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Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness

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ABSTRACT

The main purpose of this paper is three-fold. First of all, we are concerned with the limited smoothness conditions in the spirit of Hörmander on the multi-linear and multi-parameter Coifman–Meyer type Fourier multipliers studied by C. Muscalu, J. Pipher, T. Tao, C. Thiele (2004, 2006) where they established the L^r estimates for the multiplier operators under the assumption that the multiplier has smoothness of sufficiently large order. Under our limited smoothness assumption, we will prove the $L^{p_1} \times \cdots \times L^{p_n} \to L^r$ boundedness with $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}$ for $1 < p_1, \ldots, p_n < \infty$ and $0 < r < \infty$. Second, our proof of L^r estimates also offers a different and more direct approach than the one given in Muscalu et al. (2004, 2006) where they use the deep analysis of multi-linear and multi-parameter paraproducts. Third, we also prove a Hörmander type multiplier theorem in the weighted Lebesgue spaces for such operators when the Fourier multiplier is only assumed with limited smoothness.

1. Introduction

The aim of this paper is to consider the limited smoothness condition on the Fourier multipliers in the multi-parameter and multi-linear setting. This is an analogue of the well-known Hörmander–Mihlin type theorem in the linear and multi-linear cases.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^d)$ denote tempered distributions. The Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$ of $f \in \mathcal{S}(\mathbb{R}^d)$ are defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx \quad \text{and} \quad \mathcal{F}^{-1}f(\xi) = \check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi. \tag{1.1}$$

In the linear case, we first recall the following Mihlin theorem (see, e.g., [1, Corollary 8.11]):

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Theorem 1.1. *If a multiplier $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ satisfies the following condition*

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \quad \text{for all } |\alpha| \le \left[\frac{n}{2}\right] + 1 \tag{1.2}$$

then the Fourier multiplier operator $m(D)f = \mathcal{F}^{-1}[m\hat{f}]$ defined with the symbol $m(\xi)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

On the other hand, Hörmander reformulated and improved Mihlin's theorem using the Sobolev regularity of the multiplier [2]. To describe Hörmander's theorem, we let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function satisfying

$$\operatorname{supp} \Psi \subset \left\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\right\}, \qquad \sum_{j\in\mathbb{Z}} \Psi\left(\frac{\xi}{2^j}\right) = 1, \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1.3}$$

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^s} \triangleq \|(I - \triangle)^{s/2} f\|_{L^2} < \infty, \tag{1.4}$$

where $(I - \triangle)^{s/2} f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \hat{f}(\xi)]$. Then the Hörmander multiplier theorem says

Theorem 1.2. *If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} \|m(2^j \cdot)\Psi\|_{H^s(\mathbb{R}^n)} < \infty, \quad \text{for all } s > \frac{n}{2},$$

where Ψ is the same as in (1.3) when $d = n$ and $H^s(\mathbb{R}^n)$ is the Sobolev space, then the Fourier multiplier operator $m(D)$ defined with the symbol m is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Clearly, Hörmander's theorem is stronger than Mihlin's and the number $\frac{n}{2}$ cannot be improved in Hörmander's theorem.

We now turn to the weighted estimates for Fourier multipliers. We first introduce the notion of Muckenhoupt's A_p weights [3]. Let $1 < p < \infty$ and denote $p' = \frac{p}{p-1}$. We say that a weight $w \ge 0$ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$, if

$$\sup_R \left(\frac{1}{|R|}\int_R w(x)dx\right)\left(\frac{1}{|R|}\int_R w(x)^{1-p'}dx\right)^{p-1} < \infty \tag{1.5}$$

where the supremum is taken over all cubes R in $\mathbb{R}^n$. We also use the notation $\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}}$.

Then, Kurtz and Wheeden [4] extended Hörmander's theorem to weighted Lebesgue spaces and proved the following:

Theorem 1.3. *Let $\frac{n}{2} < s \le n$ and $1 < p < \infty$. Assume $\frac{n}{s} < p < \infty$ and $w \in A_{\frac{ps}{n}}$. If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} \|m(2^j \cdot)\Psi\|_{H^s(\mathbb{R}^n)} < \infty,$$

then the Fourier multiplier operator $m(D)$ defined with the symbol m is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ for all $1 < p < \infty$.

We now turn to the discussion of multi-linear Coifman–Meyer Fourier multiplier operators. We only state the bilinear case as an example for simplicity of its presentation. For $m \in L^\infty(\mathbb{R}^{2n})$, the bilinear Coifman–Meyer Fourier multiplier operator T_m is defined by

$$T_m(f, g)(x) = \frac{1}{(2\pi)^{(2n)}} \int_{\mathbb{R}^{2n}} m(\xi, \eta) e^{ix(\xi+\eta)} \hat{f}(\xi)\hat{g}(\eta) d\xi d\eta \tag{1.6}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Coifman and Meyer [5–7] first proved that if $m \in C^L(\mathbb{R}^{2n} \setminus \{0\})$ satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \le C_{\alpha\beta}(|\xi| + |\eta|)^{-(|\alpha|+|\beta|)} \tag{1.7}$$

for all $|\alpha| + |\beta| \le L$, where L is a sufficiently large natural number, then T_m is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for all $1 < p, q, r < \infty$ satisfying $1/p + 1/q = 1/r$. Results in [5–7] have been extended to multilinear Calderón–Zygmund operators by Kenig and Stein [8], Grafakos and Kalton [9], Grafakos and Torres [10], [11] to include $0 < r \le 1$ (see also recent work of generalizations to bilinear square functions and vector-valued Calderón–Zygmund operators of Hart [12]). However, in many cases where m has only limited smoothness, we cannot use this result since L is not an explicit number. Finding the best possible number of L thus becomes an interesting problem. By reducing the bilinear Fourier multiplier operators to linear Calderón–Zygmund operators, Coifman–Meyer obtained the L^r estimates under the assumption $L = 2n + 1$. In [10], the authors also proved the condition (1.7) with $L = 2n + 1$ assures the boundedness of T_m by using the bilinear $T1$ theorem. However this number seems to be too large in view of the linear case.

Recently, Tomita [13] improved this theorem for multipliers with limited smoothness in terms of the Sobolev regularity. To state the result in [13], for $m \in L^\infty(\mathbb{R}^{2n})$, we set $m_k(\xi, \eta) = m(2^k\xi, 2^k\eta)\Psi(\xi_1, \eta_1)$, where Ψ is the same as the (1.3) with $d = 2n$.

Theorem 1.4. *Let $s > n$, $1 < p, q, r < \infty$ and $1/p + 1/q = 1/r$. If $m \in L^\infty(\mathbb{R}^{2n})$ satisfies*

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(R^{2n})} < \infty$$

then T_m is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

For further improvement in this direction in the case $0 < r \le 1$ or the case where p or q can be smaller than or equal to 1, see the works in Grafakos, Miyachi and Tomita [14], Miyachi and Tomita [15] and Grafakos and Si [16].

Fujita and Tomita [17] considered the weighted norm inequalities for multilinear Fourier multiplier operators, for simplicity we only state their result in the bilinear case.

Theorem 1.5. *Let $1 < p, q < \infty$, $1/p + 1/q = 1/r$ and $n < s \le 2n$. Assume*

(i) $\min\{p, q\} > 2n/s$ *and* $w \in A_{\min\{ps/2n, qs/2n\}}$ *or*

(ii) $\min\{p, q\} < (2n/s)'$, $1 < r < \infty$ *and* $w^{1-r'} \in A_{r's/2n}$.

If $m \in L^\infty(\mathbb{R}^{2n})$ satisfies

$$\sup_{k\in\mathbb{Z}} \|m_k\|_{H^s(R^{2n})} < \infty.$$

Then T_m is bounded from $L^p(w) \times L^q(w)$ to $L^r(w)$.

This theorem can be understood as bilinear version of the results by Kurtz and Wheeden [4].

Next, we discuss the L^r estimates for the multi-linear and multi-parameter Fourier multiplier operators. In the bilinear and bi-parameter case, Muscalu, Pipher, Tao, and Thiele [18] proved the following

Theorem 1.6. *Let $1 < p, q < \infty$, $1/r = 1/p + 1/q$, $0 < r < \infty$ and $m \in L^\infty(\mathbb{R}^{4n})$ satisfy*

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2} m(\xi_1, \xi_2, \eta_1, \eta_2)| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1| + |\eta_1|)^{-(|\alpha_1|+|\beta_1|)}(|\xi_2| + |\eta_2|)^{-(|\alpha_2|+|\beta_2|)} \tag{1.8}$$

for $|\alpha_1| + |\beta_1| \le M$, and $|\alpha_2| + |\beta_2| \le N$, where M, N are sufficiently large natural numbers.

