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Calculus of Variations

Sharp weighted Trudinger–Moser–Adams inequalities on the whole space and the existence of their extremals

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Abstract

Though there have been extensive works on the existence of maximizers for sharp first order Trudinger–Moser inequalities, much less is known for that of the maximizers for higher order Adams' inequalities. In this paper, we mainly study the existence of extremals for sharp weighted Trudinger–Moser–Adams type inequalities with the Dirichlet and Sobolev norms (also known as the critical and subcritical Trudinger–Moser–Adams inequalities), see Theorems 1.1, 1.2, 1.3, 1.5, 1.7, 1.9 and 1.11. First, we employ the method based on level-sets of functions under consideration and Fourier transform to establish stronger weighted Trudinger–Moser–Adams type inequalities with the Dirichlet norm in $W^{2,\frac{n}{2}}(\mathbb{R}^n)$ and $W^{m,2}(\mathbb{R}^{2m})$ respectively. While the first order sharp weighted Trudinger–Moser inequality and its existence of extremal functions was established by Dong and the second author using a quasi-conformal type transform (Dong and Lu in Calc Var Partial Differ Equ 55:55–88, 2016), such a transform does not work for the Adams inequality involving higher order derivatives. Since the absence of the Polyá–Szegö inequality and the failure of change of variable method for higher order derivatives for weighted inequalities, we will need several compact embedding results (Lemmas 2.1, 3.1 and 5.2). Through the compact embedding and scaling invariance of the subcritical Adams inequality, we investigate the attainability of best constants. Second, we employ the method developed by Lam et al. (Adv Math 352:1253–1298, 2019) which uses the relationship between the supremums of the critical and subcritical inequalities (see also Lam in Proc Amer Math Soc 145:4885–4892, 2017) to establish the existence of extremals for weighted Adams' inequalities with the Sobolev norm. Third, using the Fourier rearrangement inequality established by Lenzmann and Sok (A sharp rearrangement principle in Fourier space and symmetry results for PDEs with arbitrary order, arXiv:1805.06294v1), we can reduce our problem to the radial case and then establish the existence of the extremal functions for the non-weighted Adams inequalities. As an application, we derive new results on high-order critical Caffarelli–Kohn–Nirenberg interpolation inequalities whose parameters extend those proved by Lin (Commun Partial Differ Equ 11:1515–1538, 1986) (see Theorems 1.13 and 1.14). Furthermore, we also establish the relationship between the best constants of the weighted Trudinger–Moser–Adams

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type inequalities and the Caffarelli–Kohn–Nirenberg inequalities in the asymptotic sense (see Theorems 1.13 and 1.14).

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1 Introduction

In this paper, we establish the weighted Adams inequality in higher order Sobolev spaces $W^{2,\frac{n}{2}}(\mathbb{R}^n)$ and $W^{m,2}(\mathbb{R}^{2m})$ and prove the existence of a maximizer associated with the singular (weighted) Adams inequality. As is well known, classical Sobolev embedding theorems on bounded domain assert that $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq p^*$ and $p < n$, where $p^* = \frac{np}{n-p}$ is called the Sobolev exponent. In the limiting case $p = n$, the Sobolev exponent becomes infinite and $W_0^{1,n}(\Omega) \subset L^q(\Omega)$ for $1 \leq q < \infty$, but $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$. To fill this gap, Trudinger [59] discovered a borderline embedding result (see also Juovič [19], Pohozaev [55]) which was subsequently sharpened by Moser [54]. This result has been known as the Trudinger–Moser inequality since then and we state it as follows.

Theorem A [54,59] *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then there exist a positive constant C_n and a sharp constant $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\alpha|u|^{\frac{n}{n-1}})dx \leq C_n,$$

for any $\alpha \leq \alpha_n$ and $u \in C_0^\infty(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$, where ω_{n-1} is the area of the surface of the unit ball.

Using a symmetrization argument, Carleson and Chang [6] reduced the existence issue to a one-dimensional problem to establish for the first time the existence of extremal functions of Trudinger–Moser inequality when Ω is a ball in $\mathbb{R}^n$. Later, results of Carleson and Chang were extended by Flucher [14] to arbitrary bounded domains in $\mathbb{R}^2$ and by Lin [41] in $\mathbb{R}^n$ for the case $n > 2$. Malchiodi and Martinazzi [49] further investigated the blow-up of a sequence of critical points of the Trudinger–Moser functionals on the planar disk.

There are many extensions of Theorem **A**. One is to extend the Trudinger–Moser inequality to the entire space, see Cao [5], do Ó [10] and Adachi and Tanaka [1], etc. We state a sharp version from [1] as follows.

Theorem B [1] *For $n \geq 2$ and $0 < \alpha < \alpha_n$, there exists a positive constant $C_{n,\alpha}$ such that*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n}|\nabla u|^n dx \leq 1} \frac{1}{\|u\|_{L^n}^n}\int_{\mathbb{R}^n} \Psi(\alpha|u(x)|^{\frac{n}{n-1}})dx \leq C_{n,\alpha}. \tag{1.1}$$

where $\Psi(t) := e^t - \sum_{i=0}^{n-2}\frac{t^i}{i!}$. Moreover, the constant α_n is sharp in the sense that if $\alpha \geq \alpha_n$, the supremum will become infinite.

When it comes to the singular Trudinger–Moser inequality in $\mathbb{R}^n$, there are several works devoted to it. Ishiwata, Nakamura and Wadade [18] investigated the scaling invariant form of the singular Trudinger–Moser inequality for radially symmetric functions and proved the existence of a maximizer. In fact, they proved

Theorem C [18] *Assume* $n \geq 2$, $-\infty < s \leq t < n$ *and* $0 < \alpha < \alpha_n := \frac{\omega_{n-1}^{\frac{1}{n-1}}}{n}$, *then there exists a positive constant* $C = C(n, s, t, \alpha)$ *such that the inequality*

$$\int_{\mathbb{R}^n} \frac{\Psi(\alpha(1-\frac{t}{n})|u(x)|^{\frac{n}{n-1}})}{|x|^t} dx \leq C \left(\int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^s} dx \right)^{\frac{n-t}{n-s}}, \tag{1.2}$$

holds for all radially symmetric functions $u \in L^n(\mathbb{R}^n; |x|^{-s}dx) \cap \dot{W}^{1,n}(\mathbb{R}^n)$ *with* $\|\nabla u\|_n \leq 1$, *where* $\dot{W}^{1,n}(\mathbb{R}^n)$ *denotes the class of functions* u *which are locally integrable and* $\|\nabla u\|_n$ *are in* $L^n(\mathbb{R}^n)$. *Moreover, the constant* $\alpha_{n,t}$ *is sharp for the inequality.*

They also showed that when $s = 0$, the constant C has an infimum and could be attained by some function $u \in W^{1,n}(\mathbb{R}^n)$. However, when $s \neq 0$, they only verified inequality (1.2) and the existence of extremals on the class of radial functions. A natural problem is whether we can remove the radially symmetric condition for functions u in inequality (1.2). Dong and Lu [13] gave an affirmative answer. Indeed, they proved

Theorem D [13] *Assume* $n \geq 2$, $-\infty < s \leq t < n$ *and* $0 < \alpha < \alpha_n$, *then there exists a positive constant* $C = C(n, s, t, \alpha)$ *such that the inequality*

$$\int_{\mathbb{R}^n} \frac{\Psi(\alpha(1-\frac{t}{n})|u(x)|^{\frac{n}{n-1}})}{|x|^t} dx \leq C \left(\int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^s} dx \right)^{\frac{n-t}{n-s}},$$

holds for all functions $u \in L^n(\mathbb{R}^n; |x|^{-s}dx) \cap \dot{W}^{1,n}(\mathbb{R}^n)$ *with* $\|\nabla u\|_n \leq 1$. *Moreover, the constant* α_n *is sharp in the sense that if* $\alpha \geq \alpha_n$ *then the above inequality cannot hold with a uniform* C *independent of* u.

By applying a new method of change of variables of quasi-conformal type in [13], Dong and the second author kept the gradient norm less than 1 and eliminated the weights of integral at the same time. Furthermore, they also established the existence of the maximizers associated with the Trudinger–Moser inequality (1.2). We also note that this method of change of variables is surprisingly simple and efficient in dealing with some weighted inequalities involving the first order derivatives. For example, this change of variable method has also been used by Lam and Lu [30] and Dong et al. [12] to obtain the existence for a wide range of parameters of the first order Caffarelli–Kohn–Nirenberg inequalities (see [9]).

Note that the Trudinger–Moser inequality (1.1) doesn't hold in the critical case $\alpha = \alpha_n$. To obtain the Trudinger–Moser inequality in the critical case, Ruf [56] (in the dimension $n = 2$) and Li and Ruf [39] (in the dimension $n \geq 3$) used the standard Sobolev norm to replace the Dirichlet norm, i.e.

$$\|u\|^n_{W_0^{1,n}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u|^n + |u|^n dx,$$

and obtained the inequality with sharp constant α_n. Furthermore, they establish the existence of a maximizer when $\alpha = \alpha_n$ by carrying out the blow-up procedure. As for the case $n = 2$ and $\alpha = \alpha_2 = 4\pi$, the existence of a maximizer was considered in Ruf [56] and Ishiwata [17]. Moreover in $n = 2$ and α is very small, the non-existence of the maximizer was also established in Ishiwata [17]. Dong and Lu [13], Lam [21–23], Dong et al. [12], Lam et al. [33] further established more existence and nonexistence result of extremal functions for more general weighted Trudinger–Moser inequalities on the whole space $\mathbb{R}^n$ and proved the radial symmetry of the extremal functions. For more related results about Trudinger–Moser inequalities and the existence of extremal functions for the Trudinger–Moser inequalities, one can also refer to [3,4,8,38,48–50] and many references therein. We note that both the proofs of the critical Trudinger–Moser inequality in [39,56] and the subcritical inequality in [1,10] use the Polyá–Szegö inequality and a symmetrization argument. A symmetrization-free argument was developed by Lam and the second author [29] (see also [28]) which gives an alternative proof of the critical Trudinger–Moser inequality (see the proof given on page 318 of [29]). A symmetrization-free argument for the subcritical Trudinger–Moser inequality of using the level sets of functions under consideration was also given by Lam, the second author and Tang [24] (see also [37,63] for use of such an argument in the concentration-compactness principle, in the proofs of Trudinger–Moser inequalities under different norms [31,34,43,58] and for Trudinger–Moser inequalities under the Lorentz–Sobolev norm).

Trudinger and Moser's results for the first order derivatives were extended to higher order derivatives by Adams [2]. To state his result, we use the symbol $\nabla^m u$ to denote

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}}, & \text{if } m \text{ is even,} \\ \nabla \Delta^{\frac{m-1}{2}}, & \text{if m is odd.} \end{cases}$$

Then, Adams' results can be stated as follows:

Theorem E [2] *Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}})dx \leq C_0, \tag{1.3}$$

for all $\beta \leq \beta(n,m)$ where

$$\beta(n,m) = \begin{cases} \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}}, & \textit{where } m \textit{ is odd.} \\ \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}}, & \textit{where } m \textit{ is even.} \end{cases}$$

Furthermore, the constant $\beta(n,m)$ is best possible in the sense that for any $\beta > \beta(n,m)$, the integral can be made as large as possible.

