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REVERSE STEIN–WEISS INEQUALITIES AND EXISTENCE OF THEIR EXTREMAL FUNCTIONS

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Abstract. In this paper, we establish the following reverse Stein–Weiss inequality, namely the reversed weighted Hardy–Littlewood–Sobolev inequality, in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x|^{\alpha}|x-y|^{\lambda}f(x)g(y)|y|^{\beta}dxdy \geq C_{n,\alpha,\beta,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p}$$

for any nonnegative functions $f \in L^{q'}(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, and $p,\ q' \in (0,1)$, $\alpha,\ \beta,\ \lambda > 0$ such that $\frac{1}{p}+\frac{1}{q'}-\frac{\alpha+\beta+\lambda}{n}=2$. We derive the existence of extremal functions for the above inequality. Moreover, some asymptotic behaviors are obtained for the corresponding Euler–Lagrange system. For an analogous weighted system, we prove necessary conditions of existence for any positive solutions by using the Pohozaev identity. Finally, we also obtain the corresponding Stein–Weiss and reverse Stein–Weiss inequalities on the n-dimensional sphere $\mathbb{S}^n$ by using the stereographic projections. Our proof of the reverse Stein–Weiss inequalities relies on techniques in harmonic analysis and differs from those used in the proof of the reverse (non-weighted) Hardy–Littlewood–Sobolev inequalities.

1. Introduction

The classical Hardy–Littlewood–Sobolev inequality that was obtained by Hardy and Littlewood [29] for $n=1$ and by Sobolev [34] for general n states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-y|^{-\lambda}f(x)g(y)dxdy \leq C_{n,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p} \tag{1}$$

with $1<q',p<\infty, 0<\lambda<n$, and $\frac{1}{q'}+\frac{1}{p}+\frac{\lambda}{n}=2$, where $q'=\frac{q}{q-1}$.

Lieb [31] showed that the sharp constant $C_{n,p,q'}$ satisfies the following estimate:

$$C_{n,p,q'} \leq \frac{n}{n-\lambda}\big(\frac{\pi^{\frac{\lambda}{2}}}{\Gamma(1+\frac{n}{2})}\big)^{\frac{\lambda}{n}}\frac{1}{q'p}\big(\big(\frac{\lambda q'}{n(q'-1)}\big)^{\frac{\lambda}{n}}+\big(\frac{\lambda p}{n(p-1)}\big)^{\frac{\lambda}{n}}\big).$$

In the diagonal case $q'=p=\frac{2n}{2n-\lambda}$, Lieb [30] obtained the best constant

$$C_{n,p,q'}=\pi^{\frac{\lambda}{2}}\frac{\Gamma(\frac{n}{2}-\frac{\lambda}{2})}{\Gamma(n-\frac{\lambda}{2})}\big(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\big)^{1-\frac{\lambda}{n}}.$$

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Lieb [30] employed the symmetric rearrangement argument to obtain the existence of the extremal functions of (1). Recently, the sharp Hardy–Littlewood–Sobolev inequality was also obtained by Frank and Lieb (including on the Heisenberg group), and by Carlen et al, without using symmetric rearrangement argument; see, e.g., [7, 19, 22, 23] and the references therein. It is also known that the sharp Hardy–Littlewood–Sobolev inequality in Euclidean spaces is closely related to the Moser–Onofri–Beckner type inequalities on the spheres (see [2, 3, 8, 31]). For more results about Hardy–Littlewood–Sobolev inequality and Hardy–Sobolev equations, we refer the reader to [2, 6, 13–15, 32, 36] and the references therein. For Hardy–Littlewood–Sobolev inequalities on the Riemannian manifolds, the upper half space $\mathbb{R}^n_+$, see, e.g, [16, 17, 19, 27].

In the 1950s, Stein and Weiss [35] proved the following doubly weighted Hardy–Littlewood–Sobolev inequality,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x|^{-\alpha}|x-y|^{-\lambda}f(x)g(y)|y|^{-\beta}dxdy \le C_{n,\alpha,\beta,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p}, \tag{2}$$

where p, q', α, β, and λ satisfy the following conditions:

$$\frac{1}{q'}+\frac{1}{p}+\frac{\alpha+\beta+\lambda}{n}=2, \quad \frac{1}{q'}+\frac{1}{p}\ge 1,$$

$$\alpha+\beta\ge 0, \quad \alpha<\frac{n}{q}, \quad \beta<\frac{n}{p'}, \quad 0<\lambda<n.$$

Concerning the best constants for the Stein–Weiss inequality, Lieb [30] obtained the sharp constants only when one of p and q' equals 2 or $p=q'$. The best constants are also obtained by Beckner when $p=q$ in [3, 4]. Stein–Weiss inequalities on the Heisenberg group were also obtained in Beckner [5] and Han, Lu, and Zhu [26].

Sharp reverse Hardy-Littlewood-Sobolev inequalities were studied by Beckner [1] and [7]) and Dou and Zhu [20]. The result of Beckner [1] was used by Carneiro [7] to establish the sharp inequality for the Strichartz norm. The reversed Hardy-Littlewood-Sobolev inequality can be seen as an extension of (1:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-y|^{\lambda}f(x)g(y)dxdy \ge C_{n,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p} \tag{3}$$

for any nonnegative functions $f\in L^{q'}(\mathbb{R}^n)$, $g\in L^p(\mathbb{R}^n)$ and $p,\ q'\in(0,1)$, $\lambda>0$ such that $\frac{1}{p}+\frac{1}{q'}-\frac{\lambda}{n}=2$. They ([1] and [26]) also derived the existence of extremal functions of (1) for the case $q'=p=\frac{2n}{2n+\lambda}$. Subsequently, Ngô and Nguyen [33] extended the results of existence of extremal functions to general p and q'. We note that the range of the exponents in the reversed Hardy-Littlewood-Sobolev inequality 3 are quite different from those in the Hardy-Littlewood-Sobolev inequality 1. They are mainly reflected in the difference that the power of the kernel $|x-y|$ is negative in 1 and positive in 3 and $p,q'>1$ in 1 and $p,q'<1$ in 3

Motivated by the work of Beckner [1] and Dou and Zhu [20], thus a natural question arises: Does there exist a reversed type Stein-Weiss inequality 2? Furthermore, does such an inequality have an extremal function for all the indices?

To answer these questions, we consider the following reversed Stein–Weiss inequality. Our first main result is the following theorem.

