}}}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}dudv, \\
I_2 &:= \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ \rho(u^{-1}\cdot v)M(|\nabla_{\mathbb{H}}f|)(v)>\frac{\delta}{2A_{Q,p}}}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}dudv.
\end{aligned}$$

Now, we estimate $I_1$.

Set

$$v = u \cdot h\sigma$$

where

$$\begin{aligned}
&\sigma \in \Sigma = \{u \in \mathbb{H} : |u| = 1\}, \\
&h \in [0, \infty),
\end{aligned}$$

then

$$\begin{aligned}
I_1 &= \iiint\limits_{\substack{\Sigma\ \mathbb{H}\ 0\\ hM(|\nabla_{\mathbb{H}}f|)(u)>\frac{\delta}{2A_{Q,p}}}}^{\infty} \frac{\delta^p}{h^{p+1}}dhdud\sigma \\
&\leq \int\limits_{\Sigma}\int\limits_{\mathbb{H}} \int\limits_{\frac{\delta}{2A_{Q,p}M|\nabla_{\mathbb{H}}f|(u)}}^{\infty} \frac{\delta^p}{h^{p+1}}dhdud\sigma \\
&= \frac{1}{p}\int\limits_{\Sigma}\int\limits_{\mathbb{H}} \left[2A_{Q,p}M|\nabla_{\mathbb{H}}f|(u)\right]^p dud\sigma
\end{aligned}$$

$$\leq \frac{(2A_{Q,p})^p}{p} \int\limits_{\Sigma} \int\limits_{\mathbb{H}} \left[M|\nabla_{\mathbb{H}} f|(u)\right]^p du d\sigma$$

$$\leq C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du.$$

Similarly, to estimate $I_2$, we put

$$u = v \cdot h\sigma.$$

Noting that $\rho(u^{-1} \cdot v) = |v^{-1}u|$, we still can get

$$I_2 \leq C_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du.$$

The proof now is completed. □

Before we state and prove the next lemma, we like to make the following remark. Let $f \in W^{1,p}(\mathbb{H})$, $1 < p < \infty$. We denote

$$I(\delta) = \int\limits_{\substack{\mathbb{H} \ \mathbb{H} \\ |f(u)-f(v)|>\delta}}\int \frac{\delta^p}{\rho(u^{-1} \cdot v)^{Q+p}} du dv$$

and

$$J(\varepsilon) = \int\limits_{\substack{\mathbb{H} \ \mathbb{H} \\ |f(u)-f(v)|\leq 1}}\int \frac{\varepsilon|f(u) - f(v)|^{p+\varepsilon}}{\rho(u^{-1} \cdot v)^{Q+p}} du dv.$$

This quantity $J(\varepsilon)$ in the Euclidean spaces was first introduced by H.M. Nguyen in Theorem 3 of [29] and played an important role in the proof of Theorem 2 in [29]. This quantity $J(\varepsilon)$ on the Heisenberg group also appears in our Theorems 1.1 and 1.2 and plays an important role in our characterizations of Sobolev spaces $W^{1,p}(\mathbb{H})$ as well.

By Lemma 2.2, $\liminf_{\delta\to 0} I(\delta)$ does exist. In the setting of Euclidean spaces [29], this limit is rather easy to evaluate. More precisely, by polar coordinates and the rotations in the Euclidean spaces, it is often assumed in [29] that $\sigma = e_N = (0, \ldots, 0, 1)$. Thus, to deal with the general case $\sigma \in \mathbb{S}^{N-1}$, the author in [29] just needs to consider the one-dimensional case. Then, using real analysis techniques such as the Maximal function, Lebesgue's dominated convergence theorem, Hoai-Minh Nguyen finds successfully the exact value of $\liminf_{\delta\to 0} I(\delta)$. In our setting of Heisenberg group $\mathbb{H}$, this approach is not available because of the underlying geometry on the Heisenberg group. Hence, we need to propose a new method in order to calculate $\liminf_{\delta\to 0} I(\delta)$. Indeed, our main idea is that we will first study the relations of $I(\delta)$ and $J(\varepsilon)$.

In fact, we can prove the following results on the Heisenberg group which are important for us to establish the characterizations of Sobolev spaces $W^{1,p}(\mathbb{H})$ on the Heisenberg group.

**Lemma 2.3.** *Let $f \in W^{1,p}(\mathbb{H})$, $1 < p < \infty$. There hold*

$$\liminf_{\varepsilon\to 0} J(\varepsilon) \geq \frac{1}{p} \liminf_{\delta\to 0} I(\delta)$$
$$\limsup_{\varepsilon\to 0} J(\varepsilon) \leq \frac{1}{p} \limsup_{\delta\to 0} I(\delta).$$

**Proof.** By Lemma 2.2, $\liminf_{\delta\to 0} I(\delta)$ and $\limsup_{\delta\to 0} I(\delta)$ exist. Assume that

$$\liminf_{\delta\to 0} I(\delta) = C$$
$$\limsup_{\delta\to 0} I(\delta) = D.$$

We first prove that

$$\liminf_{\varepsilon\to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta \geq pC. \tag{2.2}$$

Indeed, since

$$\liminf_{\delta\to 0} I(\delta) = C,$$

for every $\tau > 0$, we can find a number $X(\tau) \in (0,1)$ such that

$$I(\delta) > C - \tau \quad \text{for all } \delta \in \big(0, X(\tau)\big).$$

Then

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta \\
&= \liminf_{\varepsilon\to 0} \left[ \int_0^{X(\tau)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta + \int_{X(\tau)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta \right] \\
&\geq \liminf_{\varepsilon\to 0} \int_0^{X(\tau)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta
\end{aligned}$$

$$\geq \liminf_{\varepsilon\to 0} \int_0^{X(\tau)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}(C-\tau)d\delta$$
$$\geq p(C-\tau).$$