Much improved Hardy–Trudinger–Moser inequalities on (hyperbolic) balls and convex domains have been established by Wang and Ye [60], Lu and Yang [46], Wang [61], and the so-called Hardy–Adams inequalities have been recently established using Fourier analysis on hyperbolic spaces by the second author and Yang [36,45] (see also [62]). We also mention that the existence of extremal functions for the Adams inequality (1.3) on bounded domain in the case $n = 2m = 4$ was established by Lu and Yang [47]. The Adams inequality (1.3) on bounded domains was also extended to entire space case. Kozono [20] established the Adams type inequality in the entire space except for the critical case which was established by Ruf and Sani [57] for even integer m and Lam and Lu [26,27] for odd integer m. Indeed, Lam and Lu [29] used a symmetrization-free approach to establish the singular Adams inequality of any fractional order γ on the Sobolev space $W^{\gamma,\frac{n}{\gamma}}(\mathbb{R}^n)$ (see [29]). In particular, when $\gamma = m$ we have the following

Theorem F [29] *Let m be a positive integer less than n, $\tau > 0$ and $0 \leq \alpha < n$. Then there holds*

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \|(\tau I - \Delta)^{\frac{m}{2}} u\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \frac{\Phi_{n,m}(\beta_{\alpha,n,m}|u|^{\frac{n}{n-m}})}{|x|^\alpha} dx < \infty,$$

where

$$j_{\frac{n}{m}} = \min\left\{j \in \mathbb{Z} : j \geq \frac{n}{m}\right\} \text{ and } \Phi_{n,m}(t) = \exp(t) - \sum_{i=0}^{j_{\frac{n}{m}}-2} \frac{t^i}{i!},$$

$$\beta_{n,m} = \frac{n}{\omega_{n-1}}\left[\frac{2^m \pi^{\frac{n}{2}} \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\right]^{\frac{n}{n-m}} \text{ and } \beta_{\alpha,n,m} = \beta_{n,m}\left(1 - \frac{\alpha}{n}\right).$$

When $m = 2$, they gave another form.

Theorem G [29] *There exists a positive constant C_n such that*

$$\int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta_{n,2}(1-\frac{t}{n})|u|^{\frac{n}{n-2}})}{|x|^t} dx \leq C_n, \quad \forall\, u \in C_c^\infty(\mathbb{R}^n) \text{ with } \int_{\mathbb{R}^n} |\Delta u|^{\frac{n}{2}} + |u|^{\frac{n}{2}} dx \leq 1, \tag{1.4}$$

where $j_{\frac{n}{2}} = \min\{j \in \mathbb{Z} : j \geq \frac{n}{2}\}$ and $\beta_{n,2} = \frac{n}{\omega_{n-1}}[\frac{4\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)}]^{\frac{n}{n-2}}$.

In [32], Lam et al. established the following sharp second-order Adams inequality with the Dirichlet norm.

Theorem H [32] *For $0 < \beta < \beta_{n,2}$ and $0 \leq t < n$, then there exists a positive constant $C(n,t)$ such that for all functions $u \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ with $\|\Delta u\|_{\frac{n}{2}} = 1$, the following inequality holds.*

$$\int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})}{|x|^t} dx \leq C(n,t)\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}}, \tag{1.5}$$

where $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) = \{u \in L^1_{loc}(\mathbb{R}^n) \mid \Delta u \in L^{\frac{n}{2}}(\mathbb{R}^n)\}$. Moreover, the constant $\beta_{n,2}$ is sharp in the sense that the inequality fails if the constant β is replaced by any $\beta \geq \beta_{n,2}$.

A natural question is whether there exist extremal functions for the above inequality. To our knowledge, much less is known for that of the maximizers for Adams' inequalities. The first goal of this paper is extending Dong and Lu's work [13] to second-order Sobolev space $W^{2,\frac{n}{2}}(\mathbb{R}^n)$. Since the absence of the Polyá–Szegö inequality and the failure of change of variable method for higher order derivatives, we use the method combining the scaling invariance of the Adams inequality and the new compact imbedding $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^n; |x|^{-t}dx)$, for all $q \geq \frac{n}{2}$ and $0 < t < n$ to obtain the weighted Adams inequality with Dirichlet norm in $W^{2,\frac{n}{2}}(\mathbb{R}^n)$. This idea in spirit is similar to that in the works of Dong and Lu [13], Ishiwata, Nakamura and Wadade [18] for the first order weighted subcritical Trudinger–Moser inequality, and related works for Trudinger–Moser and Adams inequalities with exact growth by Ibrahim et al. [16], Masmoudi and Sani [51–53], Lu and Tang [42] and Lu et al. [44]. Now we start to state our first result.

Theorem 1.1 *For $0 < t < n$, $n \geq 3$, the best constant $C(n,t)$ is achieved.*

Replacing $\Phi_{n,2}(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})$ with $\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^{\frac{n}{2}}$ and $\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^q$ respectively, we establish the following stronger Adams inequality and existence of their extremals.

Theorem 1.2 *For $n \geq 3$, $0 < \beta < \beta_{n,2}$ and $0 \leq t < n$, then there exists a positive constant $C(n,t)$ such that*

$$\int_{\mathbb{R}^n} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^{\frac{n}{2}}}{|x|^t} dx \leq C(n,t)\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}}, \tag{1.6}$$

holds for all functions $u \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ *with* $\|\Delta u\|_{\frac{n}{2}} = 1$. *The constant* $\beta_{n,2}$ *is sharp in the sense that the inequality fails if the constant* β *is replaced by any* $\beta \geq \beta_{n,2}$. *Moreover, in the case* $0 < t < n$, *the best constant* $C(n,t)$ *is achieved.*

Theorem 1.3 *For* $n \geq 3$, $0 < \beta < \beta_{n,2}$, $0 \leq t < n$ *and* $q \geq \frac{n}{2}$, *then there exists a positive constant* $C(n,t)$ *such that*

$$\int_{\mathbb{R}^n} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^q}{|x|^t} dx \leq C(n,t)\left(\int_{\mathbb{R}^n} |u|^q dx\right)^{1-\frac{t}{n}}, \tag{1.7}$$

holds for all functions $u \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *with* $\|\Delta u\|_{\frac{n}{2}} = 1$. *The constant* $\beta_{n,2}$ *is sharp in the sense that the inequality fails if the constant* β *is replaced by any* $\beta \geq \beta_{n,2}$. *Moreover, in the case* $0 < t < n$, *the best constant* $C(n,t)$ *is achieved.*

Remark 1.4 In our proof of inequalities (1.6) and (1.7), the rearrangement-free argument by considering the level sets of the functions and the weighted Trudinger–Moser inequality in $W_N^{2,\frac{n}{2}}(\Omega)$ play a key role.

In 2015, Lam et al. [32] gave a precise asymptotic estimate for the Adams inequality with the Dirichlet norm. More precisely, they proved

$$\begin{aligned} ATA(\beta,t) &:= \sup_{\|\Delta u\|_{\frac{n}{2}} \leq 1} \frac{1}{\|u\|_{\frac{n}{2}}^{\frac{n}{2}(1-\frac{t}{n})}} \int_{\mathbb{R}^n} \frac{\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)}{|x|^t} dx \\ &\approx \frac{1}{\left(1-\left(\frac{\beta}{\beta_{n,2}}\right)^{\frac{n-2}{2}}\right)^{1-\frac{t}{n}}} \end{aligned}$$

with $0 < \beta < \beta_{n,2}$ and $0 \leq t < n$. Furthermore, they also established some relation of the weighted Adams inequalities with Dirichlet norms and Sobolev norms. Indeed, for any a, $b > 0$, $0 \leq t < n$ and $0 < \beta \leq \beta_{n,2}$, define

$$A_{a,b,t}(\beta) = \sup_{\|\Delta u\|_{\frac{n}{2}}^a + \|u\|_{\frac{n}{2}}^b \leq 1} \int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})}{|x|^t} dx.$$

They proved that

$$A_{a,b,t}(\beta) = \sup_{s \in (0,\beta)} \left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} ATA(s,t).$$

Thanks to this equivalence, we employ the method developed by Lam et al. [33] (see also [21]) to obtain the existence of maximizers for the weighted Adams inequality with the Sobolev norm.

Theorem 1.5 *For* $0 < t < n$, $a, b > 0$ *and* $0 < \beta \leq \beta_{n,2}$, *then there exist extremal functions for* $A_{a,b,t}(\beta)$ *in the case of* $(\beta < \beta_{n,2}, b > 0)$ *or* $(\beta = \beta_{n,2}, b < \frac{n}{2})$.

Remark 1.6 To the best knowledge of ours, our results seem to be the first result for the existence of weighted Adams inequality with the Sobolev norm on the whole space. Most proofs for the existence of maximizers of first order Trudinger–Moser inequalities with the

Sobolev norm use the rearrangement argument and the blow-up analysis. Since the absence of the Polyá–Szegö inequality for higher order derivatives, we use the subcritical way introduced by Lam et al.[1] [33] and by Lam in [21]. Roughly speaking, through combining the equivalence of subcritical and critical weighted Adams inequalities in $W^{2,\frac{n}{2}}(\mathbb{R}^n)$, and the existence of extremal functions for subcritical Adams inequalities, we can construct the maximizers of the critical weighted Adams inequalities.

Another natural thought is to establish the Adams inequality with the Dirichlet norm in $W^{m,2}(\mathbb{R}^{2m})$ for any $m \geq 2$. Since the idea of level-sets is not efficient to deal with the weighted Adams inequality in $W^{m,2}(\mathbb{R}^{2m})$ for $m \geq 3$, we use the methods based on Fourier transform to establish the following results.

Theorem 1.7 *For $0 < \beta < \beta_{2m,m}$ and $0 \leq t < 2m$, then there exists a positive constant $C(m,t)$ such that*

$$\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)}{|x|^t}dx \leq C(m,t)\left(\int_{\mathbb{R}^{2m}} |u|^2 dx\right)^{1-\frac{t}{2m}}, \tag{1.8}$$

holds for all functions $u \in \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ with $\|\nabla^m u\|_2 = 1$, where $\dot{W}^{m,2}(\mathbb{R}^{2m}) = \{u \in L^1_{loc}(\mathbb{R}^{2m}) \mid |\nabla^m u| \in L^2(\mathbb{R}^{2m})\}$. The constant $\beta_{2m,m}$ is sharp in the sense that the inequality fails if the constant β is replaced by any $\beta \geq \beta_{2m,m}$. Moreover, in the case $0 < t < 2m$, the best constant $C(m,t)$ is achieved.

Remark 1.8 In the case $t = 0$, the validity and the sharpness of inequality (1.8) were established by Lam and Lu [25]. See also Fontana and Morpurgo [15], Masmoudi and Sani [53] for more general subcritical and critical Adams inequality in $W^{m,\frac{n}{m}}(\mathbb{R}^{\frac{n}{m}})$ for general $m \geq 1$ and $n \geq m$.

In the setting of $W^{m,2}(\mathbb{R}^{2m})$, we will further prove the following

Theorem 1.9 *For $0 < \beta < \beta_{2m,m}$ and $0 \leq t < 2m$, there exists a positive constant $C(m,t)$ such that*

$$\int_{\mathbb{R}^{2m}} \frac{\exp(\beta(1-\frac{t}{2m})|u|^2)|u|^2}{|x|^t}dx \leq C(m,t)\left(\int_{\mathbb{R}^{2m}} |u|^2 dx\right)^{1-\frac{t}{2m}}, \tag{1.9}$$

holds for all functions $u \in \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ with $\|\nabla^m u\|_2 = 1$. The constant $\beta_{2m,m}$ is sharp in the sense that the inequality fails if the constant β is replaced by any $\beta \geq \beta_{2m,m}$. Moreover, in the case $0 < t < 2m$, the best constant $C(m,t)$ is achieved.

Remark 1.10 In the proof of getting the attainability of $C_{m,t}$, $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^q(\mathbb{R}^{2m}) \hookrightarrow\hookrightarrow L^r(\mathbb{R}^{2m}, \frac{dx}{|x|^t})$ for any $r \geq q$ and $t > 0$ plays an important role. It is also well-known to us that the above compact imbedding fails in the case $t = 0$. However, if u is a radial function, we are in a position to show that $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^q(\mathbb{R}^{2m})$ can be compactly imbedded into $L^r(\mathbb{R}^{2m})$ for any $r > q$. A natural question arise: Do Theorem 1.7 and 1.9 still remain true in the case $t = 0$? Using the Fourier rearrangement inequality established by Lenzmann and Sok [35], we can reduce our problem on inequality (1.8) and (1.9) to the radial case. Combining these facts, by modifying the proof of Theorems 1.7 and 1.9, we can obtain the following results.

[1] We note that the main results of [33] were described in [13].