Theorem 1. *For* $n \geq 1$, $p,\ q' \in (0,1)$, $\lambda > 0$, $0 \leq \alpha < -\frac{n}{q}$, *and* $0 \leq \beta < -\frac{n}{p'}$ *satisfying*

$$\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2,$$

there is a constant $C_{n,\alpha,\beta,p,q'} > 0$ *such that for any nonnegative functions* $f \in L^{q'}(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f(x)g(y)|y|^\beta dxdy \geq C_{n,\alpha,\beta,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p}, \tag{4}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

We remark that the constant $C_{n,\alpha,\beta,p,q'}$ above can be considered as the least one such that the above inequality holds for all nonnegative functions $f \in L^{q'}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. This constant $C_{n,\alpha,\beta,p,q'}$ is often referred to as the best constant for the reversed Stein–Weiss inequality. We also note that the range of the exponents in the reversed Stein–Weiss inequality (4) are quite different from those in the Stein–Weiss inequality (2). They are reflected in the difference that the power of the kernel $|x-y|$ is negative in (2) and positive in (4) and that $p, q' > 1$ in (2) and $p, q' < 1$ in (4). Moreover, we have power weights $|x|^{-\alpha}$ and $|y|^{-\beta}$ in (2) with $\alpha+\beta \geq 0$, $\alpha < \frac{n}{q}$, $\beta < \frac{n}{p'}$, but with power weights $|x|^\alpha$ and $|y|^\beta$ with both α and β nonnegative in (4).

Once we establish the reversed Stein–Weiss inequality, it is natural to ask whether the extremal functions for the above inequality actually exist. To answer this question, we first observe that the constant $C_{n,\alpha,\beta,p,q'}$ above is the same as the infimum of the minimizing problem

$$C_{n,\alpha,\beta,p,q'} := \inf\{\|V_\lambda(g)\|_{L^q} : g \geq 0, \|g\|_{L^p} = 1\}, \tag{5}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2$.

To understand whether this $C_{n,\alpha,\beta,p,q'}$ can be achieved, we define the following weighted operator:

$$V_\lambda(g)(x) = \int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda g(y)|y|^\beta dy. \tag{6}$$

Then we can prove that the constant $C_{n,\alpha,\beta,p,q'}$ could actually be achieved. This is stated in the following theorem.

Theorem 2. *For* $n \geq 1$, $p,\ q' \in (0,1)$, $\lambda > 0$, $0 \leq \alpha < -\frac{n}{q}$, *and* $0 \leq \beta < -\frac{n}{p'}$ *satisfying*

$$\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2, \tag{7}$$

there exists some nonnegative function $g \in L^p(\mathbb{R}^n)$ *such that* $\|g\|_{L^p} = 1$ *and* $\|V_\lambda(g)\|_{L^q} = C_{n,\alpha,\beta,p,q'}$.

Once we have obtained the reversed Stein–Weiss inequality and established the existence of extremal functions of this inequality, it is natural to consider the corresponding Euler–Lagrange system. Namely, we are interested in the equations that the extremal functions f and g of the reversed Stein–Weiss inequality satisfy. To do this, one can minimize the functional

$$J(f,g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f(x)g(y)|y|^\beta dxdy \tag{8}$$

under the constraint $\|f\|_{L^{q'}} = \|g\|_{L^p} = 1$. The Euler–Lagrange system corresponding to (8) is the following integral system:

$$(9) \qquad \begin{cases} J(f,g)f(x)^{q'-1} = \int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda g(y)|y|^\beta dy, \\ J(f,g)g(x)^{p-1} = \int_{\mathbb{R}^n} |x|^\beta |x-y|^\lambda f(y)|y|^\alpha dy. \end{cases}$$

Set $u = c_1 f^{q'-1}, v = c_2 g^{p-1}, \frac{1}{q'-1} = -p_1$, and $\frac{1}{p-1} = -p_2$. Then for a proper choice of constants c_1 and c_2, system (9) becomes

$$(10) \qquad \begin{cases} u(x) = \int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda v^{-p_2}(y)|y|^\beta dy, \\ v(x) = \int_{\mathbb{R}^n} |x|^\beta |x-y|^\lambda u^{-p_1}(y)|y|^\alpha dy, \end{cases}$$

where $\frac{1}{p_1-1} + \frac{1}{p_2-1} = \frac{\alpha+\beta+\lambda}{n}$.

Next, we consider the integral system (10) and establish the asymptotic behavior of solutions to the system (10).

Theorem 3. *Let (u,v) be a pair of positive Lebesgue measurable solutions of* (10). *Then $u(x)$ and $v(x)$ satisfy the following asymptotic behavior around the origin and near infinity:*

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{\lambda+\alpha}} = \int_{\mathbb{R}^n} v^{-p_2}(y)|y|^\beta dy, \quad \lim_{|x|\to\infty} \frac{v(x)}{|x|^{\lambda+\beta}} = \int_{\mathbb{R}^n} u^{-p_1}(y)|y|^\alpha dy,$$

$$\lim_{|x|\to 0} \frac{u(x)}{|x|^{\alpha}} = \int_{\mathbb{R}^n} v^{-p_2}(y)|y|^{\beta+\lambda} dy, \quad \lim_{|x|\to 0} \frac{v(x)}{|x|^{\beta}} = \int_{\mathbb{R}^n} u^{-p_1}(y)|y|^{\alpha+\lambda} dy.$$

Finally, it is interesting to study the following equations:

$$(11) \qquad \begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^\lambda |y|^{\nu_2} v^{-p_2}(y) dy, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^\lambda |y|^{\nu_1} u^{-p_1}(y) dy. \end{cases}$$

Then, using the Pohozaev identity we can prove the following theorem.

Theorem 4. *Given λ, ν_1, ν_2, p_1, $p_2 > 0$. Suppose that there exists a pair of positive solutions $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ of* (11). *Then the following balance condition must hold:*

$$\frac{n+\nu_1}{p_1-1} + \frac{n+\nu_2}{p_2-1} = \lambda.$$

As a corollary, we immediately conclude the following nonexistence result of any pair of positive solutions to the above integral system (11).

Corollary 5. *Given λ, ν_1, ν_2, p_1, $p_2 > 0$. If*

$$\frac{n+\nu_1}{p_1-1} + \frac{n+\nu_2}{p_2-1} \neq \lambda,$$

then there does not exist any pair of positive solutions $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ satisfying (11).

In the special case $\nu_1 = \nu_2 = 0$, the system (11) is reduced to

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^\lambda v^{-p_2}(y)dy, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^\lambda u^{-p_1}(y)dy. \end{cases} \tag{12}$$

It is clear that the system (12) corresponds to the Euler-Lagrange equations of the extremal functions of the Hardy–Littlewood–Sobolev inequality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^\lambda f(x)g(y)dxdy \geq C_{n,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p} \tag{13}$$

for any nonnegative functions $f \in L^{q'}(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, and $p,\ q' \in (0,1)$, $\lambda > 0$ such that $\frac{1}{p} + \frac{1}{q'} - \frac{\lambda}{n} = 2$ with $\frac{1}{q'-1} = -p_1$ and $\frac{1}{p-1} = -p_2$. Therefore, we can conclude from Theorems 2 and 4 the following sufficient and necessary conditions.