Then T_m is bounded from $L^p(\mathbb{R}^{2n}) \times L^q(\mathbb{R}^{2n}) \mapsto L^r(\mathbb{R}^{2n})$, where T_m is defined by

$$T_{m^1}(f, g)(x_1, x_2) = \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m(\xi_1, \xi_2, \eta_1, \eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \hat{f}(\xi_1, \xi_2)\hat{g}(\eta_1, \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2. \tag{1.9}$$

This theorem was extended to the case of multi-linear and multi-parameter setting in [19]. The method of proof of the above theorem in [18,19] is to decompose the multi-linear and multi-parameter Fourier multiplier operator into discretized multi-linear and multi-parameter paraproducts. By proving the L^r estimates for the discretized paraproducts, they establish the L^r estimates for the Fourier multipliers. The difficult part of their proof is in the quasi-Banach case when $0 < r \le 1$ where the standard duality argument for the paraproducts does not work (see also [20]). Therefore, the authors of [18,19] establish the desired result by using a new duality lemma of $L^{r,\infty}$ for $(0 < r \le 1)$, the stopping-time decompositions arguments and multi-linear interpolation. We mention in passing that the endpoint estimates of results in [18,19] were obtained by Lacey and Metcalfe [21] and L^r estimates in the above Theorem 1.6 have also been established recently in the case of multi-linear and multi-parameter pseudo-differential operators by W. Dai and the second author [22]. Furthermore, symbolic calculus has been carried out and boundedness of multi-parameter and multi-linear pseudo-differential operators in the Hörmander classes have been established by Q. Hong and the second author [23]. More recently, L^p estimates for modified bilinear and multi-parameter Hilbert transforms have also been established by W. Dai and the second author in [24], where we address the open question raised in [18].

It is worth noting that the smoothness condition for the Fourier multiplier $m(\xi_1, \xi_2, \eta_1, \eta_2)$ in [18,19] requires M and N to be sufficiently large. Thus, it is interesting to know what the limited smoothness assumption is on m to assure the L^r estimates. This is one of the main purposes of this paper.

To establish the L^r estimates of the multi-linear and multi-parameter Fourier multipliers with limited smoothness, we need to introduce the two-parameter Sobolev spaces. For $s_1, s_2 \in \mathbb{R}$, the two-parameter Sobolev space $H^{s_1,s_2}(\mathbb{R}^{4n})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$\|f\|_{H^{s_1,s_2}} = \|(I - \triangle)^{s_1/2,s_2/2} f\|_{L^2} < \infty, \tag{1.10}$$

where

$$(I-\triangle)^{s_1/2,s_2/2}f=\mathcal{F}^{-1}[(1+|\xi_1|^2+|\eta_1|^2)^{s_1/2}(1+|\xi_2|+|\eta_2|^2)^{s_2/2}\hat{f}(\xi_1,\xi_2,\eta_1,\eta_2)]$$

where $\xi_1,\xi_2,\eta_1,\eta_2\in\mathbb{R}^n$.

In this paper, we first establish a Hörmander's type theorem in the bilinear and bi-parameter setting. One of our main theorems states that:

Theorem 1.7. *Let $m\in L^\infty(\mathbb{R}^{4n})$. Set*

$$m_{j,k}(\xi_1,\xi_2,\eta_1,\eta_2)=m(2^j\xi_1,2^k\xi_2,2^j\eta_1,2^k\eta_2)\Psi_1(\xi_1,\eta_1)\Psi_2(\xi_2,\eta_2), \tag{1.11}$$

where Ψ_1,Ψ_2 are the same as (1.3) with $d=2n$. Let $s_1,s_2>n$, $s=\min(s_1,s_2)$, $1<p,q<\infty$, $p>\frac{2n}{s}$, $q>\frac{2n}{s}$ and $1/p+1/q=1/r$ with $0<r<\infty$. If $m\in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}<\infty \tag{1.12}$$

then T_m is bounded from $L^p(\mathbb{R}^{2n})\times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$.

Remark. If we allow the smoothness exponents s_1,s_2 to be close to $2n$, then p,q are allowed to be taken in the whole range of $1<p,q<\infty$. Consequently, r is allowed to be taken all $\frac{1}{2}<r<\infty$. Therefore, our theorem indeed improves the theorem of Muscalu, Pipher, Tao and Thiele [18] by requiring only limited smoothness and our proof given here provides an alternative one different than that in [18,19].

From the theorem above, we have

Theorem 1.8. *Let $1<p,q<\infty$ and $1/p+1/q=1/r$. If $m\in C^{2n+1}(\mathbb{R}^{2n}\setminus\{0\}\times\mathbb{R}^{2n}\setminus\{0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}m(\xi_1,\xi_2,\eta_1,\eta_2)|\le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1|+|\eta_1|)^{(-|\alpha_1|+|\beta_1|)}(|\xi_2|+|\eta_2|)^{(-|\alpha_2|+|\beta_2|)} \tag{1.13}$$

for all $|\alpha_1|+|\beta_1|\le n+1$, $|\alpha_2|+|\beta_2|\le n+1$ and $(\xi_1,\eta_1,\xi_2,\eta_2)\in\mathbb{R}^{2n}\setminus\{0\}\times\mathbb{R}^{2n}\setminus\{0\}$, then T_m is bounded from $L^p(\mathbb{R}^{2n})\times L^q(\mathbb{R}^{2n})$ to $L^r(\mathbb{R}^{2n})$.

Finally, we consider the weighted norm inequalities for the bilinear and bi-parameter Fourier multipliers. To this end, we first introduce the notion of product A_p weights (see [25]).

Let $1<p<\infty$. We say that a weight $w\ge0$ belongs to the product Muckenhoupt class $A_p(\mathbb{R}^n\times\mathbb{R}^n)$, if

$$\sup_R\left(\frac{1}{|R|}\int_R w(x,y)dxdy\right)\left(\frac{1}{|R|}\int_R w(x,y)^{1-p'}dxdy\right)^{p-1}<\infty \tag{1.14}$$

where the supremum is taken over all rectangles $R=I\times J$, I and J are both cubes in $\mathbb{R}^n$.

We define $A_\infty(\mathbb{R}^n\times\mathbb{R}^n)=\cup_{p>1}A_p(\mathbb{R}^n\times\mathbb{R}^n)$ as usual.

Then we can establish the following

Theorem 1.9. *Let $1<p,q<\infty$, $1/p+1/q=1/r$ and $n<s_1,s_2\le 2n$, $s=\min\{s_1,s_2\}$. Assume*

$$\text{(i) } p>2n/s_1\quad w_1\in A_{ps_1/2n} \tag{1.15}$$

$$q>2n/s_2\qquad w_2\in A_{ps_2/2n}\quad \textit{or} \tag{1.16}$$

$$\text{(ii) } \min\{p,q\}<(2n/s)',\quad 1<r<\infty \tag{1.17}$$

$$w_1^{1-r'}\in A_{r's/(2n)},\qquad w_2^{1-r'}\in A_{r's/(2n)}. \tag{1.18}$$

If $m\in L^\infty(\mathbb{R}^{4n})$ satisfies

$$\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}<\infty, \tag{1.19}$$

then T_m is bounded from $L^p(w_1)\times L^q(w_2)$ to $L^r(w)$, where $w=w_1^{r/p}w_2^{r/q}$.

The statements and their proofs of Theorems 1.7 and 1.9 can be easily generalized to multi-linear and multi-parameter settings. We also remark that the proofs of our main theorems can be viewed as alternative ones different from those given in [18,19]. Moreover, we provide weighted estimates for the multi-linear and multi-parameter Coifman–Meyer multiplier operators considered in [18,19]. We only state these results here and leave the details to the reader.

In general, any collection of n generic vectors $\xi_1=(\xi_1^i)_{i=1}^t,\dots,\xi_n=(\xi_n^i)_{i=1}^t$ in $\mathbb{R}^{t\ell}$ generates naturally the following collection of t vectors in $\mathbb{R}^{n\ell}$:

$$\bar{\xi}_1=(\xi_j^1)_{j=1}^n,\ \bar{\xi}_2=(\xi_j^2)_{j=1}^n,\ \dots,\ \bar{\xi}_t=(\xi_j^t)_{j=1}^n. \tag{1.20}$$

Let $m = m(\xi) = m(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{tn\ell})$ that is smooth away from the subspaces $\{\bar{\xi}_1 = 0\} \cup \cdots \cup \{\bar{\xi}_t = 0\}$ and satisfying

$$|\partial_{\bar{\xi}_1}^{\alpha_1} \cdots \partial_{\bar{\xi}_t}^{\alpha_t} m(\bar{\xi})| \le C_{\alpha_1,\ldots,\alpha_t} \prod_{i=1}^{t} |\bar{\xi}_i|^{-|\alpha_i|} \tag{1.21}$$

for sufficiently many multi-indices $\alpha_1, \ldots, \alpha_t$. We will naturally want to investigate the L^r estimates of the n-linear multiplier operator $T_m^{(t)}$ defined by

$$T_m^{(t)}(f_1, \ldots, f_n)(x) := \int_{\mathbb{R}^{tn}} m(\xi)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_n)}d\xi. \tag{1.22}$$

Thus, we can prove the following L^r estimates for general n-linear, t-parameter multiplier operator $T_m^{(t)}$ with limited smoothness.