Since $\tau$ is arbitrary, we now can conclude that

$$\liminf_{\varepsilon\to 0} \int_0^1 \iint\limits_{|f(u)-f(v)|>\delta} \frac{(p+\varepsilon)\varepsilon\delta^{p+\varepsilon-1}}{\rho(u^{-1}\cdot v)^{Q+p}} dudvd\delta \geq pC.$$

Using Lemma 2.1 with $\alpha = p+\varepsilon-1$, $\Phi(u,v) = |f(u)-f(v)|$, $\Psi(u,v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, we obtain

$$\liminf_{\varepsilon\to 0}\Bigg[ \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|\leq 1} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv + \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|>1} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}} dudv\Bigg] \geq pC.$$

Noting that

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|>1} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}} dudv = 0,$$

we have

$$\liminf_{\varepsilon\to 0} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|\leq 1} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \geq pC.$$

Similarly, since

$$\limsup_{\delta\to 0} I(\delta) = D,$$

for every $\tau > 0$, we can find a number $X(\tau) \in (0,1)$ such that

$$I(\delta) < D+\tau \quad \text{for all } \delta \in \big(0, X(\tau)\big).$$

Then

$$\limsup_{\varepsilon\to 0} \int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta$$
$$= \limsup_{\varepsilon\to 0}\Bigg[ \int_0^{X(\tau)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta + \int_{X(\tau)}^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1} I(\delta)d\delta\Bigg]$$

$$\leq \limsup_{\varepsilon\to 0}\Bigg[\int_0^{X(\tau)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}(D+\tau)\,d\delta$$

$$+ (p+\varepsilon)\varepsilon X(\tau)^{\varepsilon-1}C_{Q,p}\int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du \int_{X(\tau)}^1 1\,d\delta\Bigg]$$

$$\leq \limsup_{\varepsilon\to 0}\int_0^{X(\tau)} (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}(D+\tau)\,d\delta$$

$$\leq p(D+\tau).$$

Since $\tau$ is arbitrary, we now can conclude that

$$\limsup_{\varepsilon\to 0}\int_0^1 \iint_{|f(u)-f(v)|>\delta} \frac{(p+\varepsilon)\varepsilon\delta^{p+\varepsilon-1}}{\rho(u^{-1}\cdot v)^{Q+p}}\,du\,dv\,d\delta \leq pD.$$

Using Lemma 2.1 with $\alpha = p+\varepsilon-1$, $\Phi(u,v) = |f(u)-f(v)|$, $\Psi(u,v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, we obtain

$$\limsup_{\varepsilon\to 0}\Bigg[\int_{\mathbb{H}}\int_{\mathbb{H}}_{|f(u)-f(v)|\leq 1} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}\,du\,dv + \int_{\mathbb{H}}\int_{\mathbb{H}}_{|f(u)-f(v)|>1} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}}\,du\,dv\Bigg] \leq pD.$$

Noting that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}}\int_{\mathbb{H}}_{|f(u)-f(v)|>1} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}}\,du\,dv = 0,$$

we have

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{H}}\int_{\mathbb{H}}_{|f(u)-f(v)|\leq 1} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}\,du\,dv \leq pD. \qquad \square$$

**Lemma 2.4.** *There holds*

$$\frac{1}{p}K_{Q,p}\int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du \leq \liminf_{\delta\to 0}\int_{\mathbb{H}}\int_{\mathbb{H}}_{|f(u)-f(v)|>\delta} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}\,du\,dv,$$

*for any* $f \in W^{1,p}(\mathbb{H})$, $1 < p < \infty$.

**Proof.** First, we notice by the change of variable that

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>\delta}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv = \iiint\limits_{\substack{\Sigma\ \mathbb{H}\ 0\\ |\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}} dhdud\sigma.$$

Now, fix $\sigma = (\sigma', \sigma_{2n+1}) \in \Sigma$, with $\sigma' \in \mathbb{C}^n$, we will show that

$$\frac{1}{p}\int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\cdot\sigma'\right|^p du \le \liminf_{\delta\to 0} \iint\limits_{\substack{\mathbb{H}\ 0\\ |\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}} dhdu. \tag{2.3}$$

Indeed, since for a.e. $(h,u) \in (0,\infty)\times\mathbb{H}$:

$$\frac{1}{h^{p+1}}\chi_{\{|\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}|h>1\}}(h,u) \xrightarrow{\delta\to 0} \frac{1}{h^{p+1}}\chi_{\{|\langle\nabla_{\mathbb{H}} f(u),\sigma'\rangle|h>1\}}(h,u),$$

by Fatou's lemma, we have

$$\begin{aligned}\liminf_{\delta\to 0} \iint\limits_{\substack{\mathbb{H}\ 0\\ |\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}} dhdu &\ge \int\limits_{\mathbb{H}}\int\limits_{0}^{\infty} \frac{1}{h^{p+1}}\chi_{\{|\langle\nabla_{\mathbb{H}} f(u),\sigma'\rangle|h>1\}}(h,u) dhdu \\ &= \frac{1}{p}\int\limits_{\mathbb{H}} \left|\left\langle\nabla_{\mathbb{H}} f(u),\sigma'\right\rangle\right|^p du.\end{aligned}$$

Now, again by the Fatou's lemma, we obtain

$$\begin{aligned}&\liminf_{\delta\to 0} \iint\limits_{|f(u)-f(v)|>\delta} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \\ &= \liminf_{\delta\to 0} \iiint\limits_{\substack{\Sigma\ \mathbb{H}\ 0\\ |\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}} dhdud\sigma \\ &\ge \int\limits_{\Sigma} \liminf_{\delta\to 0} \iint\limits_{\substack{\mathbb{H}\ 0\\ |\frac{f(u\cdot\delta h\sigma)-f(u)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}} dhdud\sigma \\ &\ge \frac{1}{p}\int\limits_{\Sigma}\int\limits_{\mathbb{H}} \left|\left\langle\nabla_{\mathbb{H}} f(u),\sigma'\right\rangle\right|^p dud\sigma \\ &= \frac{1}{p}K_{Q,p}\int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du. \qquad \square\end{aligned}$$

