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The Moser–Trudinger inequality in unbounded domains of Heisenberg group and sub-elliptic equations

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ABSTRACT

Let $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$ be the n-dimensional Heisenberg group, $\nabla_{\mathbb{H}^n}$ be its sub-elliptic gradient operator, and $\rho(\xi) = (|z|^4 + t^2)^{1/4}$ for $\xi = (z, t) \in \mathbb{H}^n$ be the distance function in $\mathbb{H}^n$. Denote $Q = 2n + 2$ and $Q' = Q/(Q - 1)$. It is proved in this paper that there exists a positive constant α^* such that for any pair β and α satisfying $0 \le \beta < Q$ and $\frac{\alpha}{\alpha^*} + \frac{\beta}{Q} \le 1$,

$$\sup_{\|u\|_{W^{1,Q}(\mathbb{H}^n)} \le 1} \int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha |u|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u|^{kQ'}}{k!} \right\} d\xi < \infty,$$

where $W^{1,Q}(\mathbb{H}^n)$ is the Sobolev space on $\mathbb{H}^n$. When $\frac{\alpha}{\alpha^*} + \frac{\beta}{Q} > 1$, the above integral is still finite for any $u \in W^{1,Q}(\mathbb{H}^n)$. Furthermore the supremum is infinite if $\alpha/\alpha_Q + \beta/Q > 1$, where $\alpha_Q = Q\sigma_Q^{1/(Q-1)}$, $\sigma_Q = \int_{\rho(z,t)=1} |z|^Q d\mu$. Actually if we replace $\mathbb{H}^n$ and $W^{1,Q}(\mathbb{H}^n)$ by unbounded domain Ω and $W_0^{1,Q}(\Omega)$ respectively, the above inequality still holds. As an application of this inequality, a sub-elliptic equation with exponential growth is considered.

1. Introduction

Let $\mathbb{H}^n$ be the n-dimensional Heisenberg group. Recall that the Heisenberg group $\mathbb{H}^n$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(\langle y, x'\rangle) - \langle x, y'\rangle),$$

where $x, y, x', y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t}, \qquad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n.$$

These generators satisfy the non-commutative formula $[X_i, Y_i] = -4\delta_{ij} T$. We fix some notations:

$$z = (x, y) \in \mathbb{R}^{2n}, \qquad \xi = (z, t) \in \mathbb{H}^n, \qquad \rho(\xi) = (|z|^4 + t^2)^{1/4},$$

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where $\rho(\xi)$ denotes the Heisenberg distance between ξ and the origion. We now use $|\nabla_{\mathbb{H}^n} u|$ to express the norm of the sub-elliptic gradient of the function $u : \mathbb{H}^n \to \mathbb{R}$:

$$|\nabla_{\mathbb{H}^n} u| = \left(\sum_{i=1}^{n} \left((X_i u)^2 + (Y_i u)^2 \right) \right)^{1/2}.$$

Let Ω be an open set in $\mathbb{H}^n$. We use $W^{1,p}(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega \left(|\nabla_{\mathbb{H}^n} u|^p + |u|^p \right) d\xi \right)^{1/p}.$$

The following Trudinger–Moser inequality on bounded domains in the Hesenberg group $\mathbb{H}^n$ was proved by Cohn and Lu [1]:

Theorem A. *Let $\mathbb{H}^n$ be a n-dimensional Heisenberg group, $Q = 2n+2$, $Q' = Q/(Q-1)$, and $\alpha_Q = Q\sigma_Q^{1/(Q-1)}$, $\sigma_Q = \int_{\rho(z,t)=1} |z|^Q d\mu$. Then there exists a constant C_0 depending only on Q such that for all $\Omega \subset \mathbb{H}^n$, $|\Omega| < \infty$,*

$$\sup_{u \in W_0^{1,Q}(\Omega),\ \|\nabla_{\mathbb{H}^n} u\|_{L^Q} \le 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha_Q |u|^{Q'}} d\xi < \infty.$$

If α_Q is replaced by any larger number, then the supremum is infinite.

Remarks. (1) The constant σ_Q was found explicitly in [1] and it is equal to

$$\sigma_Q = \omega_{2n-1} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right)}{n!},$$

where ω_{2n-1} is the surface area of the unit sphere in $\mathbb{R}^{2n}$.

(2) When $|\Omega| = \infty$, the above inequality in Theorem A is not meaningful. It is still an open question if any type of Trudinger–Moser inequality holds on unbounded domains of $\mathbb{H}^n$. The main purpose of this paper is to establish such an inequality on any unbounded domain in $\mathbb{H}^n$. Since the validity of a Trudinger–Moser inequality on $\mathbb{H}^n$ implies the same inequality on any subdomains of $\mathbb{H}^n$, we will only prove the case $\Omega = \mathbb{H}^n$.

(3) Using similar ideas of representation formulas and rearrangement of convolutions as done on the Heisenberg group in [1], Theorem A was extended to the groups of Heisenberg type in [2] and to general Carnot groups in [3].

(4) The Euclidean version of the above sharp constant for the Moser–Trudinger inequality was obtained by Moser [10] which sharpened the results of Trudinger [11] and Pohozaev [12].

To state our main theorem, we need to introduce some preliminaries.

Let $u : \mathbb{H}^n \to \mathbb{R}$ be a nonnegative function in $W^{1,Q}(\mathbb{H}^n)$, $Q = 2n+2$, and u^* be the decreasing rearrangement of u, namely

$$u^*(\xi) := \sup\{s \ge 0 : \xi \in \{u > s\}^*\},$$

where $\{u > s\}^* = \mathbb{B}_r = \{\xi' : \rho(\xi') \le r\}$ such that $|\{u > s\}| = |\mathbb{B}_r|$. Assume u and v are two nonnegative functions on $\mathbb{H}^n$ and $uv \in L^1(\mathbb{H}^n)$. Then the Hardy–Littlewood inequality says

$$\int_{\mathbb{H}^n} (uv)^* d\xi \le \int_{\mathbb{H}^n} u^* v^* d\xi. \tag{1.1}$$

This inequality is attributed to Hardy and Littlewood (see [4,5]).

It is known from a result of Manfredi and Vera De Serio [6] that there exists a constant $c \ge 1$ depending only on Q such that

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u^*|^Q d\xi \le c \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^Q d\xi \tag{1.2}$$

for all $u \in W^{1,Q}(\mathbb{H}^n)$. Thus we can define

$$c^* = \inf\left\{ c^{\frac{1}{Q-1}} : \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u^*|^Q d\xi \le c \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^Q d\xi,\ u \in W^{1,Q}(\mathbb{H}^n) \right\}. \tag{1.3}$$

Then our main result can be stated as the following:

Theorem 1.1. *Let Q, Q' and α_Q be as in Theorem A. Let α^* be such that $\alpha^* = \alpha_Q/c^*$. Then for any pair β and α satisfying $0 \le \beta < Q$, $0 < \alpha \le \alpha^*$, and $\frac{\alpha}{\alpha^*} + \frac{\beta}{Q} \le 1$, there holds*

$$\sup_{\|u\|_{W^{1,Q}(\mathbb{H}^n)} \le 1} \int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|u|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u|^{kQ'}}{k!} \right\} d\xi < \infty. \tag{1.4}$$

When $\frac{\alpha}{\alpha^*} + \frac{\beta}{Q} > 1$, the integral in (1.4) is still finite for any $u \in W^{1,Q}(\mathbb{H}^n)$, but the supremum is infinite if further $\frac{\alpha}{\alpha_Q} + \frac{\beta}{Q} > 1$.