Theorem 1.10. *Let $m \in L^\infty(\mathbb{R}^{tn\ell})$. Set*

$$m_{j_1,\ldots,j_t}(\bar{\xi}_1, \ldots, \bar{\xi}_t) = m(2^{j_1}\bar{\xi}_1, \ldots, 2^{j_t}\bar{\xi}_t)\Psi(\bar{\xi}_1)\cdots\Psi(\bar{\xi}_t),$$

where $\Psi_1, \ldots, \Psi_t$ are the same as in (1.3) with $d = n\ell$ there. For any $n \ge 1$, $t \ge 2$, the n-linear, t-parameter multiplier operator $T_m^{(t)}$ maps $L^{p_1}(\mathbb{R}^{t\ell}) \times \cdots \times L^{p_n}(\mathbb{R}^{t\ell})$ to $L^r(\mathbb{R}^{t\ell})$, provided that $1 < p_1, \ldots, p_n < \infty$, $p_1 > \frac{t\ell}{s}, \ldots, p_n > \frac{t\ell}{s}$, where $s_1 > \frac{t\ell}{2}, \ldots, s_t > \frac{t\ell}{2}$ and $s = \min(s_1, \ldots, s_t)$ and $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} > 0$ and the multiplier m satisfies

$$\sup_{j_1,\ldots,j_t\in\mathbb{Z}} \|m_{j_1,\ldots,j_t}\|_{H^{s_1,\ldots,s_t}(\mathbb{R}^{n\ell t})} < \infty.$$

We can also establish the following weighted estimates.

Theorem 1.11. *Let $1 < p_1, \ldots, p_n < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}$ and $\frac{t\ell}{2} < s_1, \ldots, s_t \le t\ell$, $s = \min\{s_1, \ldots, s_t\}$. Assume one of the following two conditions* (i) *and* (ii) *holds, namely,*

$$\text{(i) } p_j > \frac{t\ell}{s}, \quad w_j \in A_{\frac{p_j s}{t\ell}},\ j = 1, \ldots, n,\ \text{or} \tag{1.23}$$

$$\text{(ii) } \min\{p_1, \ldots, p_n\} < \left(\frac{t\ell}{s}\right)', \quad 1 < r < \infty,\ w_j^{1-r'} \in A_{\frac{r's}{t\ell}}. \tag{1.24}$$

If $m \in L^\infty(\mathbb{R}^{tn\ell})$ satisfies

$$\sup_{j_1,\ldots,j_t\in\mathbb{Z}} \|m_{j_1,\ldots,j_t}\|_{H^{s_1,\ldots,s_t}(\mathbb{R}^{tn\ell})} < \infty. \tag{1.25}$$

Then T_m is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_n}(w_n)$ to $L^r(w)$, where $w = w_1^{\frac{r}{p_1}} \cdots w_n^{\frac{r}{p_n}}$.

The organization of this paper is as follows: In Section 2 we recall some preliminary facts and give some relevant definitions. In Section 3, we prove Theorem 1.7, namely, the L^r estimates for the multi-linear and multi-parameter Coifman–Meyer multiplier operators with limited smoothness. In Section 4, we give the proof of Theorem 1.9, i.e., the weighted version of Theorem 1.7.

2. Preliminary results

The strong maximal operator M_s for a function f on $\mathbb{R}^{2n}$ is defined by

$$M_s f(x, y) = \sup_{r_1,r_2>0} \frac{1}{{r_1}^n}\frac{1}{{r_2}^n}\int_R |f(u, v)|dudv, \tag{2.1}$$

where $R = \{(u, v) \in \mathbb{R}^{2n} \mid |u - x| < r_1, |v - y| < r_2\}$ and f is a locally integrable function on $\mathbb{R}^{2n}$. It is well known that M_s is bounded on $L^p(\mathbb{R}^{2n})$ for all $1 < p < \infty$.

Lemma 2.1. *Let $\epsilon_1, \epsilon_2 > 0$. Then there exists a constant $C > 0$ such that*

$$\sup_{r_1,r_2>0}\left({r_1}^n{r_2}^n \int_{\mathbb{R}^{2n}} \frac{|f(u, v)|}{(1 + r_1|x - u|)^{n+\epsilon_1}(1 + r_2|y - v|)^{n+\epsilon_2}}dudv\right) \le CM_s f(x, y) \tag{2.2}$$

for all locally integrable functions f on $\mathbb{R}^{2n}$.

Proof. Note that

$$r_1^{n_1} r_2^{n_2} \int_{(u,v):|u-x|<r_1^{-1},|v-y|<r_2^{-1}} \frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}} dudv \le CM_s f(x,y)$$

and

$$\begin{aligned}
&\int_{(u,v):|u-x|\ge r_1^{-1},|v-y|\ge r_2^{-1}} \frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}} dudv \\
&\quad \le \sum_{k=0}^{\infty} \int_{(u,v):2^k r_1^{-1}\le|u-x|<2^{k+1}r_1^{-1},2^k r_2^{-1}\le|v-y|<2^{k+1}r_2^{-1}} \frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}} dudv \\
&\quad \le \sum_{k=0}^{\infty} \frac{1}{(1+2^k)^{n+\epsilon}(1+2^k)^{n+\epsilon}} \int_{(u,v):|u-x|<2^{k+1}r_1^{-1},|v-y|<2^{k+1}r_2^{-1}} |f(u,v)| dudv.
\end{aligned}$$

Then it follows immediately that

$$\sup_{r_1,r_2>0} \left({r_1}^n {r_2}^n \int_{\mathbb{R}^{2n}} \frac{|f(u,v)|}{(1+r_1|x-u|)^{n+\epsilon_1}(1+r_2|y-v|)^{n+\epsilon_2}} dudv \right) \le CM_s f(x,y). \quad \square$$

Using the inequality for vector-valued Hardy–Littlewood maximal functions of C. Fefferman and Stein [26], and the fact that $M_s f(x,y) \le M_1 M_2 f(x,y)$, where M_1 and M_2 are the Hardy–Littlewood maximal functions with respect to the x and y variables respectively, we have the following inequality for the vector-valued strong maximal functions:

Lemma 2.2. *Let $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} (M_s f_k)^q \right\}^{1/q} \right\|_{L^p} \le C \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p} \tag{2.3}$$

for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^{2n}$.

Using the Littlewood–Paley inequality of L^p estimates in the product space of R. Fefferman and Stein [27], we can deduce immediately the following

Lemma 2.3. *Let $1 < p < \infty$, and let $\Psi_1, \Psi_2 \in \mathcal{S}(R^n)$ be such that* $\operatorname{supp}\psi_1 \subset \{\xi \in \mathbb{R}^n : 1/a \le |\xi| \le a\}$ *for some $a > 1$,* $\operatorname{supp}\psi_2 \subset \{\eta \in \mathbb{R}^n : 1/b \le |\eta| \le b\}$ *for some $b > 1$. Then there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p} \le C\|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^{2n}), \tag{2.4}$$

where $[\Psi_1(D/2^j)\Psi_2(D/2^k)f](\xi_1,\xi_2) = \mathcal{F}^{-1}\left[\hat{\Psi}_1(\cdot/2^j)\hat{\Psi}_2(\cdot/2^k)\hat{f}(\cdot,\cdot)\right](\xi_1,\xi_2)$. *Moreover, if $\sum_{j\in\mathbb{Z}} \Psi_i(\xi_i/2^j) = 1$ for all $\xi_i \ne 0$, for $i = 1, 2$, then*

$$\left\| \left\{ \sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p} \approx \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^{2n}). \tag{2.5}$$

Let ϕ_0 be a C^∞-function on $[0,\infty)$ satisfying

$$\phi_0(t) = 1 \quad \text{on } [0, 1/8], \ \operatorname{supp}\phi_0 \subset [0, 1/4] \tag{2.6}$$

we set $\phi_1(t) = 1 - \phi_0(t)$, and set for $\xi, \eta \in \mathbb{R}^n$ the following notations:

$$\Phi_{(1)}(\xi,\eta) = \phi_0(|\xi|/|\eta|) \qquad \Phi_{(2)}(\xi,\eta) = \phi_1(|\eta|/|\xi|) \tag{2.7}$$

$$\Phi_{(3)}(\xi,\eta) = (1-\phi_0(|\xi|/|\eta|))(1-\phi_1(|\eta|/|\xi|)). \tag{2.8}$$

Lemma 2.4 (*[17]*).

(1) *For $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$,*

$$\Phi_{(1)}(\xi,\eta) + \Phi_{(2)}(\xi,\eta) + \Phi_{(3)}(\xi,\eta) = 1. \tag{2.9}$$

(2) *Each $\Phi_{(i)}$ satisfies*

$$|\partial_\xi^{\alpha_1}\partial_\eta^{\alpha_2}\Phi_{(i)}(\xi,\eta)| \le C_{\alpha_1,\alpha_2}(|\xi|+|\eta|)^{-(|\alpha_1|+|\alpha_2|)} \tag{2.10}$$

for all multi-indices α_1, α_2.