**Lemma 2.5.** *Let $f \in C_0^1(\mathbb{H})$. Then*

$$\limsup_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv \le K_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du.$$

**Proof.** By setting $v = u \cdot h\sigma$, we have

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv = \iiint\limits_{\substack{\Sigma\ \mathbb{H}\ 0 \\ |f(u\cdot h\sigma)-f(u)|\le 1}}^{\infty} \frac{\varepsilon|f(u\cdot h\sigma)-f(u)|^{p+\varepsilon}}{h^{p+1}} dudhd\sigma.$$

In the following, $C$ will be a constant independent of $u, h, \sigma, \varepsilon$.

Since $f \in C_0^1(\mathbb{H})$, by triangle inequality and Taylor expansion [25], we have

$$\left|f(u\cdot h\sigma) - f(u)\right| \le \left|\nabla f(u).h\sigma'\right| + Ch^2 \quad \text{for } (\sigma, u, h) \in \Sigma \times B_A \times (0, R).$$

Also, we can find $M > 0$ such that $|\nabla f(u).\sigma'| \le M$ for all $(\sigma, u) \in \Sigma \times B_A$.

Hence,

$$\begin{aligned} \left|f(u\cdot h\sigma) - f(u)\right|^{p+\varepsilon} &\le \left[\left|\nabla f(u).h\sigma'\right| + Ch^2\right]^{p+\varepsilon} \\ &\le \left|\nabla f(u).h\sigma'\right|^{p+\varepsilon} + Ch^{p+\varepsilon+1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \iiint\limits_{\substack{\Sigma\ B_A\ 0 \\ |f(u\cdot h\sigma)-f(u)|\le 1}}^{R} \frac{\varepsilon|f(u\cdot h\sigma)-f(u)|^{p+\varepsilon}}{h^{p+1}} dhdud\sigma \\ &\le \limsup_{\varepsilon\to 0} \int\limits_{\Sigma}\int\limits_{B_A}\int\limits_0^R \frac{\varepsilon|\nabla f(u).h\sigma'|^{p+\varepsilon} + C\varepsilon h^{p+\varepsilon+1}}{h^{p+1}} dhdud\sigma \\ &\le \limsup_{\varepsilon\to 0} \int\limits_{\Sigma}\int\limits_{B_A}\int\limits_0^R \frac{\varepsilon|\nabla f(u).h\sigma'|^{p+\varepsilon}}{h^{p+1}} dhdud\sigma \\ &\le \limsup_{\varepsilon\to 0} \int\limits_{\Sigma}\int\limits_{B_A}\int\limits_0^R \frac{\varepsilon|\nabla f(u).\sigma'|^{p+\varepsilon}}{h^{1-\varepsilon}} dhdud\sigma \\ &\le \limsup_{\varepsilon\to 0} R^{\varepsilon} \int\limits_{\Sigma}\int\limits_{B_A} \left|\nabla f(u).\sigma'\right|^{p+\varepsilon} dud\sigma \end{aligned}$$

$$\le \limsup_{\varepsilon\to 0} R^{\varepsilon} M^{\varepsilon} \int_{\Sigma}\int_{B_A} \left|\nabla f(u).\sigma'\right|^p du d\sigma$$

$$\le K_{Q,p} \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du. \qquad \square$$

*2.2. Proof of Theorem 1.1*

(a) First, by Lemma 2.2, we have

$$\underset{|f(u)-f(v)|>\delta}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le C_{Q,p} \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du, \quad \forall \delta > 0. \tag{2.4}$$

As consequences, we get

$$\underset{|f(u)-f(v)|>1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le C_{Q,p} \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du. \tag{2.5}$$

Now, multiplying (2.4) by $\varepsilon\sigma^{\varepsilon-1}$, $0 < \varepsilon < 1$ and integrating the expression obtained with respect to $\sigma$ over $(0, 1)$, we can deduce that

$$\underset{|f(u)-f(v)|>\delta}{\int_0^1\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon\delta^{p+\varepsilon-1}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv d\delta \le C_{Q,p} \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du.$$

Using Lemma 2.1 with $\alpha = p + \varepsilon - 1$, $\Phi(u, v) = |f(u) - f(v)|$, $\Psi(u, v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, we obtain

$$\underset{|f(u)-f(v)|\le 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv + \underset{|f(u)-f(v)|>1}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}} du dv$$
$$\le C_{Q,p} \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du.$$

Thus,

$$\iint_{|f(u)-f(v)|\le 1} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le C_{Q,p} \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du. \tag{2.6}$$

By (2.5) and (2.6), we get the assertion (a).

(b) From Lemmas 2.4, 2.3, 2.5 and the density argument, we have (b).
(c) This is Lemma 2.2.
(d) This is a consequence of Lemmas 2.4, 2.3, 2.5 and the density argument.

## 3. Proof of Theorem 1.2

The proof is divided into six steps.

