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Hardy–Littlewood–Sobolev Inequalities with the Fractional Poisson Kernel and Their Applications in PDEs

Lu CHEN
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China
E-mail: *chenlu5818804@163.com*

Guozhen LU[1)]
Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail: *guozhen.lu@uconn.edu*

Chunxia TAO[2)]
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P. R. China
E-mail: *taochunxia@mail.bnu.edu.cn*

Dedicated to Carlos Kenig on his 65th birthday with admiration and appreciation

Abstract The purpose of this paper is five-fold. First, we employ the harmonic analysis techniques to establish the following Hardy–Littlewood–Sobolev inequality with the fractional Poisson kernel on the upper half space

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} f(\xi)P(x,\xi,\alpha)g(x)d\xi dx \le C_{n,\alpha,p,q'}\|g\|_{L^{q'}(\mathbb{R}^n_+)}\|f\|_{L^p(\partial\mathbb{R}^n_+)},$$

where $f\in L^p(\partial\mathbb{R}^n_+)$, $g\in L^{q'}(\mathbb{R}^n_+)$ and $p,\ q'\in(1,+\infty)$, $2\le\alpha<n$ satisfying $\frac{n-1}{np}+\frac{1}{q'}+\frac{2-\alpha}{n}=1$. Second, we utilize the technique combining the rearrangement inequality and Lorentz interpolation to show the attainability of best constant $C_{n,\alpha,p,q'}$. Third, we apply the regularity lifting method to obtain the smoothness of extremal functions of the above inequality under weaker assumptions. Furthermore, in light of the Pohozaev identity, we establish the sufficient and necessary condition for the existence of positive solutions to the integral system of the Euler–Lagrange equations associated with the extremals of the fractional Poisson kernel. Finally, by using the method of moving plane in integral forms, we prove that extremals of the Hardy–Littlewood–Sobolev inequality with the fractional Poisson kernel must be radially symmetric and decreasing about some point $\xi_0\in\partial\mathbb{R}^n_+$. Our results proved in this

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1, 2) Corresponding authors

paper play a crucial role in establishing the Stein–Weiss inequalities with the Poisson kernel in our subsequent paper.

Keywords Existence of extremal functions, Hardy–Littlewood–Sobolev inequality, Moving plane method, Poisson kernel

MR(2010) Subject Classification 35B40, 45G15

1 Introduction

The classical Hardy–Littlewood–Sobolev inequality which was first established by Hardy, Littlewood and Sobolev in [32, 47] states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-y|^{-\lambda}f(x)g(y)dxdy \leq C_{n,p,q'}\|f\|_{L^{q'}(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}, \tag{1.1}$$

where $1<q',p<\infty$, $0<\lambda<n$, $\frac{1}{q'}+\frac{1}{p}+\frac{\lambda}{n}=2$ and $q'=\frac{q}{q-1}$.

By utilizing the layer cake representation formula, Lieb and Loss [40] proved that the sharp constant $C_{n,p,q'}$ satisfies the following estimate

$$C_{n,p,q'} \leq \frac{n}{n-\lambda}\left(\frac{\pi^{\frac{\lambda}{2}}}{\Gamma(1+\frac{n}{2})}\right)^{\frac{\lambda}{n}}\frac{1}{q'p}\left(\left(\frac{\lambda q'}{n(q'-1)}\right)^{\frac{\lambda}{n}}+\left(\frac{\lambda p}{n(p-1)}\right)^{\frac{\lambda}{n}}\right).$$

Lieb [39] also employed the rearrangement inequalities to obtain the existence of the extremal functions of inequality (1.1). Furthermore, they also classified extremals of the inequality (1.1) and computed the sharp constant $C_{n,p,q'}$ only when one of p and q' is equal to 2 or $p=q'$.

The Hardy–Littlewood–Sobolev inequality was extended by Stein and Weiss to the following Stein–Weiss inequalities

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x|^{-\alpha}|x-y|^{-\lambda}f(x)g(y)|y|^{-\beta}dxdy \leq C_{n,\alpha,\beta,p,q'}\|f\|_{L^{q'}(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}, \tag{1.2}$$

where $1<p$, $q'<\infty$, α, β and λ satisfy the following conditions,

$$\begin{aligned}&\frac{1}{q'}+\frac{1}{p}+\frac{\alpha+\beta+\lambda}{n}=2, \quad \frac{1}{q'}+\frac{1}{p}\geq 1,\\ &\alpha+\beta\geq 0, \quad \alpha<\frac{n}{q}, \quad \beta<\frac{n}{p'}, \quad 0<\lambda<n.\end{aligned}$$

(see also an alternative proof of establishing the Stein–Weiss inequalities recently found in [29] by using conditions on weights to guarantee the weighted boundedness of fractional integrals given in [46] and such a method also applies to establish the Stein–Weiss inequalities on the Heisenberg groups and works for more general stratified and homogeneous groups). Lieb [39] used the method based on symmetrization argument and the Riesz rearrangement to establish the existence of extremals for the inequality (1.2) in the case $p<q$ and α, $\beta\geq 0$. Furthermore, in the case of $p=q$, the extremals can't be expected to exist (see Lieb [39] and also Herbst [33] for the case $\lambda=n-1$, $p=q=2$, $\alpha=0$, $\beta=1$). In the case of $p=q$, Beckner [2, 3] obtained the sharp constant of the Stein–Weiss inequalities (1.2) by establishing a general Stein–Weiss lemma. The precise estimate of the sharp constant of the Stein–Weiss inequalities for the case of $p\neq q$ was also established in [3]. For more results about proving precise estimates for Stein–Weiss functionals in conjunction with the study of Pitt's type inequalities and their multilinear versions, we refer the reader to the works of Beckner [4–6]. We note that the existence of

extremal functions for the Stein–Weiss inequalities in the case $p < q$ under the assumption $\alpha + \beta \geq 0$ has been established by Chen, Lu and Tao [18], which extends Lieb's result under the stronger assumption that $\alpha \geq 0$ and $\beta \geq 0$, using the concentration-compactness of Lions [41, 42].

Through the inequality (1.1), we can deduce many important geometrical inequalities such as the Gross logarithmic Sobolev inequality [26] and the Moser–Onofri–Beckner inequality [1], etc. It is also well-known that if we pick $\lambda = n - 2$, $p = q = \frac{2n}{2n-\lambda}$, then the Hardy–Littlewood–Sobolev inequality is in fact equivalent to the Sobolev inequality by Green's representation formula. By using the competing symmetry method, Carlen and Loss [10] provided a different proof from Lieb's of the sharp constants and extremal functions in the diagonal case $p = q' = \frac{2n}{2n-\lambda}$ and Frank and Lieb [22] offered a new proof using the reflection positivity of inversions in spheres in the special diagonal case. Frank and Lieb [23] further employed a rearrangement-free technique developed in [24] to recapture the best constant of inequality (1.1). Folland and Stein [21] extended the inequality (1.1) to the Heisenberg group and established the Hardy–Littlewood–Sobolev inequality on Heisenberg group. Frank and Lieb in [24] classify the extremals of this inequality in the diagonal case. This extends the earlier work of Jerison and Lee for sharp constants and extremals for the Sobolev inequality on the Heisenberg group in the conformal case in their study of CR Yamabe problem [34–36]. Furthermore, Han [27] employed the concentration-compactness principle of Lions [41, 42] to establish existence of extremals of this inequality for general indices. Han, Lu and Zhu established the double weighted Hardy–Littlewood–Sobolev inequality (namely, Stein–Weiss inequality) on the Heisenberg group and discussed the regularity and asymptotic behavior of the extremal functions. Recently, Chen, Lu and Tao [18] used the concentration-compactness principle to obtain existence of extremals of the Stein–Weiss inequality on the Heisenberg group for all indices. We also mention that when $q' = p = \frac{2n}{2n-\lambda}$, Euler–Lagrange equation of the extremals to the Hardy–Littlewood–Sobolev inequality in the Euclidean space is a conformal invariant integral equation. Using the method of moving plane or moving sphere in integral forms in Euclidean space (see [15, 38]), one can classify the positive solutions to this integral equation. The inequality (1.1) and its extensions have many applications in partial differential equations. For example, these inequalities are efficient in studying the radial symmetry and a priori estimate of positive solutions for the Hardy–Sobolev type equations and systems. For more results about the inequality (1.1) and its applications in partial differential equations, one can also see [7, 11, 12, 16, 43, 44] and the references therein.

Hang, Wang and Yan in [31] derived the following integral inequalities with the Poisson kernel,

$$\left\| \int_{\partial\mathbb{R}^n_+} P(x,\xi) f(\xi) d\xi \right\|_{L^q(\mathbb{R}^n_+)} \leq C_{n,p} \|f\|_{L^p(\partial\mathbb{R}^n_+)}, \tag{1.3}$$

where $P(x,\xi) = c_n \frac{x_n}{|x-\xi|^n}$ and $q = \frac{np}{n-1}$. They used the concentration-compactness principle to establish the existence of extremals for this inequality. For special index $p = \frac{2(n-1)}{n-2}$, by the method of moving-spheres, they classified the extremal functions of the inequality (1.3) and computed the sharp constant $C_{n,p}$. Integral inequality with the Poisson kernel is highly related with Carleman's proof of isoperimetric inequality in the plane (see [9]). By duality, one can easily see that the inequality (1.3) is in fact equivalent to the Hardy–Littlewood–Sobolev inequality with the Poisson kernel which can be stated as follows:

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} f(\xi)P(x,\xi)g(x)d\xi dx \le C_{n,p,}\|g\|_{L^{q'}(\mathbb{R}^n_+)}\|f\|_{L^p(\partial\mathbb{R}^n_+)},$$

where $1 < p < \infty$, $1 < q' < \infty$, satisfying

$$\frac{n-1}{n}\frac{1}{p} + \frac{1}{q'} = 1.$$

The Hardy–Littlewood–Sobolev inequalities are equivalent to the L^p to L^q boundedness for the convolution operators with the Riesz potential. It is well known that the Riesz potential can also be seen as the fundamental solution of the fractional Laplacian operator. On the other hand, the kernel

$$P(x,\xi,\alpha) = -\frac{1}{n-\alpha}\frac{\partial}{\partial x_n}\frac{1}{|x-\xi|^{n-\alpha}},$$

considered in [19], up to a constant, can be viewed as the fundamental solution of the fractional Laplacian operator on the upper half space. In fact, for $\alpha = 2$, this is the classical Poisson kernel.