(3) $\operatorname{supp}\Phi_{(3)} \subset \{|\xi|/8 \le |\eta| \le 8|\xi|\}$, $\operatorname{supp}\Phi_{(1)} \subset \{|\xi| \le |\eta|/2\}$ *and* $\operatorname{supp}\Phi_{(2)} \subset \{|\eta| \le |\xi|/2\}$.

With a similar proof to that of Lemma 3.2 in [13] with a little modification, we can obtain the following:

Lemma 2.5. *Assume that $m \in C^{N+M}(\mathbb{R}^{2n}\setminus\{0\}\times\mathbb{R}^{2n}\setminus\{0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}m(\xi_1,\xi_2,\eta_1,\eta_2)| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1|+|\eta_1|)^{(-|\alpha_1|+|\beta_1|)}(|\xi_2|+|\eta_2|)^{(-|\alpha_2|+|\beta_2|)} \tag{2.11}$$

for all $|\alpha_1|+|\beta_1| \le N$, $|\alpha_2|+|\beta_2| \le M$ and $(\xi_1,\eta_1,\xi_2,\eta_2) \in \mathbb{R}^{2n}\setminus\{0\}\times\mathbb{R}^{2n}\setminus\{0\}$, where N, M are non-negative integers. Let Φ_1 and $\Phi_2 \in \mathcal{S}(\mathbb{R}^{2n})$ be such that none of $\operatorname{supp}\Phi_1$, $\operatorname{supp}\Phi_2$ contains the origin, and set

$$\tilde{m}_{s,t}(\xi_1,\xi_2,\eta_1,\eta_2) = m(s\xi_1, t\xi_2, s\eta_1, t\eta_2)\Phi_1(\xi_1,\eta_1)\Phi_2(\xi_2,\eta_2). \tag{2.12}$$

Then $\sup_{s,t>0}\|\tilde{m}_{s,t}\|_{H^{N,M}(\mathbb{R}^{4n})} < \infty$.

Lemma 2.6 *([14]). Let $2 \le q < \infty$, $r > 0$ and $s \ge 0$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}\|\hat{f}\|_{L^q(w_{s,q})} &\triangleq \left(\int_{\mathbb{R}^{4n}} |f(x,y)|^q (1+x^2)^s(1+y^2)^s dxdy\right)^{1/q} \\ &\le C\|f\|_{H^{s,s}(\mathbb{R}^{2n}\times\mathbb{R}^{2n})}.\end{aligned} \tag{2.13}$$

Next, we need to establish the following

Lemma 2.7. *Let $s_1, s_2 \in \mathbb{R}$, and let $\Psi_1, \Psi_2 \in \mathcal{S}(R^{2n})$ be such that $\operatorname{supp}\Psi_1$, $\operatorname{supp}\Psi_2$ are compact and none of them contains the origin. Assume that $\Phi \in C^\infty(\mathbb{R}^{2n}\setminus\{0\}\times\mathbb{R}^{2n}\setminus\{0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\eta_1}^{\beta_1}\partial_{\eta_2}^{\beta_2}\Phi(\xi_1,\xi_2,\eta_1,\eta_2)| \le C_{\alpha_1\alpha_2\beta_1\beta_2}(|\xi_1|+|\eta_1|)^{-(|\alpha_1|+|\beta_1|)}(|\xi_2|+|\eta_2|)^{-(|\alpha_2|+|\beta_2|)}$$

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}_0^n$. Then there exists a constant $C > 0$ such that

$$\sup_{t,s>0}\|m(t\xi_1, s\xi_2, t\eta_1, s\eta_2)\Phi(t\xi_1, s\xi_2, t\eta_1, s\eta_2)\Psi_1(\xi_1,\eta_1), \Psi_2(\xi_2,\eta_2)\|_{H^{s_1,s_2}} \le C\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}}$$

for all $m \in L^\infty(\mathbb{R}^{4n})$ satisfies $\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})} < \infty$, where $m_{j,k}$ is defined by (1.11).

Proof. We mimic the proof of Lemma (3.4) in [14]. First, we assume that $\operatorname{supp}\Psi_1 \subset \{1/2^{j_0} \le |(\xi_1,\eta_1)| \le 2^{j_0}\}$ and $\operatorname{supp}\Psi_2 \subset \{1/2^{k_0} \le |(\xi_2,\eta_2)| \le 2^{k_0}\}$ for some $j_0, k_0 \in \mathbb{N}$. Given $t, s > 0$, take $j, k \in \mathbb{Z}$ satisfying $2^{j-1} \le t \le 2^j$, $2^{k-1} \le s \le 2^k$. Then, since $1 < 2^j/t \le 2$, $1 < 2^k/t \le 2$, by change of variables,

$$\|m(t\cdot, s\cdot)\Phi(t\cdot, s\cdot)\Psi_1(\cdot)\Psi_2(\cdot)\|_{H^{s_1,s_2}} \le C\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot, 2^k\cdot)\Psi_1(2^jt^{-1}\cdot)\Psi_2(2^ks^{-1}\cdot)\|_{H^{s_1,s_2}}.$$

Let $\Psi(\xi_1,\eta_1)$, $\Psi(\xi_2,\eta_2)$ be as in (1.3) with $d = 2n$. Using $\operatorname{supp}\Psi_1(2^jt^{-1}\cdot) \subset \{1/2^{j_0+1} \le |(\xi_1,\eta_1)| \le 2^{j_0}\}$ and $\operatorname{supp}\Psi_2(2^ks^{-1}\cdot) \subset \{1/2^{k_0+1} \le |(\xi_2,\eta_2)| \le 2^{k_0}\}$, we have

$$\begin{aligned}&\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot, 2^k\cdot)\Psi_1(2^jt^{-1}\cdot)\Psi_2(2^ks^{-1}\cdot)\|_{H^{s_1,s_2}} \\ &\le C\sum_{j_1=-(j_0+1)}^{j_0}\sum_{k_1=-(k_0+1)}^{k_0}\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot, 2^k\cdot)\Psi_1(2^jt^{-1}\cdot)\Psi_2(2^ks^{-1}\cdot)\Psi(\cdot/2^{j_1})\Psi(\cdot/2^{k_1})\|_{H^{s_1,s_2}} \\ &\le C\sum_{j_1=-(j_0+1)}^{j_0}\sum_{k_1=-(k_0+1)}^{k_0}\|m(2^j\cdot, 2^k\cdot)\Psi(\cdot/2^{j_1})\Psi(\cdot/2^{k_1})\|_{H^{s_1,s_2}}\|\Phi(2^j\cdot, 2^k\cdot)\Psi_1(2^jt^{-1}\cdot)\Psi_2(2^ks^{-1}\cdot)\|_{H^{s_1,s_2}} \\ &\le C\sum_{j_1=-(j_0+1)}^{j_0}\sum_{k_1=-(k_0+1)}^{k_0}\|m(2^{j+j_1}\cdot, 2^{k+k_1}\cdot)\Psi(\cdot)\Psi(\cdot)\|_{H^{s_1,s_2}}\|\Phi(t\cdot, s\cdot)\Psi_1\Psi_2\|_{H^{s_1,s_2}} \\ &\le C\left(\sup_{j,k\in\mathbb{Z}}\|m(2^{j+j_1}\cdot, 2^{k+k_1}\cdot)\Psi\Psi\|_{H^{s_1,s_2}}\right)\left(\sup_{j,s>0}\|\Phi(t\cdot, s\cdot)\Psi_1\Psi_2\|_{H^{s_1,s_2}}\right).\end{aligned}$$

By Lemma 2.5, $\sup_{j,s>0}\|\Phi(t\cdot, s\cdot)\Psi_1\Psi_2\|_{H^{s_1,s_2}} < \infty$.

The proof is then complete. □

3. Proof of Theorem 1.7

The main effort of this section is to establish the first main theorem of this paper on L^r estimates for the multi-linear and multi-parameter Fourier multipliers with limited smoothness, namely, Theorem 1.7. The proof is quite complicated and involved due to the multi-parameter structure of the Fourier multiplier m. Therefore, we will divide the proof into several steps. The main idea is to decompose the multiplier into different pieces and handle them separately in each piece.

Proof. Let $s_1, s_2 > n$ and $m \in L^\infty(\mathbb{R}^{4n})$ satisfy $\sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s_1,s_2}} < \infty$, where $m_{j,k}$ is defined by (1.11). Since $H^{s_1,s_2}(\mathbb{R}^{4n}) \hookrightarrow H^{\min\{s_1,s_2\},\min\{s_1,s_2\}}(\mathbb{R}^{4n})$, it is sufficient to consider $H^{s,s}(\mathbb{R}^{4n})$, where $s = \min\{s_1, s_2\} > n$. We rewrite m as follows:

$$\begin{aligned} m(\xi_1,\xi_2,\eta_1,\eta_2) &= m(\xi_1,\xi_2,\eta_1,\eta_2)\left(\sum_{i=1}^{3}\Phi_{(i)}(\xi_1,\eta_1)\right)\left(\sum_{j=1}^{3}\Phi_{(j)}(\xi_2,\eta_2)\right)\\ &= \sum_{i,j=1}^{3} m(\xi_1,\xi_2,\eta_1,\eta_2)\Phi_{(i)}(\xi_1,\eta_1)\Phi_{(j)}(\xi_2,\eta_2)\\ &= \sum_{i,j=1}^{3} m_{i,j}(\xi_1,\xi_2,\eta_1,\eta_2) \end{aligned} \tag{3.1}$$

where Φ_i, Φ_j $(1 \le i, j \le 3)$ are defined by (2.7) and (2.8).