Theorem 6. *Given λ, p_1, $p_2 > 0$. Then the sufficient and necessary condition for the existence of a pair of positive solutions $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ to the system (12) is*

$$\frac{n}{p_1 - 1} + \frac{n}{p_2 - 1} = \lambda.$$

As an application of Theorem 1, we give an equivalent form of reversed Stein–Weiss inequality (4) on the sphere $\mathbb{S}^n$ in the case of $q' = p$.

Theorem 7. *Let $n \geq 1$, $\lambda > 0$, $0 \leq \alpha < -\frac{n}{q}$, $0 \leq \beta < -\frac{n}{p'}$, and $q' = p = \frac{2n}{2n+\lambda+\alpha+\beta}$. There exists a constant $C_{n,\alpha,\beta,p,q'} > 0$ such that for any nonnegative functions $f \in L^{q'}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$,*

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} |\xi - S(0)|^\alpha |\xi-\eta|^\lambda F_1(\xi)G_1(\eta)|\eta - S(0)|^\beta d\xi d\eta \geq C_{n,\alpha,\beta,p,q'}\|F\|_{L^{q'}(\mathbb{S}^n)}\|G\|_{L^p(\mathbb{S}^n)}, \tag{14}$$

where

$$F(\xi) = \Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}} f(x), \quad F_1(\xi) = \Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}}\Big(\frac{1}{1+|x|^2}\Big)^{\frac{\beta}{2}} f(x),$$

$$G(\eta) = \Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}} g(y), \quad G_1(\eta) = \Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}}\Big(\frac{1}{1+|y|^2}\Big)^{\frac{\alpha}{2}} g(y),$$

$|\xi - \eta|$ is denoted as the chordal distance from ξ to η in $\mathbb{R}^{n+1}$, and S is the inverse of stereographic projection $\mathbb{S}^n \setminus \{(0,0,\dots,-1)\} \to \mathbb{R}^n$.

Remark 8. The best constant of reversed Stein–Weiss inequality on the sphere $\mathbb{S}^n$ (15) can be attained with the help of Theorem 2.

In view of the Stein–Weiss inequality (2) in $\mathbb{R}^n$, we can also obtain the following Stein–Weiss inequality on the sphere $\mathbb{S}^n$ with the help of the stereographic projection. This does not seem to be in the literature and so we include this for the interested reader.

Theorem 9. *Let $n \geq 1$, $0 < \lambda < n$, $\alpha < \frac{n}{q}$, $\beta < \frac{n}{p'}$, $\alpha + \beta \geq 0$, and $q' = p = \frac{2n}{2n-\lambda-\alpha-\beta}$. There exists a constant $C_{n,\alpha,\beta,p,q'} > 0$ such that for any functions $f \in L^{q'}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$,*

$$\begin{aligned} \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} &|\xi - S(0)|^{-\alpha}|\xi-\eta|^{-\lambda} H_1(\xi)T_1(\eta)|\eta - S(0)|^{-\beta} d\xi d\eta \\ &\leq C_{n,\alpha,\beta,p,q'}\|H\|_{L^{q'}(\mathbb{S}^n)}\|T\|_{L^p(\mathbb{S}^n)}, \end{aligned} \tag{15}$$

where

$$H(\xi)=\Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}f(x),H_1(\xi)=\Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}\Big(\frac{1}{1+|x|^2}\Big)^{-\frac{\beta}{2}}f(x),$$

$$T(\eta)=\Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}g(y),T_1(\eta)=\Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}\Big(\frac{1}{1+|y|^2}\Big)^{-\frac{\alpha}{2}}g(y).$$

We end this introduction with the following remarks. First, three of the authors have recently proved the reversed Stein-Weiss inequalities on upper half space using weighted reverse Hardy's inequality on upper half spaces and harmonic analysis techniques [11]. They are stated as follows.

Theorem A. *For $n>1$, $0<p,\ q'<1$, $\beta<\frac{1-n}{p'}$, $\lambda>0$ satisfying*

$$\frac{n-1}{np}+\frac{1}{q'}-\frac{\alpha+\beta+\lambda-1}{n}=2,$$

there exists some constant $C_{n,\alpha,\beta,p,q'}>0$ such that for any nonnegative functions $f\in L^{q'}(\mathbb{R}^n_+)$, $g\in L^p(\partial\mathbb{R}^n_+)$, there holds

$$\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}|x|^\alpha|x-y|^\lambda f(x)g(y)|y|^\beta dydx\geq C_{n,\alpha,\beta,p,q'}\|f\|_{L^{q'}(\mathbb{R}^n_+)}\|g\|_{L^p(\partial\mathbb{R}^n_+)}.$$

By considering the following minimizing problem

$$C_{n,\alpha,\beta,p,q'}:=\inf\{\|V_\lambda(g)\|_{L^q(\mathbb{R}^n_+)}:g\geq 0,\ \|g\|_{L^p(\partial\mathbb{R}^n_+)}=1\},$$

where the double weighted operator $V_\lambda(g)(x)$ is given by

$$V_\lambda(g)(x)=\int_{\partial\mathbb{R}^n_+}|x|^\alpha|x-y|^\lambda g(y)|y|^\beta dy,$$

we further proved in [11] the following.

Theorem B. *For $n>1$, $p,\ q'\in(0,1)$, $\lambda>0$, $0\leq\alpha<-\frac{n-1}{q}$ and $0\leq\beta<\frac{1-n}{p'}$ satisfying*

$$\frac{n-1}{np}+\frac{1}{q'}-\frac{\alpha+\beta+\lambda-1}{n}=2,$$

there exists some nonnegative function $g\in L^p(\partial\mathbb{R}^n_+)$ satisfying $\|g\|_{L^p(\partial\mathbb{R}^n_+)}=1$ and $\|V_\lambda(g)\|_{L^q(\mathbb{R}^n_+)}=C_{n,\alpha,\beta,p,q'}$.

Asymptotic estimates for solutions to the Euler-Lagrange equations associated with the reverse Stein-Weiss inequality on the half space and sufficient and necessary conditions for the existence of solutions to such integral systems were established in [11]. Moreover, corresponding Stein-Weiss inequality on the sphere and its reversed version were also obtained in [11].

More recently, the authors have also proved some relevant works (see [10] and [12]) on the Hardy-Littlewood-Sobolev inequality and the Stein-Weiss inequality with fractional Poisson kernel on the upper half space, which was motivated by the work [28].

This paper is organized as follows. In Sections 2 and 3, we prove the reversed Stein–Weiss inequality, namely, the weighted reversed Hardy–Littlewood–Sobolev inequality and the existence of extremal functions for the inequality. In Section 4, we obtain some asymptotic behaviors of solutions to the corresponding Euler–Lagrange system. In Section 5, we use the Pohozaev identity to obtain necessary conditions of the existence of any positive solutions of (11). In Sections 6 and 7, we establish the reversed Stein–Weiss inequality and Stein–Weiss inequality on the sphere $\mathbb{S}^n$ by stereographic projection.