From this point of view and the work of Hang, Wang and Yan, we are interested in the question whether there exists an integral inequality with the fractional Poisson kernel on the upper half space $\mathbb{R}^n_+$. Furthermore, we like to know if such an inequality has an extremal function for all the indices. In fact, the authors of [19] established the integral inequality with the fractional Poisson kernel in the special index through the methods based on conformal transformation and the moving spheres in integral forms. However, this method cannot be used to establish our inequality (1.5) for general index. We also note that the authors of [20] established the following Hardy–Littlewood–Sobolev inequality on the upper half space $\mathbb{R}^n_+$ which states

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} |x-y|^{-\lambda} f(x)g(y)dydx \le C_{n,,p,q'}\|f\|_{L^{q'}(\mathbb{R}^n_+)}\|g\|_{L^p(\partial\mathbb{R}^n_+)}, \tag{1.4}$$

where $p > 1$, $q' > 1$, $0 < \lambda < n$ with

$$\frac{1}{q'} + \frac{n-1}{n}\frac{1}{p} + \frac{\lambda+1}{n} = 2.$$

Utilizing the symmetry and rearrangement technique, they derived existence of extremals for this inequality. Furthermore, in the conformal invariant case $q' = \frac{2n}{2n-\lambda}$ and $p = \frac{2n-2}{2n-2-\lambda}$, they also classified the extremals through the moving sphere method.

In this paper, we are concerned with the Hardy–Littlewood–Sobolev inequality with the fractional Poisson kernel for general indices. Our first main result is the following:

Theorem 1.1 *For $n \ge 3$, $1 < p < \infty$, $1 < q' < \infty$ and $2 \le \alpha < n$, satisfying*

$$\frac{n-1}{n}\frac{1}{p} + \frac{1}{q'} + \frac{2-\alpha}{n} = 1,$$

there exists some constant $C_{n,\alpha,p,q'} > 0$ such that for any functions $f \in L^p(\partial\mathbb{R}^n_+)$ and $g \in L^{q'}(\mathbb{R}^n_+)$, there holds

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} f(\xi)P(x,\xi,\alpha)g(x)d\xi dx \le C_{n,\alpha,p,q'}\|g\|_{L^{q'}(\mathbb{R}^n_+)}\|f\|_{L^p(\partial\mathbb{R}^n_+)}, \tag{1.5}$$

where $P(x,\xi,\alpha)=\frac{x_n}{(|x'-\xi|^2+x_n^2)^{\frac{n+2-\alpha}{2}}}$ *is the so-called fractional Poisson kernel.*

Remark 1.2 We note that the kernel $P(x,\xi,\alpha)$ in our inequality is not L^1 integrable unlike the Poisson kernel of the integral inequality established by Hang, Wang and Yan [31]. Their proofs depend on L^1 integrability of $P(x,\xi,\alpha)$ for $\alpha=2$, which allows them to establish

$$\Big\|\int_{\partial\mathbb{R}^n_+}P(x,\xi,2)f(\xi)d\xi\Big\|_{L^\infty(\mathbb{R}^n_+)}\le\|f\|_{L^\infty(\partial\mathbb{R}^n_+)},$$

and

$$\Big\|\int_{\partial\mathbb{R}^n_+}P(x,\xi,2)f(\xi)d\xi\Big\|_{L_w^{\frac{n}{n-1}}(\mathbb{R}^n_+)}\le\|f\|_{L^1(\partial\mathbb{R}^n_+)}.$$

Given f a measurable function on $\mathbb{R}^n_+$, $0<r<+\infty$, define the weak L^r norm of f as

$$\|f\|^r_{L^r_w(\mathbb{R}^n_+)}=\sup_{t>0}t^r|\{x\in\mathbb{R}^n_+:|f|>t\}|.$$

We use the weak L^q estimate of $P(f)$ to overcome this difficulty and directly obtain that

$$\Big\|\int_{\partial\mathbb{R}^n_+}P(x,\xi,\alpha)f(\xi)d\xi\Big\|_{L^q_w(\mathbb{R}^n_+)}\le\|f\|_{L^p(\partial\mathbb{R}^n_+)}.$$

Remark 1.3 The Hardy–Littlewood–Sobolev inequality (1.5) also plays an important role in establishing our Stein–Weiss inequality with the fractional Poisson kernel by the authors [17].

By duality, it is easy to verify that the inequality (1.5) is equivalent to the following two inequalities

$$\begin{aligned}&\|P(f)\|_{L^q(\mathbb{R}^n_+)}\le C_{n,\alpha,p,q'}\|f\|_{L^p(\partial\mathbb{R}^n_+)},\\&\|T(g)\|_{L^{p'}(\partial\mathbb{R}^n_+)}\le C_{n,\alpha,p,q'}\|g\|_{L^{q'}(\mathbb{R}^n_+)},\end{aligned}\tag{1.6}$$

where p and q' satisfy the assumptions of Theorem 1.1 and

$$P(f)(x)=\int_{\partial\mathbb{R}^n_+}P(x,\xi,\alpha)f(\xi)d\xi,\quad T(g)=\int_{\mathbb{R}^n_+}P(x,\xi,\alpha)g(x)dx.$$

In order to obtain the existence of extremals of the inequality (1.5), we turn to consider the following maximizing problem

$$C_{n,\alpha,p,q'}:=\sup\{\|P(f)\|_{L^q(\mathbb{R}^n_+)}:f\ge0,\|f\|_{L^p(\partial\mathbb{R}^n_+)}=1\}.\tag{1.7}$$

It is not hard to verify that the extremals of the inequality (1.5) are those solving the maximizing problem (1.7). We use the method combining the rearrangement inequality and Lorentz interpolation to obtain the attainability of maximizers for the maximizing problem (1.7).

Theorem 1.4 *For* $n\ge3$, $1<p<\infty$, $1<q'<\infty$ *and* $2\le\alpha<n$, *satisfying*

$$\frac{n-1}{n}\frac{1}{p}+\frac{1}{q'}+\frac{2-\alpha}{n}=1,$$

there exists some nonnegative function $f\in L^p(\partial\mathbb{R}^n_+)$ *such that* $\|f\|_{L^p(\partial\mathbb{R}^n_+)}=1$ *and* $\|P(f)\|_{L^q(\mathbb{R}^n_+)}=C_{n,\alpha,p,q'}$.

Now, it is also interesting to study some properties such as the regularity and radial symmetry for the extremal functions of the inequality (1.5). By maximizing the functional

$$J(f,g)=\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}f(\xi)P(x,\xi,\alpha)g(x)d\xi dx\tag{1.8}$$

under the constraint $\|f\|_{L^p(\partial\mathbb{R}^n_+)} = \|g\|_{L^{q'}(\mathbb{R}^n_+)} = 1$, one can use the Euler–Lagrange multiplier theorem to derive that (f, g) satisfies the following integral system

$$\begin{cases} J(f,g)f(\xi)^{p-1} = \displaystyle\int_{\mathbb{R}^n_+} P(x,\xi,\alpha)g(x)dx, & \xi \in \partial\mathbb{R}^n_+, \\ J(f,g)g(x)^{q'-1} = \displaystyle\int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)f(\xi)d\xi, & x \in \mathbb{R}^n_+. \end{cases} \tag{1.9}$$

Set $u = c_1 f^{p-1}, v = c_2 g^{q'-1}, \frac{1}{p-1} = p_0$ and $\frac{1}{q'-1} = q_0$ and pick two suitable constants c_1 and c_2, then system (1.9) is simplified as

$$\begin{cases} u(\xi) = \displaystyle\int_{\mathbb{R}^n_+} P(x,\xi,\alpha)v^{q_0}(x)dx, & \xi \in \partial\mathbb{R}^n_+, \\ v(x) = \displaystyle\int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)u^{p_0}(\xi)d\xi, & x \in \mathbb{R}^n_+, \end{cases} \tag{1.10}$$

where p_0 and q_0 satisfy $\frac{n-1}{n}\frac{1}{p_0+1} + \frac{1}{q_0+1} = \frac{n-\alpha+1}{n}$.

We use the regularity lifting lemma in the spirit of Hang [31] to obtain the smoothness of positive solutions to the integral system (1.10). We also point out that this regularity lifting method is different from the usual regularity lifting method, which is basically a linear method (see [13, 16]), and can also be applied to obtain the smoothness for positive solutions to more general integral systems.

Theorem 1.5 *For* $2 \le \alpha < n$, $\frac{\alpha-2}{n+1-\alpha} < p_0 < \infty$, $\frac{\alpha-1}{n+1-\alpha} < q_0 < \infty$ *satisfying*

$$\frac{1}{p_0+1}\frac{n-1}{n} + \frac{1}{q_0+1} = \frac{n+1-\alpha}{n},$$

if we only suppose that $u(x) \in L^{p_0+1}_{\mathrm{loc}}(\partial\mathbb{R}^n_+)$ *and* (u, v) *satisfies the following integral system*

$$\begin{cases} u(\xi) = \displaystyle\int_{\mathbb{R}^n_+} P(x,\xi,\alpha)v^{q_0}(x)dx, & \xi \in \partial\mathbb{R}^n_+, \\ v(x) = \displaystyle\int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)u^{p_0}(\xi)d\xi, & x \in \mathbb{R}^n_+, \end{cases}$$

then $(u, v) \in C^\infty(\partial\mathbb{R}^n_+) \times C^\infty(\mathbb{R}^n_+)$.

Corollary 1.6 *Under the assumptions of Theorem* 1.1*, then extremals of the inequality* (1.5) *must be* C^∞ *smooth.*

Through the Pohozaev identity in integral forms, we obtain some necessary conditions for the existence of positive solutions to the integral system (1.10).

Theorem 1.7 *For* $2 \le \gamma < n$*, if we suppose that there exists a pair of positive solutions* $(u, v) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ *satisfying the integral system* (1.10)*, then the following balance condition must hold*:

$$\frac{n-1}{p_0+1} + \frac{n}{q_0+1} = n+1-\alpha.$$

As a corollary, we immediately obtain the following Liouville type theorem for positive solutions of the integral system (1.10).

Corollary 1.8 *For* $2 < \alpha < n$*, suppose that*

$$\frac{n-1}{p_0+1} + \frac{n}{q_0+1} \neq n+1-\alpha,$$

then there does not exist any pair of positive solutions $(u, v) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ *satisfying the integral system* (1.10).

Obviously, extremals (f, g) of inequality (1.5) satisfies the integral system (1.9). In light of Theorem 1.4 and Theorem 1.5, we obtain the sufficient and necessary conditions for existence of positive solutions to the integral system (1.10).

Theorem 1.9 *For* $p_0 > 0$, $q_0 > 0$, *then the sufficient and necessary condition for the existence of a pair of positive solutions* $(u, v) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ *to the system* (1.10) *is*

$$\frac{n-1}{p_0+1} + \frac{n}{q_0+1} = n + 1 - \alpha.$$

We also employ the method of moving plane in integral forms to investigate the radial symmetry of positive solution of the integral system (1.10).