Theorem 1.11 *In the case $t = 0$, the best constant $C(m, 0)$ in inequalities (1.8) and (1.9) is achieved.*

Finally, as an application of the above theorems, we obtain the higher order Caffarelli–Kohn–Nirenberg (CKN) inequalities in the critical case which was not included in Lin's work [40] and investigate the asymptotic behavior of the best constants of Caffarelli–Kohn–Nirenberg inequalities. We note that the existence of extremal functions for higher order CKN inequalities have been established by Dong [11]. Indeed, we obtain the following results.

Theorem 1.12 *Suppose $n \geq 2$ and $0 \leq t < n$, there exists a constant $c(n, t, q)$ such that for any $u \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$, there holds*

$$\|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq c(n, t, q)\|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}. \tag{1.10}$$

Furthermore, if we assume $\alpha > (\beta_{n,2}(1 - \frac{t}{n})en')^{-\frac{1}{n'}}$, then there exists a sharp constant $q_1(n, t, \alpha) \geq \frac{n}{2}$ such that for $u \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ and $q \geq q_1$, there holds

$$\|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq \alpha q^{\frac{1}{n'}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}. \tag{1.11}$$

Theorem 1.13 *Suppose $m \geq 2$ and $0 \leq t < 2m$, there exists a constant $c(m, t, q)$ such that for any $u \in \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$, there holds*

$$\|u\|_{L^q(\mathbb{R}^{2m};|x|^{-t}dx)} \leq c(m, t, q)\|u\|_2^{\frac{2m-t}{qm}} \|\nabla^m u\|_2^{1-\frac{2m-t}{qm}}. \tag{1.12}$$

Furthermore, if we assume $\alpha > (\beta_{2m,m}(1 - \frac{t}{2m})2e)^{-\frac{1}{2}}$, there exists a sharp constant $q_1(m, t, \beta) \geq 2$ such that for any $u \in \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ and $q \geq q_1$, there holds

$$\|u\|_{L^q(\mathbb{R}^{2m};|x|^{-t}dx)} \leq \alpha q^{\frac{1}{n'}} \|u\|_2^{\frac{2m-t}{qm}} \|\nabla^m u\|_2^{1-\frac{2m-t}{qm}}. \tag{1.13}$$

For the convenience of the statement, we give some notations. We define the sharp constant $\mu_{k_1k_2,k_2,t,\beta}(\mathbb{R}^{k_1k_2})$ by

$$\mu_{k_1k_2,k_2,t,\beta}(\mathbb{R}^{k_1k_2}) := \sup_{u\in \dot{W}^{k_2,k_1}(\mathbb{R}^{k_1k_2}),\|\nabla^{k_2}u\|_{k_1}=1} F_{k_1k_2,k_2,t,\beta}(u),$$

where

$$F_{k_1k_2,k_2,t,\beta}(u) := \frac{\int_{\mathbb{R}^{k_1k_2}} \frac{\Phi_{k_1k_2,k_2}(\beta|u|^{\frac{k_1}{k_1-1}})}{|x|^t}dx}{\|u\|_{k_1}^{\frac{k_1k_2-t}{k_2}}}.$$

This paper is organized as follows. In Sects. 2 and 3, we establish a new compact imbedding theorem. By applying the rearrangement-free argument in the spirit of the work [29] and the weighted Adams' inequalities in $W_N^{2,\frac{n}{2}}(\Omega)$, we establish inequalities (1.6) and (1.7). We also employ the scaling invariant form of the weighted Adams inequality and a new compact imbedding to establish the existence of extremals for inequalities (1.6) and (1.7). With the help of the equivalence results of the weighted Adams' inequalities with Dirichlet norms (subcritical case) and Sobolev norms (critical case) in [32], we derive the first result for the existence of the Adams inequality with the Sobolev norm in Sect. 4. Section 5 is devoted to obtaining the Adams inequalities with the Dirichlet norm and the existence of their extremals in Sobolev space $W^{m,2}(\mathbb{R}^{2m})$. As an application of Theorems 1.1 and 1.7, in Sect. 6, we establish the critical higher order Caffarelli–Kohn–Nirenberg inequalities and investigate some relationship between the best constants of the weighted Adams inequality and the Caffarelli–Kohn–Nirenberg inequality in the asymptotic sense.

2 The proof of Theorem 1.1

In this section, we give the attainability of sharp constant $C(n,t)$ for Adams inequality (1.5) which equipped with the Dirichlet norm. For this purpose, we need the following compact imbedding lemma which also plays a crucial role in obtaining extremal functions for inequalities (1.6) and (1.7).

Lemma 2.1 *Let $n \geq 3$ and $0 < t < n$, then $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ can be compactly embedded into $L^p(\mathbb{R}^n, |x|^{-t}dx)$ for $p \geq \frac{n}{2}$.*

Proof To begin with, we show that $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ can be continuously imbedded into $L^p(\mathbb{R}^n, |x|^{-t}dx)$. For $p \geq \frac{n}{n-2}(j_{\frac{n}{2}} - 1)$, the continuous embedding is a direct result of inequality (1.5). For $p = \frac{n}{2}$, one can employ the following inequality

$$\int_{\mathbb{R}^n} \frac{|u|^{\frac{n}{2}}}{|x|^t} dx \leq \int_{\mathbb{R}^n} |u|^{\frac{n}{2}} dx + \left(\int_{B_1(0)} \frac{|u|^q}{|x|^t} dx \right)^{\frac{n}{2q}} \left(\int_{B_1(0)} \frac{1}{|x|^t} dx \right)^{1-\frac{n}{2q}} \tag{2.1}$$

to obtain the desired continuous imbedding. For $\frac{n}{2} < p < \frac{n}{n-2}(j_{\frac{n}{2}} - 1)$, it follows from the general interpolation inequality. Next it suffices to verify that the above continuous embedding is compact, i.e. for any sequence (u_k) bounded in $W^{2,\frac{n}{2}}(\mathbb{R}^n)$, there exists a subsequence which we still denote as (u_k) such that $\|u_k - u\|_{L^p(\mathbb{R}^n;|x|^{-t}dx)} \to 0$ as $k \to \infty$. We conclude it through two steps.

Step 1 We show that there exists a subsequence still denoted by (u_k) such that $u_k \to u$ for almost $x \in \mathbb{R}^n$. Through Sobolev interpolation inequalities with weights (see Lin's work [41]) and the $L^p(\mathbb{R}^n)$ boundedness of Riesz transform, we have

$$\|\nabla u\|_{\frac{n}{2}} \leq \|D^2 u\|_{\frac{n}{2}}^{\frac{1}{2}} \|u\|_{\frac{n}{2}}^{\frac{1}{2}} \leq \|\Delta u\|_{\frac{n}{2}}^{\frac{1}{2}} \|u\|_{\frac{n}{2}}^{\frac{1}{2}},$$

which implies that

$$\int_{\Omega} |\nabla u|^{\frac{n}{2}} + |u|^{\frac{n}{2}} dx \leq C(\Omega).$$

Due to the classical Sobolev compact embedding $W^{1,\frac{n}{2}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ for $1 \leq r < n$ and the diagonal trick, one can obtain that there exists a subsequence (we still denote by (u_k)) such that

$$\begin{aligned} &u_k(x) \to u(x), \quad \text{strongly in } L^r_{loc}(\mathbb{R}^n), \\ &u_k(x) \to u(x), \quad \text{for almost everywhere } x \in \mathbb{R}^n. \end{aligned}$$

Step 2 We claim that for any $p \geq \frac{n}{2}$, $u_k \to u$ in $L^p(\mathbb{R}^n; |x|^{-t}dx)$. For any $R > 0$, by applying the Egoroff theorem, one can find that for any $B_R(0)$ and $\delta > 0$,

$$\exists\, E_\delta \subset B_R(0) \quad \text{satisfying} \quad m(E_\delta) < \delta,$$

such that

$$u_k \text{ uniformlyconvergesto } u \text{ in } B_R(0) \setminus E_\delta.$$

Thus, we split the integral into three parts.

$$\begin{aligned}
&\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{\mathbb{R}^n}\frac{|u_k-u|^p}{|x|^t}dx\\
&=\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{E_\delta}\frac{|u_k-u|^p}{|x|^t}dx+\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{B_R(0)\setminus E_\delta}\frac{|u_k-u|^p}{|x|^t}dx\\
&+\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_R(0)}\frac{|u_k-u|^p}{|x|^t}dx\\
&=: I_1+I_2+I_3.
\end{aligned} \tag{2.2}$$

For I_1, the Hölder inequality and the classical Sobolev continuous embedding lead to

$$\begin{aligned}
I_1 &\le \lim_{\delta\to 0}\lim_{k\to+\infty}\left(\int_{E_\delta}1dx\right)^{\frac{1}{s}}\left(\int_{E_\delta}\frac{|u_k-u|^{ps'}}{|x|^{ts'}}dx\right)^{\frac{1}{s'}}\\
&\lesssim \lim_{\delta\to 0}(m(E_\delta))^{\frac{1}{s}}\\
&=0,
\end{aligned} \tag{2.3}$$

where $s > 1$ and $ts' < n$. As for I_2, it follows from the uniform convergence of u_k in $B_R(0)\setminus E_\delta$ that

$$\begin{aligned}
I_2 &= \lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\int_{B_R(0)\setminus E_\delta}\frac{|u_k-u|^p}{|x|^t}dx\\
&=\lim_{R\to+\infty}\lim_{\delta\to 0}\int_{B_R(0)\setminus E_\delta}\lim_{k\to+\infty}\frac{|u_k-u|^p}{|x|^t}dx\\
&=0.
\end{aligned} \tag{2.4}$$

For I_3, using continuous imbedding $W^{2,\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $p \ge \frac{n}{2}$, we obtain that

$$\begin{aligned}
I_3 &\le \lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{k\to+\infty}\frac{1}{R^t}\int_{\mathbb{R}^n\setminus B_R(0)}|u_k-u|^p dx\\
&\lesssim \lim_{R\to+\infty}\frac{1}{R^t}\\
&=0.
\end{aligned} \tag{2.5}$$

Combining (2.3), (2.4) and (2.5), we get a result which states that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\frac{|u_k-u|^p}{|x|^t}=0. \tag{2.6}$$

Thus we finish the proof of Lemma 2.1. □

Lemma 2.2 *For $0 < t < n$ and $0 < \beta < \beta_{n,2}$, let $(u_k) \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ satisfying $u_k \rightharpoonup u$ in $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ as $k \to +\infty$. Then, we have the following convergence.*

$$\int_{\mathbb{R}^n} \left(\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right) - \frac{\left(\beta\left(1-\frac{t}{n}\right)\right)^{j_{\frac{n}{2}}-1}}{\left(j_{\frac{n}{2}}-1\right)!}|u_k|^{\frac{n}{n-2}\left(j_{\frac{n}{2}}-1\right)} \right) \frac{dx}{|x|^t}$$
$$\to \int_{\mathbb{R}^n} \left(\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right) - \frac{\left(\beta\left(1-\frac{t}{n}\right)\right)^{j_{\frac{n}{2}}-1}}{\left(j_{\frac{n}{2}}-1\right)!}|u|^{\frac{n}{n-2}\left(j_{\frac{n}{2}}-1\right)} \right) \frac{dx}{|x|^t} \text{ as } k \to \infty. \tag{2.7}$$

Proof For simplicity, we define $\Psi_{n,2}(\tau) := \exp(\tau) - \sum_{j=0}^{j_{\frac{n}{2}}-1} \frac{\tau^{n' j}}{j!}$ for $\tau > 0, n \geq 2, k \in \mathbb{N}\cup\{0\}$, where $n' = \frac{n}{n-2}$ and $0 < \beta < \beta_{n,2}$. Then we can rewrite (2.7) as

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right)\frac{dx}{|x|^t} = \int_{\mathbb{R}^n} \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\frac{dx}{|x|^t}. \tag{2.8}$$

Hence, it follows from the mean value theorem and the convexity of the function $\Psi_{n,2}$ that

$$\begin{aligned}
&\left|\Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right) - \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\right| \\
&\lesssim \Phi_{n,2}\left(\theta\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}} + (1-\theta)\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\left(|u|^{\frac{2}{n-2}} + |u_k|^{\frac{2}{n-2}}\right)|u_k - u| \\
&\lesssim (|u_k| + |u|)^{\frac{2}{n-2}}\left(\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right) + \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\right)|u_k - u|,
\end{aligned} \tag{2.9}$$

where $\theta \in [0, 1]$.