2. The proof of Theorem 1

Let G be a locally compact group. It is known that G possesses a positive measure μ on the Borel set that is nonzero on all nonempty open sets and is left invariant, i.e.,

$$\mu(tA) = \mu(A),$$

for any $t \in G$ and $A \subseteq G$. Such a measure μ is called a left Haar measure on G.

The convolution of two functions $g, h \in L^1(G)$ is defined as

$$(g * h)(x) = \int_G g(y)h(y^{-1}x)d\mu(y),$$

where y^{-1} denotes the inverse of y in the group G.

To prove Theorem 1, we need the following reversed Young's inequality.

Lemma 10. *Let G be a locally compact group with left Haar measure μ that satisfies $\mu(A) = \mu(A^{-1})$ for all measurable sets $A \subseteq G$. Assume that p, q, and s satisfy*

$$0 < p < 1, \quad q, s < 0, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{s}.$$

Then for any nonnegative $g \in L^p(G, \mu)$ and $h \in L^s(G, \mu)$, we have

$$\|g * h\|_{L^q(G,\mu)} \geq \|g\|_{L^p(G,\mu)} \|h\|_{L^s(G,\mu)}.$$

Proof. One can check that

$$\frac{1}{q} + \frac{1}{p'} + \frac{1}{s'} = 1, \quad \frac{p}{q} + \frac{p}{s'} = 1, \quad \frac{s}{q} + \frac{s}{p'} = 1.$$

Using the reversed Hölder inequality with respect to exponents q, p', and s', we obtain

$$\begin{aligned}
(g * h)(x) &= \int_G g(y)h(y^{-1}x)d\mu(y) \\
&= \int_G g^{\frac{p}{s'}}(y) g^{\frac{p}{q}}(y) h^{\frac{s}{q}}(y^{-1}x) h^{\frac{s}{p'}}(y^{-1}x)d\mu(y) \\
&\geq \|g\|^{\frac{p}{s'}}_{L^p(G,\mu)} \Big(\int_G g^p(y)h^s(y^{-1}x)d\mu(y) \Big)^{\frac{1}{q}} \Big(\int_G h^s(y^{-1}x)d\mu(y) \Big)^{\frac{1}{p'}} \\
&= \|g\|^{\frac{p}{s'}}_{L^p(G,\mu)} \|h\|^{\frac{s}{p'}}_{L^s(G,\mu)} \Big(\int_G g^p(y)h^s(y^{-1}x)d\mu(y) \Big)^{\frac{1}{q}}.
\end{aligned}$$

Now take L^q norms in the variable x and apply Fubini's theorem to deduce that

$$\begin{aligned}
\|g * h\|_{L^q(G,\mu)} &\geq \|g\|^{\frac{p}{s'}}_{L^p(G,\mu)} \|h\|^{\frac{s}{p'}}_{L^s(G,\mu)} \Big(\int_G \int_G g^p(y)h^s(y^{-1}x)d\mu(x)d\mu(y) \Big)^{\frac{1}{q}} \\
&= \|g\|^{\frac{p}{s'}}_{L^p(G,\mu)} \|h\|^{\frac{s}{p'}}_{L^s} \|g\|^{\frac{p}{q}}_{L^p(G,\mu)} \|h\|^{\frac{s}{q}}_{L^s(G,\mu)} \\
&= \|g\|_{L^p(G,\mu)} \|h\|_{L^s(G,\mu)}.
\end{aligned}$$

□

It is easy to check that $\mu = \frac{dx}{|x|}$ is a left Haar measure and satisfies $\mu(A) = \mu(A^{-1})$ for all measurable sets $A \subseteq G = \mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and that $\mu = \frac{dx}{x}$ is a left Haar measure and satisfies $\mu(A) = \mu(A^{-1})$ for all measurable sets $A \subseteq G = \mathbb{R}^+ = (0, \infty)$.

Next, we will introduce some properties about the rearrangement which will be used in the proof of the main theorem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let u be a nonnegative function. Define a radially symmetric function $u^* : \Omega^* = B_R(0) \to \mathbb{R}$ satisfying $|B_R(0)| = |\Omega|$, and for any $s > 0$

$$|\{x \in B_R(0) : u^*(x) > s\}| = |\{x \in \Omega : u(x) > s\}|.$$

Then u be a decreasing function and is called the rearrangement of u. Assume $F : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function. By construction

$$\int_\Omega F(u)dx = \int_{B_R(0)} F(u^*)dx.$$

When $\Omega = \mathbb{R}^n$, the rearrangement can be defined in a similar way. Let u be a nonnegative function. The rearrangement of u is defined by

$$u^* = \int_0^\infty \chi_{\{u>t\}^*}(x)dt,$$

where $\chi_{\{u>t\}^*}$ is the characteristic function of the set $\{u > t\}^* = B_r(0)$ with $|B_r(0)| = |\{u > t\}|$. Then u^* is radially decreasing and satisfies

$$\int_{\mathbb{R}^n} F(u)dx = \int_{\mathbb{R}^n} F(u^*)dx.$$

For $\lambda > 0$, let

$$(T_\lambda g)(x) = \int_{\mathbb{R}^n} g(y)|x-y|^\lambda dy.$$

In order to prove the reversed Stein–Weiss inequality, it is equivalent to prove

$$\||x|^\alpha T_\lambda g\|_{L^q(\mathbb{R}^n)} \geq C_{n,\alpha,\beta,p,q'}\||x|^{-\beta} g\|_{L^p(\mathbb{R}^n)}.$$

The following lemma is a direct result of [30, 33].

Lemma 11.

(i) *For any nonnegative functions $f(x)$ and $g(x)$ defined on $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f(x)g(y)|y|^\beta dxdy \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f^*(x)g^*(y)|y|^\beta dxdy,$$

where f^* and g^* are rearrangements of f and g.

(ii) *If g is radially decreasing, then $V_\lambda(g)$ is radially increasing.*

(iii) *For any nonnegative function $g \in L^p(\mathbb{R}^n)$, there holds*

$$\|V_\lambda(g)\|_{L^q(\mathbb{R}^n)} \geq \|V_\lambda(g^*)\|\|_{L^q(\mathbb{R}^n)}.$$

We now begin the proof of Theorem 1.

Proof. We can distinguish two cases as follows.

Case 1. For $n = 1$, one only needs to prove

$$\||x|^{\alpha+\frac{1}{q}} T_\lambda g\|_{L^q(\mu)} \geq C_{n,\alpha,\beta,p,q'}\||x|^{-\beta+\frac{1}{p}} g\|_{L^p(\mu)},$$

where $d\mu(x) = \frac{dx}{|x|}$.