Theorem 1.10 *For* $2 \le \alpha < n$, $0 < p_0 < \infty$, $0 < q_0 < \infty$ *satisfying* $\frac{n-1}{n}\frac{1}{p_0+1} + \frac{1}{q_0+1} = \frac{n-\alpha+1}{n}$, *if* $(u, v) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ *is a pair of positive solutions of the integral system* (1.10), *then* $u(\xi)$ *and* $v(x)|_{\partial\mathbb{R}^n_+}$ *are radially symmetric and monotone decreasing about some point* $\xi_0 \in \partial\mathbb{R}^n_+$.

Corollary 1.11 *Under the assumptions of Theorem* 1.1, *extremals of inequality* (1.5) *must be radially decreasing about some point* $\xi_0 \in \partial\mathbb{R}^n_+$.

This paper is organized as follows. In Section 2, we employ the harmonic analysis technique to establish the Hardy–Litttlewood–Sobolev inequality with the fractional Poisson kernel on the upper half space. In Section 3, by the rearrangement inequality and Lorentz interpolation, we obtain the existence of extremals of the inequality (1.5). Section 4 and Section 6 are devoted to the regularity estimate and the radial symmetry of extremals of the Hardy–Litttlewood–Sobolev inequality with the fractional Poisson kernel. In Section 5, using the Pohozaev identity in integral forms, we give sufficient and necessary conditions for the existence of positive solutions of the integral system (1.10).

2 The Proof of Theorem 1.1

In this section, we will use the Marcinkiewicz interpolation theorem and weak type estimate to establish the Hardy–Littlewood–Sobolev inequality with the fractional Poisson kernel.

For $n \ge 3$, $2 \le \alpha < n$, $t > 0$ and $x' \in \mathbb{R}^{n-1}$, we define

$$P_t(x') = \frac{t}{(|x'|^2 + t^2)^{\frac{n+2-\alpha}{2}}}.$$

Clearly, we have

$$\begin{aligned} P(x, \xi, \alpha) &= P_{x_n}(x' - \xi) \quad \text{for } x = (x', x_n) \in \mathbb{R}^n_+,\ \xi \in \partial\mathbb{R}^n_+, \\ P(f)(x) &= (P_{x_n} * f)(x') \quad \text{for } x \in \mathbb{R}^n_+. \end{aligned}$$

By Young's inequality, we derive the following estimate.

Lemma 2.1 *For* $2 \le \alpha < n$ *and* $1 < p < \frac{n-1}{\alpha-2}$, *pick* $r < \frac{np}{n-1+(2-\alpha)p}$ *sufficiently close to* $\frac{np}{n-1+(2-\alpha)p}$ *and* $s \ge 1$ *satisfying* $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{s}$, *there holds*

$$\int_{0<x_n<a} (Pf)^r(x)dx \lesssim a^{\frac{(n-1)r}{s}+1-(n+1-\alpha)r} \|f\|^r_{L^p(\partial\mathbb{R}^n_+)}.$$

Proof By Young inequality, it follows

$$\begin{aligned}\int_{0<x_n<a}(Pf)^r(x)dx &= \int_0^a\int_{\mathbb{R}^{n-1}}|(P_{x_n}*f)(x')|^r dx'dx_n\\ &\le \|f\|^r_{L^p(\mathbb{R}^{n-1})}\int_0^a\|P_{x_n}\|^r_{L^s(\mathbb{R}^{n-1})}dx_n\\ &= \|f\|^r_{L^p(\mathbb{R}^{n-1})}\int_0^a\left(\int_{\mathbb{R}^{n-1}}\left(\frac{|x_n|}{(|x'|^2+x_n^2)^{\frac{n+2-\alpha}{2}}}\right)^s dx'\right)^{\frac{r}{s}}dx_n\\ &= \|f\|^r_{L^p(\mathbb{R}^{n-1})}\int_0^a|x_n|^{\frac{(n-1)r}{s}-(n+1-\alpha)r}dx_n\int_{\mathbb{R}^{n-1}}\frac{1}{(|x'|^2+1)^{\frac{(n+2-\alpha)s}{2}}}dx'.\end{aligned}$$

Since $r<\frac{np}{n-1+(2-\alpha)p}$ sufficiently close to $\frac{np}{n-1+(2-\alpha)p}$, thus

$$\int_{\mathbb{R}^{n-1}}\frac{1}{(|x'|^2+1)^{\frac{(n+2-\alpha)s}{2}}}dx'<\infty \quad\text{and}\quad \int_0^a|x_n|^{\frac{(n-1)r}{s}-(n+1-\alpha)r}dx_n<\infty.$$

Thus we complete the proof of Lemma 2.1. □

Proof of Theorem 1.1 By the Marcinkiewicz interpolation theorem (see [50]), we only need to prove the weak-type estimate. Namely, we will prove that

$$\|P(f)\|_{L^q_w(\mathbb{R}^n_+)}\le C_{n,\alpha,p,q'}\|f\|_{L^p(\partial\mathbb{R}^n_+)}. \tag{2.1}$$

Without the loss of generality, we may assume that $\|f\|_p=1$. In view of the Holder inequality and the integration of fractional Poisson kernel, we can see that

$$P(f)(x)\le C(n,\alpha,p)x_n^{\frac{n-1}{p'}-(n+1-\alpha)}.$$

Hence for any $t>0$,

$$\begin{aligned}|\{x\in\mathbb{R}^+_n:P(f)(x)>t\}| &= \left|\left\{x\in\mathbb{R}^+_n:0<x_n<C(n,\alpha,p)\left(\frac{1}{t}\right)^{\frac{p'}{p'(n+1-\alpha)-(n-1)}},\ P(f)(x)>t\right\}\right|\\ &\le\frac{1}{t^r}\int_{x\in\mathbb{R}^n_+,0<x_n<C(n,\alpha,p)(\frac{1}{t})^{\frac{p'}{p'(n+1-\alpha)-(n-1)}}}(Pf)^r(x)dx.\end{aligned}\tag{2.2}$$

Pick $r<\frac{np}{n-1+(2-\alpha)p}$ sufficiently close to $\frac{np}{n-1+(2-\alpha)p}$ and $q\ge1$ satisfying $\frac{1}{r}+1=\frac{1}{p}+\frac{1}{s}$, by Lemma 2.1, we obtain

$$|\{x\in\mathbb{R}^+_n:\ P(f)(x)>t\}|\lesssim\left(\frac{1}{t}\right)^{\frac{np}{n-1+(2-\alpha)p}},$$

which implies that

$$\|P(f)\|_{L^q_w(\mathbb{R}^n_+)}\le C_{n,\alpha,\beta,p,q}\|f\|_{L^p(\partial\mathbb{R}^n_+)}.$$

3 The Proof of Theorem 1.4

Throughout this section, we will employ the method based on the Lorentz interpolation and rearrangement inequality to investigate the existence of maximizers for the maximizing problem

$$C_{n,\alpha,p,q'}:=\sup\{\|P(f)\|_{L^q(\mathbb{R}^n_+)}:f\ge0,\|f\|_{L^p(\partial\mathbb{R}^n_+)}=1\}. \tag{3.1}$$

Proof Assume that $\{f_i\}_i$ is a maximizing sequence for the problem (3.1), namely $\|f_i\|_{L^p(\partial\mathbb{R}^n_+)}=1$ and $\lim_{i\to+\infty}\|P(f_i)\|_{L^q(\mathbb{R}^n_+)}=C_{n,\alpha,p,q'}$. By the Riesz rearrangement inequality [40], we obtain

$$\|f_i^*\|_{L^p(\partial\mathbb{R}^n_+)}=\|f_i\|_{L^p\partial\mathbb{R}^n_+)}=1,\quad \|P(f_i)\|^q_{L^q(\mathbb{R}^n_+)}\le\|P(f_i^*)\|^q_{L^q(\mathbb{R}^n_+)}.$$

Hence we may assume $\{f_i\}_i$ is a nonnegative radially decreasing sequence.

For any $f_i \in L^p(\partial\mathbb{R}^n_+)$ and $\lambda > 0$, set $f_i^\lambda(\xi) = \lambda^{-\frac{n-1}{p}} f_i(\frac{\xi}{\lambda})$, then it is clear that $\|f_i^\lambda\|_{L^p(\partial\mathbb{R}^n_+)} = \|f_i\|_{L^p(\partial\mathbb{R}^n_+)}$ and $\|P(f_i^\lambda)\|_{L^q(\mathbb{R}^n_+)} = \|P(f_i)\|_{L^q(\mathbb{R}^n_+)}$. Hence $\{f_i^\lambda\}_i$ is also a maximizing sequence for problem (3.1). For convenience, denote

$$e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \quad \text{and} \quad a_i = \sup_{\lambda>0} f_i^\lambda(e_1) = \sup_{\lambda>0} \lambda^{-\frac{n-1}{p}} f_i\left(\frac{e_1}{\lambda}\right).$$

It follows that

$$0 \le f_i(\xi) \le a_i|\xi|^{-\frac{n-1}{p}} \quad \text{and} \quad \|f_i\|_{L^{p,\infty}(\partial\mathbb{R}^n_+)} \le w_{n-2}^{\frac{1}{p}} a_i.$$

In the proof of Theorem 1.1, we have obtained that for $2 \le \alpha < n$ and $1 < p < \frac{n-1}{\alpha-2}$, there holds

$$\|P(f)\|_{L^q_w(\mathbb{R}^n_+)} \le C_{n,\alpha,p,q'}\|f\|_{L^p(\partial\mathbb{R}^n_+)}. \tag{3.2}$$

Given f a measurable function on $\partial\mathbb{R}^n_+$, $0 < r, s < +\infty$, define the Lorentz norm with indices r and s as

$$\|f\|_{L^{r,s}(\partial\mathbb{R}^n_+)} = \begin{cases} \left(\displaystyle\int_0^\infty (t^{\frac{1}{r}} f^\sharp(t))^s \frac{dt}{t}\right)^{\frac{1}{s}}, & \text{if } s < \infty,\\ \sup_{t>0} t^{\frac{1}{r}} f^\sharp(t), & \text{if } s = \infty,\end{cases}$$

where $f^\sharp(t)$ denotes the decreasing rearrangement of f. By the Lorentz interpolation theorem, we have

$$\|P(f)\|_{L^q(\mathbb{R}^n_+)} \le C_{n,\alpha,p,q'}\|f\|_{L^{p,q}(\partial\mathbb{R}^n_+)}. \tag{3.3}$$