An analogous result to Theorem 1.1 in the Euclidean space has been recently derived in [7]. It is an easy consequence of Theorem 1.1 that (1.4) still holds if we replace $\mathbb{H}^n$ and $W^{1,Q}(\mathbb{H}^n)$ by unbounded domain Ω and $W_0^{1,Q}(\Omega)$ respectively. This is due to the monotonicity of the function $\psi(s) = e^s - \sum_{k=0}^{Q-2} \frac{s^k}{k!}$ for $s \ge 0$. The proof of Theorem 1.1 is based on an rearrangement argument, and the inequalities (1.1) and (1.2) and Theorem A on bounded domains, which lead to $\alpha^* \le \alpha_Q$.

Though substantial works have been done for subelliptic equations with polynomial growth using Sobolev embeddings, it has been absent in the literature on the study of subelliptic equations of exponential growth. This paper is an attempt to investigate such type of equations in the subelliptic setting by employing the Moser–Trudinger inequality on the Heisenberg group. As an applications of Theorem 1.1, we consider the existence of weak solutions for the nonhomogeneous singular problem

$$-\operatorname{div}_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n} u|^{Q-2} \nabla_{\mathbb{H}^n} u) + V|u|^{Q-2}u = \frac{f(\xi, u)}{\rho(\xi)^\beta} + \varepsilon h(u), \tag{1.5}$$

where $V : \mathbb{H}^n \to \mathbb{R}$ is a continuous function satisfying $V(\xi) \ge V_0 > 0$ for all $\xi \in \mathbb{H}^n$, $f(\xi, s)$ is continuous in $\mathbb{H}^n \times \mathbb{R}$ and behaves like $e^{\alpha|s|^{Q'}}$ as $|s| \to \infty$, $h \in (W^{1,Q}(\mathbb{H}^n))^*$, $h \ne 0$, and $\varepsilon > 0$ is a small parameter. Problem (1.5) in the Euclidean space was studied in [7–9].

Since we are interested in positive solutions, we may assume $f(\xi, s) = 0$ for all $(\xi, s) \in \mathbb{H}^n \times (-\infty, 0]$. Moreover we assume the following growth condition on the nonlinearity $f(\xi, s)$:

(H_1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(\xi, s) \in \mathbb{H}^n \times \mathbb{R}^+$,

$$|f(\xi, s)| \le b_1 s^{Q-1} + b_2 \left\{ e^{\alpha_0 |s|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha_0^k s^{kQ'}}{k!} \right\};$$

(H_2) There exists $\mu > Q$ such that for all $\xi \in \mathbb{H}^n$ and $s > 0$,

$$0 < \mu F(\xi, s) \equiv \mu \int_0^s f(\xi, s) ds \le s f(\xi, s);$$

(H_3) There exist constants $R_0, M_0 > 0$ such that for all $\xi \in \mathbb{H}^n$ and $s \ge R_0$,

$$F(\xi, s) \le M_0 f(\xi, s).$$

Define a function space

$$E = \left\{ u \in W^{1,Q}(\mathbb{H}^n) : \int_{\mathbb{H}^n} V(\xi)|u(\xi)|^Q d\xi < \infty \right\}.$$

We say that $u \in E$ is a weak solution of problem (1.5) if for all $\varphi \in C_0^\infty(\mathbb{H}^n)$ we have

$$\int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}^n} u|^{Q-2} \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} \varphi + V|u|^{Q-2} u\varphi \right) d\xi = \int_{\mathbb{H}^n} \frac{f(\xi, u)}{\rho(\xi)^\beta} \varphi d\xi + \varepsilon \int_{\mathbb{H}^n} h(\xi) \varphi d\xi.$$

The assumption $V(\xi) \ge V_0 > 0$ implies that E is a reflexive Banach space when equipped with the norm

$$\|u\| \equiv \left\{ \int_{\mathbb{H}^n} \left(|\nabla_{\mathbb{H}^n} u|^Q + V|u|^Q \right) d\xi \right\}^{\frac{1}{Q}} \tag{1.6}$$

and for all $q \ge Q$, the embedding

$$E \hookrightarrow W^{1,Q}(\mathbb{H}^n) \hookrightarrow L^q(\mathbb{H}^n)$$

is continuous. For any $0 \le \beta < Q$, we define a singular eigenvalue by

$$\lambda_\beta = \inf_{u \in E,\, u \ne 0} \frac{\|u\|^Q}{\int_{\mathbb{H}^n} \frac{|u(\xi)|^Q}{\rho(\xi)^\beta} d\xi}. \tag{1.7}$$

The continuous embedding of $W^{1,Q}(\mathbb{H}^n) \hookrightarrow L^q(\mathbb{H}^n)$ for all $q \ge Q$ together with the Hölder inequality implies that $\lambda_\beta > 0$ for any $0 \le \beta < Q$.

Now we can state a result as an application of Theorem 1.1 as follows:

Theorem 1.2. *Suppose that $f(\xi, s)$ is continuous in $\mathbb{H}^n \times \mathbb{R}$, $f(\xi, s) = 0$ in $\mathbb{H}^n \times (-\infty, 0]$, V is continuous in $\mathbb{H}^n$, $V(\xi) \geq V_0 > 0$, $V(\xi) \to \infty$ as $\rho(\xi) \to \infty$, and (H_1), (H_2) and (H_3) are satisfied. Furthermore we assume*

$$(H_4) \quad \limsup_{s\to 0+} \frac{QF(\xi, s)}{|s|^Q} < \lambda_\beta \quad \text{uniformly with respect to } \xi \in \mathbb{H}^n.$$

Then there exists $\epsilon > 0$ such that if $0 < \varepsilon < \epsilon$, then the Eq. (1.5) has a nontrivial weak solution of the mountain-pass type.

We finally remark that the ideas and methods used in this paper can be applied to more general stratified groups (for definitions of stratified groups see for example [13–15]). Therefore, the results derived in this paper hold in that setting as well. Nevertheless, for the clarity and simplicity of presentation, we have chosen to present it only on the Heisenberg group.

We further remark that multiplicity of solutions can be derived for the non-uniformly subelliptic equations of Q-Laplacian type. Moreover, we can establish the existence and multiplicity of solutions of such class of subelliptic equations when the nonlinear term f does not satisfy the well-known Ambrosetti–Rabinowitz condition (H2). We refer the reader to [16] for these results.

The proof of Theorem 1.2 is based on the conclusion of Theorem 1.1 and the mountain-pass theorem. In the remaining part of this paper, Theorem 1.1 is proved in Section 2, and Theorem 1.2 is proved in Section 3.

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The method we used here is combining the Hardy–Littlewood inequality [4,5], the radial lemma [17], the Young inequality with Theorem A and a rearrangement argument.

Proof of Theorem 1.1. We first prove for any fixed $\alpha > 0$, $\beta : 0 \leq \beta < Q$, and $u \in W^{1,Q}(\mathbb{H}^n)$ that

$$\int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|u|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u|^{kQ'}}{k!} \right\} d\xi < \infty. \tag{2.1}$$

Let u^* be the decreasing rearrangement of $|u|$. Notice that $(\rho(\xi)^{-\beta})^* = \rho(\xi)^{-\beta}$, it follows from (1.1) and (1.2) that $u^* \in W^{1,Q}(\mathbb{H}^n)$ and

$$\int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|u|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u|^{kQ'}}{k!} \right\} d\xi \leq \int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|u^*|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u^*|^{kQ'}}{k!} \right\} d\xi. \tag{2.2}$$

A straightforward calculation shows for any $r > 0$,

$$\begin{aligned} \int_{\mathbb{H}^n} u^*(\xi)^Q d\xi &\geq \int_{\rho(\xi)\leq r} u^*(\xi)^Q d\xi \\ &= \int_0^r s^{Q-1} u^*(s)^Q \omega_{Q-1} ds \\ &\geq \frac{\omega_{Q-1}}{Q} r^Q u^*(r)^Q. \end{aligned} \tag{2.3}$$

Here and in the sequel ω_{Q-1} stands for the area of the unit sphere in $\mathbb{H}^n$, namely

$$\omega_{Q-1} = \int_{\rho(\xi)=1} d\xi.$$

It follows from (2.3) that

$$u^*(\xi)^Q \leq \frac{Q}{\omega_{Q-1}} \frac{\|u^*\|^Q_{L^Q(\mathbb{H}^n)}}{\rho(\xi)^Q}, \quad \forall \xi \in \mathbb{H}^n \setminus \{(0, 0)\}.$$

Note that this is known as the Radial Lemma in the Euclidean case [17].