**Step 1:** (1) ⇒ (2). This is a consequence of part (a) of Theorem 1.1 and the fact that $f \in W^{1,p}(\mathbb{H})$.

**Step 2:** (2) ⇒ (1). First, we will assume further that $f \in L^\infty(\mathbb{H})$. Then from the assumption

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}_{|f(u)-f(v)|\le 1} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv + \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}_{|f(u)-f(v)|>1} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} du dv < \infty,$$

it is easy to deduce that

$$L(f) = \sup_{0<\varepsilon<1} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv < \infty.$$

Now, let $(\gamma_k)$ be a sequence of smooth mollifiers on $\mathbb{H}$ and set

$$f_k := f * \gamma_k.$$

Since $f_k \in L^p(\mathbb{H}) \cap C^\infty(\mathbb{H})$, so $f_k \in W^{1,p}(\mathbb{H})$. By using Theorem 1.1(b), we can conclude that

$$\begin{aligned} K_{Q,p} \int\limits_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f_k(u)\right|^p du &\le \liminf_{\varepsilon\to 0} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}_{|f_k(u)-f_k(v)|\le 1} \frac{\varepsilon|f_k(u)-f_k(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \\ &\le \liminf_{\varepsilon\to 0} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f_k(u)-f_k(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv. \end{aligned}$$

From Jensen's inequality and the convexity of the function $x^{p+\varepsilon}$, we can obtain

$$\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f_k(u)-f_k(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le L(f).$$

Hence,

$$K_{Q,p}\int_{\mathbb{H}}\left|\nabla_{\mathbb{H}} f_k(u)\right|^p du \le \liminf_{\varepsilon\to 0}\int_{\mathbb{H}}\int_{\mathbb{H}}\frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \le L(f).$$

Thus, with an extra assumption $f \in L^\infty(\mathbb{H})$, we have that (2) ⇒ (1).

For the general case, we make use of the truncated function. For $R > 0$, define

$$f_R(u) = \begin{cases} [c]cf(u), & \text{if } |f(u)| < R, \\ \frac{Rf(u)}{|f(u)|}, & \text{otherwise.} \end{cases}$$

It is clear that $f_R \in L^\infty(\mathbb{H})$ and $f_R(u) \overset{R\to\infty}{\longrightarrow} f(u)$ pointwise for a.e. $u \in \mathbb{H}$. Moreover, it can be checked that

$$\left|f_R(u) - f_R(v)\right| \le \left|f(u) - f(v)\right| \quad \text{for all } u, v \in \mathbb{H}.$$

As a consequence, one has

$$\begin{aligned}
&\iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f_R(u)-f_R(v)|\le 1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \\
&= \iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f_R(u)-f_R(v)|\le 1\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \\
&\quad + \iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f_R(u)-f_R(v)|\le 1\\ |f(u)-f(v)|> 1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \\
&\le \iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv + \iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f(u)-f(v)|> 1}} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}}dudv.
\end{aligned}$$

Also,

$$\iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f_R(u)-f_R(v)|> 1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \le \iint\limits_{\substack{\mathbb{H}\,\mathbb{H}\\ |f(u)-f(v)|> 1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}dudv.$$

Thus, we have $f_R \in W^{1,p}(\mathbb{H})$. Moreover, by part (b) of Theorem 1.1, one has

$$
\begin{aligned}
&K_{Q,p}\int\limits_{\mathbb{H}}\left|\nabla_{\mathbb{H}}f_R(u)\right|^p du \\
&\quad\le \liminf_{\varepsilon\to 0}\underset{|f_R(u)-f_R(v)|\le 1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{\varepsilon|f_R(u)-f_R(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \\
&\quad\le \liminf_{\varepsilon\to 0}\Bigg[\underset{|f(u)-f(v)|\le 1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv+\underset{|f(u)-f(v)|>1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+p}}dudv\Bigg] \\
&\quad= \liminf_{\varepsilon\to 0}\underset{|f(u)-f(v)|\le 1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv.
\end{aligned}
$$

Since $R > 0$ is arbitrary, we can deduce that $f \in W^{1,p}(\mathbb{H})$.

**Step 3:** (1) $\Rightarrow$ (3). This is a consequence of part (c) of Theorem 1.1 and the fact that $f \in W^{1,p}(\mathbb{H})$.

**Step 4:** (3) $\Rightarrow$ (1). Suppose that $f \in L^p(\mathbb{H})$ and there is a constant $C > 0$ such that for all $\delta \in (0,1)$:

$$
\underset{|f(u)-f(v)|>\delta}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{\delta^p}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \le C. \tag{3.1}
$$

Multiplying (3.1) by $\varepsilon\delta^{\varepsilon-1}$, $0 < \varepsilon < 1$, and integrating with respect to $\delta$ over $(0,1)$, by Lemma 2.1 with $\alpha = p+\varepsilon-1$, $\Phi(u,v) = |f(u)-f(v)|$, $\Psi(u,v) = \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}$, one has

$$
\underset{|f(u)-f(v)|\le 1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \le C(p+1).
$$

Also, by Fatou's lemma, we also get

$$
\underset{|f(u)-f(v)|>1}{\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}}\frac{1}{\rho(u^{-1}\cdot v)^{Q+p}}dudv < \infty.
$$

As a consequence of Step 2, we have $f \in W^{1,p}(\mathbb{H})$.

The proof is now completed.

## 4. The case $p = 1$ and BV functions on the Heisenberg group

In this section, we will investigate the special case $p = 1$. First, we recall the definition of the space $BV(\Omega)$ of functions with bounded variation in $\Omega \subset \mathbb{H}$.

**Definition 4.1** *(Horizontal vector fields).* The space of smooth sections of $H\Omega$, the horizontal subbundle on $\Omega$, is denoted by $\Gamma(H\Omega)$. The space $\Gamma_c(H\Omega)$ denotes all the elements of $\Gamma(H\Omega)$ with support contained in $\Omega$. Elements of $\Gamma(H\Omega)$ are called horizontal vector fields.

**Definition 4.2** *(H-BV functions).* We say that a function $u \in L^1(\Omega)$ is a function of $H$-bounded variation if

$$|D_H u|(\Omega) = \sup\left\{\int_\Omega u \operatorname{div} \phi d\xi : \phi \in \Gamma_c(H\Omega),\ |\phi| \le 1\right\} < \infty,$$

where the symbol div denotes the Riemannian divergence. We denote by $BV(\Omega)$ the space of all functions of $H$-bounded variation.