By Lemma 2.4, we divide these $m_{j,k}$ into four groups and estimate the bilinear and bi-parameter Fourier multiplier operator defined by each symbol $m_{j,k}$. Since the Fourier multiplier operator corresponding to every symbol $m_{j,k}$ in the same group can be estimated in the similar way, we just choose one to handle in each group.

- Group 1:
  - $m_{1,1}$, where $\operatorname{supp} m_{1,1} \in \{|\xi_1| \le |\eta_1|/2, |\xi_2| \le |\eta_2|/2\}$
  - $m_{2,2}$, where $\operatorname{supp} m_{1,1} \in \{|\eta_1| \le |\xi_1|/2, |\eta_2| \le |\xi_2|/2\}$.
- Group 2:
  - $m_{1,3}$, where $\operatorname{supp} m_{1,3} \in \{|\xi_1| \le |\eta_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$
  - $m_{2,3}$, where $\operatorname{supp} m_{1,3} \in \{|\eta_1| \le |\xi_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$
  - $m_{3,1}$, where $\operatorname{supp} m_{1,3} \in \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\xi_2| \le |\eta_2|/2\}$
  - $m_{3,2}$, where $\operatorname{supp} m_{1,3} \in \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\eta_2| \le |\xi_2|/2\}$.
- Group 3:
  - $m_{1,2}$, where $\operatorname{supp} m_{1,2} \in \{|\xi_1| \le |\eta_1|/2, |\eta_2| \le |\xi_2|/2\}$
  - $m_{2,1}$, where $\operatorname{supp} m_{2,1} \in \{|\eta_1| \le |\xi_1|/2, |\xi_2| \le |\eta_2|/2\}$.
- Group 4:
  - $m_{3,3}$, where $\operatorname{supp} m_{3,3} \in \{|\eta_1|/8 \le |\xi_1| \le 8|\eta_1|, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$.

In the following proof, we assume $2n/s < p, q$.

Estimates for Fourier multiplier corresponding to a symbol $m_{j,k}$ in Group 1.

First, we consider $m_{2,2}$, for simplicity we denote it as m^1 instead of $m_{2,2}$. Using the fact that L^p norm is bounded by the H^p norm in the multi-parameter setting established, e.g., in [28–30], and the equivalence of the definition of the multi-parameter Hardy space, we have for all $0 < r < \infty$

$$\begin{aligned} \|T_m(f,g)\|_{L^p} &\le \|\sup_{s,t>0} |\Phi_{s,t} * T_m(f,g)|\|_{L^r}\\ &\approx \left\| \left\{ \sum_{j,k\in\mathbb{Z}} |\psi_1(D/2^j)\psi_2(D/2^k)T_m(f,g)|^2 \right\}^{1/2} \right\|_{L^r} \end{aligned} \tag{3.2}$$

for $0 < p < \infty$, where $\Phi_{s,t}(x,y) = 2^{sn}\phi(2^{sn}x)2^{tn}\phi(2^{tn}y)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\phi}$ does not contain the origin, Ψ is the same as (1.3) with $d = n$.

Let $f, g \in \mathcal{S}(\mathbb{R}^{2n})$, since $\sum_{j\in\mathbb{Z}} \Psi_j(\xi) = 1$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$\begin{aligned} A_{j,k} &\triangleq \Psi(D/2^j)\Psi(D/2^k)T_{m^1}(f,g)(x_1,x_2)\\ &= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^1(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\hat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\ &= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^1(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\\ &\quad\times \Psi_j(\xi_1+\eta_1)\tilde{\Psi}_j(\xi_1)\hat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\tilde{\Psi}_k(\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2 \end{aligned}$$

$$= \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^1(\xi_1, \xi_2, \eta_1, \eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}$$
$$\times \Psi_j(\xi_1+\eta_1)\tilde{\Psi}_j(\xi_1)\hat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\tilde{\Psi}_k(\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2$$
$$= \int_{\mathbb{R}^{4n}} 2^{(2jn+2kn)}(\mathcal{F}^{-1}m^1_{j,k})(2^j(x_1-y_1), 2^k(x_2-y_2), 2^j(x_1-z_1), 2^k(x_2-z_2))$$
$$\times (\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)(y_1,y_2)g(z_1,z_2)dy_1dy_2dz_1dz_2 \tag{3.3}$$

where $\Psi_k(\xi) = \Psi(\xi/2^k)$ and $\tilde{\Psi}(\xi_1) \in \mathcal{S}(\mathbb{R}^n)$ such that $\tilde{\Psi}(\xi_1)\Psi(\xi_1+\eta_1) = \Psi(\xi_1+\eta_1)$, on the supp m^1, since $|\xi_1+\eta_1| \approx |\xi_1|$. The same is true for $\tilde{\Psi}(\xi_2)$, i.e., $\tilde{\Psi}(\xi_2)\Psi(\xi_2+\eta_2) = \Psi(\xi_2+\eta_2)$, on the supp m^1, since $|\xi_2+\eta_2| \approx |\xi_2|$.

$$m^1_{j,k} = m^1(2^j\xi_1, 2^k\xi_2, 2^j\eta_1, 2^k\eta_2)\Psi(\xi_1+\eta_1)\Psi(\xi_2+\eta_2). \tag{3.4}$$

Take $1 < t < 2$ satisfying $2n/s < t < \min\{2, p, q\}$.

$$|A_{j,k}| \le 2^{2jn+2kn} \int_{\mathbb{R}^{4n}} (1+2^j|x_1-y_1|+2^j|x_1-z_1|)^s(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^s$$
$$\times (\mathcal{F}^{-1}m^1_{j,k})(2^j(x_1-y_1), 2^k(x_2-y_2), 2^j(x_1-z_1), 2^k(x_2-z_2))$$
$$\times (1+2^j|x_1-y_1|+2^j|x_1-z_1|)^{-s}(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^{-s}$$
$$\times (\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)(y_1,y_2)g(z_1,z_2)dy_1dy_2dz_1dz_2$$
$$\le \left(\int_{\mathbb{R}^{4n}} (1+|y_1|+|z_1|)^{t's}(1+|y_2|+|z_2|)^{t's}|(\mathcal{F}^{-1}m^1_{j,k})(y_1,y_2,z_1,z_2)|^{t'}dy_1dy_2dz_1dz_2\right)^{1/t'}$$
$$\times \left(\int_{\mathbb{R}^{4n}} 2^{2jn+2kn}(1+2^j|x_1-y_1|+2^j|x_1-z_1|)^{-ts}(1+2^k|x_2-y_2|+2^k|x_2-z_2|)^{-ts}\right.$$
$$\left.\times |(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)(y_1,y_2)g(z_1,z_2)|^t dy_1dy_2dz_1dz_2\right)^{1/t}$$
$$\lesssim \|m^1_{j,k}\|_{L^{t'}(w_{s,t'})} \left(\int_{\mathbb{R}^{2n}} 2^{jn+kn}|g(z_1,z_2)|^t(1+2^k|x_2-z_2|)^{-st/2}(1+2^j|x_1-z_1|)^{-st/2}dz_1dz_2\right)^{1/t}$$
$$\times \left(\int_{\mathbb{R}^{2n}} 2^{jn+kn}|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)(y_1,y_2)|^t(1+2^j|x_1-y_1|)^{-st/2}(1+2^k|x_2-y_2|)^{-st/2}dy_1dy_2\right)^{1/t}$$
$$\lesssim \|m^1_{j,k}\|_{H^{s,s}} \left(M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t)(x_1,x_2)\right)^{1/t} \left(M_s(|g|^t)(x_1,x_2)\right)^{1/t}. \tag{3.5}$$

The last inequality is from Lemmas 2.1 and 2.7 since $st/2 > n$.

Then by Hölder's inequality, (3.2) and (3.5), we have

$$\|T^1_m(f,g)(x_1,x_2)\|_{L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k} (M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t))^{2/t} \right\}^{1/2} \right\|_{L^p} \|\{(M_s(|g|^t))^{2/t}\}^{1/2}\|_{L^q}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k} (M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t))^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{p/t}} \|\{(M_s(|g|^t))^{2/t}\}^{t/2}\|^{1/t}_{L^{q/t}}$$
$$\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s,s}} \|f\|_{L^p} \|g\|_{L^q}. \tag{3.6}$$

Using supp $m^1 \in \{1/a \le \sqrt{|\xi_1|^2+|\eta_1|^2} \le a,\ 1/b \le \sqrt{|\xi_2|^2+|\eta_2|^2} \le b\}$ for some $a, b > 1$, by Lemma 2.7 we have

$$\sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s,s}}. \tag{3.7}$$

Consequently

$$\|T_{m^1}\|_{L^p\times L^q\to L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{s_1,s_2}. \tag{3.8}$$

Changing the roles ξ_1, η_1 and ξ_2, η_2, we can prove

$$\|T_{m^1}\|_{L^p\times L^q\to L^r} \le \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s_1,s_2}} \tag{3.9}$$

where $m^1 = m_{1,1}$ this time.