Choosing R_0 sufficiently large such that $u^*(\xi) < 1$ for all $\rho(\xi) \geq R_0$, we obtain

$$\begin{aligned} \int_{\rho(\xi)>R_0} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|u|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u|^{kQ'}}{k!} \right\} d\xi &\leq \frac{1}{R_0^\beta} \int_{\rho(\xi)>R_0} \left\{ \frac{\alpha^{Q-1}|u^*|^Q}{(Q-1)!} + \sum_{k=Q}^{\infty} \frac{\alpha^k |u^*|^{kQ'}}{k!} \right\} d\xi \\ &\leq \frac{\|u^*\|^Q_{L^Q(\mathbb{H}^n)}}{R_0^\beta} \sum_{k=Q-1}^{\infty} \frac{\alpha^k}{k!}. \end{aligned} \tag{2.4}$$

On the other hand, we have by the Hölder inequality and the Young inequality,

$$\int_{\rho(\xi)\le R_0}\frac{1}{\rho(\xi)^\beta}\left\{e^{\alpha|u|^{Q'}}-\sum_{k=0}^{Q-2}\frac{\alpha^k|u|^{kQ'}}{k!}\right\}d\xi \le \left(\int_{\rho(\xi)\le R_0}\frac{1}{\rho(\xi)^{\beta p'}}d\xi\right)^{1/p'}\left(\int_{\rho(\xi)\le R_0}e^{\alpha p|u^*|^{Q'}}d\xi\right)^{1/p}$$
$$\le C\left(\int_{\rho(\xi)\le R_0}e^{\alpha p(1+\epsilon)|u^*-u^*(R_0)|^{Q'}}d\xi\right)^{1/p} \tag{2.5}$$

for some constant C depending only on n, α, β, p' and ϵ, where $1/p+1/p'=1$, $1<p'<Q/\beta$, and $\epsilon>0$. Since $u^*-u^*(R_0)\in W_0^{1,Q}(\mathbb{B}_{R_0})$, where $\mathbb{B}_{R_0}=\{\xi\in\mathbb{H}^n\colon \rho(\xi)\le R_0\}$, the integral on the left hand side of (2.5) is bounded thanks to the Trudinger–Moser inequality on bounded domain of $\mathbb{H}^n$ (Theorem A). Combining (2.2), (2.4) and (2.5), we conclude (2.1).

Next we prove the uniform estimate (1.4) for $\alpha\le(1-\beta/Q)\alpha^*$, where $\alpha^*=\alpha_Q/c^*$ and c^* is defined in (1.3). Let $\widetilde{u}=u/\|u\|_{W^{1,Q}(\mathbb{H}^n)}$. When $\alpha>0$, it is easy to see that

$$\int_{\mathbb{H}^n}\frac{1}{\rho(\xi)^\beta}\left\{e^{\alpha|u|^{Q'}}-\sum_{k=0}^{Q-2}\frac{\alpha^k|u|^{kQ'}}{k!}\right\}d\xi\le\int_{\mathbb{H}^n}\frac{1}{\rho(\xi)^\beta}\left\{e^{\alpha|\widetilde{u}|^{Q'}}-\sum_{k=0}^{Q-2}\frac{\alpha^k|\widetilde{u}|^{kQ'}}{k!}\right\}d\xi,$$

provided that $\|u\|_{W^{1,Q}(\mathbb{H}^n)}\le 1$. This together with the inequality (1.1) implies that it suffices to prove there exists a uniform constant C such that for all radially decreasing symmetric functions $u\in W^{1,Q}(\mathbb{H}^n)$ with $\|u\|_{W^{1,Q}(\mathbb{H}^n)}=1$,

$$\int_{\mathbb{H}^n}\frac{1}{\rho(\xi)^\beta}\left\{e^{\alpha_0|u|^{Q'}}-\sum_{k=0}^{Q-2}\frac{{\alpha_0}^k|u|^{kQ'}}{k!}\right\}d\xi\le C, \tag{2.6}$$

where $\alpha_0=(1-\beta/Q)\alpha^*$. In the following, we assume that u is radially decreasing in $\mathbb{H}^n$ and $\|u\|_{W^{1,Q}(\mathbb{H}^n)}=1$. Take $R_0>(Q/\omega_{Q-1})^{1/Q}$. Thanks to (2.4), there holds

$$\int_{\rho(\xi)>R_0}\frac{1}{\rho(\xi)^\beta}\left\{e^{\alpha_0|u|^{Q'}}-\sum_{k=0}^{Q-2}\frac{{\alpha_0}^k|u|^{kQ'}}{k!}\right\}d\xi\le C. \tag{2.7}$$

Define the set $\mathcal{S}=\{\xi\in\mathbb{B}_{R_0}\colon |u(\xi)-u(R_0)|>2|u(R_0)|\}$. We can assume $\mathcal{S}$ is nonempty for otherwise (2.6) already holds in view of (2.7). Then a straightforward calculation shows for all $\xi\in\mathcal{S}$ and $\epsilon>0$,

$$\begin{aligned}|u(\xi)|^{Q'} &= |u(\xi)-u(R_0)+u(R_0)|^{Q'}\\ &= |u(\xi)-u(R_0)|^{Q'}\left(1+\frac{|u(R_0)|}{|u(\xi)-u(R_0)|}\right)^{Q'}\\ &\le |u(\xi)-u(R_0)|^{Q'}+C|u(R_0)||u(\xi)-u(R_0)|^{\frac{1}{Q-1}}\\ &\le (1+\epsilon)|u(\xi)-u(R_0)|^{Q'}+C\frac{|u(R_0)|^{Q'}}{\epsilon^{1/(Q-1)}}.\end{aligned}$$

Choosing ϵ such that

$$1+\epsilon=\frac{1}{\|\nabla_{\mathbb{H}^n}u\|_{L^Q(\mathbb{H}^n)}^{Q'}}=\left(\frac{1}{1-\|u\|_{L^Q(\mathbb{H}^n)}^Q}\right)^{1/(Q-1)}.$$

Applying the mean value theorem to the function $\varphi(t)=t^{1/(Q-1)}$, we can find some $\zeta:1-\|u\|_{L^Q(\mathbb{H}^n)}^Q\le\zeta\le1$ such that

$$1-\left(1-\|u\|_{L^Q(\mathbb{H}^n)}^Q\right)^{\frac{1}{Q-1}}=\frac{1}{Q-1}\zeta^{\frac{2-Q}{Q-1}}\|u\|_{L^Q(\mathbb{H}^n)}^Q.$$

Hence

$$\epsilon=\frac{\|u\|_{L^Q(\mathbb{H}^n)}^Q}{(Q-1)\zeta^{\frac{Q-2}{Q-1}}\left(1-\|u\|_{L^Q(\mathbb{H}^n)}^Q\right)^{\frac{1}{Q-1}}}\ge\frac{\|u\|_{L^Q(\mathbb{H}^n)}^Q}{Q-1}.$$

This together with the fact that $|u(R_0)|\le(Q/\omega_{2n})^{1/Q}\|u\|_{L^Q(\mathbb{H}^n)}/R_0$ leads to

$$\frac{|u(R_0)|^{Q'}}{\epsilon^{1/(Q-1)}}\le C,$$

and thus for all $\xi \in \mathcal{S}$,

$$|u(\xi)|^{Q'} \le \frac{|u(\xi)-u(R_0)|^{Q'}}{\|\nabla_{\mathbb{H}^n} u\|^{Q'}_{L^Q(\mathbb{H}^n)}} + C.$$