See [1,8,9] for definitions of $BV$ spaces on more general settings.

In this section, we will prove the following property:

**Theorem 4.1.** *Let $f$ be a function in $L^1(\mathbb{H})$ satisfying*

$$\sup_{0<\delta<1} \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|>\delta} \frac{\delta}{\rho(u^{-1}\cdot v)^{Q+1}} dudv < \infty.$$

*Then $f \in BV(\mathbb{H})$.*

**Proof.** Assume that $f \in L^1(\mathbb{H})$ and

$$\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|>\delta} \frac{\delta}{\rho(u^{-1}\cdot v)^{Q+1}} dudv < C \tag{4.1}$$

for some positive constant $C > 0$.

Proceeding similarly as in Step 4 of the proof of Theorem 1.2, multiplying (4.1) by $\varepsilon\delta^{\varepsilon-1}, 0 < \varepsilon < 1$, integrating with respect to $\delta$ over $(0, 1)$, and then using Lemma 2.1, we have

$$\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|\le 1} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}} dudv \le 2C.$$

By Fatou's lemma, from (4.1), we also get

$$\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}}\limits_{|f(u)-f(v)|>1} \frac{1}{\rho(u^{-1}\cdot v)^{Q+1}} dudv < C.$$

Now, we also split the proof into two steps:

**Step 1:** We suppose further that $f \in L^\infty(\mathbb{H})$. Now, we define $f_k$ as in Step 2 of the proof of Theorem 1.2. Noting that the function $t^{1+\varepsilon}$ is still convex on $\mathbb{R}^+$, we can also have

$$\begin{aligned}
&\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f_k(u)-f_k(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\quad\le \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\quad\le \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\qquad + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|> 1}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\quad\le 2C+\varepsilon\big[2\|f\|_\infty\big]^{1+\varepsilon}C.
\end{aligned}$$

Now, we can repeat the proofs of (b) in Theorem 1.1 and Step 2 in Theorem 1.2 to conclude that $f_k \in BV(\mathbb{H})$ and

$$\begin{aligned}
K_{Q,1}\|\nabla_{\mathbb{H}} f_k\| &\le \liminf_{\varepsilon\to 0}\int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\varepsilon|f_k(u)-f_k(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\le \liminf_{\varepsilon\to 0}\big\{2C+\varepsilon\big[2\|f\|_\infty\big]^{1+\varepsilon}C\big\} \\
&= 2C.
\end{aligned}$$

Hence, $f \in BV(\mathbb{H})$.

**Step 2:** The general case. Similarly, we also introduce the truncated function

$$f_R(u) = \begin{cases} [c]cf(u) & \text{if } |f(u)| < R, \\ \frac{Rf(u)}{|f(u)|} & \text{otherwise} \end{cases} \qquad \text{for } R>0.$$

Then one has $f_R \in L^\infty(\mathbb{H})$, $f_R(u) \overset{R\to\infty}{\longrightarrow} f(u)$ pointwise for a.e. $u \in \mathbb{H}$, and

$$\big|f_R(u)-f_R(v)\big| \le \big|f(u)-f(v)\big| \quad \text{for all } u,v\in\mathbb{H}.$$

As a consequence, one gets

$$\begin{aligned}
&\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\le 1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&= \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\le 1\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\quad + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\le 1\\ |f(u)-f(v)|>1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\le \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+1}}dudv.
\end{aligned}$$

Also,

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|>1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \le \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+1}}dudv.$$

Thus, we have $f_R \in BV(\mathbb{H})$. Moreover,

$$\begin{aligned}
&K_{Q,1}\|\nabla_{\mathbb{H}} f_R\| \\
&\le \liminf_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f_R(u)-f_R(v)|\le 1}} \frac{\varepsilon|f_R(u)-f_R(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\le \liminf_{\varepsilon\to 0} \Bigg[ \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \\
&\quad + \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|>1}} \frac{\varepsilon}{\rho(u^{-1}\cdot v)^{Q+1}}dudv \Bigg] \\
&= \liminf_{\varepsilon\to 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H}\\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{1+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+1}}dudv.
\end{aligned}$$

Since $R > 0$ is arbitrary, we can deduce that $f \in BV(\mathbb{H})$. □

Using Theorem 4.1, we can also have the following Lipschitz type characterization of $BV$ space:

**Theorem 4.2.** *Let $f \in L^1(\mathbb{H})$ be such that there exists a nonnegative function $F \in L^1(\mathbb{H})$ satisfying*

$$\left|f(u) - f(v)\right| \le \rho\left(u^{-1} \cdot v\right)\left(F(u) + F(v)\right) \quad \text{for a.e. } u, v \in \mathbb{H}. \tag{4.2}$$

*Then $f \in BV(\mathbb{H})$.*

Before we begin our proof of this theorem, we like to make some remarks. We note that in our Theorem 1.2, part (4), the Sobolev spaces $W^{1,p}(\mathbb{H})$ for $p > 1$ was characterized if the above estimate (4.2) holds for $F \in L^p(\mathbb{H})$. But this characterization does not hold for $p = 1$ (see also the paper [17]). Therefore, our theorem can be viewed as the borderline case of the Sobolev space when $p = 1$ on the Heisenberg group $\mathbb{H}$. More recently, it has been shown in [32] that if the above estimate (4.2) holds for $F \in L^1(\mathbb{H})$, then $f \in W^{1,1}(\mathbb{H})$.