Combining the above estimate, we derive that

$$\begin{aligned}\|P(f)\|_{L^q(\mathbb{R}^n_+)} &\le C_{n,\alpha,p,q'}\|f\|_{L^{p,q}(\partial\mathbb{R}^n_+)}\\ &\le C_{n,\alpha,p,q'}\|f\|^{1-\frac{p}{q}}_{L^{p,\infty}}\|f\|^{\frac{p}{q}}_{L^p}\\ &\le C_{n,\alpha,p,q'} a_i^{1-\frac{p}{q}},\end{aligned}$$

which implies $a_i \ge C_{n,\alpha,p,q'} > 0$. Then, we may choose $\lambda_i > 0$ such that $f_i^{\lambda_i}(e_1) \ge C_{n,\alpha,p,q'} > 0$. Hence, we may demand our maximizing sequence $\{f_i\}_i$ satisfying $f_i(1) \ge C_{n,\alpha,p,q'} > 0$. On the other hand, for any $R > 0$, direct calculation yields

$$\begin{aligned}\upsilon_{n-1} f_i^p(R)R^{n-1} &\le \omega_{n-2}\int_0^R f_i^p(r)r^{n-2}dr\\ &\le \omega_{n-2}\int_0^{+\infty} f_i^p(r)r^{n-2}dr\\ &= \int_{\partial\mathbb{R}^n_+} f_i^p(x)dx\\ &= 1,\end{aligned}$$

which implies that

$$0 \le f_i(\xi) \le \upsilon_{n-1}^{-\frac{1}{p}}|\xi|^{-\frac{n-1}{p}}. \tag{3.4}$$

Following Lieb's argument [39] based on the Helly theorem, after passing to a subsequence we may find a nonnegative, radially decreasing function f such that $f_i \to f$ almost everywhere in $\partial\mathbb{R}^n_+$. Clearly, we have $f(\xi) \geq C_{n,\alpha,p,q'} > 0$ for $|\xi| \leq 1$. With the help of the Brezis–Lieb theorem (see [8]), we obtain

$$\begin{aligned}\|f_i - f\|^p_{L^p(\partial\mathbb{R}^n_+)} &= \|f_i\|^p_{L^p(\mathbb{R}^{n-1})} - \|f\|^p_{L^p(\partial\mathbb{R}^n_+)} + o(1)\\ &= 1 - \|f\|^p_{L^p(\partial\mathbb{R}^n_+)} + o(1).\end{aligned} \tag{3.5}$$

From (3.4), we know

$$P(f_i)(x) \leq \int_{\partial\mathbb{R}^n_+} \frac{x_n}{(|\xi - x'|^2 + x_n^2)^{\frac{n+2-\alpha}{2}}} \frac{1}{|\xi|^{\frac{n-1}{p}}} d\xi. \tag{3.6}$$

According to the assumptions of Theorem 1.1, we can obtain that $\alpha < \frac{n-1}{p} + 2$, which implies that the right hand of (3.6) is finite. Then it follows from the dominated convergence theorem that $P(f_i)(x) \to P(f)(x)$ for $x \in \mathbb{R}^n_+$. Note that

$$\begin{aligned}\lim_{i\to\infty} \|P(f_i)\|^q_{L^q(\mathbb{R}^n_+)} &= \|P(f)\|^q_{L^q(\mathbb{R}^n_+)} + \lim_{i\to\infty} \|P(f_i) - P(f)\|^q_{L^q(\mathbb{R}^n_+)}\\ &\leq C_{n,\alpha,p,q'}\|f\|^q_{L^p(\partial\mathbb{R}^n_+)} + C_{n,\alpha,p,q'} \lim_{i\to\infty} \|f_i - f\|^q_{L^p(\partial\mathbb{R}^n_+)},\end{aligned}$$

which implies that

$$1 \leq \|f\|^q_{L^p(\partial\mathbb{R}^n_+)} + (1 - \|f\|^p_{L^p(\partial\mathbb{R}^n_+)})^{\frac{q}{p}}.$$

Since $q > p$ and $f \neq 0$, $\|f\|_{L^p(\mathbb{R}^{n-1})}$ must be equal to 1. Hence $f_i \to f$ in $L^p(\mathbb{R}^{n-1})$ and f is a actually maximizer for the problem (3.1). Then we complete the proof of Theorem 1.4. □

4 The Proof of Theorem 1.5

Through out this section, we will give the regularity estimate for extremal function of the integral inequality (1.5). For this purpose, we need the following two regularity lifting lemmas. The main idea of this proof is similar to that of regularity lifting proved by Hang [30]. Our case is more complicated and we give a detailed proof here. For simplicity, we give the following notation. Define

$$\begin{aligned}&B_R(x) = \{y \in \mathbb{R}^n : | y - x |< R,\ x \in \mathbb{R}^n\},\\ &B^{n-1}_R(x) = \{y \in \partial\mathbb{R}^n_+ : | y - x |< R,\ x \in \partial\mathbb{R}^n_+\},\\ &B^+_R(x) = \{y = (y_1, y_2, \dots, y_n) \in B_R(x) :\ y_n > 0,\ x \in \mathbb{R}^n_+\}.\end{aligned}$$

For $x = 0$, we write $B_R = B_R(0)$, $B^{n-1}_R = B^{n-1}_R(0)$, $B^+_R = B^+_R(0)$.

Lemma 4.1 *For $2 \leq \alpha < n$, $n \geq 3$, $1 < a, b \leq +\infty$, $1 \leq r < +\infty$, $\frac{n}{n-\alpha+1} < p < q < +\infty$ satisfying*

$$\frac{\alpha-1}{n} < \frac{r}{q} + \frac{1}{a} < \frac{r}{p} + \frac{1}{a} \leq 1, \quad \text{and} \quad \frac{n-1}{b} + \frac{n}{ar} + (2-\alpha) = \frac{\alpha-1}{r},$$

if we suppose that $u, v \in L^p(B^+_R), U \in L^a(B^+_R), F \in L^b(B^{n-1}_R)$ are all nonnegative functions satisfying $v|_{B^+_{R/2}} \in L^q(B^+_{R/2})$,

$$\|U\|^{1/r}_{L^a(B^+_R)}\|F\|_{L^b(B^{n-1}_R)} \leq \epsilon(n, \alpha, p, q, r, a, b) \ \textit{small}$$

and

$$u(x) \leq \int_{B^{n-1}_R} P(x, \xi, \alpha)F(\xi)\left(\int_{B^+_R} P(y, \xi, \alpha)U(y)u(y)^r dy\right)^{1/r} d\xi + v(x) \tag{4.1}$$

for $x \in B_R^+$, *then we have* $u|_{B_{R/4}^+} \in L^q(B_{R/4}^+)$ *and*

$$\|u\|_{L^q(B_{R/4}^+)} \leq c(n, \alpha, p, q, r, a, b)(R^{\frac{n}{q}-\frac{n}{p}}\|u\|_{L^p(B_R^+)} + \|v\|_{L^q(B_{R/2}^+)}).$$

Proof By scaling, we may assume $R = 1$. We carry out the proof of Lemma 4.1 by two steps.

Step 1 $u, v \in L^q(B_1^+)$. Denote

$$f(\xi) = \int_{B_1^+} P(x, \xi, \alpha)U(x)u(x)^r dx \quad \text{for } \xi \in B_1^{n-1}.$$

Let p_1 and q_1 be the numbers satisfying

$$\frac{n-1}{p_1} = \frac{nr}{p} + \frac{n}{a} - (\alpha - 1), \frac{n-1}{q_1} = \frac{nr}{q} + \frac{n}{a} - (\alpha - 1).$$

Clearly

$$\begin{aligned} &\|f\|_{L^{p_1}(B_1^{n-1})} \leq C(n, p, r, a)\|U\|_{L^a(B_1^+)}\|u\|_{L^p(B_1^+)}^r, \\ &\|f\|_{L^{q_1}(B_1^{n-1})} \leq C(n, q, r, a)\|U\|_{L^a(B_1^+)}\|u\|_{L^q(B_1^+)}^r, \end{aligned} \tag{4.2}$$

with the help of the integral inequality (1.6). Straightforward calculations yield that for $0 < s < t \leq \frac{1}{2}$ and $x \in B_s^+$, there holds

$$\begin{aligned} u(x) &\leq \int_{B_{(s+t)/2}^{n-1}} P(x, \xi, \alpha)F(\xi)f(\xi)^{1/r}d\xi + \int_{B_1^{n-1}\setminus B_{(s+t)/2}^{n-1}} P(x, \xi, \alpha)F(\xi)f(\xi)^{1/r}d\xi + v(x) \\ &\leq \int_{B_{(s+t)/2}^{n-1}} P(x, \xi, \alpha)F(\xi)f(\xi)^{1/r}d\xi + \frac{c(n, \alpha)}{(t-s)^{n+1-\alpha}}\int_{B_1^{n-1}\setminus B_{(s+t)/2}^{n-1}} F(\xi)f(\xi)^{1/r}d\xi + v(x) \\ &\leq \int_{B_{(s+t)/2}^{n-1}} P(x, \xi, \alpha)F(\xi)f(\xi)^{1/r}d\xi + \frac{c(n, \alpha, p)}{(t-s)^{n+1-\alpha}}\|F\|_{L^b(B_1^{n-1})}\|f\|_{L^{p_1}(B_1^{n-1})}^{1/r} + v(x) \\ &\leq \int_{B_{(s+t)/2}^{n-1}} P(x, \xi, \alpha)F(\xi)f(\xi)^{1/r}d\xi + \frac{c(n, \alpha, p, q, r, a, b)}{(t-s)^{n+1-\alpha}}\|u\|_{L^p(B_1^+)} + v(x). \end{aligned}$$

Combining this and the inequality (4.2), we obtain

$$\begin{aligned} \|u\|_{L^q(B_s^+)} \leq{}& c(n, \alpha, q, r, a)\|F\|_{L^b(B_1^{n-1})}\|f\|_{L^{q_1}(B_{(s+t)/2}^{n-1})}^{1/r} \\ &+ \frac{c(n, \alpha, p, q, r, a, b)}{(t-s)^{n+1-\alpha}}\|u\|_{L^p(B_1^+)} + \|v\|_{L^q(B_{1/2}^+)}. \end{aligned} \tag{4.3}$$

On the other hand, for $\xi \in B_{(s+t)/2}^{n-1}$, we also have

$$\begin{aligned} f(\xi) &= \int_{B_t^+} P(x, \xi, \alpha)U(x)u(x)^r dx + \int_{B_1^+\setminus B_t^+} P(x, \xi, \alpha)U(x)u(x)^r dx \\ &\leq \int_{B_t^+} P(x, \xi, \alpha)U(x)u(x)^r dx + \frac{c(n)}{(t-s)^{n+1-\alpha}}\int_{B_1^+\setminus B_t^+} U(x)u(x)^r dx \\ &\leq \int_{B_t^+} P(x, \xi, \alpha)U(x)u(x)^r dx + \frac{c(n, p, r, a)}{(t-s)^{n+1-\alpha}}\|U\|_{L^a(B_1^+)}\|u\|_{L^p(B_1^+)}^r. \end{aligned}$$

Then it follows

$$\begin{aligned} \|f\|_{L^{q_1}(B_{(s+t)/2}^{n-1})} \leq{}& c(n, \alpha, q, r, a)\|U\|_{L^a(B_1^+)}\|u\|_{L^q(B_1^+)}^r \\ &+ \frac{c(n, \alpha, p, r, a)}{(t-s)^{n+1-\alpha}}\|U\|_{L^a(B_1^+)}\|u\|_{L^p(B_1^+)}^r. \end{aligned} \tag{4.4}$$