Obviously $u - u(R_0) \in W_0^{1,Q}(\mathbb{B}_{R_0})$ and

$$\int_{\mathbb{B}_{R_0}} |\nabla_{\mathbb{H}^n}(u-u(R_0))|^Q d\xi \le \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^Q d\xi \le 1.$$

Denote $\widetilde{u}(\xi) = (u(\xi)-u(R_0))/\|\nabla(u-u(R_0))\|_{L^Q(\mathbb{H}^n)}$. It is easy to see that

$$\begin{aligned}\int_{\mathbb{B}_{R_0}} \frac{e^{\alpha_0|u|^{Q'}}}{\rho(\xi)^\beta} d\xi &= \int_{\mathcal{S}} \frac{e^{\alpha_0|u|^{Q'}}}{\rho(\xi)^\beta} d\xi + \int_{\mathbb{B}_{R_0}\setminus\mathcal{S}} \frac{e^{\alpha_0|u|^{Q'}}}{\rho(\xi)^\beta} d\xi \\ &\le \int_{\mathbb{B}_{R_0}} \frac{e^{\alpha_0|\widetilde{u}|^{Q'}}}{\rho(\xi)^\beta} d\xi + C(Q,\beta) \le C. \end{aligned} \tag{2.8}$$

Notice that $\alpha_0 < (1-\beta/Q)\alpha_Q$, in the last inequality above, we have used the Hölder inequality and Theorem A. Thus (2.8) together with (2.7) implies (2.6). Hence, for all $\alpha : 0 < \alpha \le (1-\beta/Q)\alpha^*$, we get the uniform estimate (1.4).

Finally we prove for any $\beta : 0 \le \beta < Q$ and $\alpha > (1-\beta/Q)\alpha_Q$,

$$\sup_{\|u\|_{W^{1,Q}(\mathbb{H}^n)}\le 1} \int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|u|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |u|^{kQ'}}{k!} \right\} d\xi = \infty. \tag{2.9}$$

We employ the following Moser function sequence:

$$M_l(\xi, r) = \frac{1}{\sigma_Q^{1/Q}} \begin{cases} (\log l)^{(Q-1)/Q} & \text{when } \rho(\xi) \le r/l, \\ (\log l)^{-1/Q} \log(r/\rho(\xi)) & \text{when } r/l < \rho(\xi) < r, \\ 0 & \text{when } \rho(\xi) \ge r. \end{cases} \tag{2.10}$$

Notice that $|\nabla_{\mathbb{H}^n}\rho(\xi)| = \frac{|z|}{\rho(\xi)}$, where $\xi = (z,t) \in \mathbb{H}^n$, we immediately have

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} M_l|^Q d\xi = 1,$$

and thus

$$\|M_l\|_{W^{1,Q}(\mathbb{H}^n)} = 1 + O(1/\log l).$$

Let $\widetilde{M}_l = M_l/\|M_l\|_{W^{1,Q}(\mathbb{H}^n)}$. It follows that

$$\begin{aligned}\int_{\mathbb{H}^n} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|\widetilde{M}_l|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |\widetilde{M}_l|^{kQ'}}{k!} \right\} d\xi &\ge \int_{\rho\le\frac{r}{l}} \frac{1}{\rho(\xi)^\beta} \left\{ e^{\alpha|\widetilde{M}_l|^{Q'}} - \sum_{k=0}^{Q-2} \frac{\alpha^k |\widetilde{M}_l|^{kQ'}}{k!} \right\} d\xi \\ &\ge \left(l^{\frac{\alpha}{\sigma_Q^{1/(Q-1)}}} e^{O(1)} + O\left((\log l)^{Q-2}\right)\right) \frac{\omega_{Q-1} r^{Q-\beta}}{(Q-\beta) l^{Q-\beta}}. \end{aligned}$$

The last term in the above inequality tends to infinity as $l \to \infty$, thanks to $\alpha > (1-\beta/Q)\alpha_Q$. Therefore (2.9) holds, and thus the proof of Theorem 1.1 is completely finished. □

3. Proof of Theorem 1.2

In this section, we will prove the existence of weak solution to Eq. (1.5). This problem is solved via variational method. The concrete tool we used here is Theorem 1.1 and the mountain-pass theorem.

3.1. The functional

For β: $0 \le \beta < Q$, we define the functional $J_\beta : E \to \mathbb{R}$ by

$$J_\beta(u) = \frac{1}{Q}\|u\|^Q - \int_{\mathbb{H}^n} \frac{F(\xi,u)}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}^n} h(\xi) u d\xi,$$

where $\|u\|$ is defined by (1.6) and $F(\xi,s)=\int_0^s f(\xi,\tau)d\tau$ is the primitive of $f(\xi,s)$. Assume f satisfies the hypothesis (H_1). Then there exist some positive constants α_1 and b_3 such that for all $(\xi,s)\in\mathbb{H}^n\times\mathbb{R}$,

$$F(\xi,s)\le b_3\left\{e^{\alpha_1|s|^{Q/(Q-1)}}-S_{Q-2}(\alpha_1,s)\right\}, \tag{3.1}$$

where

$$S_{Q-2}(\alpha_1,s)=\sum_{k=0}^{Q-2}\frac{\alpha_1^k s^{kQ'}}{k!}.$$

Thus the functional J_β is well defined thanks to Theorem 1.1. It is not difficult to check that $J_\beta\in\mathcal{C}^1(E,\mathbb{R})$. A straightforward calculation shows

$$\langle J_\beta'(u),\phi\rangle=\int_{\mathbb{H}^n}\left(|\nabla_{\mathbb{H}^n}u|^{Q-2}\nabla_{\mathbb{H}^n}u\nabla_{\mathbb{H}^n}\phi+V(z,t)|u|^{Q-2}u\phi\right)d\xi-\int_{\mathbb{H}^n}\frac{f(\xi,u)}{\rho(\xi)^\beta}\phi d\xi-\varepsilon\int_{\mathbb{H}^n}h(\xi)\phi d\xi \tag{3.2}$$

for all $\phi\in E$. Hence a weak solution of (1.5) is a critical point of J_β.

3.2. The geometry of the functional J_β

In this subsection, we check that J_β satisfies the geometric conditions of the mountain-pass theorem without the Palais–Smale condition. For simplicity, here and in the sequel, we write

$$R(\alpha,u)=e^{\alpha|u|^{Q/(Q-1)}}-S_{Q-2}(\alpha u)=\sum_{k=Q-1}^{\infty}\frac{\alpha^k|u|^{kQ'}}{k!}.$$

Lemma 3.1. *Assume that $V(\xi)\ge V_0$ for all $\xi\in\mathbb{H}^n$, (H_1), (H_2), and (H_3) are satisfied. Then for any nonnegative, compactly supported function $u\in W^{1,Q}(\mathbb{H}^n)\setminus\{0\}$, there holds $J_\beta(\tau u)\to-\infty$ as $\tau\to+\infty$.*

Proof. By (H_2) and (H_3), there exists $R_0>0$ such that for all $(\xi,s)\in\mathbb{H}^n\times[R_0,\infty)$, $F(\xi,s)>0$ and $\mu F(\xi,s)\le s\frac{\partial}{\partial s}F(\xi,s)$. This implies $\frac{\partial}{\partial s}(\ln F(\xi,s))\ge\frac{\mu}{s}$, and thus $F(\xi,s)\ge F(\xi,R_0)R_0^{-\mu}s^\mu$. Assume u is supported in a bounded domain Ω. Then, for all $(\xi,s)\in\Omega\times[0,\infty)$, there exist $c_1,c_2>0$ such that $F(\xi,s)\ge c_1s^\mu-c_2$. It follows that

$$\begin{aligned}J_\beta(\tau u)&=\frac{\tau^Q}{Q}\|u\|^Q-\int_\Omega\frac{F(\xi,\tau u)}{\rho(\xi)^\beta}d\xi-\varepsilon\int_\Omega h(\xi)\tau u d\xi\\&\le\frac{\tau^Q}{Q}\|u\|^Q-c_1\tau^\mu\int_\Omega\frac{|u|^\mu}{\rho(\xi)^\beta}d\xi+\tau\int_\Omega|\varepsilon h(\xi)u|d\xi+O(1).\end{aligned}$$