Estimates for the Fourier multiplier operators with a symbol in Group 2:

We write m^2 instead of $m_{1,3}$ for simplicity. Since supp $m_{1,3} \in \{|\xi_1| \le |\eta_1|/2, |\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$, then there exists $\Psi^1 \in \mathcal{S}(\mathbb{R}^n)$, such that $\Psi(\xi_2)\Psi^1(\eta_2) = \Psi(\xi_2)$ on $\{|\eta_2|/8 \le |\xi_2| \le 8|\eta_2|\}$, where Ψ is the function which is the same as case 1. Hence,

$$\begin{aligned}
&\Psi(D/2^j)T_{m^2}(f,g)(x_1,x_2)\\
&\quad= \frac{1}{(2\pi)^{(4n)}}\int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\hat{f}(\xi_1,\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&\quad= \frac{1}{(2\pi)^{(4n)}}\sum_k\int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\\
&\qquad\times \Psi_j(\xi_1+\eta_1)\tilde{\Psi}_j(\eta_1)\Psi_k(\xi_2)\hat{f}(\xi_1,\xi_2)\Psi_k^1(\eta_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&\quad= \frac{1}{(2\pi)^{(4n)}}\sum_k\int_{\mathbb{R}^{4n}} m^2(\xi_1,\xi_2,\eta_1,\eta_2)e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)}\Psi_j(\xi_1+\eta_1)\\
&\qquad\times \tilde{\Psi}_j(\xi_1)\Psi_k(\xi_2)\Psi_k^2(\xi_2)\hat{f}(\xi_1,\xi_2)\Psi_k^1(\eta_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2\\
&\quad= \sum_k\int_{\mathbb{R}^{4n}} 2^{(2jn+2kn)}(\mathcal{F}^{-1}m_{j,k}^2)(2^j(x_1-y_1),2^k(x_2-y_2),2^j(x_1-z_1),2^k(x_2-z_2))\\
&\qquad\times(\tilde{\Psi}_j(D)\Psi_k^2(D)f)(y_1,y_2)(\Psi_k^1(D)g)(z_1,z_2)dy_1dy_2dz_1dz_2\\
&\quad\triangleq \sum_k A_{j,k}
\end{aligned}\tag{3.10}$$

where $\tilde{\Psi}$ is the same as we used in *Estimates for symbols in Group* 1 and $\Psi(\xi_2)\Psi^2(\xi_2) = \Psi(\xi_2)$.

$$m_{j,k}^2 = m^2(2^j\xi_1, 2^k\eta_1, 2^j\xi_2, 2^k\eta_2)\Psi(\xi_1+\eta_1)\Psi(\xi_2).\tag{3.11}$$

Take $1 < t < 2$ satisfying $2n/s < t < \min\{2,p,q\}$. Arguing in the same way as deriving (3.5), we can prove

$$|A_{j,k}| \lesssim \|m_{j,k}^2\|_{H^{s,s}}\left(M_s(|(\tilde{\Psi}_j(D)\Psi_k^2(D)f)|^t)(x_1,x_2)\right)^{1/t}\left(M_s(|\Psi_k^1(D)g|^t)(x_1,x_2)\right)^{1/t}.\tag{3.12}$$

Moreover we can assume $f(\xi_1,\xi_2) = f_1(\xi_1)f_2(\xi_2)$, where $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, since $f_1 \otimes f_2$ is dense in $L^p(\mathbb{R}^{2n})$, $1 \le p < \infty$. Then we have

$$|A_{j,k}| \lesssim \|m_{j,k}^2\|_{H^{s,s}}\left(M(|g_1|^t)(x_1)M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1)\right)^{1/t}\left(M(|\Psi_k^1(D)g_2|^t)(x_2)M(|\Psi_k^2(D)f_2|^t)(x_2)\right)^{1/t}.\tag{3.13}$$

Then from (3.10) and (3.13), we have

$$\begin{aligned}
|\Psi(D/2^j)T_{m^2}(f,g)(x_1,x_2)| &\lesssim \sum_k \|m_{j,k}^2\|_{H^{s,s}}\left(M(|g_1|^t)(x_1)M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1)\right)^{1/t}\\
&\quad\times\left(M(|\Psi_k^1(D)g_2|^t)(x_2)M(|\Psi_k^2(D)f_2|^t)(x_2)\right)^{1/t}\\
&\lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}^2\|_{H^{s,s}}\left(M(|g_1|^t)(x_1)M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1)\right)^{1/t}\\
&\quad\times\left\{\sum_k\left[M(|\Psi_k^1(D)g_2|^t)(x_2)M(|\Psi_k^2(D)f_2|^t)(x_2)\right]^{1/t}\right\}.
\end{aligned}\tag{3.14}$$

Then

$$\begin{aligned}
\left(\sum_j\left|\Psi(D/2^j)T_m^2(f,g)(x_1,x_2)\right|^2\right)^{1/2} &\lesssim \sup_{j,k\in\mathbb{Z}}\|m_{j,k}^2\|_{H^{s,s}}\left\{\sum_j\left[M(|g_1|^t)(x_1)M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1)\right]^{2/t}\right.\\
&\quad\left.\times\left[\sum_k(M(|\Psi_k^1(D)g_2|^t)(x_2)M(|\Psi_k^2(D)f_2|^t)(x_2))^{1/t}\right]^2\right\}^{1/2}\\
&= \sup_{j,k\in\mathbb{Z}}\|m_{j,k}^2\|_{H^{s,s}}\left\{\sum_j\left[M(|g_1|^t)(x_1)M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1)\right]^{2/t}\right\}^{1/2}\\
&\quad\times\left\{\sum_k\left[M(|\Psi_k^1(D)g_2|^t)(x_2)M(|\Psi_k^2(D)f_2|^t)(x_2)\right]^{1/t}\right\}.
\end{aligned}\tag{3.15}$$

Since $p/t, q/t, 2/t > 1$, by Hölder's inequality, Lemmas 2.2, 2.3 and (3.15)

$$\begin{aligned}
\|T_m^2(f,g)(x_1,x_2)\|_{L^r} &\lesssim \left\| \left(\sum_j |\Psi(D/2^j)T_m^2(f,g)(x_1,x_2)|^2 \right)^{1/2} \right\|_{L^r} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{H^{s,s}} \left\| \left\{ \sum_j \left[M(|g_1|^t)(x_1) M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1) \right]^{2/t} \right\}^{1/2} \right\|_{L^r(\mathbb{R}^n)} \\
&\quad \times \left\| \left[\sum_k \left(M(|\Psi_k^1(D)g_2|^t)(x_2) M(|\Psi_k(D)f_2|^t)(x_2) \right)^{1/t} \right] \right\|_{L^r(\mathbb{R}^n)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{H^{s,s}} \left\| \left\{ \sum_j \left[M(|(\tilde{\Psi}_j(D)f_1)|^t)(x_1) \right]^{2/t} \right\}^{1/2} \right\|_{L^p} \|(M(|g_1|^t))^{1/t}\|_{L^q} \\
&\quad \times \left\| \left(\sum_k (M(|\Psi_k^1(D)g_2|^t)(x_2))^{2/t} \right)^{1/2} \left(\sum_k (M(|\Psi_k(D)f_2|^t)(x_2))^{2/t} \right)^{1/2} \right\|_{L^r(\mathbb{R}^n)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{H^{s,s}} \|f_1\|_{L^p} \|g_1\|_{L^q} \\
&\quad \times \left\| \left(\sum_k (M(|\Psi_k^1(D)g_2|^t)(x_2))^{2/t} \right)^{1/2} \right\|_{L^q} \left\| \left(\sum_k (M(|\Psi_k^1(D)f_2|^t)(x_2))^{2/t} \right)^{1/2} \right\|_{L^p} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{H^{s,s}} \|f_1\|_{L^p} \|f_2\|_{L^p} \|g_1\|_{L^q} \|g_2\|_{L^q}.
\end{aligned} \tag{3.16}$$

Using $\operatorname{supp} m_{j,k}^2 \in \{1/a \le \sqrt{|\xi_1|^2 + |\eta_1|^2} \le a,\ 1/b \le \sqrt{|\xi_2|^2 + |\eta_2|^2} \le b\}$ for some $a, b > 1$, by Lemma 2.7 we have

$$\sup_{j,k\in\mathbb{Z}} \|m_{j,k}^2\|_{H^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s,s}}. \tag{3.17}$$

Consequently

$$\|T_{m^2}\|_{L^p\times L^q\to L^r} \le \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s,s}}. \tag{3.18}$$

By changing the roles of ξ_1 and η_1 or (ξ_1, η_1) and (ξ_2, η_2), we can prove other situations in Group 2.