Thanks to the inequalities (4.3) and (4.4), one can derive

$$\|u\|_{L^q(B_s^+)} \le \frac{1}{2}\|u\|_{L^q(B_t^+)} + \frac{c(n,\alpha,p,q,r,a,b)}{(t-s)^{n+1-\alpha}}\|u\|_{L^p(B_1^+)} + \|v\|_{L^q(B_{1/2}^+)}.$$

For sufficiently small $\epsilon(n,\alpha,p,q,r,a,b)$, one can employ the usual iteration procedure (see [28]) to obtain

$$\|u\|_{L^q(B_{1/4}^+)} \le c(n,\alpha,p,q,r,a,b)(\|u\|_{L^p(B_1^+)} + \|v\|_{L^q(B_{1/2}^+)}).$$

Step 2 We will show the above estimate still holds if we only assume $u \in L^p(B_1^+)$, $v \in L^q(B_{1/2}^+)$. According to the inequality (4.1), we can find that there exists a function $0 \le \eta(x) \le 1$ such that

$$u(x) \le \eta(x)\int_{B_1^{n-1}} P(x,\xi,\alpha)F(\xi)\left(\int_{B_1^+} P(y,\xi,\alpha)U(y)u(y)^r dy\right)^{1/r} d\xi + \eta(x)v(x).$$

We may define a map T by

$$T(\varphi) = \eta(x)\int_{B_1^{n-1}} P(x,\xi,\alpha)F(\xi)\left(\int_{B_1^+} P(y,\xi,\alpha)U(y)|\varphi(y)|^r dy\right)^{1/r} d\xi.$$

Choosing small enough $\varepsilon(n,\alpha,p,q,r,a,b)$, in view of the integral inequality (1.6), we have

$$\|T(\varphi)\|_{L^p(B_1^+)} \le c(n,\alpha,p,q,r,a,b)\|U\|_{L^a(B_1^+)}^{1/r}\|F\|_{L^b(B_1^{n-1})}\|\varphi\|_{L^p(B_1^+)} \le \frac{1}{2}\|\varphi\|_{L^p(B_1^+)},$$
$$\|T(\varphi)\|_{L^q(B_1^+)} \le c(n,\alpha,p,q,r,a,b)\|U\|_{L^a(B_1^+)}^{1/r}\|F\|_{L^b(B_1^{n-1})}\|\varphi\|_{L^q(B_1^+)} \le \frac{1}{2}\|\varphi\|_{L^q(B_1^+)}.$$

Furthermore, one can utilize the Minkowski inequality to obtain that for $\varphi, \psi \in L^p(B_1^+)$,

$$|T(\varphi)(x) - T(\psi)(x)| \le T(|\varphi-\psi|)(x) \quad \text{for } x \in B_1^+,$$

which implies

$$\|T(\varphi) - T(\psi)\|_{L^p(B_1^+)} \le \|T(|\varphi-\psi|)\|_{L^p(B_1^+)} \le \frac{1}{2}\|\varphi-\psi\|_{L^p(B_1^+)}.$$

Similarly, we also obtain

$$\|T(\varphi) - T(\psi)\|_{L^q(B_1^+)} \le \frac{1}{2}\|\varphi-\psi\|_{L^q(B_1^+)}$$

for any φ and $\psi \in L^q(B_1^+)$. Set $v_k(x) = \min\{v(x), k\}$, using the regular lifting theorem with contracting operators which can be seen in [13, 45], we may find a unique $u_k \in L^q(B_1^+)$ such that

$$\begin{aligned} u_k(x) &= T(u_k)(x)\eta(x)v_k(x) \\ &= \eta(x)\int_{B_1} P(x,\xi,\alpha)F(\xi)\left(\int_{B_1^+} P(y,\xi,\alpha)U(y)u(y)^r dy\right)^{1/r} d\xi + \eta(x)v_k(x). \end{aligned}$$

Applying a priori estimate to u_k, we obtain

$$\|u_k\|_{L^q(B_{1/4}^+)} \le c(n,p,r,a)(\|u_k\|_{L^p(B_1^+)} + \|v\|_{L^q(B_{1/2}^+)}). \tag{4.5}$$

Observing that

$$u(x) = T(u)(x) + \eta(x)v(x),$$

and using the contraction of operator T, we have

$$\begin{aligned}\|u_k - u\|_{L^p(B_1^+)} &\le \|T(u_k - T(u))\|_{L^p(B_1^+)} + \|v_k - v\|_{L^p(B_1^+)} \\ &\le \frac{1}{2}\|u_k - u\|_{L^p(B_1^+)} + \|v_k - v\|_{L^p(B_1^+)}.\end{aligned}$$

Hence $\|u_k - u\|_{L^p(B_1^+)} \le 2\|v_k - v\|_{L^p(B_1^+)} \to 0$ as $k \to \infty$. Taking a limit process in the inequality (4.5), we conclude that

$$\|u\|_{L^q(B_{1/4}^+)} \le c(n,\alpha,p,q,r,a,b)(\|u\|_{L^p(B_1^+)} + \|v\|_{L^q(B_{1/2}^+)})$$

for $u \in L^p(B_1^+)$, $v \in L^q(B_{1/2}^+)$. Then we accomplish the proof of Lemma 4.1. □

Lemma 4.2 *For $2 \le \alpha < n$, $n \ge 3$, $1 < a, b \le +\infty$, $1 \le r < +\infty$, $\frac{n-1}{n-\alpha+1} < p < q < +\infty$ satisfying*

$$\frac{\alpha-2}{n-1} < \frac{r}{q} + \frac{1}{a} < \frac{r}{p} + \frac{1}{a} \le 1, \quad \text{and} \quad \frac{n}{b} + \frac{n-1}{ar} + \frac{2-\alpha}{r} = \alpha - 1,$$

if we suppose that f and $g \in L^p(B_R^{n-1}), F \in L^a(B_R^{n-1}), U \in L^b(B_R^+)$ are all nonnegative functions with $g|_{B_{R/2}} \in L^q(B_{R/2}^{n-1})$,

$$\|F\|_{L^a(B_R^{n-1})}^{1/r}\|U\|_{L^b(B_R^+)} \le \epsilon(n,p,\alpha,q,r,a,b) \text{ small}$$

and

$$f(\xi) \le \int_{B_R^+} P(x,\xi,\alpha)U(x)\bigg(\int_{B_R^{n-1}} P(x,\xi,\alpha)F(\xi)f(\xi)^r d\xi\bigg)^{1/r} dx + g(\xi)$$

for $\xi \in B_R$, then we have $f|_{B_{R/4}^{n-1}} \in L^q(B_{R/4})$ and

$$\|f\|_{L^q(B_{R/4}^{n-1})} \le c(n,\alpha,p,q,r,a,b)(R^{\frac{n-1}{q}-\frac{n-1}{p}}\|f\|_{L^p(B_R^{n-1})} + \|g\|_{L^q(B_{R/2}^{n-1})}).$$

Proof Without loss of generality, we may assume that $R = 1$. As we did in Lemma 4.1, we first suppose that f and $g \in L^q(B_1^{n-1})$. For $x \in B_1^+$, define

$$u(x) = \int_{B_1^{n-1}} P(x,\xi,\alpha)F(\xi)f(\xi)^r d\xi.$$

By the integral inequality (1.6) again, we derive that

$$\begin{aligned}\|u\|_{L^{p_1}(B_1^+)} &\le C(n,p,r,a)\|F\|_{L^a(B_1^{n-1})}\|f\|_{L^p(B_1^{n-1})}^r, \\ \|u\|_{L^{q_1}(B_1^+)} &\le C(n,q,r,a)\|F\|_{L^a(B_1^{n-1})}\|f\|_{L^q(B_1^{n-1})}^r,\end{aligned}$$

where

$$\frac{n}{p_1} = \frac{n-1}{a} + \frac{(n-1)r}{p} + (2-\alpha), \quad \frac{n}{q_1} = \frac{n-1}{a} + \frac{(n-1)r}{q} + (2-\alpha).$$

For $0 < s < t \le \frac{1}{2}$ and $\xi \in B_s^{n-1}$, a similar argument as we did in the proof of Lemma 4.1 leads to

$$f(\xi) \le \int_{B_{(s+t)/2}^+} P(x,\xi,\alpha)U(x)u(x)^{1/r}dx + \frac{c(n,\alpha,p,q,r,a,b)}{(t-s)^{n+1-\alpha}}\|f\|_{L^p(B_1^{n-1})} + g(\xi).$$

Then it follows

$$\|f\|_{L^q(B_s^{n-1})} \le c(n,\alpha,q,r,b)\|U\|_{L^b(B_1^+)}\|u\|_{L^{q_1}(B_{(s+t)/2}^+)}^{1/r}$$

$$+\frac{c(n,p,q,r,a)}{(t-s)^{n+1-\alpha}}\|f\|_{L^p(B_1^{n-1})}+\|g\|_{L^q(B_{1/2}^{n-1})}. \tag{4.6}$$

On the other hand, for $x \in B^+_{(s+t)/2}$, similarly, we also obtain

$$u(x) \le \int_{B_t} P(x,\xi)F(\xi)f(\xi)^r d\xi + \frac{c(n,\alpha,p,r,a)}{(t-s)^{n+1-\alpha}}\|F\|_{L^a(B_1^{n-1})}\|f\|^r_{L^p(B_1^{n-1})}$$

and

$$\|u\|_{L^{q_1}(B^+_{(s+t)/2})} \le c(n,\alpha,q,r,a)\|F\|_{L^a(B_1^{n-1})}\|f\|^r_{L^q(B_1^{n-1})} + \frac{c(n,\alpha,p,r,a)}{(t-s)^{n+1-\alpha}}\|F\|_{L^a(B_1^{n-1})}\|f\|^r_{L^p(B_1^{n-1})}. \tag{4.7}$$

Gathering the inequalities (4.6) and (4.7), we see

$$\|f\|_{L^q(B_s^{n-1})} \le \frac{1}{2}\|f\|_{L^q(B_t^{n-1})} + \frac{c(n,\alpha,p,q,r,a,b)}{(t-s)^{n+1-\alpha}}\|f\|_{L^p(B_1^{n-1})} + \|g\|_{L^q(B_{1/2}^{n-1})},$$

which implies

$$\|f\|_{L^q(B_{1/4}^{n-1})} \le c(n,\alpha,p,q,r,a,b)(\|f\|_{L^p(B_1^{n-1})} + \|g\|_{L^q(B_{1/2}^{n-1})}).$$

if we choose sufficiently small $\varepsilon(n,\alpha,p,q,r,a,b)$. With this a priori estimate at hands, we may proceed in the same way as the proof of Lemma 4.1 to obtain the above estimate still holds if we suppose that $f \in L^p(B_1^{n-1})$, $v \in L^q(B_{1/2}^{n-1})$. □