This together with the singular Adams inequality (1.4) leads to

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left(\Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right) - \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\right)\frac{dx}{|x|^t}\right| \\
&\lesssim \int_{\mathbb{R}^n}(|u_k| + |u|)^{\frac{2}{n-2}}\left(\Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right) + \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\right)|u_k - u|\frac{dx}{|x|^t} \\
&\lesssim \||u_k| + |u|\|^{\frac{2}{n-2}}_{L^{\frac{2a}{n-2}}(\mathbb{R}^n;|x|^{-t}dx)} \\
&\times \left\|\Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u_k|^{\frac{n}{n-2}}\right) + \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)\right\|_{L^b(\mathbb{R}^n;|x|^{-t}dx)}\|u_k - u\|_{L^c(\mathbb{R}^n;|x|^{-t}dx)} \\
&\lesssim \|u_k - u\|_{L^c(\mathbb{R}^n;|x|^{-t}dx)}
\end{aligned} \tag{2.10}$$

where the constants $\frac{2}{n-2}a \geq \frac{n}{2}$, $b > 1$ sufficiently close to 1 and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Moreover, thanks to the compact result of Lemma 2.1, we obtain (2.7). □

Keeping the previous result in mind, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let (u_k) be a bounded function sequence in $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n)$ such that $\|\Delta u_k\|_{\frac{n}{2}} = 1$ and $F_{n,2,t,\beta}(u_k) \to \mu_{n,2,t,\beta}(\mathbb{R}^n)$ as $k \to \infty$. Denote a new sequence (v_k) by $v_k(x) := u_k(\|u_k\|^{\frac{1}{2}}_{\frac{n}{2}} x)$ for $x \in \mathbb{R}^n$. Then it is easy to check that

$$\|\Delta v_k\|_{\frac{n}{2}} = 1, \quad \|v_k\|_{\frac{n}{2}} = 1$$

and

$$F_{n,2,t,\beta}(v_k) = F_{n,2,t,\beta}(u_k) \to \mu_{n,2,t,\beta}(\mathbb{R}^n) \text{ as } k \to \infty.$$

Thus we obtain a new maximizing sequence for $\mu_{n,2,t,\beta}(\mathbb{R}^n)$ satisfying that (v_k) is bounded in $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$. As a consequence, there exists a subsequence (still denoted by (v_k)) such that

$$v_k \rightharpoonup v \text{ in } \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n).$$

By the weak semi-continuity of the norm in $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$, we derive that

$$\|\Delta v\|_{\frac{n}{2}} \leq 1, \quad \|v\|_{\frac{n}{2}} \leq 1. \tag{2.11}$$

Up to a sequence, we can apply Lemmas 2.1 and 2.2 to obtain that

$$\begin{aligned} \mu_{n,2,t,\beta}(\mathbb{R}^n) &= \lim_{k\to\infty} F_{n,2,t,\beta}(v_k) \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \Psi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|v_k|^{\frac{n}{n-2}}\right)\frac{dx}{|x|^t} \\ &\quad + \int_{\mathbb{R}^n} \frac{\left(\beta\left(1-\frac{t}{n}\right)\right)^{j_{\frac{n}{2}}-1}}{\left(j_{\frac{n}{2}}-1\right)!} |u_k|^{\frac{n}{n-2}\left(j_{\frac{n}{2}}-1\right)} \frac{dx}{|x|^t} \\ &= \int_{\mathbb{R}^n} \Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|v|^{\frac{n}{n-2}}\right)\frac{dx}{|x|^t}, \end{aligned} \tag{2.12}$$

which implies that

$$\mu_{n,2,t,\beta}(\mathbb{R}^n) \leq F_{n,2,t,\beta}(v). \tag{2.13}$$

On the other hand, through the definition of $\mu_{n,2,t,\beta}(\mathbb{R}^n)$ and (2.11), we can write

$$\begin{aligned} \mu_{n,2,t,\beta}(\mathbb{R}^n) &\geq F_{n,2,t,\beta}\left(\frac{v}{\|\Delta v\|_{\frac{n}{2}}}\right) \\ &= \frac{\|\Delta v\|_{\frac{n}{2}}^{\frac{n-t}{2}}}{\|v\|_{\frac{n}{2}}^{\frac{n-t}{2}}} \sum_{i=j_{\frac{n}{2}}-1}^{\infty} \frac{\beta^i}{i!} \frac{\|v\|_{n'i(\mathbb{R}^n;|x|^{-t}dx)}^{n'i}}{\|\Delta v\|_{\frac{n}{2}}^{n'i}} \\ &\geq F_{n,2,t,\beta}(v) + \left(\frac{1}{\|\Delta v\|_{\frac{n}{2}}^{n'(j_{\frac{n}{2}}-1)-\frac{n-t}{2}}} - 1\right) F_{n,2,t,\beta}(v), \end{aligned}$$

which implies that $\|v\|_{\frac{n}{2}} = \|\Delta v\|_{\frac{n}{2}} = 1$ and $\mu_{n,2,t,\beta}(\mathbb{R}^n) = F_{n,2,t,\beta}(v)$. Then we accomplish the proof of Theorem 1.1. □

Corollary 2.3 *For $q \geq \frac{n}{2}$, there exists a constant $C(q,n,t)$ such that*

$$\int_{\mathbb{R}^n} \frac{|u|^q}{|x|^t} dx \lesssim \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}} \tag{2.14}$$

holds for all functions $u \in \dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ with $\|\Delta u\|_{\frac{n}{2}} = 1$.

Proof For $q \geq \frac{n}{n-2}(j_{\frac{n}{2}} - 1)$, inequality (2.14) is a direct consequence of Theorem 1.1. We only need to verify that inequality (2.14) holds for $q = \frac{n}{2}$. We can split the integral in inequality (2.14) into two parts.

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|u|^{\frac{n}{2}}}{|x|^t} dx &= \int_{\Omega^c(u)} \frac{|u|^{\frac{n}{2}}}{|x|^t} dx + \int_{\Omega(u)} \frac{|u|^{\frac{n}{2}}}{|x|^t} dx. \\ &= I_1 + I_2,\end{aligned} \tag{2.15}$$

where $\Omega(u) = \{x | u(x) > 1\}$. For I_1, by dividing the integral into two parts, one can obtain that

$$\begin{aligned} I_1 &= \int_{\Omega^c(u)\cap\left\{|x|\leq\|u\|_{\frac{n}{2}}^{\frac{1}{2}}\right\}} \frac{|u|^{\frac{n}{2}}}{|x|^t} dx + \int_{\Omega^c(u)\cap\left\{|x|>\|u\|_{\frac{n}{2}}^{\frac{1}{2}}\right\}} \frac{|u|^{\frac{n}{2}}}{|x|^t} dx \\ &\leq \int_{\left\{|x|\leq\|u\|_{\frac{n}{2}}^{\frac{1}{2}}\right\}} \frac{1}{|x|^t} dx + \int_{\Omega^c(u)} \frac{|u|^{\frac{n}{2}}}{\|u\|_{\frac{n}{2}}^{\frac{t}{2}}} dx \\ &\lesssim \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}}. \end{aligned} \tag{2.16}$$

As for I_2, by setting $|u| = v+1$, it follows from the singular Adams inequality in $W_N^{2,\frac{n}{2}}(\Omega(u))$ that

$$I_2 \lesssim |\Omega(u)|^{1-\frac{t}{n}} \lesssim \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}}. \tag{2.17}$$

Then the proof of Corollary 2.3 is completed. □

3 Proofs of Theorems 1.2 and 1.3

In this section, we utilize the arrangement-free argument introduced in [28,29] together with the singular Adams inequality with the Navier boundary condition to establish inequalities (1.6) and (1.7). By exploring the scaling invariant form of singular Adams' inequalities (1.6) and (1.7), we also establish the existence of their extremals.

We start the proof of Theorem 1.2. We first show that inequality (1.6) holds. By splitting the integral in inequality (1.6) into two parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^{\frac{n}{2}}}{|x|^t} dx \\ &= \int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^{\frac{n}{2}}}{|x|^t} + \sum_{i=0}^{j_{\frac{n}{2}}-2} \frac{(\beta(1-\frac{t}{n}))^i |u|^{\frac{in}{n-2}} |u|^{\frac{n}{2}}}{|x|^t} dx \\ &=: I_1 + I_2. \end{aligned} \tag{3.1}$$

For I_1, choose $p > 1$ sufficiently close to 1 such that $p\beta < \beta_{n,2}$, then it follows from the Hölder inequality, Theorem 1.1 and Corollary 2.3 that

$$\begin{aligned} I_1 &\leq \left(\int_{\mathbb{R}^n} \frac{\Phi_{n,2}(p\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})}{|x|^t} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|u|^{p'\frac{n}{2}}}{|x|^t} dx\right)^{\frac{1}{p'}} \\ &\leq \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{2}}\right)^{\frac{n-t}{n}}. \end{aligned} \tag{3.2}$$

For I_2, note the fact that I_2 consists of $j_{\frac{n}{2}} - 1$ terms and the power of every term is larger than $\frac{n}{2}$. Thus we can apply Corollary 2.3 to employ that

$$\begin{aligned} I_2 &= \sum_{i=0}^{j_{\frac{n}{2}}-2} \left(\beta\left(1-\frac{t}{n}\right)\right)^i \int_{\mathbb{R}^n} \frac{|u|^{\frac{in}{n-2}}|u|^{\frac{n}{2}}}{|x|^t} dx \\ &\lesssim \sum_{i=0}^{j_{\frac{n}{2}}-2} \left(\beta\left(1-\frac{t}{n}\right)\right)^i \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}} \\ &\lesssim \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{2}} dx\right)^{1-\frac{t}{n}}. \end{aligned} \tag{3.3}$$

This together with (3.1) and (3.2) yields inequality (1.6).

In order to obtain the sharpness of inequality (1.6), we use the test sequence (u_k) introduced in [32]. Its definition is given by

$$u_k = \begin{cases} (\frac{1}{\beta_{n,2}} \ln k)^{1-\frac{2}{n}} - \frac{n\beta_{n,2}^{\frac{2}{n}-1}}{2} \frac{|x|^2}{(\frac{\ln k}{k})^{\frac{2}{n}}} + \frac{n\beta_{n,2}^{\frac{2}{n}-1}}{2} \frac{1}{(\frac{\ln k}{k})^{\frac{2}{n}}}, & \text{if } 0 \leq |x| \leq (\frac{1}{k})^{\frac{1}{n}}, \\ n\beta_{n,2}^{\frac{2}{n}-1} (\ln k)^{-\frac{2}{n}} \ln \frac{1}{|x|}, & \text{if } (\frac{1}{k})^{\frac{1}{n}} \leq |x| \leq 1, \\ \varsigma_k, & \text{if } |x| > 1, \end{cases}$$

where ς_k is a smooth function satisfying $\text{supp}(\varsigma_k) \subset \{|x| < 2\}$,

$$\varsigma_k|_{|x|=1} = 0,\ \frac{\partial \varsigma_k}{\partial v}|_{|x|=1} = n\beta_{n,2}^{\frac{2}{n}-1}(\ln_k)^{-\frac{2}{n}},\ \varsigma_k = O((\ln_k)^{\frac{2}{n}}),\ \Delta\varsigma_k = O((\ln_k)^{\frac{2}{n}}).$$

Directly computations yield that

$$1 \leq \|\Delta u_k\|_{\frac{n}{2}}^{\frac{n}{2}} \leq 1 + O\left(\frac{1}{\ln k}\right).$$