Since $\frac{1}{p} - \frac{1}{q} - (\alpha + \beta + \lambda) = 1$, we have

$$|x|^{\alpha+\frac{1}{q}} T_\lambda g(x) = \int_{-\infty}^{\infty} \frac{|x|^{\alpha+\frac{1}{q}} g(y)|y|^{-\beta+\frac{1}{p}}}{|y|^{\alpha+\frac{1}{q}}|1-\frac{x}{y}|^{-\lambda}} \frac{dy}{|y|} = (\bar{g} * h)(x),$$

where $\bar{g}(x) = g(x)|x|^{-\beta+\frac{1}{p}}$ and $h(x) = |x|^{\alpha+\frac{1}{q}}|1-x|^{\lambda}$.

By Lemma 10, since $s < 0$, $0 < \alpha < -\frac{1}{q}$ and $0 < \beta < -\frac{1}{p'}$, we conclude that

$$\begin{aligned} \||\cdot|^{\alpha+\frac{1}{q}} T_\lambda g\|_{L^q(\mu)} &\geq \||\cdot|^{-\beta+\frac{1}{p}} g\|_{L^p(\mu)} \|h\|_{L^s(\mu)} \\ &\geq C_{n,\alpha,\beta,p,q'} \||\cdot|^{-\beta+\frac{1}{p}} g\|_{L^p(\mu)}, \end{aligned} \tag{16}$$

where $\frac{1}{p} + \frac{1}{s} = 1 + \frac{1}{q}$.

Case 2. According to Lemma 11, we may assume that g is radially symmetric and decreasing. For $n \geq 2$, it suffices to prove

$$\||x|^{\alpha} T_\lambda g\|_{L^q(\mathbb{R}^n)} \geq C_{n,\alpha,\beta,p,q'} \||x|^{-\beta} g\|_{L^p(\mathbb{R}^n)}. \tag{17}$$

To show this, we can write the fractional integral operator acting on a radial function as a convolution in the group $\mathbb{R}^+$ with Haar measure $\mu = \frac{dx}{x}$. This is a useful technique from harmonic analysis and has also been used in, e.g., [36], [18]. To this end, we will need the following lemma whose proof can be found on, e.g., p. 420 in [24].

Lemma 12. *Let $x \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and denote*

$$I(x) = \int_{S^{n-1}} \varphi(x \cdot y) dy.$$

Then $I(x)$ is a constant independent of x and

$$I(x) = \omega_{n-2} \int_{-1}^{1} \varphi(t)(1-t^2)^{\frac{n-3}{2}} dt,$$

where ω_{n-2} denotes the area of S^{n-2}.

Set $|x| = \rho$; by Lemma 12,

$$\begin{aligned} T_\lambda g(x) &= \int_{\mathbb{R}^n} g(y)|x-y|^{\lambda} dy \\ &= \int_0^\infty \int_{S^{n-1}} g(r)|x - ry'|^{\lambda} r^{n-1} dr dy' \\ &= \int_0^\infty \int_{S^{n-1}} g(r)(r^2 - 2r\rho x'y' + \rho^2)^{\frac{\lambda}{2}} r^{n-1} dr dy' \\ &= \omega_{n-2} \int_0^\infty \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} (r^2 - 2r\rho t + \rho^2)^{\frac{\lambda}{2}} dt g(r) r^{n-1} dr \\ &= \omega_{n-2} \int_0^\infty \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \Big(1 - 2(\frac{\rho}{r})t + (\frac{\rho}{r})^2\Big)^{\frac{\lambda}{2}} dt g(r) r^{n+\lambda-1} dr \\ &= \omega_{n-2} \int_0^\infty g(r) r^{n+\lambda-1} I_\lambda(\frac{\rho}{r}) dr, \end{aligned} \tag{18}$$

where

$$I_\lambda(a) = \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} (1 - 2at + a^2)^{\frac{\lambda}{2}} dt, \quad a > 0.$$

It is obvious that $I_\lambda(a)$ is well defined and continuous for any $a>0$. Then, we can apply (18) to derive that

$$\begin{aligned}\rho^{\frac{n}{q}+\alpha}T_\lambda g(x) &= \omega_{n-2}\int_0^\infty g(r)r^{n+\lambda+\frac{n}{q}+\alpha}\frac{\rho^{\frac{n}{q}+\alpha}}{r^{\frac{n}{q}+\alpha}}I_\lambda(\frac{\rho}{r})\frac{dr}{r}\\ &= \omega_{n-2}\big(g(r)r^{n+\lambda+\frac{n}{q}+\alpha}\big)*\big(r^{\frac{n}{q}+\alpha}I_\lambda(r)\big)(\rho).\end{aligned}$$

Since $\frac{n}{p}-\frac{n}{q}-(\alpha+\beta+\lambda)=n$, one can calculate

$$\begin{aligned}&\omega_{n-1}^{\frac{1}{p}}\|g(r)r^{n+\lambda+\frac{n}{q}+\alpha}\|_{L^p(\mu)}\\ &=\omega_{n-1}^{\frac{1}{p}}\Big(\int_0^{+\infty}(g(r))^p r^{(n+\lambda+\frac{n}{q}+\alpha)p-n}r^n\frac{dr}{r}\Big)^{\frac{1}{p}}\\ &=\|g(x)|x|^{n+\lambda+\frac{n}{q}+\alpha-\frac{n}{p}}\|_{L^p(\mathbb{R}^n)}\\ &=\|g(x)|x|^{-\beta}\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

By the dominated convergence theorem, one can obtain

$$\frac{I_\lambda(r)}{r^\lambda}\sim\int_{-1}^{1}(1-t^2)^{\frac{n-3}{2}}dt,\quad\text{as}\quad r\to\infty.$$

Now we check that $\|r^{\frac{n}{q}+\alpha}I_\lambda(r)\|_{L^s(\mu)}\neq 0$. Since $s<0$, we only need to prove

$$\int_0^{+\infty}\big(r^{\frac{n}{q}+\alpha}I_\lambda(r)\big)^s r^{-1}dr<+\infty. \tag{19}$$

Note that $\big(r^{\frac{n}{q}+\alpha}I_\lambda(r)\big)^s r^{-1}\sim Cr^{(\frac{n}{q}+\alpha)s-1}$ as $\lambda\to 0$ and $\big(r^{\frac{n}{q}+\alpha}I_\lambda(r)\big)^s r^{-1}\sim Cr^{(\frac{n}{q}+\alpha+\lambda)s-1}$ as $\lambda\to+\infty$. In order to guarantee that (19) is finite, we only need to verify that

$$(\frac{n}{q}+\alpha)s-1>-1\ \text{ and }\ (\frac{n}{q}+\alpha+\lambda)s-1<-1.$$

In fact, it is equivalent to check that

$$\frac{n}{q}+\alpha<0\ \text{ and }\ \frac{n}{q}+\alpha+\lambda>0.$$

Recall from the assumption of Theorem 2 that $0\le\alpha<-\frac{n}{q}$, $0\le\beta<-\frac{n}{p'}$, and $\frac{n}{p}-\frac{n}{q}-(\alpha+\beta+\lambda)=n$. Thus $\frac{n}{q}+\alpha+\lambda=\frac{n}{p}-n-\beta=-\frac{n}{p'}-\beta$, which implies that $\frac{n}{q}+\alpha<0$ and $\frac{n}{q}+\alpha+\lambda>0$. So we obtain (19). Therefore, we accomplish the proof of Theorem 1. □

3. The proof of Theorem 2

In this section, we will prove Theorem 2. We can divide our proof into two steps.