Since $\mu>Q$, this gives the desired result. □

Lemma 3.2. *Assume that $V(\xi)\ge V_0$ for all $\xi\in\mathbb{H}^n$, (H_1), and (H_4) hold. Then there exist $\epsilon>0$ such that for any $\varepsilon:0<\varepsilon<\epsilon$, there exist $r_\varepsilon>0$ and $\vartheta_\varepsilon>0$ such that $J_\beta(u)\ge\vartheta_\varepsilon$ for all u with $\|u\|=r_\varepsilon$.*

Proof. By (H_4), there exist $\tau,\delta>0$ such that if $|s|\le\delta$, then

$$F(\xi,s)\le\frac{\lambda_\beta-\tau}{Q}|s|^Q \tag{3.3}$$

for all $\xi\in\mathbb{H}^n$. By (H_1), we have for $|s|\ge\delta$,

$$\begin{aligned}F(\xi,s)&\le\int_0^{|s|}\left\{b_1t^{Q-1}+b_2R(\alpha_0,t)\right\}dt\\&\le\frac{b_1}{Q}|s|^Q+b_2R(\alpha_0,s)|s|\\&\le c_\delta|s|^{Q+1}R(\alpha_0,s),\end{aligned} \tag{3.4}$$

where $c_\delta=\frac{b_1}{\delta QR(\alpha_0,\delta)}+\frac{b_2}{\delta^Q}$. Combining (3.3) and (3.4), we have for all $(\xi,s)\in\mathbb{H}^n\times\mathbb{R}$,

$$F(\xi,s)\le\frac{\lambda_\beta-\tau}{Q}|s|^Q+C|s|^{Q+1}R(\alpha_0,s). \tag{3.5}$$

Now we *claim* the following inequality

$$\int_{\mathbb{H}^n}\frac{|u|^{Q+1}R(\alpha_0,u)}{\rho(\xi)^\beta}d\xi\le C\|u\|^{Q+1},\quad\forall u\in W^{1,Q}(\mathbb{H}^n). \tag{3.6}$$

To this end, we use the symmetrization argument. Assume u^* is the decreasing rearrangement of $|u|$. By the Hardy–Littlewood inequality (1.1), we have

$$\int_{\mathbb{H}^n} \frac{|u|^{Q+1} R(\alpha_0, u)}{\rho(\xi)^\beta} d\xi \le \int_{\mathbb{H}^n} \frac{|u^*|^{Q+1} R(\alpha_0, u^*)}{\rho(\xi)^\beta} d\xi. \tag{3.7}$$

Let γ be a positive number to be chosen later, we estimate

$$\begin{aligned}
\int_{\rho\le\gamma} \frac{|u^*|^{Q+1} R(\alpha_0, u^*)}{\rho(\xi)^\beta} d\xi &\le \int_{\rho\le\gamma} \frac{|u^*|^{Q+1} e^{\alpha_0 |u^*|^{\frac{Q}{Q-1}}}}{\rho(\xi)^\beta} d\xi \\
&\le \left(\int_{\rho\le\gamma} \frac{e^{p\alpha_0 |u^*|^{Q'}}}{\rho(\xi)^\beta} d\xi \right)^{1/p} \left(\int_{\rho\le\gamma} \frac{1}{\rho(\xi)^{\beta s}} d\xi \right)^{\frac{1}{p's}} \left(\int_{\rho\le\gamma} |u^*|^{(Q+1)p's'} d\xi \right)^{\frac{1}{p's'}} \\
&\le C \left(\int_{\mathbb{H}^n} \frac{R(p\alpha_0, u^*)}{\rho(\xi)^\beta} d\xi \right)^{1/p} \left(\int_{\mathbb{H}^n} |u^*|^{(Q+1)p's'} d\xi \right)^{\frac{1}{p's'}},
\end{aligned}$$

where $p > 1$, $1 < s < \frac{Q}{\beta}$, $1/p + 1/p' = 1$, and $1/s + 1/s' = 1$. This together with Theorem 1.1 and the continuous embedding of $E \hookrightarrow L^q(\mathbb{H}^n)$ $(q \ge Q)$ implies

$$\int_{\rho\le\gamma} \frac{|u^*|^{Q+1} R(\alpha_0, u^*)}{\rho(\xi)^\beta} d\xi \le C\|u\|^{Q+1} \tag{3.8}$$

for some constant C depending only on Q, β and γ, provided that $\|u\|$ is sufficiently small such that $p\alpha_0\|u\|^{Q'} \le \alpha^*$.

On the other hand, taking γ suitably large such that $(Q/\omega_{Q-1})^{1/Q}\gamma^{-1}\|u\|_{L^Q(\mathbb{H}^n)} < 1/2$, we obtain by the radial lemma and the continuous embedding of $E \hookrightarrow L^{Q+1}(\mathbb{H}^n)$,

$$\begin{aligned}
\int_{\rho\ge\gamma} \frac{|u^*|^{Q+1} R(\alpha_0, u^*)}{\rho(\xi)^\beta} d\xi &\le \frac{R(\alpha_0, u^*(\gamma))}{\gamma^\beta} \int_{\rho\ge\gamma} |u^*|^{Q+1} d\xi \\
&\le \frac{R\left(\alpha_0, \frac{1}{2}\right)}{\gamma^\beta} \|u^*\|^{Q+1}_{L^{Q+1}(\mathbb{H}^n)} \\
&\le C\|u\|^{Q+1}
\end{aligned} \tag{3.9}$$

for some constant C. Combining (3.7)–(3.9), we arrive at (3.6), and thus the above claim follows.

Thanks to (3.5), (3.6), and the definition of λ_β,

$$\begin{aligned}
J_\beta(u) &\ge \frac{1}{Q}\|u\|^Q - \frac{\lambda_\beta - \tau}{Q} \int_{\mathbb{H}^n} \frac{|u|^Q}{\rho(\xi)^\beta} d\xi - C\|u\|^{Q+1} - \varepsilon \int_{\mathbb{H}^n} h(\xi) u d\xi \\
&\ge \frac{\tau}{Q\lambda_\beta}\|u\|^Q - C\|u\|^{Q+1} - \varepsilon\|h\|_{E'}\|u\| \\
&= \|u\| \left(\frac{\tau}{Q\lambda_\beta}\|u\|^{Q-1} - C\|u\|^Q - \varepsilon\|h\|_{E'} \right).
\end{aligned}$$

Since $\tau > 0$, there holds for sufficiently small $r > 0$,

$$\frac{\tau}{Q\lambda_\beta} r^{Q-1} - Cr^Q \ge \frac{\tau}{2Q\lambda_\beta} r^{Q-1}.$$

So if we choose ϵ small enough, the conclusion of the lemma follows immediately. □

3.3. Palais–Smale sequence

In this subsection, we analyze the compactness of Palais–Smale sequences of J_β. This is the key step in the study of existence results. First we need the following inequality (for Euclidean or Riemannian cases, see [18,19,11]):

Lemma 3.3. *Let $\mathbb{B}_r = \mathbb{B}_r(\xi^*)$ be a Heisenberg ball centered at $(\xi^*) \in \mathbb{H}^n$ with radius r. Then there exists a positive constant ϵ_0 depending only on n such that*

$$\sup_{\int_{\mathbb{B}_r} |\nabla_{\mathbb{H}^n} u|^Q d\xi \le 1,\ \int_{\mathbb{B}_r} u d\xi = 0} \frac{1}{|\mathbb{B}_r|} \int_{\mathbb{B}_r} e^{\epsilon_0 |u|^{Q'}} d\xi \le C_0 \tag{3.10}$$

for some constant C_0 depending only on n.