**Proof of Theorem 4.2.** First, we note here that for all $\delta \in (0, 1)$:

$$\begin{aligned}\left\{\left|f(u) - f(v)\right| > \delta\right\} &\subset \left\{\rho\left(u^{-1} \cdot v\right)\left(F(u) + F(v)\right) > \delta\right\} \\ &\subset \left\{\rho\left(u^{-1} \cdot v\right)F(u) > \frac{\delta}{2}\right\} \cup \left\{\rho\left(u^{-1} \cdot v\right)F(v) > \frac{\delta}{2}\right\}.\end{aligned}$$

Hence, one receives

$$\begin{aligned}&\underset{|f(u)-f(v)|>\delta}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta}{\rho(u^{-1} \cdot v)^{Q+1}} dudv \\ &\le \underset{\rho(u^{-1}\cdot v)F(u)>\frac{\delta}{2}}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta}{\rho(u^{-1} \cdot v)^{Q+1}} dudv + \underset{\rho(u^{-1}\cdot v)F(v)>\frac{\delta}{2}}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta}{\rho(u^{-1} \cdot v)^{Q+1}} dudv.\end{aligned}$$

We denote

$$I_1 := \underset{\rho(u^{-1}\cdot v)F(u)>\frac{\delta}{2}}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta}{\rho(u^{-1} \cdot v)^{Q+1}} dudv,$$

$$I_2 := \underset{\rho(u^{-1}\cdot v)F(v)>\frac{\delta}{2}}{\int_{\mathbb{H}}\int_{\mathbb{H}}} \frac{\delta}{\rho(u^{-1} \cdot v)^{Q+1}} dudv.$$

We now estimate $I_1$.

Setting

$$v = u \cdot h\sigma$$

where

$$\sigma \in \Sigma = \{u \in \mathbb{H} : |u| = 1\},$$
$$h \in [0, \infty),$$

one has

$$\begin{aligned}
I_1 &= \int\limits_{\Sigma} \int\limits_{\mathbb{H}} \int\limits_{\substack{0 \\ hF(u)>\frac{\delta}{2}}}^{\infty} \frac{\delta}{h^2} dh du d\sigma \\
&\leq \int\limits_{\Sigma} \int\limits_{\mathbb{H}} \int\limits_{\frac{\delta}{2F(u)}}^{\infty} \frac{\delta}{h^2} dh du d\sigma \\
&= 2 \int\limits_{\Sigma} \int\limits_{\mathbb{H}} F(u) du d\sigma \\
&= C_Q \int\limits_{\mathbb{H}} F(u) du.
\end{aligned}$$

Similarly, by noting that $\rho(u^{-1} \cdot v) = |v^{-1}u|$, we also have

$$I_2 \leq C_Q \int\limits_{\mathbb{H}} F(u) du.$$

Hence, we have

$$\sup_{0<\delta<1} \int\limits_{\mathbb{H}} \int\limits_{\substack{\mathbb{H} \\ |f(u)-f(v)|>\delta}} \frac{\delta}{\rho(u^{-1} \cdot v)^{Q+1}} du dv < \infty.$$

By Theorem 4.1, we obtain $f \in BV(\mathbb{H})$. □

## 5. Some generalizations and variants of characterizations

In this section, we will study some generalizations of the above results. The next Theorem is a generalized result of Theorem 1.2:

**Theorem 5.1.** *Let $f \in L^p(\mathbb{H})$, $1 < p < \infty$ and $F : [0, \infty) \to [0, \infty)$ be continuous such that*

$$\int\limits_0^{\infty} F(t) t^{-p-1} dt = 1.$$

*Set*

$$F_\delta(t) = \delta^p F\left(\frac{t}{\delta}\right), \quad \delta > 0.$$

*Then we have:*

(a) *If*

$$\sup_{0<\delta<1} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} du dv < \infty$$

*and*

$$\int_{\mathbb{H}} \int_{\mathbb{H}}_{|f(u)-f(v)|>\delta} \frac{1}{\rho(u^{-1} \cdot v)^{Q+p}} du dv < \infty, \quad \forall \delta > 0,$$

*then* $g \in W^{1,p}(\mathbb{H})$.

(b) *If* $g \in W^{1,p}(\mathbb{H})$ *and*

$$\int_0^\infty F_\delta(t) t^{-p-1} dt < \infty, \quad \forall \delta > 0,$$

*then*

$$\int_{\mathbb{H}} \int_{\mathbb{H}} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} du dv \le C_{Q,p} \int_0^\infty F_\delta(t) t^{-p-1} dt \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} f(u)\right|^p du, \quad \forall \delta > 0.$$

**Proof.** (b) Setting

$$D_1(f) = \left\{(u, v) \in \mathbb{H} \times \mathbb{H} : M\left(|\nabla_{\mathbb{H}} f|\right)(u) \ge M\left(|\nabla_{\mathbb{H}} f|\right)(v)\right\}$$
$$D_2(f) = \left\{(u, v) \in \mathbb{H} \times \mathbb{H} : M\left(|\nabla_{\mathbb{H}} f|\right)(u) < M\left(|\nabla_{\mathbb{H}} f|\right)(v)\right\},$$

then

$$\int_{\mathbb{H}} \int_{\mathbb{H}} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} du dv$$
$$= \iint_{D_1(f)} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} du dv + \iint_{D_2(f)} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} du dv.$$

Now, we will first concern

$$I_1 = \iint_{D_1(f)} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} dudv.$$

Using (2.1), (i) and noting that on the domain $D_1(f)$, one has

$$M\big(|\nabla_{\mathbb{H}} f|\big)(u) \geq M\big(|\nabla_{\mathbb{H}} f|\big)(v),$$

we get

$$I_1 \leq \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{F_\delta(2A_{Q,p}\rho(u^{-1} \cdot v)M(|\nabla_{\mathbb{H}} f|)(u))}{\rho(u^{-1} \cdot v)^{Q+p}} dudv.$$