Estimates for Fourier multiplier with symbols in Group 3:

We write m^3 instead of $m_{1,2}$, the proof is similar to case 1 with necessary modification. Since $|\xi_1 + \eta_1| \approx |\eta_1|$ and $|\xi_2 + \eta_2| \approx |\xi_2|$, we have

$$\begin{aligned}
&\Psi(D/2^j)\Psi(D/2^k)T_{m^3}(f,g)(x_1,x_2) = \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1,\xi_2,\eta_1,\eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \\
&\quad \times \Psi_j(\xi_1+\eta_1)\hat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2 \\
&\quad = \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1,\xi_2,\eta_1,\eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \\
&\quad \times \Psi_j(\xi_1+\eta_1)\tilde{\Psi}_j(\eta_1)\hat{f}(\xi_1,\xi_2)\Psi_k(\xi_2+\eta_2)\tilde{\Psi}_k(\xi_2)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2 \\
&\quad = \frac{1}{(2\pi)^{(4n)}} \int_{\mathbb{R}^{4n}} m^3(\xi_1,\xi_2,\eta_1,\eta_2) e^{ix_1(\xi_1+\eta_1)+ix_2(\xi_2+\eta_2)} \\
&\quad \times \Psi_k(\xi_2+\eta_2)\tilde{\Psi}_k(\xi_2)\hat{f}(\xi_1,\xi_2)\Psi_j(\xi_1+\eta_1)\tilde{\Psi}_j(\eta_1)\hat{g}(\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2 \\
&\quad = \int_{\mathbb{R}^{4n}} 2^{(2jn+2kn)}(\mathcal{F}^{-1}m_{j,k}^3)(2^j(x_1-y_1), 2^k(x_2-y_2), 2^j(x_1-z_1), 2^k(x_2-z_2)) \\
&\quad \times (\tilde{\Psi}_k(D)f)(y_1,y_2)\tilde{\Psi}_j(D)g(z_1,z_2)dy_1 dy_2 dz_1 dz_2 \\
&\quad \triangleq A_{j,k}
\end{aligned} \tag{3.19}$$

where $\Psi, \tilde{\Psi}$ are defined the same way as we deal with symbols in *Group* 1 and

$$m_{j,k}^3 = m^3(2^j\xi_1, 2^k\xi_2, 2^j\eta_1, 2^k\eta_2)\Psi(\xi_1+\eta_1)\Psi(\xi_2+\eta_2). \tag{3.20}$$

As we did in dealing with symbols in *Group* 1, we can easily prove

$$|A_{j,k}| \lesssim \|m^3_{j,k}\|_{H^{s,s}} (M_s(|(\tilde{\Psi}_j(D)f)|^t)(x_1,x_2))^{1/t} (M_s(|(\tilde{\Psi}_k(D)g)|^t)(x_1,x_2))^{1/t} \tag{3.21}$$

where $\max\{1, 2n/s\} < t < 2$.

Since the rest of the proof is similar to that of case 1, we omit the details. Thus we obtain

$$\|T_{m^3}\|_{L^p\times L^q\to L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \|m^3_{j,k}\|_{H^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s,s}}. \tag{3.22}$$

By changing the roles of (ξ_1,η_1) and (ξ_2,η_2), we can get the same conclusion for $m_{2,1}$.

Estimates for Fourier multipliers with symbols in Group 4:

We write m^4 instead of $m_{3,3}$. Since the proof is similar to the case dealing with symbols in *Group* 2, we will outline the main estimates and omit the details here.

First, we can easily prove

$$\begin{aligned} |T_{m^4}(f,g)(x_1,x_2)| &\lesssim \sup_{j,k\in\mathbb{Z}} \|m^4_{j,k}\|_{H^{s,s}} \left\{ \sum_{j,k} (M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)f)|^t)(x_1,x_2))^{2/t} \right\}^{1/2} \\ &\quad \times \left\{ \sum_{j,k} (M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)f)|^t)(x_1,x_2))^{t/2} \right\}^{1/2} \end{aligned} \tag{3.23}$$

where $\max\{1, 2n/s\} < t < 2$.

$$m^4_{j,k} = m^4(2^j\xi_1, 2^k\eta_1, 2^j\xi_2, 2^k\eta_2)\Psi(\xi_1+\eta_1)\tilde{\Psi}(\xi_1)\Psi(\xi_2+\eta_2)\tilde{\Psi}(\xi_2). \tag{3.24}$$

Since $p/t, q/t, 2/t > 1$, by Hölder's inequality, Lemmas 2.2 and 2.3, we have

$$\begin{aligned} \|T^4_m(f,g)(x_1,x_2)\|_{L^r} &\lesssim \sup_{j,k\in\mathbb{Z}} \|m^4_{j,k}\|_{H^{s,s}} \left\| \left(\sum_{j,k} M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)f)|^t)^{2/t} \right)^{1/2} \right\|_{L^p} \\ &\quad \times \left\| \left(\sum_{j,k} M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)g)|^t)^{2/t} \right)^{1/2} \right\|_{L^q} \\ &\lesssim \sup_{j,k\in\mathbb{Z}} \|m^4_{j,k}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k} M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)f)|^t)^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{p/t}} \\ &\quad \times \left\| \left\{ \sum_{j,k} M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)g)|^t)^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{p/t}} \\ &\lesssim \sup_{j,k\in\mathbb{Z}} \|m^4_{j,k}\|_{H^{s,s}} \left\| \left\{ \sum_{j,k} |(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)f)|^2 \right\}^{1/2} \right\|_{L^p} \left\| \left\{ \sum_{j,k} |(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)g)|^2 \right\}^{1/2} \right\|_{L^q} \\ &\lesssim \sup_{j,k\in\mathbb{Z}} \|m^4_{j,k}\|_{H^{s,s}} \|f\|_{L^p} \|g\|_{L^q}. \end{aligned} \tag{3.25}$$

Since $\operatorname{supp} m^4 \in \{1/a \le \sqrt{|\xi_1|^2+|\eta_1|^2} \le a, 1/b \le \sqrt{|\xi_2|^2+|\eta_2|^2} \le b\}$ for some $a, b > 1$, by Lemma 2.7 we have

$$\|T_{m^4}\|_{L^p\times L^q\to L^r} \lesssim \sup_{j,k\in\mathbb{Z}} \|m^3_{j,k}\|_{H^{s,s}} \lesssim \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{H^{s,s}}. \tag{3.26}$$

Next, we consider $T_{m^{*1}}, T_{m^{*2}}$, the dual operator of T_m, which are defined by

$$\int_{\mathbb{R}^{2n}} T_m(f,g)h\,dx = \int_{\mathbb{R}^{2n}} T_{m^{*1}}(h,g)f\,dx = \int_{\mathbb{R}^{2n}} T_{m^{*2}}(f,h)g\,dx \tag{3.27}$$

for all $f, g, h \in \mathcal{S}(\mathbb{R}^{2n})$.

If we have proved the same conclusion for $T_{m^{*1}}, T_{m^{*2}}$ as T_m, then using the same proof as in the bilinear case in [13], we complete the proof of Theorem 1.7 by multi-linear and multi-parameter duality and interpolation. We omit the details here.

To finish the proof of Theorem 1.7, we only need to show

$$\begin{aligned}\sup_{j,k\in\mathbb{Z}}\|m^{*1}_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}&\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}\\ \sup_{j,k\in\mathbb{Z}}\|m^{*2}_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}&\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{H^{s_1,s_2}(\mathbb{R}^{4n})}\end{aligned}\tag{3.28}$$

where $m^{*1}(\xi_1,\eta_1,\xi_2,\eta_2)=m(-(\xi_1+\eta_1),\eta_1,-(\xi_2+\eta_2),\eta_2)$ and $m(\xi_1,\eta_1,\xi_2,\eta_2)=m^{*1}(\xi_1,-(\xi_1+\eta_1),\xi_2,-(\xi_2+\eta_2))$.
We only choose one case to prove, the remaining cases are the same.
By a change of variables,

$$\begin{aligned}\|m^{*1}_{j,k}\|_{H^{s_1,s_2}}&=\|m(-2^j(\xi_1+\eta_1),-2^k(\xi_2+\eta_2),2^j\eta_1,2^k\eta_2)\Psi_1(\xi_1,\eta_1)\Psi_2(\xi_2,\eta_2)\|_{H^{s_1,s_2}}\\ &\approx\|m(2^j\xi_1,2^k\xi_2,2^j\eta_1,2^k\eta_2)\Psi_1(-(\xi_1+\eta_1),\eta_1)\Psi_2(-(\xi_2+\eta_2),\eta_2)\|_{H^{s_1,s_2}}.\end{aligned}\tag{3.29}$$

Since $\sqrt{|\xi+\eta|^2+|\eta|^2}\approx\sqrt{|\xi|^2+|\eta|^2}$, then we can obtain

$$\sup_{j,k\in\mathbb{Z}}\|m^{*1}_{j,k}\|_{H^{s_1,s_2}}\lesssim\sup_{j,k\in\mathbb{Z}}\|m^{*1}_{j,k}\|_{H^{s_1,s_2}}.\tag{3.30}$$

Therefore, we have finished the proof of Theorem 1.7. □

Remark 3.1. In the proof of Theorem 1.7, we only assume $p,q>2n/s,\ s>n$, it implies that the target space L^r may be the quasi Banach space, where r depends on s. □

4. Proof of Theorem 1.9

This section is devoted to establishing the second main theorem of this paper on weighted estimates for the multi-linear and multi-parameter Fourier multipliers with limited smoothness, namely, Theorem 1.9. Before we prove Theorem 1.9, we recall some useful facts about product $A_p(\mathbb{R}^n\times\mathbb{R}^n)$ weights.