Proof of Theorem 1.5 Define that

$$v_R(x) = \int_{\partial\mathbb{R}^n_+ \setminus B_R^{n-1}} P(x,\xi,\alpha)u(\xi)^{p_0}d\xi, \quad u_R(\xi) = \int_{\mathbb{R}^n_+\setminus B_R^+} P(x,\xi,\alpha)v(x)^{q_0}dx.$$

We first verify that $v \in L^{q_0+1}_{\rm loc}(\overline{\mathbb{R}^n_+})$ and $v_R \in L^{q_0+1}(B_R^+) \cap L^\infty_{\rm loc}(B_R^+ \cup B_R^{n-1})$. In fact, from $u \in L^{p_0+1}_{\rm loc}(\mathbb{R}^{n-1})$, one can see that $u < \infty$ a.e. on $\mathbb{R}^{n-1}$. The integral system (1.10) implies $v(x) < \infty$ a.e. on $\mathbb{R}^n_+$, then there exists $x_0 \in B_R^+$ such that $v(x_0) < \infty$. Then, it follows that

$$\int_{\partial\mathbb{R}^n_+\setminus B_R^{n-1}} \frac{u(\xi)^{p_0}}{|x_0-\xi|^{n+2-\alpha}}d\xi < \infty \quad \text{and} \quad \int_{\partial\mathbb{R}^n_+\setminus B_R^{n-1}} \frac{u(\xi)^{p_0}}{|\xi|^{n+2-\alpha}}d\xi < \infty.$$

For $0<\theta<1$, $x \in B^+_{\theta R}$, there holds

$$v_R(x) \le \frac{c(n,\alpha)R}{(1-\theta)^{n+2-\alpha}} \int_{\partial\mathbb{R}^n_+\setminus B_R^{n-1}} \frac{u(\xi)^{p_0}}{|\xi|^{n+2-\alpha}}d\xi,$$

which implies that $v_R \in L^\infty_{\rm loc}(B_R^+ \cup B_R^{n-1})$. Thanks to the integral inequality (1.6), we derive that

$$\int_{B_R^{n-1}} P(x,\xi,\alpha)u(\xi)^{p_0}d\xi \in L^{q_0+1}(\mathbb{R}^n_+).$$

Hence

$$v \in L^{q_0+1}_{\rm loc}(B_R^+ \cup B_R^{n-1}).$$

For sufficiently large R, we have

$$v \in L^{q_0+1}_{\rm loc}(\overline{\mathbb{R}^n_+}) \quad \text{and} \quad v_R \in L^{q_0+1}(B_R^+).$$

We now turn to verify that $u_R \in L^{p_0+1}(B_R^{n-1}) \cap L^\infty_{\mathrm{loc}}(B_R^{n-1})$. Since $u(x) \in L^{p_0+1}_{\mathrm{loc}}(\partial\mathbb{R}^n_+)$, one can find $\xi_0 \in B_R^{n-1}$ such that

$$\int_{\mathbb{R}^n_+} P(x,\xi_0,\alpha)v(x)^{q_0}dx < \infty,$$

which implies

$$\int_{\mathbb{R}^n_+\setminus B_R^+} \frac{x_n}{|x|^{n+2-\alpha}}v(x)^{q_0}dx < \infty.$$

When $0<\theta<1$, $\xi \in B_{\theta R}^{n-1}$, applying similar estimate as v_R, one can calculate that

$$u_R(\xi) \le \frac{c(n,\alpha)}{(1-\theta)^{n+2-\alpha}}\int_{\mathbb{R}^n_+\setminus B_R^+} \frac{x_n}{|x|^{n+2-\alpha}}v(x)^{q_0}dx,$$

which leads to $u_R \in L^\infty_{\mathrm{loc}}(B_R^{n-1})$. Now we are going to establish the regularity of u. Our proof can be divided into two cases.

Case 1 For $\frac{n+2}{2} \le \alpha < n$ or $0<\alpha<\frac{n+2}{2}$, $0<p_0<\frac{n+2\alpha-4}{n+2-2\alpha}$. In this case, we have $q_0>1$. Pick a number r such that

$$1 \le r < q_0 \quad \text{and} \quad r > \frac{1}{p_0},$$

then it follows that

$$u(\xi)^{\frac{1}{r}} \le \left(\int_{B_R^+} P(x,\xi,\alpha)v(x)^{q_0}dx\right)^{\frac{1}{r}} + u_R(\xi)^{\frac{1}{r}}.$$

Hence

$$\begin{aligned} v(x) &= \int_{B_R^{n-1}} P(x,\xi,\alpha)u(\xi)^{p_0-\frac{1}{r}}u(\xi)^{\frac{1}{r}}d\xi + v_R(x) \\ &\le \int_{B_R^{n-1}} P(x,\xi,\alpha)u(\xi)^{p_0-\frac{1}{r}}\left(\int_{B_R^+} P(y,\xi,\alpha)v(y)^{q_0-r}v(y)^r dy\right)^{\frac{1}{r}}d\xi + \tilde{v}_R(x), \end{aligned}$$

where

$$\tilde{v}_R(x) = \int_{B_R^{n-1}} P(x,\xi,\alpha)u(\xi)^{p_0-\frac{1}{r}}u_R(\xi)^{\frac{1}{r}}d\xi + v_R(x).$$

Since $u(\xi) \in L^{p_0+1}(B_R^{n-1})$ and $u_R(\xi) \le u(\xi)$, then it follows from the inequality (1.6) that $\tilde{v}_R \in L^{q_0+1}(B_R^+)$. On the other hand, for $0<\theta<1$, $x \in B_{\theta R}^+$, we have

$$\begin{aligned} &\int_{B_R^{n-1}} P(x,\xi,\alpha)u(\xi)^{p_0-\frac{1}{r}}u_R(\xi)^{\frac{1}{r}}d\xi \\ &\le \|u_R\|^{\frac{1}{r}}_{L^\infty(B^{n-1}_{(1+\theta)R/2})}\int_{B^{n-1}_{(1+\theta)R/2}} P(x,\xi,\alpha)u(\xi)^{p_0-\frac{1}{r}}d\xi \\ &\quad + \frac{c(n,\alpha)}{(1-\theta)^{n+2-\alpha}R^{n+1-\alpha}}\int_{B_R^{n-1}\setminus B^{n-1}_{(1+\theta)R/2}} u(\xi)^{p_0-\frac{1}{r}}u_R(\xi)^{\frac{1}{r}}d\xi \\ &\le \|u_R\|^{\frac{1}{r}}_{L^\infty(B^{n-1}_{(1+\theta)R/2})}\int_{B^{n-1}_{(1+\theta)R/2}} P(x,\xi,\alpha)u(\xi)^{p_0-\frac{1}{r}}d\xi \\ &\quad + \frac{c(n,\alpha,p_0)}{(1-\theta)^{n+2-\alpha}R^{\frac{(n+1-\alpha)p_0}{p_0+1}}}\|u\|^{p_0}_{L^{p_0+1}(B_R^{n-1})}. \end{aligned}$$

If $(n+1-\alpha)p_0r+(2-\alpha)r-(n-1)\le 0$, then $\tilde v_R\in L^q_{\mathrm{loc}}(B_R^+)$ for any $q_0+1<q<\infty$. Set

$$a=\frac{q_0+1}{q_0-r},\quad b=\frac{(n-1)(q_0+1)r}{(\alpha-1)(q_0+1)+(\alpha-2)(q_0+1)r-n(q_0-r)}.$$

Simple computations lead to

$$\frac{n}{ra}+\frac{n-1}{b}+(2-\alpha)=\frac{\alpha-1}{r}\quad\text{and}\quad\frac{r}{q_0+1}+\frac{1}{a}=\frac{(\alpha-1)p_0+(n+\alpha-2)}{n(p_0+1)}<1.$$

For $q_0+1<q<\infty$, it follows from Lemma 4.1 that $v|_{B^+_{R/4}}\in L^q(B^+_{R/4})$. Hence for any $\xi\in B_{R/8}$,

$$\begin{aligned}u(\xi)&=\int_{B^+_{R/4}}P(x,\xi,\alpha)v^{q_0}(x)dx+u_{R/4}(\xi)\\&\le C(n,q)\|v\|^{q_0}_{L^q(B^+_{R/4})}+u_{R/4}(\xi)\lesssim 1.\end{aligned}$$

Since every point may be viewed as a center, we deduce that $u\in L^\infty_{\mathrm{loc}}(\partial\mathbb{R}^n_+)$ and $v\in L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$. For any $R>0$, $x\in B_R$, $\xi\in B_R^+$, one can apply

$$\int_{\partial\mathbb{R}^n_+\setminus B_R^{n-1}}\frac{u(\xi)^{p_0}}{|\xi|^{n+2-\alpha}}d\xi<\infty\quad\text{and}\quad\int_{\mathbb{R}^n_+\setminus B_R^+}\frac{x_n}{|x|^{n+2-\alpha}}v(x)^{q_0}dx<\infty$$

to obtain $v_R\in C^\infty(B_R^+\cup B_R^{n-1})$ and $u_R\in C^\infty(B_R)$, which yields that $u\in C^\gamma_{\mathrm{loc}}(\mathbb{R}^{n-1})$ for $0<\gamma<1$. By the standard potential theory (see [25, Chap. 4]) and bootstrap method, we see that $(u,v)\in C^\infty(\partial\mathbb{R}^n_+)\times C^\infty(\mathbb{R}^n_+)$.

If $(n+1-\alpha)p_0r+(2-\alpha)r-(n-1)>0$, then with the help of the integral inequality (1.6), we derive that $\tilde v_R\in L_{\mathrm{loc}}^{\frac{n(p_0+1)r}{(n+1-\alpha)p_0r+(2-\alpha)r-(n-1)}}(B_R^+\cup B_R^{n-1})$. Denote

$$a=\frac{q_0+1}{q_0-r},\quad b=\frac{(n-1)(q_0+1)r}{(\alpha-1)(q_0+1)+(\alpha-2)(q_0+1)r-n(q_0-r)}.$$

Then

$$\frac{n}{ra}+\frac{n-1}{b}+(2-\alpha)=\frac{\alpha-1}{r}\quad\text{and}\quad\frac{r}{q_0+1}+\frac{1}{a}=\frac{(\alpha-1)p_0+(n+\alpha-2)}{n(p_0+1)}<1.$$

For

$$q_0+1<q<\frac{n(p_0+1)r}{(n+1-\alpha)p_0r+(2-\alpha)r-(n-1)},$$

we have $\frac{r}{q}+\frac{1}{a}>\frac{\alpha-1}{n}$. This together with Lemma 4.1 yields that $v|_{B^+_{R/4}}\in L^q(B^+_{R/4})$. Since $\frac{n(p_0+1)r}{(n+1-\alpha)p_0r+(2-\alpha)r-(n-1)}>\frac{n}{\alpha-1}q_0$, we may choose that $q>\frac{n}{\alpha-1}q_0$ such that

$$\begin{aligned}u(\xi)&=\int_{B^+_{R/4}}P(x,\xi,\alpha)v^{q_0}(x)dx+u_{R/4}(\xi)\\&\le C(n,q,\alpha)\|v\|^{q_0}_{L^q(B^+_{R/4})}+u_{R/4}(\xi).\end{aligned}$$

In the same way as we did in previous argument, we conclude that $(u,v)\in C^\infty(\partial\mathbb{R}^n_+)\times C^\infty(\mathbb{R}^n_+)$.