Now, by the change of variables and Fubini's theorem, we obtain

$$I_1 \leq \int_{\Sigma} \int_{\mathbb{H}} \int_0^\infty \frac{F_\delta(2A_{Q,p}hM(|\nabla_{\mathbb{H}} f|)(u))}{h^{p+1}} dhdud\sigma.$$

Now, for every $\sigma \in \Sigma$, we can have the following estimate:

$$\begin{aligned} \int_{\mathbb{H}} \int_0^\infty \frac{F_\delta(2A_{Q,p}hM(|\nabla_{\mathbb{H}} f|)(u))}{h^{p+1}} dhdu &= \int_{\mathbb{H}} \big[2A_{Q,p}M\big(|\nabla_{\mathbb{H}} f|\big)(u)\big]^p du \int_0^\infty F_\delta(t)t^{-p-1}dt \\ &\leq C_{Q,p} \int_0^\infty F_\delta(t)t^{-p-1}dt \int_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du. \end{aligned}$$

Similarly, by noting that $\rho(u^{-1} \cdot v) = |v^{-1}u|$, we can also receive

$$I_2 \leq C_{Q,p} \int_0^\infty F_\delta(t)t^{-p-1}dt \int_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du.$$

Hence, we can conclude that

$$\int_{\mathbb{H}} \int_{\mathbb{H}} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} dudv \leq C_{Q,p} \int_0^\infty F_\delta(t)t^{-p-1}dt \int_{\mathbb{H}} \big|\nabla_{\mathbb{H}} f(u)\big|^p du.$$

(a) The assumptions on $F$, we can find four positive constants $m, M, \lambda$ and $\sigma$ with $m < M$ such that

$$\big|\big\{t \in [m, M] : F(t) \geq \lambda\big\}\big| \geq \sigma.$$

Since $F$ is continuous on $[0, \infty)$, there exists an interval $A \neq \emptyset$ such that

$$A \subset \big\{t \in [m, M] : F(t) \geq \lambda\big\}.$$

Since

$$\sup_{0<\delta<1} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{F_\delta(|f(u) - f(v)|)}{\rho(u^{-1} \cdot v)^{Q+p}} dudv < \infty,$$

we get

$$\sup_{0<\delta<1} \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{\delta^p \chi_A(\frac{|f(u)-f(v)|}{\delta})}{\rho(u^{-1} \cdot v)^{Q+p}} dudv < \infty.$$

This implies

$$\sup_{0<\varepsilon<1} \int_0^1 \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{\varepsilon \delta^{\varepsilon+p-1} \chi_A(\frac{|f(u)-f(v)|}{\delta})}{\rho(u^{-1} \cdot v)^{Q+p}} dudvd\delta < \infty.$$

By Fubini's theorem,

$$\sup_{0<\varepsilon<1} \int_{\mathbb{H}} \int_{\mathbb{H}}\limits_{|f(u)-f(v)| \leq m} \int_0^1 \frac{\varepsilon \delta^{\varepsilon+p-1} \chi_A(\frac{|f(u)-f(v)|}{\delta})}{\rho(u^{-1} \cdot v)^{Q+p}} d\delta dudv < \infty.$$

Since

$$\frac{|f(u) - f(v)|}{\delta} \leq M,$$

we have

$$\delta^{\varepsilon+p-1} \geq M^{-p-\varepsilon+1} \big|f(u) - f(v)\big|^{\varepsilon+p-1}.$$

Hence,

$$\sup_{0<\varepsilon<1} \int_{\mathbb{H}} \int_{\mathbb{H}}\limits_{|f(u)-f(v)| \leq m} \frac{\varepsilon |f(u) - f(v)|^{\varepsilon+p-1}}{\rho(u^{-1} \cdot v)^{Q+p}} \int_0^1 \chi_A\left(\frac{|f(u) - f(v)|}{\delta}\right) d\delta dudv < \infty. \tag{5.1}$$

Moreover, since $A \subset [m, M]$,

$$\int_0^1 \chi_A\left(\frac{t}{\delta}\right) d\delta = \int_0^\infty \chi_A\left(\frac{t}{\delta}\right) d\delta = t \int_0^\infty \chi_A\left(\frac{1}{\delta}\right) d\delta = C(A)t, \quad t \leq m. \tag{5.2}$$

Here

$$C(A) = \int_0^\infty \chi_A\left(\frac{1}{\delta}\right) d\delta > 0.$$

From (5.1) and (5.2), we get

$$\sup_{0<\varepsilon<1} \int\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\leq m}}\int \frac{\varepsilon |f(u)-f(v)|^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < \infty.$$

Also,

$$\int\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>m}}\int \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < \infty.$$

Setting

$$\widetilde{f} = \frac{f}{m},$$

and using Theorem 1.2, we get $\widetilde{f} \in W^{1,p}(\mathbb{H})$. Hence, $f \in W^{1,p}(\mathbb{H})$. □

The second result in this section is to weaken the statement (3) in Theorem 1.2. More precisely, we will prove that

**Theorem 5.2.** *Let $1 < p < \infty$ and $f \in L^p(\mathbb{H})$ be such that*

$$\sup_{n\in\mathbb{N}} \int\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta_n}}\int \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < \infty.$$

*Here $(\delta_n)_{n\in\mathbb{N}}$ is some arbitrary sequence of positive numbers with*

$$\delta_0 = 1$$
$$\delta_n \leq \delta_{n-1} \leq 2\delta_n$$
$$\lim_{n\to\infty} \delta_n = 0.$$

*Then $f \in W^{1,p}(\mathbb{H})$.*

We notice that one could replace number 2 in the condition of the sequence $(\delta_n)$ by an arbitrary number $c > 1$.