Lemma 4.1 ([31]). *Let $1<p<\infty$ and $w\in A_p(\mathbb{R}^n\times\mathbb{R}^n)$. Then*

(1) $w^{1-p'}\in A_{p'}(\mathbb{R}^n\times\mathbb{R}^n)$
(2) *there exists $1<q<p$ such that $w\in A_q(\mathbb{R}^n\times\mathbb{R}^n)$.*

Lemma 4.2. *Suppose that $w_j\in A_{p_j}(\mathbb{R}^n\times\mathbb{R}^n)$ with $1\le j\le m$ for some $1\le p_1,\dots,p_m\le\infty$ and let $0<\theta_1,\dots,\theta_m<1$ be such that $\theta_1+\cdots+\theta_m=1$. Then*

$${w_1}^{\theta_1}\cdots{w_m}^{\theta_m}\in A_{\max\{p_1,\dots,p_m\}}.\tag{4.1}$$

Proof. First note that $w_j\in A_{\max\{p_1,\dots,p_m\}}$, for $j=1,\dots,m$, then apply Hölder's inequality, we can obtain the conclusion.

Lemma 4.3 ([26]). *Let $1<p,q<\infty$ and $w\in A_p(\mathbb{R}^n\times\mathbb{R}^n)$. Then there exists a constant $C>0$ such that*

$$\left\|\left\{\sum_{k\in\mathbb{Z}}(M_sf_k)^q\right\}^{1/q}\right\|_{L^p(w)}\le C\left\|\left\{\sum_{k\in\mathbb{Z}}(f_k)^q\right\}^{1/q}\right\|_{L^p(w)}\tag{4.2}$$

for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^{2n}$.

Lemma 4.4 ([27]). *Let $1<p<\infty$ $w\in A_p(\mathbb{R}^n\times\mathbb{R}^n)$, and let $\Psi_1,\Psi_2\in\mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}\Psi_1\subset\{\xi\in\mathbb{R}^n:1/a\le|\xi|\le a\}$ for some $a>1$, $\operatorname{supp}\Psi_2\subset\{\xi\in\mathbb{R}^n:1/b\le|\xi|\le b\}$ for some $b>1$. Then there exists a constant $C>0$ such that*

$$\left\|\left\{\sum_{j,k\in\mathbb{Z}}|\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2\right\}^{1/2}\right\|_{L^p(w)}\le C\|f\|_{L^p(w)}\quad\textit{for all } f\in L^p_w(\mathbb{R}^n).\tag{4.3}$$

Moreover, if $\sum_{j\in\mathbb{Z}}\Psi_i(\xi/2^j)=1$ for all $\xi\ne0$, for $i=1,2$, then

$$\left\|\left\{\sum_{j,k\in\mathbb{Z}}|\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2\right\}^{1/2}\right\|_{L^p(w)}\approx\|f\|_{L^p(w)}\quad\textit{for all } f\in L^p(w).\tag{4.4}$$

Lemma 4.5 (*[32]*). *If* $0 < p < \infty$, $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, *f is a local integrable function in* $H^p_w(\mathbb{R}^n \times \mathbb{R}^n)$. *Then*

$$\|f\|_{L^p(w)} \le \left\| \left\{ \sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p(w)}. \tag{4.5}$$

We first prove Theorem 1.9 under assumption (i) in Theorem 1.9. Since $2n/s_1 < \min\{2, p\}$ and $w_1 \in A_{ps_1/2n}$, by Lemma 4.1, we can take $2n/s_1 < p_1 < \min\{2, p\}$ satisfying $w_1 \in A_{p/p_1}$, the same is for w_2. Then

$$\begin{aligned}
\|T_{m^1}(f,g)\|_{L^p(w)} &\le \left\| \left\{ \sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)T_m(f,g)|^2 \right\}^{1/2} \right\|_{L^p(w)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}} \left\| \left\{ \sum_{j,k} (M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t))^{2/t} \right\}^{1/2} \right\|_{L^p(w_1)} \|\{(M_s(|g|^t))^{2/t}\}^{1/2}\|_{L^q(w_2)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}} \left\| \left\{ \sum_{j,k} (M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t))^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{p/t}({w_1}^{p/t})} \|\{(M_s(|g|^t))^{2/t}\}^{t/2}\|^{1/t}_{L^{q/t}({w_2}^{q/t})} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s,s}} \|f\|_{L^p(w_1)} \|g\|_{L^q(w_2)}
\end{aligned} \tag{4.6}$$

where we take $t = \max\{p_1, q_1\}$, then $w_1 \in A_{p/t}$ and $w_2 \in A_{q/t}$.

To conclude the weighted estimates for the Fourier multipliers m, we need to do estimates corresponding to other symbols. Since the estimates for the remaining symbols in other groups are similar to that of m^1, we omit the details here.

Next, we give the proof of Theorem 1.9 under condition (ii) we consider case $p = \min\{p, q\}$. Since $p' < (2n/s)'$, then $\max\{1/r', 1/q\} < 1/r' + 1/q = 1/p < s/2n$, that is, $r', q > 2n/s$. Hence $2n/s < \min\{2, r', q\}$.

Since $1/2 < s/2n \le 1$ and $w_1^{1-r'} \in A_{r's/(2n)}$, $w_2^{1-r'} \in A_{r's/(2n)}$, by Lemma 4.1 we have

$$w_1^{1-r'} \in A_{r's/(2n)} \subset A_{r'}, \quad \text{then } w_1 \in A_r \tag{4.7}$$

$$w_2^{1-r'} \in A_{r's/(2n)} \subset A_{r'}, \quad \text{then } w_2 \in A_r \tag{4.8}$$

$$w^{1-r'} = w_1^{(1-r')r/p} w_2^{(1-r')r/q} \in A_{r's/(2n)} \tag{4.9}$$

where (4.9) is from Lemma 4.2.

It is from the assumption that $p \le q$, we also have $r \le q/2$, then $w_2 \in A_r \subset A_{q/2} \subset A_{qs/2n}$. Since $w^{1-r'} \in A_{r's/(2n)}$, $w_2 \in A_r \subset A_{qs/2n}$, by Lemma 4.2 we can take $2n/s < t < \min\{2, r', q\}$ such that

$$w^{1-r'} \in A_{r'/t}, \qquad w_2 \in A_{q/t}. \tag{4.10}$$

By duality and (3.29), it is enough to prove

$$\|T_{m^{*1}}\|_{L^{r'}(w^{1-r'})\times L^q(w_2)\to L^{p'}(w_1^{1-p'})} \le C \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}}. \tag{4.11}$$

From the proof of Theorem 1.7, we have

$$\begin{aligned}
\|T_{m^{*1}}(f,g)\|_{L^{p'}(w_1^{1-p'})} &\le \left\| \left\{ \sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)T_m(f,g)|^2 \right\}^{1/2} \right\|_{L^p(w)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}} \Bigg\| \left\{ \sum_{j,k} (M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t)(x_1,x_2))^{2/t} \right\}^{1/2} w^{-1/r} \\
&\quad \times (M_s(|g|^t)(x_1,x_2))^{1/t} w_2^{1/q} \Bigg\|_{L^{p'}} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}} \left\| \left\{ \sum_{j,k} (M_s(|(\tilde{\tilde{\Psi}}_j(D)\tilde{\tilde{\Psi}}_k(D)f)|^t)(x_1,x_2))^{2/t} \right\}^{1/2} \right\|_{L^{r'}(w^{1-r'})}
\end{aligned}$$

$$\begin{aligned}
&\times \|\{(M_s(|g|^t)(x_1,x_2))^{2/t}\}^{1/2}\|_{L^q(w_2)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}} \left\| \left\{ \sum_{j,k} (M_s(|(\tilde{\Psi}_j(D)\tilde{\Psi}_k(D)f)|^t)(x_1,x_2))^{2/t} \right\}^{t/2} \right\|^{1/t}_{L^{r'/t}(w^{1-r'})} \\
&\quad \times \|\{(M_s(|g|^t)(x_1,x_2))^{2/t}\}^{t/2}\|^{1/t}_{L^{q/t}(w_2)} \\
&\lesssim \sup_{j,k\in\mathbb{Z}} \|m^1_{j,k}\|_{H^{s_1,s_2}} \|f\|_{L^p(w_1)} \|g\|_{L^q(w_2)}. \qquad (4.12)
\end{aligned}$$

The weighted estimates for the Fourier multiplier operators corresponding to the remaining symbols are the same as with T_{m^1}, thus we finish the proof of Theorem 1.9.

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