Let $\widetilde{v}_k = \frac{u_k}{\|\Delta u_k\|_{\frac{n}{2}}}$, we derive that

$$\|\Delta \widetilde{v}_k\|_{\frac{n}{2}} = 1 \text{ and } \|\widetilde{v}_k\|_{\frac{n}{2}}^{\frac{n}{2}} \leq \|\Delta u_k\|_{\frac{n}{2}}^{\frac{n}{2}} \leq A(\ln k)^{-1} + B(\ln k)^{\frac{n-2}{2}} \frac{1}{k}.$$

Then we manage the calculation as follows:

$$
\begin{aligned}
&\|\widetilde{v}_k\|_{\frac{n}{2}}^{-\frac{n}{2}(1-\frac{t}{n})} \int_{\mathbb{R}^n} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)|\widetilde{v}_k|^{\frac{n}{2}}}{|x|^t}dx \\
&\geq \|\widetilde{v}_k\|_{\frac{n}{2}}^{-\frac{n}{2}(1-\frac{t}{n})} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{n}}} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)|\widetilde{v}_k|^{\frac{n}{2}}}{|x|^t}dx \\
&\geq \left(\frac{1}{\beta_{n,2}}\ln k\right)^{\frac{n}{2}-1} \|\widetilde{v}_k\|_{\frac{n}{2}}^{-\frac{n}{2}(1-\frac{t}{n})} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{n}}} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)}{|x|^t}dx \\
&\geq \left(\frac{1}{\beta_{n,2}}\ln k\right)^{\frac{n}{2}-1} \|\widetilde{v}_k\|_{\frac{n}{2}}^{-\frac{n}{2}(1-\frac{t}{n})} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{n}}} \frac{\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)}{|x|^t}dx \\
&\geq \left(\frac{1}{\beta_{n,2}}\ln k\right)^{\frac{n}{2}-1} \left(\frac{1}{1-\frac{\beta}{\beta_{n,2}}}\right)^{1-\frac{t}{n}} \\
&\to \infty \text{ as } \beta \to \beta_{n,2},
\end{aligned}
\tag{3.4}
$$

which completes the proof of the sharpness of inequality (1.6).

The proof of the attainability of the best constant $C(n,t)$ for inequality (1.6) is similar to that of Theorem 1.1. In fact, by the scaling invariant form of the weighted Trudinger–Moser inequality, we can choose a maximizing sequence (v_k) for $C(n,t)$ satisfying that (v_k) is bounded in $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$. Following the same procedure as that of Lemma 2.2 and Theorem 1.1, we can obtain

$$
\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{(\exp(\beta(1-\frac{t}{n})|v_k|^{\frac{n}{n-2}})-1)|v_k|^{\frac{n}{2}}}{|x|^t}dx = \int_{\mathbb{R}^n} \frac{(\exp(\beta(1-\frac{t}{n})|v|^{\frac{n}{n-2}})-1)|v|^{\frac{n}{2}}}{|x|^t}dx
$$

and $\|\Delta v\|_{\frac{n}{2}} = \|v\|_{\frac{n}{2}} = 1$, which implies the attainability of the best constant $C(n,t)$.

We now start to prove Theorem 1.3. We first apply the arrangement-free argument introduced in [29] and the weighted Adams inequality in $W_N^{\frac{n}{2}}(\Omega)$ to obtain inequality (1.7). Indeed, by dividing the integral into two parts, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^q}{|x|^t}dx &= \int_{|u|\leq 1} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^q}{|x|^t}dx \\
&\quad + \int_{|u|>1} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^q}{|x|^t}dx \\
&=: I_1 + I_2.
\end{aligned}
\tag{3.5}
$$

For I_2, setting $|u| = v+1$ and using an elementary inequality

$$
|u|^{\frac{n}{n-2}} \leq (1+\varepsilon)v^{\frac{n}{n-2}} + C_\varepsilon,\ \forall \varepsilon > 0,
$$

one can obtain

$$
\begin{aligned}
I_2 &\leq \int_{|u|>1} \frac{\exp(\beta(1-\frac{t}{n})(1+\varepsilon)v^{\frac{n}{n-2}})\exp(\beta(1-\frac{t}{n})C_\varepsilon)|u|^q}{|x|^t}dx \\
&\lesssim \left(\int_{|u|>1} \frac{\exp(\beta(1-\frac{t}{n})p(1+\varepsilon)v^{\frac{n}{n-2}})}{|x|^t}dx\right)^{\frac{1}{p}} \left(\int_{|u|>1} \frac{(v+1)^{qp'}}{|x|^t}dx\right)^{\frac{1}{p'}} \\
&\lesssim |\{|u|>1\}|^{\frac{n-t}{np}} \left(\int_{|u|>1} \frac{|v|^{qp'}}{|x|^t} + \frac{1}{|x|^t}dx\right)^{\frac{1}{p'}} \\
&\lesssim |\{|u|>1\}|^{\frac{n-t}{np}} |\{|u|>1\}|^{\frac{n-t}{np'}} + |\{|u|>1\}|^{\frac{n-t}{np}} \left(\int_{|u|>1} \frac{1}{|x|^t}dx\right)^{\frac{1}{p'}} \\
&\lesssim \left(\int_{\mathbb{R}^n} |u|^q dx\right)^{1-\frac{t}{n}}.
\end{aligned}
\tag{3.6}
$$

For I_1, direct computations show that

$$
\begin{aligned}
&\int_{|u|\leq 1} \frac{\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})|u|^q}{|x|^t}dx \\
&\quad\lesssim \int_{\{|u|\leq 1\}\cap\left\{|x|\leq\|u\|_q^{\frac{q}{n}}\right\}} \frac{|u|^q}{|x|^t}dx + \int_{\{|u|\leq 1\}\cap\left\{|x|>\|u\|_q^{\frac{q}{n}}\right\}} \frac{|u|^q}{|x|^t}dx \\
&\quad=: I_{11} + I_{12}.
\end{aligned}
\tag{3.7}
$$

We can estimate I_{11} as follows

$$
\int_{\{|u|\leq 1\}\cap\left\{|x|\leq\|u\|_q^{\frac{q}{n}}\right\}} \frac{|u|^q}{|x|^t}dx \leq \int_{\left\{|x|\leq\|u\|_q^{\frac{q}{n}}\right\}} |x|^{-t}dx = \|u\|_q^{\frac{q(n-t)}{n}}. \tag{3.8}
$$

Similarly, we also derive that

$$
\int_{\{|u|\leq 1\}\cap\left\{|x|\geq\|u\|_q^{\frac{q}{n}}\right\}} \frac{|u|^q}{|x|^t}dx \leq \|u\|_q^{-\frac{qt}{n}} \int_{\{|u|\leq 1\}} |u|^q dx = \|u\|_q^{\frac{q(n-t)}{n}}. \tag{3.9}
$$

Combining inequalities (3.5), (3.6), (3.8) with (3.9), we obtain the required inequality (1.7).

Next, we show the sharpness of inequality (1.7). Using the same test function sequence $(u_k)_k$ as that of Theorem 1.2, one can easily calculate that

$$
\|\widetilde{v}_k\|_q^q \leq A(\ln k)^{-\frac{2}{n}q} + B\frac{(\ln k)^{(1-\frac{2}{n})q}}{k} + C\frac{(\ln k)^{-\frac{2}{n}q}}{k}.
$$

Then, it follows that

$$
\begin{aligned}
&\|\widetilde{v}_k\|_q^{-q(1-\frac{t}{n})} \int_{\mathbb{R}^n} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)|\widetilde{v}_k|^q}{|x|^t}dx \\
&\geq \|\widetilde{v}_k\|_q^{-q(1-\frac{t}{n})} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{n}}} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)|\widetilde{v}_k|^q}{|x|^t}dx \\
&\geq \left(\frac{1}{\beta_{n,2}}\ln k\right)^{q\left(1-\frac{2}{n}\right)} \|\widetilde{v}_k\|_{\frac{n}{2}}^{-q(1-\frac{t}{n})} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{n}}} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)}{|x|^t}dx \qquad (3.10)\\
&\geq \left(\frac{1}{\beta_{n,2}}\ln k\right)^{q\left(1-\frac{2}{n}\right)} \|\widetilde{v}_k\|_{\frac{n}{2}}^{-\frac{n}{2}(1-\frac{t}{n})} \int_{|x|\leq\left(\frac{1}{k}\right)^{\frac{1}{n}}} \frac{\exp\left(\beta\left(1-\frac{t}{n}\right)|\widetilde{v}_k|^{\frac{n}{n-2}}\right)}{|x|^t}dx \\
&\to \infty \text{ as } \beta \to \beta_{n,2}.
\end{aligned}
$$

Then, we show the attainability of the sharp constant $C(n,t)$ for inequality (1.7). We need the following compact imbedding.

Lemma 3.1 *For $n \geq 3$, $r \geq q$ and $0 < t < n$, $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ can be embedded compactly into $L^r(\mathbb{R}^n; |x|^{-t}dx)$.*

For the continuity of the proof, we postpone the proof of Lemma 3.1. With the help of Lemma 3.1, applying the same method in Lemma 2.2, we can derive the following required convergence.

$$
\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{(\exp(\beta(1-\frac{t}{n})|u_k|^{\frac{n}{n-2}})-1)|u_k|^q}{|x|^t}dx = \int_{\mathbb{R}^n} \frac{(\exp(\beta(1-\frac{t}{n})|u|^{\frac{n}{n-2}})-1)|u|^q}{|x|^t}dx. \tag{3.11}
$$

Then, we can use the same procedure as Theorem 1.1 to obtain the attainability of the best constant. At last, we focus on the proof of Lemma 3.1.

The continuity of the embedding is a direct result of inequality (1.7) and the Hölder inequality. Then it is sufficient to show that for any bounded sequence (u_k) in $\dot{W}^{2,\frac{n}{2}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, there exists a subsequence which we still denote as (u_k) such that

$$
\lim_{k\to\infty} \|u_k - u\|_{L^r(\mathbb{R}^n;|x|^{-t}dx)} = 0 \text{ as } k \to \infty \text{ for } r \geq q.
$$

Similar to the proof of Lemma 2.1, we carry out the process of proof by two steps.

Step 1 We first show that there exists a subsequence still denoted by (u_k) such that $u_k \to u$ for almost $x \in \mathbb{R}^n$. In fact, through Sobolev interpolation inequalities with weights (see Lin's work [41]), we can obtain

$$
\|\nabla u\|_{2q} \leq \|\triangle u\|_{\frac{n}{2}}^{\frac{1}{2}} \|u\|_q^{\frac{1}{2}}.
$$

Then it follows from the Hölder inequality that

$$
\int_\Omega |\nabla u|^{\frac{n}{2}} + |u|^{\frac{n}{2}} dx \leq C(\Omega).
$$

According to the classical Sobolev compact embedding $W^{1,\frac{n}{2}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ for $1 \leq r < n$ and the diagonal trick, one can obtain that there exists a subsequence (we still denote by (u_k)) such that

$$u_k(x) \to u(x), \quad \text{strongly in } L^r_{loc}(\mathbb{R}^n),$$
$$u_k(x) \to u(x), \quad \text{for almost everywhere } x \in \mathbb{R}^n.$$

Step 2 We claim that for any $r \geq q$, $u_k \to u$ in$L^r(\mathbb{R}^n; |x|^{-t}dx)$. Since the process of the proof is similar to that of Lemma 2.1, we omit the details.

4 Proof of Theorem 1.5

Throughout this section, we will employ the method of using the relationship between the supremums of the subcritical and critical inequalities developed by Lam et al. [33] to establish the existence of maximizers for the singular Adams inequality with the Sobolev norm. We need the following lemmas whose proofs can be found in Lam [21], Lam et al. [33].

Lemma 4.1 *For* $0 < t < n,\ a, b > 0$, *then* $ATA(\cdot, t)$ *is continuous on* $(0, \beta_{n,2})$.