Proof. The proof is more or less standard by now [20,21,11] as long as we have the representation formula for functions without the compact support on the Heisenberg group first derived in [22]. For completeness, we give the details here.

Assume $g \in L^Q(\mathbb{B}_r)$ such that $g \geq 0$ and $\|g\|_{L^Q(\mathbb{B}_r)} = 1$. Define an operator $\mathbf{T}$ by

$$\mathbf{T}g(\xi) = \int_{\mathbb{H}^n} \frac{g(\xi')\chi_{\mathbb{B}_r}(\xi')}{\rho_\xi(\xi')^{Q-1}} d\xi',$$

where $\xi = (z, t)$, $\xi' = (z', t')$, $d\xi' = dz'dt'$, and $\rho_\xi(\xi')$ denotes the Heisenberg distance between ξ and ξ'. Without loss of generality, we assume the support of g is a subset of $\mathbb{B}_r$. To estimate $\mathbf{T}g(\xi)$, we set $0 < \delta < R = 2r$. Then

$$\mathbf{T}g(\xi) \leq \int_{\rho_\xi \leq \delta} \frac{g(\xi')}{\rho_\xi(\xi')^{Q-1}} d\xi' + \int_{\delta < \rho_\xi \leq R} \frac{g(\xi')}{\rho_\xi(\xi')^{Q-1}} d\xi'. \tag{3.11}$$

The first integral in the above inequality can be estimated by

$$\begin{aligned} \int_{\rho_\xi \leq \delta} \frac{g(\xi')}{\rho_\xi(\xi')^{Q-1}} d\xi' &= \sum_{k=0}^{\infty} \int_{2^{-k-1}\delta \leq \rho_\xi \leq 2^{-k}\delta} \frac{g(\xi')}{\rho_\xi(\xi')^{Q-1}} d\xi' \\ &\leq \sum_{k=0}^{\infty} (2^{-k-1}\delta)^{1-Q} \int_{\rho_\xi \leq 2^{-k}\delta} g(\xi') d\xi' \\ &\leq C\delta(\mathbf{M}g)(\xi), \end{aligned} \tag{3.12}$$

where $\mathbf{M}g$ is the Hardy–Littlewood maximum function, C is a constant depending only on Q. Notice that $\|g\|_{L^Q(\mathbb{B}_r)} = 1$, we have by using the Hölder inequality

$$\begin{aligned} \int_{\delta < \rho_\xi \leq R} \frac{g(\xi')}{\rho_\xi(\xi')^{Q-1}} d\xi' &\leq \left(\int_{\delta < \rho_\xi \leq R} \rho_\xi(\xi')^{-Q} d\xi' \right)^{1/Q'} \\ &\leq \left(\sum_{k=0}^{\left[\frac{\log R/\delta}{\log 2}\right]} \int_{2^{-k-1}R \leq \rho_\xi \leq 2^{-k}R} (\rho_\xi(\xi')^{-Q}) d\xi' \right)^{1/Q'} \\ &\leq C\,(\log R/\delta)^{1/Q'}. \end{aligned} \tag{3.13}$$

Inserting (3.12) and (3.13) into (3.11), we obtain

$$\mathbf{T}g(\xi) \leq C\,(\log R/\delta)^{1/Q'} + C\delta\mathbf{M}g(\xi).$$

Take $\delta = \delta(\xi) = \min\{(2C\mathbf{M}g(\xi))^{-1}, R\}$. If $\mathbf{T}g(\xi) > 1$, then

$$\mathbf{T}g(\xi) \leq 2C\,(\log R/\delta)^{1/Q'}.$$

Define a set $E = \{\xi \in \mathbb{B}_r : \mathbf{T}g(\xi) > 1\}$. Noticing $\|\mathbf{M}g\|_{L^Q(\mathbb{B}_r)} \leq A\|g\|_{L^Q(\mathbb{B}_r)}$ for some constant A depending only on Q, and $R/\delta \leq 1 + 2CR\mathbf{M}g(\xi)$, we estimate

$$\begin{aligned} \frac{1}{|\mathbb{B}_r|} \int_E e^{\left(\frac{1}{2C}\mathbf{T}g(\xi)\right)^{Q'}} d\xi &\leq \frac{1}{|\mathbb{B}_r|} \int_E \frac{R}{\delta} d\xi \\ &\leq 1 + \frac{2CR}{|\mathbb{B}_r|} \int_{\mathbb{B}_r} \mathbf{M}g(\xi) d\xi \\ &\leq 1 + \frac{2CR}{|\mathbb{B}_r|^{1/Q}} \left(\int_{\mathbb{B}_r} (\mathbf{M}g(\xi))^Q d\xi \right)^{1/Q} \\ &\leq 1 + \frac{2CR}{|\mathbb{B}_r|} A\|g\|_{L^Q(\mathbb{B}_r)}. \end{aligned}$$

Recall $R = 2r$ and $\|g\|_{L^Q(\mathbb{B}_r)} = 1$, we can find a constant C_1 depending only on Q such that

$$\frac{1}{|\mathbb{B}_r|} \int_E e^{\left(\frac{1}{2C}\mathbf{T}g(\xi)\right)^{Q'}} d\xi \leq C_1.$$

On the other hand, there holds

$$\frac{1}{|\mathbb{B}_r|} \int_{\mathbb{B}_r \setminus E} e^{\left(\frac{1}{2C}\mathbf{T}g(\xi)\right)^{Q'}} d\xi \leq e^{1/(2C)^{Q'}}.$$

Therefore we obtain for some constant C_2 depending only on Q,

$$\frac{1}{|\mathbb{B}_r|}\int_{\mathbb{B}_r} e^{\left(\frac{1}{2C}\mathbf{T}g(\xi)\right)^{Q'}} d\xi \le C_2. \tag{3.14}$$

To prove the lemma, it suffices to prove the integrals in (3.10) are bounded for all functions $u \in W^{1,Q}(\mathbb{B}_r)$ with $\|\nabla_{\mathbb{H}^n} u\|_{L^Q(\mathbb{B}_r)} = 1$ and $\int_{\mathbb{B}_r} u d\xi = 0$. For such u, it was shown in [22] that

$$|u(\xi)| \le C_3 \int_{\mathbb{B}_r} \frac{|\nabla_{\mathbb{H}^n} u(\xi')|}{\rho_\xi(\xi')^{Q-1}} d\xi', \quad \forall \xi \in \mathbb{B}_r.$$

Set $g(\xi) = |\nabla_{\mathbb{H}^n} u(\xi)|$. Then $g \ge 0$ and $\|g\|_{L^Q(\mathbb{B}_r)} = 1$. Hence we get the desired result from (3.14). □

Lemma 3.4. *Assume $V \ge V_0 > 0$ in $\mathbb{H}^n$, $V(\xi) \to \infty$ as $\rho(\xi) \to \infty$, (H_1) and (H_2) are satisfied. Let $(u_k) \subset E$ be an arbitrary Palais–Smale sequence of J_β, i.e.,*

$$J_\beta(u_k) \to c, \qquad J'_\beta(u_k) \to 0 \quad \text{in } E' \text{ as } k \to \infty.$$

Then there exists a subsequence of (u_k) (still denoted by (u_k)) and $u \in E$ such that

$$\begin{cases} \dfrac{f(\xi,u_k)}{\rho(\xi)^\beta} \to \dfrac{f(\xi,u)}{\rho(\xi)^\beta} & \text{strongly in } L^1_{loc}(\mathbb{H}^n) \\ \nabla u_k(\xi) \to \nabla u(\xi) & \text{almost everywhere in } \mathbb{H}^n \\ |\nabla u_k|^{Q-2}\nabla u_k \rightharpoonup |\nabla u|^{Q-2}\nabla u & \text{weakly in } \left(L^{Q'}(\mathbb{H}^n)\right)^{Q-2}. \end{cases}$$

Furthermore u is a weak solution of (1.5).