Step 1. We will choose a suitable minimizing sequence $\{g_j\}_j$ for (5) satisfying $g_j(1)>c_0$.

Let $\{g_j\}_j$ be a minimizing sequence for the problem (5). According to Lemma 11, we can assume that $\{g_j\}_j$ is a nonnegative radially symmetric and decreasing minimizing sequence.

For any $R > 0$, we have

$$\begin{aligned} v_n g_j^p(R)R^n &\le \omega_{n-1}\int_0^R g_j^p(r)r^{n-1}dr \\ &\le \omega_{n-1}\int_0^{+\infty} g_j^p(r)r^{n-1}dr \\ &= \int_{\mathbb{R}^n} g_j^p(x)dx = 1. \end{aligned}$$

We can apply the above estimate to conclude that

$$0 \le g_j(R) \le CR^{-\frac{n}{p}}$$

for any $R > 0$ and for some constant C independent of j. We also need the following lemma proved in [20, 30].

Lemma 13. *Suppose that $g \in L^p(\mathbb{R}^n)$ is nonnegative, radially symmetric, and satisfies $g(|x|) \le \varepsilon |x|^{-\frac{n}{p}}$ for all $|x| > 0$. Then for any $0 < t < p$, there exists a constant $C > 0$, independent of g and ε such that*

$$\|V_\lambda(g)\|_{L^q(\mathbb{R}^n)} \ge C\varepsilon^{1-\frac{p}{t}}\|g\|_{L^p(\mathbb{R}^n)}^{\frac{p}{t}}.$$

Let

$$a_j = \sup_{r>0} r^{\frac{n}{p}} g_j(r) \le C.$$

Since $\|g_j\|_{L^p(\mathbb{R}^n)} = 1$ and $\|V_\lambda(g_j)\|_{L^q(\mathbb{R}^n)} \to C_{n,\alpha,\beta,p,q'} < \infty$, it follows from Lemma 13 that $a_j \ge 2c_0$ for some $c_0 > 0$. Therefore, we can choose $\lambda_j > 0$ such that $\lambda_j^{\frac{n}{p}} g_j(\lambda_j) > c_0$. Then we set

$$\tilde{g}_j(x) = \lambda_j^{\frac{n}{p}} g_j(\lambda_j x).$$

It is easy to check that $\|\tilde{g}_j\|_{L^p} = \|g_j\|_{L^p} = 1$ and $\|V_\lambda \tilde{g}_j\|_{L^q} = \|V_\lambda g_j\|_{L^q}$. Then $\{\tilde{g}_j(x)\}_j$ is also a minimizing sequence. Consequently, replacing the sequence $\{g_j(x)\}_j$ with the new sequence $\{\tilde{g}_j(x)\}_j$, if necessary, one can further assume that our sequence $\{g_j(x)\}_j$ obeys $g_j(1) \ge c_0$ for any j.

Similar to Lieb's argument which is based on the Helly theorem, by passing to a subsequence, we have $g_j \to g$ a.e. in $\mathbb{R}^n$. It is evident that g is nonnegative radially symmetric and decreasing. The rest of our arguments is to show that g is indeed the desired minimizer for (5).

Step 2. We will show that g is actually a minimizer of (5).

One can apply Lemma 11 to derive that $(V_\lambda g_j)(x)$ is radially symmetric and increasing for any j. Moreover, for all $x \in \mathbb{R}^n$, we have

$$(V_\lambda g_j)(x) \ge c_0|x|^\alpha \int_{|y|\le 1} |x-y|^\lambda |y|^\beta dy \ge C(1+|x|^{\lambda+\alpha}) \tag{20}$$

for some constant C independent of j.

Since $\lim_{j\to+\infty} \|V_\lambda g_j\|_{L^q} \to C_{n,\alpha,\beta,p,q'}$, there exists some constant $C > 0$ such that $\|V_\lambda g_j\|_{L^q}^q \le C$ for any j. Then for any $R > 0$, one can estimate

$$\begin{aligned} v_n (V_\lambda g_j)^q(R)R^n &\le \int_{|x|\le R} (V_\lambda g_j)^q(x)dx \\ &\le \int_{\mathbb{R}^n} (V_\lambda g_j)^q(x)dx \le C. \end{aligned}$$

Consequently, we have

$$0 \leq \big(V_\lambda g_j(R)\big)^{-1} \leq C_4 R^{\frac{n}{q}}$$

for all $R > 0$. Since $(V_\lambda g_j(x))^{-1}$ is radially symmetric and decreasing, we can use the Helly theorem to conclude that $(V_\lambda g_j(x))^{-1} \to k(x)$ a.e. in $\mathbb{R}^n$ for some function $k(x)$. By (20) and the dominated convergence theorem, we derive that

$$\lim_{j\to\infty}\Big(\int_{\mathbb{R}^n}(V_\lambda g_j)^q(x)dx\Big)^{\frac{1}{q}} = \Big(\int_{\mathbb{R}^n}\lim_{j\to\infty}(V_\lambda g_j)^q(x)dx\Big)^{\frac{1}{q}}.$$

For $|x| > 2R$, we can employ the reversed Hölder inequality to obtain that

$$\begin{aligned}\int_{\mathbb{R}^n}|x|^\alpha|x-y|^\lambda g_j(y)|y|^\beta dy &\geq C(R)\int_{|y|\leq R} g_j(y)|y|^\beta dy\\ &\geq C(R)\Big(\int_{|y|\leq R} g_j^\tau(y)dy\Big)^{\frac{1}{\tau}}\Big(\int_{|y|\leq R}|y|^{\tau'\beta}dy\Big)^{\frac{1}{\tau'}},\end{aligned}$$

where $0 < \tau < 1$ and $\frac{1}{\tau} + \frac{1}{\tau'} = 1$. Since $0 < \beta < -\frac{n}{p'}$, one can choose τ satisfying $\tau > p$ and $\tau'\beta > -n$. Then, there exists some constant $C(R)$ such that

$$\int_{|y|\leq R} g_j^\tau(y)dy < C(R). \tag{21}$$

Select $\bar{x}$ satisfying that $1 < |\bar{x}| < \frac{R}{2}$; then we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|\bar{x}|^\alpha|\bar{x}-y|^\lambda g_j(y)|y|^\beta dy\\ &\geq (\frac{R}{4})^\lambda\int_{\frac{3}{4}R\leq|y|\leq R} g_j(y)|y|^\beta dy\\ &\geq (\frac{R}{4})^\lambda g_j(R)\int_{\frac{3}{4}R\leq|y|\leq R}|y|^\beta dy\\ &\geq (\frac{R}{4})^{\lambda+n+\beta} g_j(R).\end{aligned}$$