**Proof.** Setting

$$\sup_{n\in\mathbb{N}} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta_n}} \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} du dv < C,$$

then it is clear that

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>1}} \frac{1}{\rho(u^{-1}\cdot v)^{Q+p}} du dv < C. \tag{5.3}$$

So we just need to prove for all $\varepsilon \in (0,1)$,

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le C_{Q,p}$$

since by Theorem 1.2, we get the assertion.

Now, for every $\varepsilon \in (0,1)$, since

$$\iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta_n}} \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} du dv < C,$$

we get for every $n \ge 0$:

$$\varepsilon(\delta_n-\delta_{n+1})\delta_n^{\varepsilon-1} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta_n}} \frac{\delta_n^p}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le C\varepsilon(\delta_n-\delta_{n+1})\delta_n^{\varepsilon-1}.$$

Hence,

$$\sum_{n\ge 0} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|>\delta_n}} \frac{\varepsilon(\delta_n-\delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \le C\sum_{n\ge 0} \varepsilon(\delta_n-\delta_{n+1})\delta_n^{\varepsilon-1}. \tag{5.4}$$

Now, if we denote $h(\delta) = \varepsilon\delta^{\varepsilon-1}$, then we have by the Lebesgue Dominated Convergence Theorem and noting that $h$ is a decreasing function:

$$\begin{aligned} 1 &= \int_0^1 h(\delta) d\delta \\ &= \sum_{n\ge 0} \int_{\delta_{n+1}}^{\delta_n} h(\delta) d\delta \end{aligned}$$

$$\geq \sum_{n\geq 0}(\delta_n - \delta_{n+1})h(\delta_n)$$
$$= \sum_{n\geq 0}\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon-1}. \tag{5.5}$$

Thus, from (5.4), one has

$$\sum_{n\geq 0}\underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}}\frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv \leq C. \tag{5.6}$$

Note that

$$\sum_{n\geq 0}\underset{|f(u)-f(v)|>\delta_n}{\int_{\mathbb{H}}\int_{\mathbb{H}}}\frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}dudv$$
$$\geq \underset{|f(u)-f(v)|\leq 1}{\int_{\mathbb{H}}\int_{\mathbb{H}}}\sum_{n\geq 0}\frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}\chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v)dudv. \tag{5.7}$$

Now, fix $(u, v)$ such that

$$0 < \left|f(u) - f(v)\right| \leq 1$$

and denote $n_{(u,v)}$ the smallest integer number such that

$$\delta_{n_{(u,v)}} < \left|f(u) - f(v)\right|.$$

Then

$$\sum_{n\geq 0}\frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}\chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v)$$
$$= \sum_{n\geq n_{(u,v)}}\frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}\chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v)$$
$$= \sum_{n\geq n_{(u,v)}}\frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}. \tag{5.8}$$

We claim that

$$\frac{|f(u) - f(v)|}{2} \leq \delta_{n_{(u,v)}} < \left|f(u) - f(v)\right|. \tag{5.9}$$

Indeed, we could suppose by contradiction that

$$\delta_{n_{(u,v)}} < \frac{|f(u)-f(v)|}{2} < \left|f(u)-f(v)\right| \leq \delta_{n_{(u,v)}-1},$$

then

$$\begin{aligned}\delta_{n_{(u,v)}-1} - \delta_{n_{(u,v)}} &> \left|f(u)-f(v)\right| - \frac{|f(u)-f(v)|}{2} \\ &= \frac{|f(u)-f(v)|}{2} \\ &> \delta_{n_{(u,v)}}\end{aligned}$$

which is impossible by the assumption of the sequence $(\delta_n)$.

Set

$$k(\delta) = \frac{\varepsilon \delta^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} \quad \text{on the interval } 0 \leq \delta < \left|f(u)-f(v)\right|.$$

Noting that this function is increasing, arguing as (5.5), we obtain by (5.9):

$$\begin{aligned}\frac{1}{(p+1)2^{p+1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} &\leq \frac{1}{(p+\varepsilon)2^{p+\varepsilon}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} \\ &= \int_0^{\frac{|f(u)-f(v)|}{2}} k(\delta)\,d\delta \\ &\leq \int_0^{\delta_{n_{(u,v)}}} k(\delta)\,d\delta \\ &= \sum_{n\geq n_{(u,v)}} \int_{\delta_{n+1}}^{\delta_n} k(\delta)\,d\delta \\ &\leq \sum_{n\geq n_{(u,v)}} (\delta_n - \delta_{n+1}) k(\delta_n) \\ &= \sum_{n\geq n_{(u,v)}} \frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}}.\end{aligned} \tag{5.10}$$

Hence, by (5.6), (5.7), (5.8) and (5.10), we get

$$\begin{aligned}C &\geq \sum_{n\geq 0} \int_{\mathbb{H}}\int_{\mathbb{H}}\limits_{|f(u)-f(v)|>\delta_n} \frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} du dv \\ &\geq \int_{\mathbb{H}}\int_{\mathbb{H}}\limits_{|f(u)-f(v)|\leq 1} \sum_{n\geq 0} \frac{\varepsilon(\delta_n - \delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} \chi_{\{|f(u)-f(v)|>\delta_n\}}(u,v) du dv\end{aligned}$$

$$= \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \sum_{n\ge n_{(u,v)}} \frac{\varepsilon(\delta_n-\delta_{n+1})\delta_n^{\varepsilon+p-1}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv$$

$$\ge \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{1}{(p+1)2^{p+1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv.$$

Thus, we can conclude the assertion

$$\sup_{0<\varepsilon<1} \iint\limits_{\substack{\mathbb{H}\ \mathbb{H} \\ |f(u)-f(v)|\le 1}} \frac{\varepsilon|f(u)-f(v)|^{p+\varepsilon}}{\rho(u^{-1}\cdot v)^{Q+p}} dudv < \infty.$$

By Theorem 1.2 (statement 2), we have $f \in W^{1,p}(\mathbb{H})$. □

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