Proof. Let (u_k) be a Palais–Smale sequence of J_β, i.e.,

$$\frac{1}{Q}\|u_k\|^Q - \int_{\mathbb{H}^n} \frac{F(\xi,u_k)}{\rho(\xi)^\beta} d\xi - \varepsilon \int_{\mathbb{H}^n} h(\xi) u_k d\xi \to c \quad \text{as } k \to \infty, \tag{3.15}$$

$$\left|\langle J'_\beta(u_k), \varphi\rangle\right| \le \tau_k \|\varphi\| \quad \text{for all } \varphi \in E, \tag{3.16}$$

where $\tau_k \to 0$ as $k \to \infty$. Taking $\varphi = u_k$ in (3.16), we have

$$\int_{\mathbb{H}^n} \frac{f(\xi,u_k)u_k}{\rho(\xi)^\beta} d\xi + \varepsilon \int_{\mathbb{H}^n} h(\xi) u_k d\xi - \|u_k\|^Q \le \tau_k \|u_k\|.$$

This together with (3.15) and the hypothesis (H_2) leads to

$$\left(\frac{\mu}{Q} - 1\right)\|u_k\|^Q \le C(1 + \|u_k\|).$$

Hence we conclude that $\|u_k\|$ is bounded, and thus

$$\int_{\mathbb{H}^n} \frac{f(\xi,u_k)u_k}{\rho(\xi)^\beta} d\xi \le C, \qquad \int_{\mathbb{H}^n} \frac{F(\xi,u_k)}{\rho(\xi)^\beta} d\xi \le C. \tag{3.17}$$

Here we have used the hypothesis (H_2) again. Thanks to the assumptions on the potential V, the embedding $E \hookrightarrow L^q(\mathbb{H}^n)$ is compact for all $q \ge Q$, and thus we can assume without loss of generality that $u_k \rightharpoonup u$ weakly in E, $u_k \to u$ strongly in $L^q(\mathbb{H}^n)$ for all $q \ge Q$, and $u_k \to u$ almost everywhere in $\mathbb{H}^n$. In view of (H_1), we have by the Trudinger–Moser inequality and the Hölder inequality that $\frac{f(\xi,u)}{\rho(\xi)^\beta} \in L^1_{loc}(\mathbb{H}^n)$. Noticing that Lemma 2.1 in [23] is applicable in our case, we conclude

$$\frac{f(\xi,u_k)}{\rho(\xi)^\beta} \to \frac{f(\xi,u)}{\rho(\xi)^\beta} \quad \text{strongly in } L^1_{loc}(\mathbb{H}^n). \tag{3.18}$$

Now we are proving the remaining part of the lemma. Up to a subsequence, we can define an energy concentration set for any fixed $\delta > 0$,

$$\Sigma_\delta = \left\{\xi \in \mathbb{H}^n : \lim_{r\to 0}\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi)} (|\nabla_{\mathbb{H}^n} u_k|^Q + |u_k|^Q) d\xi' \ge \delta\right\}.$$

Since (u_k) is bounded in E, Σ_δ must be a finite set. For any $\xi^* \in \mathbb{H}^n \setminus \Sigma_\delta$, there exists $r : 0 < r < dist(\xi^*, \Sigma_\delta)$ such that

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)} (|\nabla_{\mathbb{H}^n} u_k|^Q + |u_k|^Q) d\xi < \delta.$$

It follows that for large k,

$$\int_{\mathbb{B}_r(\xi^*)} (|\nabla_{\mathbb{H}^n} u_k|^Q + |u_k|^Q) d\xi < \delta. \tag{3.19}$$

Thanks to Lemma 3.3, for sufficiently small $\delta > 0$, there exists some $q > 1$ such that

$$\int_{\mathbb{B}_r(\xi^*)} \frac{|f(\xi, u_k)|^q}{\rho(\xi)^\beta} d\xi \le C. \tag{3.20}$$

For any $M > 0$, we denote

$$A_M = \{\xi \in \mathbb{B}_r(\xi^*) : |u(\xi)| \ge M\}.$$

It can be estimated that

$$\begin{aligned}
\int_{A_M} \frac{|f(\xi, u_k) - f(\xi, u) \parallel u|}{\rho(\xi)^\beta} d\xi &\le \left(\int_{A_M} \frac{|f(\xi, u_k) - f(\xi, u)|^q}{\rho(\xi)^\beta} d\xi\right)^{1/q} \left(\int_{A_M} \frac{|u|^{q'}}{\rho(\xi)^\beta} d\xi\right)^{1/q'} \\
&\le \left\{ \left\| \frac{f(\xi, u_k)}{\rho(\xi)^{\beta/q}} \right\|_{L^q(\mathbb{B}_r(z^*, t^*))} + \left\| \frac{f(\xi, u)}{\rho(\xi)^{\beta/q}} \right\|_{L^q(\mathbb{B}_r(\xi^*))} \right\} \\
&\quad \times \left\| \frac{1}{\rho(\xi)^\beta} \right\|_{L^s(\mathbb{B}_r(\xi^*))}^{1/q'} \left(\int_{A_M} |u|^{q's'} d\xi\right)^{1/(q's')} \\
&\le C \left(\int_{A_M} |u|^{q's'} d\xi\right)^{1/(q's')},
\end{aligned}$$

where $1/q + 1/q' = 1$, $1/s + 1/s' = 1$, and $0 < s < Q/\beta$. Here we have used (3.20) in the last inequality. Since $u \in L^{q's'}(\mathbb{B}_r(\xi^*))$, we have for any $\nu > 0$,

$$\int_{A_M} \frac{|f(\xi, u_k) - f(\xi, u) \parallel u|}{\rho(\xi)^\beta} d\xi < \nu, \tag{3.21}$$

provided that M is chosen sufficiently large. It follows from (3.18) that

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)\setminus A_M} \frac{|f(\xi, u_k) - f(\xi, u) \parallel u|}{\rho(\xi)^\beta} d\xi = 0. \tag{3.22}$$

Combining (3.21) and (3.22), we have

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)} \frac{|f(\xi, u_k) - f(\xi, u) \parallel u|}{\rho(\xi)^\beta} d\xi \le \nu,$$

and thanks to the fact that $\nu > 0$ is arbitrary,

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)} \frac{|f(\xi, u_k) - f(\xi, u) \parallel u|}{\rho(\xi)^\beta} d\xi = 0. \tag{3.23}$$

On the other hand, we have by using the Hölder inequality, (3.18) and (3.20),

$$\begin{aligned}
\int_{\mathbb{B}_r(\xi^*)} \frac{|f(\xi, u_k) \parallel u_k - u|}{\rho(\xi)^\beta} d\xi &\le \left\| \frac{f(\xi, u_k)}{\rho(\xi)^{\beta/q}} \right\|_{L^q} \left\| \frac{1}{\rho(\xi)^\beta} \right\|_{L^s}^{\frac{1}{q'}} \|u_k - u\|_{L^{q's'}} \\
&\le C\|u_k - u\|_{L^{q's'}} \to 0,
\end{aligned} \tag{3.24}$$

where $1/q + 1/q' = 1$, $1/s + 1/s' = 1$, and $0 < s < Q/\beta$. Combining (3.23) and (3.24), we get

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)} \frac{|f(\xi, u_k)u_k - f(\xi, u)u|}{\rho(\xi)^\beta} d\xi = 0.$$