Lemma 4.2 *If* $t > 0$, $a, b > 0$, *then*

$$\lim_{s\to 0}\left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} ATA(s,t) = 0.$$

Lemma 4.3 *For* $a > 0$, *if* $(\beta < \beta_{n,2}, b > 0)$ *or* $(\beta = \beta_{n,2}, 0 < b < \frac{n}{2})$, *then*

$$\lim_{s\to \beta}\left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} ATA(s,t) = 0.$$

Proof of Theorem 1.5 With the help of Theorem 1.1, Lemmas 4.1, 4.2 and 4.3, we are in a position to establish the existence of extremals for the singular Adams inequality with the Sobolev norm. We only need to prove that there exists an extremal function for $A_{a,b,t}(\beta)$ in the case of $(\beta < \beta_{n,2}, b > 0)$ or $(\beta = \beta_{n,2}, b < \frac{n}{2})$. It is easy to check that

$$\lim_{s\to 0}\left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{m-t}{2b}} ATA(\beta, s) < A_{a,b,t}(\beta)$$

and

$$\lim_{s\to \alpha}\left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} ATA(\beta, s) < A_{a,b,t}(\beta).$$

On the other hand, we also have

$$A_{a,b,t}(\beta) = \sup_{s\in(0,\beta)}\left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} ATA(s,t).$$

This together with Lemma 4.2 and Lemma 4.3 yields that there exists $s \in (0, \beta)$ such that

$$\left(\frac{1-(\frac{s}{\beta})^{\frac{n-2}{n}a}}{(\frac{s}{\beta})^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} ATA(s,t) = A_{a,b,t}(\beta).$$

Assume that $u \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ with $\|\Delta u\|_{\frac{n}{2}} \leq 1 = \|u\|_{\frac{n}{2}}$ is the maximizer for $ATA(s,t)$. Define

$$v(x) = \left(\frac{s}{\beta}\right)^{\frac{n-2}{n}} u(\lambda x),$$

$$\lambda = \left(\frac{\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}b}}{1-\left(\frac{s}{\alpha}\right)^{\frac{n-2}{n}a}}\right)^{\frac{1}{2b}},$$

then it follows that

$$\begin{aligned}\|\Delta v\|_{\frac{n}{2}}^{a} &= \left(\frac{s}{\beta}\right)^{\frac{n-2}{n}a} \|\Delta u\|_{\frac{n}{2}}^{a} \leq \left(\frac{s}{\beta}\right)^{\frac{n-2}{n}a},\\ \|v\|_{\frac{n}{2}}^{b} &= \left(\frac{s}{\beta}\right)^{\frac{n-2}{n}b} \frac{1}{\lambda^b}\|u\|_{\frac{n}{2}}^{b} = 1-\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}a}.\end{aligned}$$

which implies that $\|\Delta v\|_{\frac{n}{2}}^{a} + \|v\|_{\frac{n}{2}}^{b} \leq 1$. Hence,

$$\begin{aligned}A_{a,b,t}(\beta) &= \left(\frac{1-\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}a}}{\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} \int_{\mathbb{R}^n} \frac{\Phi_{n,2}\left(s\left(1-\frac{t}{n}\right)|u|^{\frac{n}{n-2}}\right)}{|x|^t}dx\\ &= \left(\frac{1-\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}a}}{\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} \int_{\mathbb{R}^n} \frac{\Phi_{n,2}\left(s\left(1-\frac{t}{n}\right)|u(\lambda x)|^{\frac{n}{n-2}}\right)}{|\lambda x|^t}d(\lambda x)\\ &= \left(\frac{1-\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}a}}{\left(\frac{s}{\beta}\right)^{\frac{n-2}{n}b}}\right)^{\frac{n-t}{2b}} \lambda^{n-t}\int_{\mathbb{R}^n} \frac{\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|v|^{\frac{n}{n-2}}\right)}{|x|^t}dx\\ &= \int_{\mathbb{R}^n} \frac{\Phi_{n,2}\left(\beta\left(1-\frac{t}{n}\right)|v|^{\frac{n}{n-2}}\right)}{|x|^t}dx.\end{aligned} \tag{4.1}$$

This implies that v is actually a maximizer for $A_{a,b,t}(\beta)$. □

5 Proofs of Theorems 1.7, 1.9 and 1.11

In this section, we establish weighted Adams' inequalities (1.8) and (1.9) which equipped with the Dirichlet norm and the existence of their extremal functions. It is well known that the arrangement-free argument introduced in [29] is a useful tool in dealing with the Trudinger–Moser inequality and the second-order Adams inequality. However, this method may fail when we come to consider the higher order inequalities. Therefore, we use the method based on Fourier transform to establish inequalities (1.8) and (1.9). We need the following lemma.

Lemma 5.1 *For any $\beta \in (0, \beta_{2m,m})$, there exists a positive constant C_β such that*

$$\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)}{|x|^t}dx \le C_\beta, \tag{5.1}$$

where $u \in W^{m,2}(\mathbb{R}^{2m})$, $\|\nabla^m u\|_2 \le 1$ and $\|u\|_2 = 1$.

Proof We first claim that for any fixed $\beta \in (0, \beta_{2m,m})$, there exists sufficient small $\tau > 0$ such that for all $u \in \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ with $\|\nabla^m u\|_2 \le 1$ and $\|u\|_2 = 1$, there holds

$$\|(\tau I - \Delta)^{\frac{m}{2}} u\|_2^2 \le \frac{\beta_{2m,m}}{\beta}. \tag{5.2}$$

Indeed, by Fourier transform, we have

$$\|(\tau I - \Delta)^{\frac{m}{2}} u\|_2^2 = \sum_{j=0}^{m} C_m^j \tau^{m-j} \|\nabla^j u\|_2^2.$$

Thanks to the Sobolev interpolation inequalities, one can derive that for every $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that

$$\|(\tau I - \Delta)^{\frac{m}{2}} u\|_2^2 \le (1+\varepsilon)\|\nabla^m u\|_2^2 + C_\varepsilon \tau \|u\|_2^2,$$

which implies inequality (5.2). With the help of Theorem D in [29], we derive that

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)}{|x|^t}dx &= \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\left(\beta(1-\frac{t}{2m})\|(\tau I - \Delta)^{\frac{m}{2}} u\|_2^2 \left|\frac{u}{\|(\tau I-\Delta)^{\frac{m}{2}} u\|_2}\right|^2\right)}{|x|^t}dx \\
&\le \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\left(\beta_{2m,m}(1-\frac{t}{2m}) \left|\frac{u}{\|(\tau I-\Delta)^{\frac{m}{2}} u\|_2}\right|^2\right)}{|x|^t}dx \\
&\le C_\beta,
\end{aligned} \tag{5.3}$$

which finishes the proof. □

With the help of Lemma 5.1, we start the proof of inequality (1.8). In fact, for any $u \in W^{m,2}(\mathbb{R}^{2m})$ satisfying $\|\nabla^m u\|_2 \le 1$, we define $u_\lambda(x) = u(\lambda x)$ with $\lambda = \|u\|_2^{\frac{1}{m}}$. Through direct calculations, we derive that

$$\begin{aligned}
&\|u_\lambda\|_2^2 = \lambda^{-2m}\|u\|_2^2 = 1, \\
&\|\nabla^m u_\lambda\|_2 = \|\nabla^m u\|_2 = 1, \\
&\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u_\lambda|^2)}{|x|^t}dx = \lambda^{t-2m}\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)}{|x|^t}dx.
\end{aligned}$$

Then it follows from inequality (5.1) that

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)}{|x|^t}dx &= \lambda^{2m-t}\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u_\lambda|^2)}{|x|^t}dx \\
&\le \lambda^{2m-t} C_\beta \\
&= C_\beta \|u\|_2^{2(1-\frac{t}{2m})}.
\end{aligned} \tag{5.4}$$

Next, we show the sharpness of inequality (1.8).

We will modify the idea of constructing test functions for the Adams inequality on domains of finite measure in Euclidean spaces [2]. Let $\phi \in \mathcal{C}_0^\infty([0,1])$ such that

$$\begin{aligned}&\phi(0)=\phi'(0)=\cdots=\phi^{m-1}(0)=0, \quad \phi(1)=\phi'(1)=1,\\ &\phi''(1)=\cdots=\phi^{m-1}(1)=0.\end{aligned}$$

For $0<\varepsilon<\frac{1}{2}$, set

$$H(t):=\begin{cases}\varepsilon\phi(\frac{t}{\varepsilon}), & \text{if } 0<t\le\varepsilon,\\ t, & \text{if } \varepsilon<t\le 1-\varepsilon,\\ 1-\varepsilon\phi(\frac{1-t}{\varepsilon}), & \text{if } 1-\varepsilon<t\le 1,\\ 1, & \text{if } t\ge 1.\end{cases}$$

For any fixed $r>0$ sufficiently small, we define

$$\psi_r(|x|):=H_{\varepsilon(r)}\left(\frac{\log\frac{1}{|x|}}{\log\frac{1}{r}}\right),$$

where $\varepsilon(r)=\frac{1}{\log\frac{1}{r}}$. Obviously, $\psi_r\in W_0^{m,2}(B_1)$ and

$$\psi_r=1 \ \text{ on } B_r.$$

It was proved in Adams [2] that

$$\|\nabla^m\psi_r\|_2^2\le(2m)^{-1}\beta_{2m,m}\left(\log\frac{1}{r}\right)^{-1}A_r,$$

where

$$A_r:=1+O\left(\frac{1}{\log\frac{1}{r}}\right).$$

Moreover, direct computations show

$$\|\psi_r\|_2\lesssim\frac{1}{\log\frac{1}{r}}.$$

Define

$$u_r:=\frac{\psi_r}{\left((2m)^{-1}\beta_{2m,m}(\log\frac{1}{r})^{-1}A_r\right)^{\frac{1}{2}}}.$$

By direct calculations, one can obtain that

$$\|\nabla^m u_r\|_2\le 1$$

and

$$\|u_r\|_2^2\sim\frac{\|\psi_r\|_2^2}{(\log\frac{1}{r})^{-1}A_r}\lesssim\frac{1}{\log\frac{1}{r}}.$$

Let $r \to 0$, it follows that

$$\begin{aligned}
&\lim_{r\to 0} \frac{\int_{\mathbb{R}^{2m}} \Phi_{2m,m}\left(\beta_{2m,m}\left(1-\frac{t}{2m}\right)|u_r|^2\right)|x|^{-t}dx}{\|u_r\|_2^{2(1-\frac{t}{2m})}} \\
&\gtrsim \lim_{r\to 0}\left(\log\frac{1}{r}\right)^{1-\frac{t}{2m}} \int_{B_r} \exp\left(\beta_{2m,m}\left(1-\frac{t}{2m}\right)|u_r|^2\right)|x|^{-t}dx \\
&= \lim_{r\to 0}\left(\log\frac{1}{r}\right)^{1-\frac{t}{2m}} \int_{B_r} \exp\left((2m-t)\log\frac{1}{r}A_r^{-1}\right)|x|^{-t}dx \\
&\gtrsim \lim_{r\to 0}\left(\log\frac{1}{r}\right)^{1-\frac{t}{2m}} r^{2m-t}\exp\left((2m-t)\log\frac{1}{r}A_r^{-1}\right) \\
&\gtrsim \lim_{r\to 0}\left(\log\frac{1}{r}\right)^{1-\frac{t}{2m}} \exp\left((2m-t)\log\frac{1}{r}\left(A_r^{-1}-1\right)\right) \\
&\gtrsim \lim_{r\to 0}\left(\log\frac{1}{r}\right)^{1-\frac{t}{2m}} \to \infty,
\end{aligned} \tag{5.5}$$

which completes the proof of sharpness.

At last, we show the attainability of $\mu_{2m,m,t,\beta}$. Just as what we did in Theorem 1.1, we need the following compactness lemma.