Therefore, there exists some constant C such that $g_j(R) < CR^{-n-\lambda-\beta}$ for sufficiently large R. One can combine this fact with $0 < \alpha < -\frac{n}{q}$ to conclude that

$$\lim_{R\to\infty}\lim_{j\to\infty}\int_{|y|\geq R} g_j^p(y)dy = 0. \tag{22}$$

Combining (21) and (22), we derive that

$$\begin{aligned}\lim_{j\to\infty}\int_{\mathbb{R}^n} g_j^p(y)dy &= \lim_{R\to\infty}\lim_{j\to\infty}\int_{|y|\leq R} g_j^p(y)dy + \lim_{R\to\infty}\lim_{j\to\infty}\int_{|y|\geq R} g_j^p(y)dy\\ &= \lim_{R\to\infty}\lim_{j\to\infty}\int_{|y|\leq R} g_j^p(y)dy\\ &= \lim_{R\to\infty}\int_{|y|\leq R} g^p(y)dy\\ &= \int_{\mathbb{R}^n} g^p(y)dy.\end{aligned}$$

Therefore, $\|g\|_{L^q} = 1$. We can use Fatou's lemma to obtain that

$$\lim_{j\to\infty}\big(V_\lambda g_j(x)\big)^q = \big(\lim_{j\to\infty}V_\lambda g_j(x)\big)^q \leq \big(V_\lambda g(x)\big)^q.$$

Then

$$
\begin{aligned}
C_{n,\alpha,\beta,p,q'} &= \lim_{j\to\infty}\Big(\int_{\mathbb{R}^n}(V_\lambda g_j)^q(x)dx\Big)^{\frac{1}{q}}\\
&= \Big(\lim_{j\to\infty}\int_{\mathbb{R}^n}(V_\lambda g_j)^q(x)dx\Big)^{\frac{1}{q}}\\
&= \Big(\int_{\mathbb{R}^n}\lim_{j\to\infty}(V_\lambda g_j)^q(x)dx\Big)^{\frac{1}{q}}\\
&\geq \Big(\int_{\mathbb{R}^n}(V_\lambda g)^q(x)dx\Big)^{\frac{1}{q}}\\
&\geq C_{n,\alpha,\beta,p,q'}\|g\|_{L^q} = C_{n,\alpha,\beta,p,q'}.
\end{aligned} \tag{23}
$$

Therefore, by $\|g\|_{L^q} = 1$ and (23), one concludes that g is actually a minimizer of (5). This completes the proof of Theorem 2.

4. The proof of Theorem 3

In this section, we consider some asymptotic behavior of positive solutions for the weighted integral system (10). We first prove the following lemma.

Lemma 14. *For α, β, p_1, p_2, and $\lambda > 0$, let (u, v) be a pair of positive Lebesgue measurable solutions of* (10). *Then*

$$\int_{\mathbb{R}^n}(1+|y|^\lambda)v^{-p_2}(y)|y|^\beta dy < \infty, \quad \int_{\mathbb{R}^n}(1+|y|^\lambda)u^{-p_1}(y)|y|^\alpha dy < \infty, \tag{24}$$

and for some constant $C_1, C_2 \geq 1$,

$$\frac{1}{C_1}(1+|x|^\lambda) \leq \frac{u(x)}{|x|^\alpha} \leq C_1(1+|x|^\lambda), \quad \frac{1}{C_2}(1+|x|^\lambda) \leq \frac{v(x)}{|x|^\beta} \leq C_2(1+|x|^\lambda). \tag{25}$$

Proof. We only deal with

$$\frac{1}{C_1}(1+|x|^\lambda) \leq \frac{u(x)}{|x|^\alpha} \leq C_1(1+|x|^\lambda) \tag{26}$$

and

$$\int_{\mathbb{R}^n}(1+|y|^\lambda)v^{-p_2}(y)|y|^\beta dy < \infty. \tag{27}$$

Since (u, v) is a pair of positive Lebesgue measurable solutions of (10), we have

$$\text{meas}\{x \in \mathbb{R}^n | u(x) < +\infty\} > 0, \quad \text{meas}\{x \in \mathbb{R}^n | v(x) < +\infty\} > 0.$$

Moreover, there exists $R > 1$ and some measurable set E such that

$$E \subset \{x \in \mathbb{R}^n | u(x), v(x) < R\} \cap B_R$$

with $|E| > \frac{1}{R}$.

For $|x| > 2R > 2$,

$$\begin{aligned}\frac{u(x)}{|x|^\alpha} &= \int_{\mathbb{R}^n} |x-y|^\lambda v^{-p_2}(y)|y|^\beta dy \\ &\geq C\int_E (1+|x|^\lambda)v^{-p_2}(y)|y|^\beta dy \\ &\geq C(1+|x|^\lambda)R^{-p_2}\int_E |y|^\beta dy \\ &\geq C_R(1+|x|^\lambda).\end{aligned} \tag{28}$$

For $0 < |x| \leq 2R$,

$$\begin{aligned}\frac{u(x)}{|x|^\alpha(1+|x|^\lambda)} &\geq \frac{1}{1+(2R)^\lambda}R^{-p_2}\int_E |x-y|^\lambda |y|^\beta dy \\ &\geq \frac{1}{1+(2R)^\lambda}R^{-p_2}\int_{E\cap\{|x-y|\leq|y|\}} |x-y|^{\lambda+\beta}dy \\ &\quad + \frac{1}{1+(2R)^\lambda}R^{-p_2}\int_{E\cap\{|x-y|\geq|y|\}} |y|^{\lambda+\beta}dy \\ &\geq \frac{c}{1+(2R)^\lambda}R^{-p_2}\Big(|E\cap\{|x-y|\leq|y|\}|^{1+\frac{\lambda+\beta}{n}} \\ &\quad + |E\cap\{|x-y|\geq|y|\}|^{1+\frac{\lambda+\beta}{n}}\Big) \\ &\geq \frac{c_0}{1+(2R)^\lambda}R^{-p_2-1-\frac{\lambda+\beta}{n}}.\end{aligned}$$

Then, for any $x\in\mathbb{R}^n\setminus\{0\}$,

$$\frac{u(x)}{|x|^\alpha} \geq \frac{1}{C_1}(1+|x|^\lambda).$$

Thus we obtain the left-hand side of the inequality in (26).