Case 2 $2\le\alpha<\frac{n+2}{2}$, $p_0\ge\frac{n+2\alpha-4}{n+2-2\alpha}$. Choosing a number r satisfying

$$1\le r\le p_0\quad\text{and}\quad r\ge\frac{1}{q_0},$$

then one can get

$$v(x)^{\frac{1}{r}}\le\left(\int_{B_R^{n-1}}P(x,\xi,\alpha)u(\xi)^{p_0}d\xi\right)^{\frac{1}{r}}+v_R(x)^{\frac{1}{r}}.$$

Hence

$$u(\xi) \le \int_{B_R^+} P(x,\xi,\alpha)v(x)^{q_0-r^{-1}}\Big(\int_{B_R} P(x,\xi)u(\xi)^{p_0-r}u(\xi)^r d\xi\Big)^{\frac{1}{r}}dx + g_R(\xi),$$

where

$$g_R(\xi) = \int_{B_R^+} P(x,\xi,\alpha)v(x)^{q_0-r^{-1}}v_R(x)^{\frac{1}{r}}dx + u_R(\xi).$$

In view of $v \in L^{q_0+1}(B_R^+)$, $v_R \le v$ and the inequality (1.6), we derive that $g_R \in L^{p_0+1}(B_R)$. On the other hand, for $0<\theta<1$ and $\xi \in B_{\theta R}$, we also have

$$\begin{aligned}
&\int_{B_R^+} P(x,\xi,\alpha)v(x)^{q_0-r^{-1}}v_R(x)^{\frac{1}{r}}dx\\
&\le \|v_R\|^{\frac{1}{r}}_{L^\infty(B^+_{(1+\theta)R/2})}\int_{B^+_{(1+\theta)R/2}} P(x,\xi,\alpha)v(x)^{q_0-r^{-1}}dx\\
&\quad + \frac{c(n)}{(1-\theta)^{n+2-\alpha}R^{n+1-\alpha}}\int_{B_R^+\setminus B^+_{(1+\theta)R/2}} v(x)^{q_0-r^{-1}}v_R(x)^{\frac{1}{r}}dx\\
&\le \|v_R\|^{\frac{1}{r}}_{L^\infty(B^+_{\frac{1+\theta}{2}R})}\int_{B_{(1+\theta)R/2}} P(x,\xi,\alpha)v(x)^{q_0-r^{-1}}dx\\
&\quad + \frac{c(n,p_0)}{(1-\theta)^{n+2-\alpha}R^{\frac{(n+1-\alpha)q_0}{q_0+1}}}\|v\|^{q_0}_{L^{q_0+1}(B_R^+)}.
\end{aligned}$$

If $n(q_0-r^{-1})-(\alpha-1)(q_0+1)<0$, then $g_R \in L^q_{\mathrm{loc}}(B_R^{n-1})$ for any $p_0+1<q<\infty$. Set

$$a = \frac{p_0+1}{p_0-r}, \quad b = \frac{n(p_0+1)r}{(p_0+1)r+(\alpha-2)(p_0+1)-(n-1)(p_0-r)}.$$

Direct computations show that $\frac{n-1}{ra}+\frac{n}{b}+\frac{2-\alpha}{r}=\alpha-1$, $\frac{r}{p_0+1}+\frac{1}{a}=\frac{p_0}{p_0+1}\in(\frac{\alpha-2}{n-1},1)$. Hence, one can apply Lemma 4.2 to obtain that $u \in L^q(B^{n-1}_{R/4})$. Then for any $x \in B^+_{R/8}$,

$$\begin{aligned}
v(x) &= \int_{B^{n-1}_{R/4}} P(x,\xi,\alpha)u^{p_0}(\xi)d\xi + v_{R/4}(\xi)\\
&\le C(n,q,\alpha)\|u\|^{p_0}_{L^q(B^{n-1}_{R/4})} + v_{R/4}(\xi) \lesssim 1.
\end{aligned}$$

If $n(q_0-r^{-1})-(\alpha-1)(q_0+1)>0$, then $g_R \in L_{\mathrm{loc}}^{\frac{(n-1)(q_0+1)}{n(q_0-r^{-1})-(\alpha-1)(q_0+1)}}(B_R^{n-1})$. Arguing this as we did in Case 1, we can also $v(x)\in L^\infty(B^+_{R/8})$. Hence by the standard bootstrap method, we conclude that $u \in C^\infty(\partial\mathbb{R}^n_+)$ and $v(x)\in C^\infty(\mathbb{R}^n_+)$.

5 The Proof of Theorem 1.7

In this section, we will give some necessary and sufficient conditions for the existence of positive solutions to the integral system (1.10). For $2<\alpha<n$, we suppose that $(u,v)\in L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)$ is a pair of positive solutions of the following integral system associated with the fractional Poisson kernel

$$\begin{cases}
u(\xi) = \displaystyle\int_{\mathbb{R}^n_+} P(x,\xi,\alpha)v^{q_0}(x)dx, & \xi \in \partial\mathbb{R}^n_+,\\
v(x) = \displaystyle\int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)u^{p_0}(\xi)d\xi, & x \in \mathbb{R}^n_+.
\end{cases}$$

Applying integration by parts formula, we obtain

$$\begin{aligned}
&\int_{B_R^{n-1}} u^{p_0}(\xi)(\xi\cdot\nabla u(\xi))d\xi \\
&\quad= \frac{1}{1+p_0}\int_{B_R^{n-1}} \xi\cdot\nabla(u^{1+p_0}(\xi))d\xi \\
&\quad= \frac{R}{1+p_0}\int_{\partial B_R^{n-1}} u^{p_0+1}(\xi)d\sigma - \frac{n-1}{1+p_0}\int_{B_R^{n-1}} u^{p_0+1}(\xi)d\xi
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_R^+} v^{q_0}(x)(x\cdot\nabla v(x))dx \\
&\quad= \frac{R}{q_0+1}\int_{\partial B_R^+} v^{q_0+1}(x)d\sigma - \frac{n}{q_0+1}\int_{B_R^+} v^{q_0+1}(x)dx.
\end{aligned}$$

Then it follows from $(u,v)\in L^{p_0+1}(\partial\mathbb{R}^n_+)\times L^{q_0+1}(\mathbb{R}^n_+)$ that there exists $R_j\to+\infty$ such that

$$R\int_{\partial B_{R_j}^{n-1}} u^{p_0+1}(\xi)d\sigma\to 0,\quad R\int_{\partial B_{R_j}^+} v^{q_0+1}(x)d\sigma\to 0.$$

Combining the above estimate, we derive that

$$\begin{aligned}
&\int_{\partial\mathbb{R}^n_+} u^{p_0}(\xi)(\xi\cdot\nabla u(\xi))d\xi + \int_{\mathbb{R}^n_+} v^{q_0+1}(x)(x\cdot\nabla v(x))dx \\
&\quad= -\frac{n-1}{p_0+1}\int_{\partial\mathbb{R}^n_+} u^{p_0+1}(\xi)d\xi - \frac{n}{q_0+1}\int_{\mathbb{R}^n_+} v^{q_0+1}(x)dx.
\end{aligned}\tag{5.1}$$

Thanks to the integral system (1.5), one can write

$$\begin{aligned}
\nabla u(\xi)\cdot\xi &= \frac{du(\rho x)}{dp}|_{\rho=0} \\
&= -(n+2-\alpha)\int_{\mathbb{R}^n_+} P(x,\xi,\alpha)|x-\xi|^{-2}(\xi-x)\cdot\xi v^{q_0}(x)dx
\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}
\nabla v(x)\cdot x &= \frac{dv(\rho x)}{d\rho}\bigg|_{\rho=0} \\
&= -(n+2-\alpha)\int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)|x-\xi|^{-2}(x-\xi)\cdot x u^{p_0}(x)d\xi \\
&\quad + \int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)u^{p_0}(\xi)d\xi.
\end{aligned}\tag{5.3}$$

Hence, it follows that

$$\begin{aligned}
&\int_{\partial\mathbb{R}^n_+} u^{p_0}(\xi)(\xi\cdot\nabla u(\xi))d\xi + \int_{\mathbb{R}^n_+} v^{q_0}(x)(x\cdot\nabla v(x))dx \\
&\quad= -(n+1-\alpha)\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} P(x,\xi,\alpha)u^{p_0}(\xi)v^{q_0}(x)d\xi dx \\
&\quad= -(n+1-\alpha)\int_{\partial\mathbb{R}^n_+} u^{p_0+1}(\xi)d\xi
\end{aligned}$$

$$= -(n+1-\alpha)\int_{\mathbb{R}^n_+} v^{q_0+1}(x)dx.$$

This together with (5.1) implies that $\frac{n-1}{p_0+1} + \frac{n}{q_0+1} = n+1-\alpha$.

6 The Proof of Theorem 1.10

Throughout this section, we will utilize the method of moving plane in integral forms developed by Chen, Li and Ou [15] to establish the radial symmetry for each pair of solutions (u,v) of the integral system (1.10). In order to state our result, we first introduce some notations. For $\lambda \in \mathbb{R}$, $\xi = (\xi_1, \xi'') \in \partial\mathbb{R}^n_+$ and $x = (x_1, x'') \in \mathbb{R}^n_+$, set

$$H_\lambda = \{\xi \in \partial\mathbb{R}^n_+ : \xi_1 < \lambda\}, \quad Q_\lambda = \{x \in \mathbb{R}^n_+ : x_1 < \lambda\} \quad \xi_\lambda = (2\lambda - \xi_1, \xi'') \quad x_\lambda = (2\lambda - x_1, x'').$$

We also write $u_\lambda(\xi) = u(\xi^\lambda)$ and $v_\lambda(x) = v(x)$. Let $(u,v) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ be a pair of positive solutions of the integral system (1.10)

Through Theorem 1.5, we can see that $(u,v) \in C^\infty(\partial\mathbb{R}^n_+) \times C^\infty(\mathbb{R}^n_+)$.