Lemma 5.2 *For $m \geq 2$, $p \geq 2$ and $0 < t < 2m$, then $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ can be compactly embedded into $L^p(\mathbb{R}^{2m}, |x|^{-t}dx)$.*

Proof The proof is similar to that of Lemma 2.1 once we prove the equivalence between the space $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ and the standard Sobolev space $W^{m,2}(\mathbb{R}^{2m})$. Indeed, it suffices to show that

$$\|\partial^\alpha u\|_2^2 \lesssim \|u\|_2^2 + \|\nabla^m u\|_2^2, \quad \forall\, |\alpha| \leq m. \tag{5.6}$$

We first prove the

$$\|\nabla^k u\|_2^2 \lesssim \|u\|_2^2 + \|\nabla^m u\|_2^2, \quad \forall\, 1 \leq k \leq m, k \in \mathbb{N}. \tag{5.7}$$

In fact, by the Fourier transform, we have

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} |\nabla^k u|^2 dx &= \int_{\mathbb{R}^{2m}} |\xi|^{2k}|\hat{u}(\xi)|^2 d\xi \\
&\leq \int_{\mathbb{R}^{2m}} (1+|\xi|^{2m})|\hat{u}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^{2m}} |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^{2m}} |\xi|^{2m}|\hat{u}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^{2m}} |u|^2 dx + \int_{\mathbb{R}^{2m}} |\nabla^m u|^2 dx.
\end{aligned}$$

Combining this result, in order to obtain the equivalence result, we only need to show that

$$\|\partial^\alpha u\|_2^2 \leq \|\nabla^{|\alpha|} u\|_2^2. \tag{5.8}$$

One can derive it by induction. For $|\alpha| \geq 2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, there exist $\alpha_j + \alpha_k \geq 2$ such that $\partial^\alpha = \frac{\partial^2}{\partial x_j \partial x_k}\partial^\beta$. Hence, it follows from the Fourier transform and the Riesz transform that

$$\begin{aligned}
\|\partial^\alpha u\|_2^2 &= \int_{\mathbb{R}^{2m}} \left| \left(\frac{\partial^2}{\partial x_j \partial x_k} \partial^\beta u \right)^\wedge (\xi) \right|^2 d\xi \\
&= \int_{\mathbb{R}^{2m}} |4\pi^2 \xi_j \xi_k \widehat{\partial^\beta u}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^{2m}} \left| \left(-i\frac{\xi_j}{|\xi|}\right) \left(-i\frac{\xi_k}{|\xi|}\right) (4\pi^2|\xi|^2) \widehat{\partial^\beta u}(\xi) \right|^2 d\xi \\
&= \int_{\mathbb{R}^{2m}} |(R_j R_k \Delta(\partial^\beta u))^\wedge(\xi)|^2 d\xi.
\end{aligned}$$

Then, with the help of the induction and the definition of ∇^m, one can get

$$\int_{\mathbb{R}^{2m}} |(R_j R_k \Delta(\partial^\beta u))^\wedge(\xi)|^2 d\xi \le \|\partial^\beta(\Delta u)\|_2^2 \le \|\nabla^{|\beta|}\Delta u\|_2^2 = \|\nabla^{|\alpha|}|u\|_2^2,$$

which proves the required equivalence. □

Now we show that the best constant $\mu_{2m,m,t,\beta}$ could be attained by a function in $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$. Assume that $(u_k) \subset \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ satisfying

$$\|\nabla^m u_k\|_2 = 1 \text{ and } F_{2m,m,t,\beta}(u_k) \to \mu_{2m,m,t,\beta}(\mathbb{R}^{2m}) \text{ as } k \to \infty.$$

Constructing a new function sequence (v_k) defined by $v_k(x) := u_k(\|u_k\|_2^{\frac{1}{m}} x)$ for $x \in \mathbb{R}^{2m}$, one can easily verify that

$$\|\nabla^m v_k\|_2 = 1, \quad \|v_k\|_2 = 1,$$

and

$$F_{2m,m,t,\beta}(v_k) = F_{2m,m,t,\beta}(u_k) \to \mu_{2m,m,t,\beta}(\mathbb{R}^{2m}) \ \text{ as } k \to \infty.$$

Hence, (v_k) is also a maximizing sequence for $\mu_{2m,m,t,\beta}(\mathbb{R}^{2m})$. Note that (v_k) is bounded in $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$, thus up to a sequence, we may assume that

$$v_k \rightharpoonup v \text{ in } \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m}).$$

It follows from weak semicontinuity of the norm in $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ that

$$\|\nabla^m v\|_2 \le 1, \quad \|v\|_2 \le 1. \tag{5.9}$$

Then, implementing same procedures as we did in Lemma 2.2, we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\mathbb{R}^{2m}} \left(\Phi_{2m,m}\left(\beta\left(1-\frac{t}{2m}\right)|u_k|^2\right) - \beta\left(1-\frac{t}{2m}\right)|u_k|^2 \right) \frac{dx}{|x|^t} \\
&= \int_{\mathbb{R}^{2m}} \left(\Phi_{2m,m}\left(\beta\left(1-\frac{t}{2m}\right)|u|^2\right) - \beta\left(1-\frac{t}{2m}\right)|u|^2 \right) \frac{dx}{|x|^t}.
\end{aligned} \tag{5.10}$$

Combining (5.10) with Lemma 5.2, we derive that up to a sequence,

$$\begin{aligned}
\mu_{2m,m,t,\beta}(\mathbb{R}^{2m}) &= \int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta(1-\frac{t}{2m})|v_k|^2)\frac{dx}{|x|^t} + o(1) \\
&= \int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta(1-\frac{t}{2m})|v|^2)\frac{dx}{|x|^t},
\end{aligned} \tag{5.11}$$

which implies $v \neq 0$. Then we can deduce from (5.9) and (5.11) that

$$\mu_{2m,m,t,\beta}(\mathbb{R}^{2m}) \leq \frac{\int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta(1-\frac{t}{2m})|v|^2)\frac{dx}{|x|^t}}{\|v\|_2^{2-\frac{t}{m}}} = F_{2m,m,t,\beta}(v). \tag{5.12}$$

Therefore, it remains to show that $\|\nabla^m v\|_2 = 1$. By the definition of $\mu_{2m,m,t,\beta}(\mathbb{R}^{2m})$ and (5.9), we see that

$$\begin{aligned}
\mu_{2m,m,t,\beta}(\mathbb{R}^{2m}) &\geq F_{2m,m,t,\beta}\left(\frac{v}{\|\nabla^m v\|_2}\right) \\
&= \sum_{i=1}^{\infty} \frac{\beta^i}{i!} \|v\|_{L^{2i}(\mathbb{R}^{2m};|x|^{-t}dx)}^{2i} \|v\|_2^{\frac{t}{m}-2} \|\nabla^m v\|_2^{2-\frac{t}{m}-2i} \\
&\geq F_{2m,m,t,\beta}(v) + (\|\nabla^m v\|_2^{-\frac{t}{m}} - 1)F_{2m,m,t,\beta}(v).
\end{aligned} \tag{5.13}$$

This together with (5.9) and (5.12) implies that $\|\nabla^m v\|_2 = 1$. Then we complete the proof of Theorem 1.7.

The Proof of Theorem 1.9 We first establish inequality (1.9). Just as what we did in Theorem 1.2, we divide the integral in inequality (1.9) into two parts.

$$\begin{aligned}
&\int_{\mathbb{R}^{2m}} \frac{\exp(\beta(1-\frac{t}{2m})|u|^2)|u|^2}{|x|^t} dx \\
&= \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)|u|^2}{|x|^t} dx + \int_{\mathbb{R}^{2m}} \frac{|u|^2}{|x|^t} dx \\
&=: I_1 + I_2.
\end{aligned} \tag{5.14}$$

By applying the Hölder inequality and inequality (1.8), one can estimate I_1 as follows

$$\begin{aligned}
I_1 &\leq \left(\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\left(\beta p(1-\frac{t}{2m})|u|^2\right)}{|x|^t} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{2m}} \frac{|u|^{2p'}}{|x|^t} dx\right)^{\frac{1}{p'}} \\
&\leq \left(\int_{\mathbb{R}^{2m}} |u|^2 dx\right)^{\frac{1}{p}(1-\frac{t}{2m})} \left(\int_{\mathbb{R}^{2m}} |u|^2 dx\right)^{\frac{1}{p'}(1-\frac{t}{2m})} \\
&= \left(\int_{\mathbb{R}^{2m}} |u|^2 dx\right)^{(1-\frac{t}{2m})},
\end{aligned} \tag{5.15}$$

where $p > 1$ and $\beta p < \beta_{2m,m}$. As for I_2, it is an immediate result of inequality (1.8).

One can deduce the sharpness of inequality (1.9) from the sharpness of inequality (1.13). In fact, one only needs to observe the following fact

$$\int_{\mathbb{R}^{2m}} \frac{\exp(\beta(1-\frac{t}{2m})|u|^2)|u|^2}{|x|^t} dx \geq \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\beta(1-\frac{t}{2m})|u|^2)}{|x|^t} dx.$$

For the attainability of the best constant $C(m,t)$ of inequality (1.9), one can manage the same steps as what we do in Theorem 1.7 to obtain the required results. □

The Proof of Theorem 1.11 We first employ the Fourier rearrangement tools to prove there exists radially maximizing sequence for $\mu_{2m,m,0,\beta}(\mathbb{R}^{2m})$. In fact, assume that (u_k) is a maximizing sequence for $\mu_{2m,m,0,\beta}(\mathbb{R}^{2m})$, that is

$$\|(-\Delta)^{\frac{m}{2}} u_k\|_2 = 1, \quad \lim_{k\to\infty} F_{2m,m,0,\beta}(u_k) \to \mu_{2m,m,0,\beta}(\mathbb{R}^{2m}).$$

Define $u_k^\sharp$ by $u_k^\sharp = F^{-1}\{(F(u_k))^*\}$, where F denotes the Fourier transform on $\mathbb{R}^{2m}$ (with its inverse F^{-1}) and f^* stands for the Schwarz symmetrization of f. Using the property of the Fourier rearrangement from [35], one can derive that

$$\|(-\Delta)^{\frac{m}{2}} u_k^\sharp\|_2 \le \|(-\Delta)^{\frac{m}{2}} u_k\|_2,\ \|u_k^\sharp\|_2 = \|u_k\|_2,\ \|u_k^\sharp\|_q \ge \|u_k\|_q.$$

Hence, $\lim\limits_{k\to\infty} F_{2m,m,0,\beta}(u_k) \le \lim\limits_{k\to\infty} F_{2m,m,0,\beta}(u_k^\sharp)$, which implies that $(u_k^\sharp)$ is also the maximizing sequence for $\mu_{2m,m,0,\beta}(\mathbb{R}^{2m})$. Constructing a new function sequence (v_k) defined by $v_k(x) := u_k(\|u_k\|_2^{\frac{1}{m}} x)$ for $x \in \mathbb{R}^{2m}$, one can easily verify that (v_k) is also a maximizing sequence for $\mu_{2m,m,0,\beta}(\mathbb{R}^{2m})$ with $\|\nabla^m v_k\|_2 = 1$ and $\|v_k\|_2 = 1$. Note (v_k) is bounded in $W^{m,2}(\mathbb{R}^{2m})$, up to a sequence, we may assume that

$$v_k \rightharpoonup v \text{ in } \dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m}),$$

thus v satisfies that $\|v\|_2 \le 1$ and $\|\nabla^m v\|_2^2 \le 1$. Since $W^{m,2}(\mathbb{R}^{2m})$ can be compactly imbedded into $L^r(\mathbb{R}^{2m})$ for any $r > 2$ (please refer to [7], Lemma 5.3), implementing same procedures as what we did in Lemma 2.2, one can deduce that

$$\lim_{k\to\infty} \int_{\mathbb{R}^{2m}} \left(\Phi_{2m,m}(\beta|u_k|^2) - \beta|u_k|^2\right) = \int_{\mathbb{R}^{2m}} \left(\Phi_{2m,m}(\beta|u|^2) - \beta|u|^2\right). \tag{5.16}$$

Then it follows that

$$\begin{aligned}
\mu_{2m,m,0,\beta}(\mathbb{R}^{2m}) &= F_{2m,m,0,\beta}(v_k) + o(1)\\
&= \int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta|v_k|^2)dx + o(1)\\
&= \beta + \int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta|v_k|^2) - \beta|v_k|^2 dx + o(1)\\
&= \beta + \int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta|v|^2) - \beta|v|^2 dx.
\end{aligned} \tag{5.17}$$