A covering argument implies that for any compact set $K \subset\subset \mathbb{H}^n \setminus \Sigma_\delta$,

$$\lim_{k\to\infty} \int_K \frac{|f(\xi, u_n)u_n - f(\xi, u)u|}{\rho(\xi)^\beta} d\xi = 0. \tag{3.25}$$

Next we will prove for any compact set $K \subset\subset \mathbb{H}^n \setminus \Sigma_\delta$,

$$\lim_{k\to\infty} \int_K |\nabla_{\mathbb{H}^n} u_k - \nabla_{\mathbb{H}^n} u|^Q d\xi = 0. \tag{3.26}$$

It suffices to prove for any $(\xi^*) \in \mathbb{H}^n \setminus \Sigma_\delta$, and r given by (3.19), there holds

$$\lim_{k\to\infty} \int_{\mathbb{B}_{r/2}(\xi^*)} |\nabla_{\mathbb{H}^n} u_k - \nabla_{\mathbb{H}^n} u|^Q d\xi = 0. \tag{3.27}$$

For this purpose, we take $\phi \in C_0^\infty(\mathbb{B}_r(\xi^*))$ with $0 \le \phi \le 1$ and $\phi \equiv 1$ on $\mathbb{B}_{r/2}(\xi^*)$. Obviously ϕu_k is a bounded sequence in E. Inserting $\varphi = \phi u_k$ and $\varphi = \phi u$ into (3.16) respectively, we have

$$\begin{aligned}
&\int_{\mathbb{B}_r(\xi^*)} \phi(|\nabla_{\mathbb{H}^n} u_k|^{Q-2}\nabla_{\mathbb{H}^n} u_k - |\nabla_{\mathbb{H}^n} u|^{Q-2}\nabla_{\mathbb{H}^n} u)(\nabla_{\mathbb{H}^n} u_k - \nabla_{\mathbb{H}^n} u)d\xi \\
&\quad\le \int_{\mathbb{B}_r(\xi^*)} |\nabla_{\mathbb{H}^n} u_k|^{Q-2}\nabla_{\mathbb{H}^n} u_k \nabla_{\mathbb{H}^n}\phi(u - u_k)d\xi + \int_{\mathbb{B}_r(\xi^*)} \phi|\nabla_{\mathbb{H}^n} u|^{Q-2}\nabla_{\mathbb{H}^n} u(\nabla_{\mathbb{H}^n} u - \nabla_{\mathbb{H}^n} u_k)d\xi \\
&\qquad + \int_{\mathbb{B}_r(\xi^*)} \phi(u_k - u)\frac{f(\xi, u_k)}{\rho(\xi)^\beta}d\xi + \tau_k\|\phi u_k\| + \tau_k\|\phi u\| - \varepsilon\int_{\mathbb{B}_r(\xi^*)} \phi h(u_k - u)d\xi.
\end{aligned} \tag{3.28}$$

The integrals on the right side of this inequality can be estimated as below. By the Hölder inequality and the compact embedding of $E \hookrightarrow L^Q(\mathbb{H}^n)$, we have

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)} |\nabla_{\mathbb{H}^n} u_k|^{Q-2}\nabla_{\mathbb{H}^n} u_k \nabla_{\mathbb{H}^n}\phi(u - u_k)d\xi = 0. \tag{3.29}$$

Since $\nabla_{\mathbb{H}^n} u_k \rightharpoonup \nabla_{\mathbb{H}^n} u$ weakly in $(L^Q(\mathbb{H}^n))^{Q-2}$, there holds

$$\lim_{k\to\infty} \int_{\mathbb{B}_r(\xi^*)} \phi|\nabla_{\mathbb{H}^n} u|^{Q-2}\nabla_{\mathbb{H}^n} u(\nabla_{\mathbb{H}^n} u - \nabla_{\mathbb{H}^n} u_k)d\xi = 0. \tag{3.30}$$

The Hölder inequality and (3.24) implies that $\int_{\mathbb{B}_r(\xi^*)} \phi(u_k - u)\frac{f(\xi,u_k)}{\rho(\xi)^\beta}d\xi \to 0$ as $k \to \infty$. This together with (3.29), (3.30), $u_k \rightharpoonup u$ weakly in E, and $\tau_k \to 0$ implies that the integral sequence on the left side of (3.28) tends to zero as $k \to \infty$. Using an elementary inequality

$$2^{2-Q}|b - a|^Q \le \langle |b|^{Q-2}b - |a|^{Q-2}a, b - a\rangle, \quad \forall a,\ b \in \mathbb{R}^{Q-2},$$

we derive (3.27) from (3.28). Hence (3.26) holds thanks to a covering argument. Since Σ_δ is a finite set, it follows that $\nabla_{\mathbb{H}^n} u_k$ converges to $\nabla_{\mathbb{H}^n} u$ almost everywhere in $\mathbb{H}^n$. This immediately implies, up to a subsequence, $|\nabla_{\mathbb{H}^n} u_k|^{Q-2}\nabla_{\mathbb{H}^n} u_k \rightharpoonup |\nabla_{\mathbb{H}^n} u|^{Q-2}\nabla_{\mathbb{H}^n} u$ weakly in $(L^{Q'}(\mathbb{B}_R))^{Q-2}$ for any $R > 0$. For any fixed $\varphi \in C_0^\infty(\mathbb{H}^n)$, there exists some $R_0 > 0$ such that the support of φ is contained in the ball $\mathbb{B}_{R_0}$. Hence

$$\lim_{k\to\infty} \int_{\mathbb{H}^n} (|\nabla_{\mathbb{H}^n} u_k|^{Q-2}\nabla_{\mathbb{H}^n} u_k - |\nabla_{\mathbb{H}^n} u|^{Q-2}\nabla_{\mathbb{H}^n} u)\varphi d\xi = 0.$$

This equality holds for all $\varphi \in L^{Q'}(\mathbb{H}^n)$, thanks to the density of $C_0^\infty(\mathbb{H}^n)$ in $L^{Q'}(\mathbb{H}^n)$. Hence we obtain

$$|\nabla_{\mathbb{H}^n} u_k|^{Q-2}\nabla_{\mathbb{H}^n} u_k \rightharpoonup |\nabla_{\mathbb{H}^n} u|^{Q-2}\nabla_{\mathbb{H}^n} u \quad \text{weakly in } (L^{Q'}(\mathbb{H}^n))^{Q-2}. \tag{3.31}$$

Passing to the limit $k \to \infty$ in (3.16), we obtain by combining (3.18) and (3.31),

$$\langle J_\beta'(u), \varphi\rangle = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{H}^n).$$

Since $C_0^\infty(\mathbb{H}^n)$ is dense in E, the above equation implies that u is a weak solution of (1.5). This completes the proof of the lemma. □

3.4. Completion of the proof of Theorem 1.2

By Lemmas 3.1 and 3.2, there exists $\epsilon > 0$ such that for all $0 < \varepsilon < \epsilon$, J_β satisfies all the hypotheses of the mountain-pass theorem except for the Palais–Smale condition: $J_\beta \in \mathcal{C}^1(E, \mathbb{R})$; $J_\beta(0) = 0$; $J_\beta(u) \ge \vartheta > 0$ when $\|u\| = r$; $J_\beta(e) < 0$ for some $e \in E$ with $\|e\| > r$. Then using the mountain-pass theorem without the Palais–Smale condition, we can find a sequence (u_n) of E such that

$$J_\beta(u_n) \to c > 0, \qquad J_\beta'(u_n) \to 0 \quad \text{in } E',$$

where

$$c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J_\beta(u) \geq \vartheta$$

is the mountain-pass level of J_β, where $\Gamma = \{g \in \mathcal{C}([0,1], E) : g(0) = 0, g(1) = e\}$. By Lemma 3.4, there exists a subsequence of (u_n) converges weakly to a solution of (1.5) in E. Finally, this solution must be nontrivial since $h \neq 0$.

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