Lemma 6.1 *If (u,v) is a pair of nonnegative solutions of the integral system* (1.10)*, for any $\xi \in \partial\mathbb{R}^n_+$ and $x \in \mathbb{R}^n_+$, we have*

$$u_\lambda(\xi) - u(\xi) = \int_{Q_\lambda} (P(x,\xi,\alpha) - P(x,\xi^\lambda,\alpha))(v_\lambda^{q_0}(x) - v^{q_0}(x))dx$$

and

$$v_\lambda(x) - v(x) = \int_{H_\lambda} (P(x,\xi,\alpha) - P(x,\xi^\lambda,\alpha))(u^{p_0}(\xi) - u_\lambda^{p_0}(\xi))d\xi.$$

Proof Thanks to the integral system (1.10) and the change of variable, we have

$$u(\xi) = \int_{Q_\lambda} P(x,\xi,\alpha)v^{q_0}(x)dx + \int_{Q_\lambda} P(x^\lambda,\xi,\alpha)v_\lambda^{q_0}(x)dx$$

and

$$u_\lambda(\xi) = \int_{Q_\lambda} P(x,\xi^\lambda,\alpha)v^{q_0}(x)dx + \int_{Q_\lambda} P(x^\lambda,\xi^\lambda,\alpha)v_\lambda^{q_0}(x)dx.$$

Since $P(x,\xi^\lambda,\alpha) = P(x^\lambda,\xi,\alpha)$ and $P(x^\lambda,\xi^\lambda,\alpha) = P(x,\xi,\alpha)$, one can write

$$\begin{aligned} u_\lambda(\xi) - u(\xi) &= \int_{Q_\lambda} P(x,\xi,\alpha)(v_\lambda^{q_0}(x) - v^{q_0}(x)) + \int_{Q_\lambda} P(x,\xi^\lambda,\alpha)(v^{q_0}(x) - v_\lambda^{q_0}(x)) \\ &= \int_{Q_\lambda} (P(x,\xi,\alpha) - P(x,\xi^\lambda,\alpha))(v_\lambda^{q_0}(x) - v^{q_0}(x))dx. \end{aligned}$$

Similarly, we can also obtain

$$v_\lambda(x) - v(x) = \int_{H_\lambda} (P(x,\xi,\alpha) - P(x,\xi^\lambda,\alpha))(u_\lambda^{p_0}(\xi) - u^{p_0}(\xi))d\xi.$$

Then we accomplish the proof of Lemma 6.1. □

Now we continue with the proof of Theorem 1.10, the proof will be separated from two steps.

Step 1 We are going to show that

$$u_\lambda(\xi) - u(\xi) \geq 0, \quad v_\lambda(x) - v(x) \geq 0, \quad \forall\, \xi \in H_\lambda, \quad x \in Q_\lambda \tag{6.1}$$

for λ sufficiently negative. Define

$$H_\lambda^u = \{\xi \in H_\lambda \mid u_\lambda(x) - u(x) < 0\}, \quad Q_\lambda^v = \{x \in Q_\lambda \mid v_\lambda(x) - v(x) < 0\}.$$

It suffices to show that for sufficiently negative λ, both H_λ^u and Q_λ^v must be empty set.

One can utilize the mean value theorem and Lemma 6.1 to obtain for any $\xi \in H_\lambda^u$ and $x \in Q_\lambda^v$, there holds

$$\begin{aligned} u(\xi) - u_\lambda(\xi) &= \int_{Q_\lambda} (P(x,\xi,\alpha) - P(x,\xi^\lambda,\alpha))(v^{q_0}(x) - v_\lambda^{q_0}(x))dx \\ &\le \int_{Q_\lambda^v} P(x,\xi,\alpha)(v^{q_0}(x) - v_\lambda^{q_0}(x))dx \\ &\le q_0 \int_{Q_\lambda^v} P(x,\xi,\alpha)v^{q_0-1}(x)(v(x) - v_\lambda(x))dx \end{aligned}$$

and

$$\begin{aligned} v(x) - v_\lambda(x) &= \int_{H_\lambda} (P(x,\xi,\alpha) - P(x,\xi^\lambda,\alpha))(u^{p_0}(\xi) - u_\lambda^{q_0}(\xi))d\xi \\ &\le \int_{H_\lambda^u} P(x,\xi,\alpha)(u^{p_0}(\xi) - v_\lambda^{p_0}(\xi))d\xi \\ &\le p_0 \int_{H_\lambda^v} P(x,\xi,\alpha)u^{p_0-1}(\xi)(u(\xi) - u_\lambda(\xi))d\xi. \end{aligned}$$

By the integrable condition $(u,v) \in L^{p_0+1}(\partial\mathbb{R}_+^n) \times L^{q_0+1}(\mathbb{R}_+^n)$ and the inequalities (1.5), it is easy to see that

$$\|u_\lambda - u\|_{L^{p_0+1}(H_\lambda^u)} \le q_0 \|v\|_{L^{q_0+1}(Q_\lambda)}^{q_0-1} \|v_\lambda - v\|_{L^{q_0+1}(Q_\lambda^v)} \tag{6.2}$$

and

$$\|v_\lambda - v\|_{L^{q_0+1}(Q_\lambda^v)} \le p_0 \|u\|_{L^{p_0+1}(H_\lambda)}^{p_0-1} \|u_\lambda - u\|_{L^{p_0+1}(H_\lambda^u)}, \tag{6.3}$$

which implies

$$\|u_\lambda - u\|_{L^{p_0+1}(H_\lambda^u)} \le p_0 q_0 \|v\|_{L^{q_0+1}(Q_\lambda)}^{q_0-1} \|u\|_{L^{p_0+1}(H_\lambda)}^{p_0-1} \|u_\lambda - u\|_{L^{p_0+1}(H_\lambda^u)} \tag{6.4}$$

and

$$\|v_\lambda - v\|_{L^{q_0+1}(Q_\lambda^v)} \le p_0 q_0 \|v\|_{L^{q_0+1}(Q_\lambda)}^{q_0-1} \|u\|_{L^{p_0+1}(H_\lambda)}^{p_0-1} \|v_\lambda - v\|_{L^{q_0+1}(Q_\lambda^u)}. \tag{6.5}$$

According to the conditions $(u,v) \in L^{p_0+1}(\partial\mathbb{R}_+^n) \times L^{q_0+1}(\mathbb{R}_+^n)$, it is possible for us to choose sufficiently negative λ such that

$$\|u_\lambda - u\|_{L^{p_0+1}(H_\lambda^u)} \le \frac{1}{2} \|u_\lambda - u\|_{L^{p_0+1}(H_\lambda^u)}, \quad \|v_\lambda - v\|_{L^{q_0+1}(Q_\lambda^v)} \le \frac{1}{2} \|v_\lambda - v\|_{L^{q_0+1}(Q_\lambda^v)}, \tag{6.6}$$

which implies that H_λ^u and Q_λ^v must be empty sets.

Step 2 The inequality (6.1) provides a starting point to move the plane $T_\lambda = \{x \in \mathbb{R}^n : x_1 = \lambda\}$. Now we start from the negative infinity of x_1-axis and move the plane to the right as long as (6.1) holds. Set

$$\lambda_0 = \sup\{\lambda \mid u_\mu(\xi) \ge u(\xi),\, v_\mu(x) \ge v(x),\, \mu \le \lambda,\, \forall\, \xi \in H_\mu,\ x \in Q_\mu\}.$$

Suppose that $\lambda_0 < 0$, we will show that u and v must be symmetric about the plane T_{λ_0}, that is

$$u_{\lambda_0}(\xi) \equiv u(\xi), \quad v_{\lambda_0}(x) \equiv v(x), \quad \forall \xi \in H_{\lambda_0},\ x \in Q_{\lambda_0}. \tag{6.7}$$

We argue this by contradiction. We suppose

$$u_{\lambda_0}(\xi) \geq u(\xi), \quad v_{\lambda_0}(x) \geq v(x), \quad \text{but } u_{\lambda_0}(\xi) \not\equiv u(\xi) \quad \text{or } v_{\lambda_0}(x) \not\equiv v(x) \quad \forall \xi \in H_{\lambda_0},\ x \in Q_{\lambda_0}.$$

Since $u_{\lambda_0}(\xi) \not\equiv u(\xi)$ on H_{λ_0}, through Lemma 6.1, we have $u_{\lambda_0}(\xi) > u(\xi)$ and $v_{\lambda_0}(x) > v(x)$ in the interior of H_{λ_0} and Q_{λ_0} respectively.

Next, we are going to illustrate that the plane can be moved further to the right. Equivalently, there exists an $\varepsilon > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$,

$$u_\lambda(\xi) \geq u(\xi), \quad v_\lambda(x) \geq v(x), \quad \forall \xi \in H_\lambda,\ x \in Q_\lambda. \tag{6.8}$$

Let

$$\overline{H^u_{\lambda_0}} = \{x \in H_{\lambda_0} | u(\xi) \geq u_{\lambda_0}(\xi)\}, \quad \overline{Q^v_{\lambda_0}} = \{x \in Q_{\lambda_0} | v(x) \geq v_{\lambda_0}(x)\}.$$

It is not hard to obtain that $\overline{H^u_{\lambda_0}}$ and $\overline{Q^v_{\lambda_0}}$ are empty sets and $\lim_{\lambda\to\lambda_0} H^u_\lambda \subset \overline{H^u_{\lambda_0}}$, $\lim_{\lambda\to\lambda_0} Q^v_\lambda \subset \overline{Q^v_{\lambda_0}}$. Then it follows from $(u, v) \in L^{p_0+1}(\partial\mathbb{R}^n_+) \times L^{q_0+1}(\mathbb{R}^n_+)$ that one can pick sufficiently small ε such that

$$p_0 q_0 \|u\|^{p_0-1}_{L^{p_0+1}(H^u_\lambda)} \|v\|^{q_0-1}_{L^{q_0+1}(Q^v_\lambda)} \leq \frac{1}{2}$$

for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. Similar estimates as (6.6) yields that

$$\|u - u_\lambda\|_{L^{p_0+1}(H^u_\lambda)} = \|v - v_\lambda\|_{L^{q_0+1}(Q^v_\lambda)} = 0.$$

Therefore H^u_λ and Q^v_λ must be measure zero, which implies (6.7).

If $\lambda_0 = 0$, then we can carry out previous procedure in the opposite direction, namely we move the plan in the negative x_1 direction from positive infinity toward the origin. If our planes T_λ stop somewhere before the origin, we derive the symmetry and monotonicity in x_1 direction by the above argument. If they stop at the origin again, we also obtain the symmetry and monotonicity in the $x_1 = 0$ by combing two inequalities obtained in the two opposite directions. Replacing the x_1 direction with x_i direction for $i = 1, 2, \dots, n-1$, we derive that $u(\xi)$ and $v(x)|_{\partial\mathbb{R}^n_+}$ must be radially symmetric and monotone decreasing about some point $\xi_0 \in \partial\mathbb{R}^n_+$.

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