Next, we show $v \ne 0$. Indeed, one can pick u_0 in $\dot{W}^{m,2}(\mathbb{R}^{2m}) \cap L^2(\mathbb{R}^{2m})$ satisfying $\|\nabla^m u_0\|_2 = 1$ arbitrarily. Then, we have

$$\begin{aligned}
\mu_{2m,m,0,\beta}(\mathbb{R}^{2m}) \ge F_{2m,m,0,\beta}(u_0) &= \frac{\int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta|u_0|^2)dx}{\|u_0\|_2^2}\\
&= \frac{\sum_{j=1}^{\infty} \frac{\beta^j}{j!} \|u_0\|_{2j}^{2j}}{\|u_0\|_2^2}\\
&= \beta + \frac{\sum_{j=2}^{\infty} \frac{\beta^j}{j!} \|u_0\|_{2j}^{2j}}{\|u_0\|_2^2} > \beta.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_{2m,m,0,\beta}(\mathbb{R}^{2m}) &\le \beta + \frac{\int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta|v|^2) - \beta|v|^2 dx}{\|v\|_2^2}\\
&= \frac{\int_{\mathbb{R}^{2m}} \Phi_{2m,m}(\beta|v|^2)dx}{\|v\|_2^2} = F_{2m,m,0,\beta}(v).
\end{aligned}$$

Therefore, it remains to show $\|\nabla^m v\|_2^2 = 1$. Recall that $\|\nabla^m v\|_2^2 \leq 1$, it suffices to show that $\|\nabla^m v\|_2^2 \geq 1$. Through the definition of $\mu_{2m,m,0,\beta}(\mathbb{R}^{2m})$, one can obtain that

$$
\begin{aligned}
\mu_{2m,m,0,\beta}(\mathbb{R}^{2m}) &\geq F_{2m,m,0,\beta}(\frac{v}{\|\nabla^m v\|_2}) \\
&= \sum_{j=1}^{\infty} \frac{\beta^j}{j!} \frac{\|v\|_{2j}^{2j}}{\|v\|_2^2} \|\nabla^m v\|_2^{2-2j} \\
&\geq \beta + \frac{\beta^2}{2} \frac{\|v\|_4^4}{\|v\|_2^2} \|\nabla^m v\|_2^{-2} + \sum_{j=2}^{\infty} \frac{\beta^j}{j!} \frac{\|v\|_{2j}^{2j}}{\|v\|_2^2} \\
&= F_{2m,m,0,\beta}(v) + \frac{\beta^2}{2} \frac{\|v\|_4^4}{\|v\|_2^2} (\|\nabla^m v\|_2^{-2} - 1) \\
&\geq \mu_{2m,m,0,\beta}(\mathbb{R}^{2m}) + \frac{\beta^2}{2} \frac{\|v\|_4^4}{\|v\|_2^2} (\|\nabla^m v\|_2^{-2} - 1)
\end{aligned} \tag{5.18}
$$

which implies that $\|\nabla^m v\|_2^2 \geq 1$. Thus, v is a maximizer for $\mu_{2m,m,0,\beta}(\mathbb{R}^{2m})$ which completes the proof of Theorem 1.11. □

6 Proofs of Theorems 1.12 and 1.13

In this section, we give some applications of Theorem 1.1 and Theorem 1.7. We first establish the higher order critical Caffarelli–Kohn–Nirenberg inequalities which are not included in Lin's work [40]. Moreover, we also investigate the relationship between the best constants of the singular Adams inequality and the Caffarelli–Kohn–Nirenberg inequality in the asymptotic sense.

Proof of Theorem 1.12 We first give the proof of inequality (1.10). Denoting

$$
\begin{aligned}
\beta_0 := \sup \Bigg\{ \beta : &\int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta |u|^{\frac{n}{n-2}})}{|x|^t} dx \\
\leq\ & C(n,t) \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{2}} dx \right)^{1-\frac{t}{n}},\ \forall u \in W^{2,\frac{n}{2}}(\mathbb{R}^n) \text{ with } \|\Delta u\|_{\frac{n}{2}} \leq 1 \Bigg\},
\end{aligned}
$$

then for any $\beta < \beta_0$, there exists a constant $C(n,t) > 0$ such that for $u \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ and $k \geq j_{\frac{n}{2}} - 1$, there holds

$$
\begin{aligned}
C(n,t) \left(\frac{\|u\|_{\frac{n}{2}}}{\|\Delta u\|_{\frac{n}{2}}} \right)^{\frac{n-t}{2}} &\geq \int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta (\frac{|u|}{\|\Delta u\|_{\frac{n}{2}}})^{\frac{n}{n-2}})}{|x|^t} dx \\
&\geq \frac{\beta^k}{k!} \left(\frac{\|u\|_{L^{n'k}(\mathbb{R}^n; |x|^{-t}dx)}}{\|\Delta u\|_{\frac{n}{2}}} \right)^{n'k},
\end{aligned} \tag{6.1}
$$

which implies that for $u \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ and $k \geq j_{\frac{n}{2}} - 1$,

$$
\|u\|_{L^{n'k}(\mathbb{R}^n; |x|^{-t}dx)} \leq \left(C(n,t) \frac{k!}{\beta^k} \right)^{\frac{1}{n'k}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2n'k}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2n'k}}. \tag{6.2}
$$

For any $q \geq n'(j_{\frac{n}{2}} - 1)$, there exists $k \geq j_{\frac{n}{2}} - 1$ satisfying $n'k \leq q < n'(k+1)$ such that

$$\|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq \|u\|^{\theta}_{L^{n'k}(\mathbb{R}^n;|x|^{-t}dx)} \|u\|^{1-\theta}_{L^{n'(k+1)}(\mathbb{R}^n;|x|^{-t}dx)}. \tag{6.3}$$

Combining (6.2) with (6.3) and the fact $\frac{1}{q} = \frac{\theta}{n'k} + \frac{1-\theta}{n'(k+1)}$, one can conclude that

$$\|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq C(n,t)^{\frac{1}{q}} \beta^{-\frac{1}{n'}} \left((k+1)!\right)^{\frac{1}{q}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}. \tag{6.4}$$

Since $\frac{q}{n'} \geq k$, we have

$$\left((k+1)!\right)^{\frac{1}{q}} \leq \left(\Gamma(\frac{q}{n'}+2)\right)^{\frac{1}{q}}. \tag{6.5}$$

Combining inequalities (6.4) and (6.5), one can derive inequality (1.10) with estimating $c(n,q,t) \approx C(n,t)^{\frac{1}{q}} \beta^{-\frac{1}{n'}} \left(\Gamma(\frac{q}{n'}+2)\right)^{\frac{1}{q}}$.

Next, we claim that there exists $\alpha > 0$ such that $c(n,q,t)$ behaves like $c(n,q,t) \simeq \alpha q^{\frac{1}{n'}}$ as $q \to +\infty$ which is equivalent to say

$$\exists\, q_1 \geq j_{\frac{n}{2}},\ \forall\, q \geq q_1,\ \|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq \alpha q^{\frac{1}{n'}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}.$$

By recalling Stirling's asymptotic formula, we see that as $q \to \infty$,

$$\left(\Gamma(\frac{q}{n'}+2)\right)^{\frac{1}{q}} = (1+o(1))(\frac{q}{en'})^{\frac{1}{n'}}.$$

Therefore, we derive that

$$\|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq (1+o(1))(\frac{q}{\beta en'})^{\frac{1}{n'}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}, \tag{6.6}$$

which accomplishes the claim.

At last, we show the relationship between β_0 and $\alpha_{n,t}$, where

$$\alpha_{n,t} := \inf\left\{\alpha > 0 : \exists q_1 \geq j_{\frac{n}{2}},\ \forall q \geq q_1,\ \|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq \alpha q^{\frac{1}{n'}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}\right\}.$$

According to the definition of $\alpha_{n,t}$, combining inequality (6.6), one can derive that $\alpha_{n,t} \leq (\frac{1}{\beta en'})^{\frac{1}{n'}}$. Then it follows from the definition of β_0 that

$$\alpha_{n,t} \leq \left(\frac{1}{\beta_0 en'}\right)^{\frac{1}{n'}}. \tag{6.7}$$

Then it suffices to show that

$$\alpha_{n,t} \geq \left(\frac{1}{\beta_0 en'}\right)^{\frac{1}{n'}}.$$

Pick any $\alpha > \alpha_{n,t}$, through the definition of $\alpha_{n,t}$, there exists $q_0 \geq j_{\frac{n}{2}}$ such that for any $q \geq q_0$,

$$\|u\|_{L^q(\mathbb{R}^n;|x|^{-t}dx)} \leq \alpha q^{\frac{1}{n'}} \|u\|_{\frac{n}{2}}^{\frac{n-t}{2q}} \|\Delta u\|_{\frac{n}{2}}^{1-\frac{n-t}{2q}}. \tag{6.8}$$

Then for $u \in W^{2,\frac{n}{2}}(\mathbb{R}^n)$ and $\|\Delta u\|_{\frac{n}{2}} \le 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\Phi_{n,2}(\beta|u|^{\frac{n}{n-2}})}{|x|^t}dx \\
&= \int_{\mathbb{R}^n} \left(\sum_{j\frac{n}{2} \le n'k < q_0} \frac{\beta^k}{k!}|u(x)|^{n'k} \right) \frac{dx}{|x|^t} + \int_{\mathbb{R}^n} \left(\sum_{n'k \ge q_0} \frac{\beta^k}{k!}|u(x)|^{n'k} \right) \frac{dx}{|x|^t} \\
&=: J_1 + J_2.
\end{aligned} \tag{6.9}$$

Since J_1 consists of finite weighted norms and $\frac{n}{2} \le n'k < q_0$, one can get

$$\|u\|_{L^{n'k}(\mathbb{R}^n;|x|^{-t}dx)} \le \|u\|^{\theta}_{L^{\frac{n}{2}}(\mathbb{R}^n;|x|^{-t}dx)} \|u\|^{1-\theta}_{L^{q_0}(\mathbb{R}^n;|x|^{-t}dx)} \tag{6.10}$$

through using the Hölder inequality. Taking (6.8) and (6.10) into consideration, we get that for all $\frac{n}{2} \le n'k < q_0$,

$$\|u\|_{L^{n'k}(\mathbb{R}^n;|x|^{-t}dx)} \le C\|u\|_{\frac{n}{2}}^{\frac{n-t}{2(n'k)}}, \tag{6.11}$$

where we used the fact that $\|\Delta u\|_{\frac{n}{2}} \le 1$. Then it follows from (6.11) that

$$J_1 \le C \left(\sum_{\frac{n}{2} \le n'k < q_0} \frac{\beta^k}{k!} \right) \|u\|_{\frac{n}{2}}^{\frac{n-t}{2}}. \tag{6.12}$$

For J_2, inequality (6.8) leads to

$$J_2 \le \left(\sum_{n'k \ge q_0} \frac{k^k}{k!} (\beta n' \alpha^{n'})^k \right) \|u\|_{\frac{n}{2}}^{\frac{n-t}{2}}. \tag{6.13}$$

Then it follows from the Stirling's asymptotic formula that the power in (6.13) converges if $\beta n' \alpha^{n'} < \frac{1}{e}$, which implies that $\beta \in (0, \frac{1}{en'\alpha^{n'}})$. Hence, the definition of β_0 leads to $\beta_0 \ge \frac{1}{en'\alpha^{n'}}$. Moreover, through the definition of $\alpha_{n,t}$, we get that

$$\beta_0 \ge \frac{1}{en'\alpha_{n,t}^{n'}},$$

which is equivalent to

$$\alpha_{n,t} \ge \left(\frac{1}{en'\beta_0} \right)^{\frac{1}{n'}}. \tag{6.14}$$

Combining (6.7) and (6.14), we complete the proof. □

Remark 6.1 The proof of Theorem 1.13 is similar to that of Theorem 1.12, we omit the details.

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