Similarly, for any $x\in\mathbb{R}^n\setminus\{0\}$, we also have

$$\frac{v(x)}{|x|^\beta} \geq \frac{1}{C_2}(1+|x|^\lambda). \tag{29}$$

Next, we show that

$$\int_{\mathbb{R}^n}(1+|y|^\lambda)v^{-p_2}(y)|y|^\beta dy < \infty.$$

There exists some $\bar{x}\in E\setminus\{0\}$ such that

$$u(\bar{x}) = \int_{\mathbb{R}^n}|\bar{x}|^\alpha|x-y|^\lambda v^{-p_2}(y)|y|^\beta dy < +\infty.$$

Moreover,

$$\begin{aligned}&\int_{\mathbb{R}^n}(1+|y|^\lambda)v^{-p_2}(y)|y|^\beta dy \\ &\leq C_{\bar{x}}\int_{|y|<\frac{1}{2}|\bar{x}|}|\bar{x}-y|^\lambda v^{-p_2}(y)|y|^\beta dy + C_{\bar{x}}\int_{|y|>2|\bar{x}|}|\bar{x}-y|^\lambda v^{-p_2}(y)|y|^\beta dy \\ &\quad + \int_{\frac{1}{2}|\bar{x}|\leq|y|\leq 2|\bar{x}|}(1+|y|^\lambda)v^{-p_2}(y)|y|^\beta dy,\end{aligned}$$

which combines with (29), and we obtain (27).

For $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} \frac{u(x)}{|x|^{\alpha}(1+|x|^{\lambda})} &= \int_{\mathbb{R}^n} \frac{|x-y|^{\lambda}}{(1+|x|^{\lambda})} v^{-p_2}(y)|y|^{\beta} dy \\ &\leq \int_{\mathbb{R}^n} (1+|y|^{\lambda}) v^{-p_2}(y)|y|^{\beta} dy < +\infty. \end{aligned} \tag{30}$$

Then the right-hand side of the inequality in (26) follows from (27) and (30). This completes the proof of Lemma 14. □

Now, we start our proof of Theorem 3. The proof is carried out in two parts.

Part I. We show the asymptotic behavior of u, v around infinity.

For $|x| > 1$, by Lemma 14, we have

$$|x|^{-\lambda} \int_{\mathbb{R}^n} |x-y|^{\lambda} |y|^{\beta} v^{-p_2}(y) dy \leq C_{\lambda} \int_{\mathbb{R}^n} (1+|y|^{\lambda}) v^{-p_2}(y)|y|^{\beta} dy < \infty$$

and

$$|x|^{-\lambda} \int_{\mathbb{R}^n} |x-y|^{\lambda} |y|^{\alpha} u^{-p_1}(y) dy \leq C_{\lambda} \int_{\mathbb{R}^n} (1+|y|^{\lambda}) u^{-p_1}(y)|y|^{\alpha} dy < \infty.$$

Then, we can apply the dominated convergence theorem to obtain

$$\begin{aligned} \lim_{|x|\to\infty} \frac{u(x)}{|x|^{\lambda+\alpha}} &= \lim_{|x|\to\infty} |x|^{-\lambda-\alpha} \int_{\mathbb{R}^n} |x|^{\alpha} |x-y|^{\lambda} |y|^{\beta} v^{-p_2}(y) dy \\ &= \lim_{|x|\to\infty} |x|^{-\lambda} \int_{\mathbb{R}^n} |x-y|^{\lambda} |y|^{\beta} v^{-p_2}(y) dy \\ &= \int_{\mathbb{R}^n} v^{-p_2}(y)|y|^{\beta} dy \end{aligned}$$

and

$$\begin{aligned} \lim_{|x|\to\infty} \frac{v(x)}{|x|^{\lambda+\beta}} &= \lim_{|x|\to\infty} |x|^{-\lambda-\beta} \int_{\mathbb{R}^n} |x|^{\beta} |x-y|^{\lambda} |y|^{\alpha} u^{-p_1}(y) dy \\ &= \lim_{|x|\to\infty} |x|^{-\lambda} \int_{\mathbb{R}^n} |x-y|^{\lambda} |y|^{\alpha} u^{-p_1}(y) dy \\ &= \int_{\mathbb{R}^n} u^{-p_1}(y)|y|^{\alpha} dy. \end{aligned}$$

Then we accomplish the proof of Part I.

Part II. We show the asymptotic behavior of u, v around the origin.

For $0 < |x| < 1$, by Lemma 14, we have

$$\int_{\mathbb{R}^n} |x-y|^{\lambda} v^{-p_2}(y)|y|^{\beta} dy \leq C_{\lambda} \int_{\mathbb{R}^n} (1+|y|^{\lambda}) v^{-p_2}(y)|y|^{\beta} dy < \infty$$

and

$$\int_{\mathbb{R}^n} |x-y|^{\lambda} u^{-p_1}(y)|y|^{\alpha} dy \leq C_{\lambda} \int_{\mathbb{R}^n} (1+|y|^{\lambda}) u^{-p_1}(y)|y|^{\alpha} dy < \infty.$$

Then, we employ the dominated convergence theorem to obtain

$$\begin{aligned} \lim_{|x|\to 0} \frac{u(x)}{|x|^{\alpha}} &= \lim_{|x|\to 0} \int_{\mathbb{R}^n} |x-y|^{\lambda} v^{-p_2}(y)|y|^{\beta} dy \\ &= \int_{\mathbb{R}^n} v^{-p_2}(y)|y|^{\lambda+\beta} dy \end{aligned}$$

and

$$\lim_{|x|\to 0}\frac{v(x)}{|x|^{\beta}}=\lim_{|x|\to 0}\int_{\mathbb{R}^n}|x-y|^{\lambda}u^{-p_1}(y)|y|^{\alpha}dy$$
$$=\int_{\mathbb{R}^n}u^{-p_1}(y)|y|^{\lambda+\alpha}dy.$$

This accomplishes the proof of Part II.

5. The proof of Theorem 4

In this section, we will prove Theorem 4. For λ, ν_1, ν_2, p_1, $p_2>0$, assume that (u,v) is a pair of positive solutions of the following integral system:

$$\begin{cases} u(x)=\int_{\mathbb{R}^n}|x-y|^{\lambda}|y|^{\nu_2}v^{-p_2}(y)dy,\\ v(x)=\int_{\mathbb{R}^n}|x-y|^{\lambda}|y|^{\nu_1}u^{-p_1}(y)dy. \end{cases} \tag{31}$$

We can apply the integration by parts to obtain

$$\begin{aligned}
&\int_{B_R}|x|^{\nu_1}u^{-p_1}(x)(x\cdot\nabla u(x))dx\\
&\quad=\frac{1}{1-p_1}\int_{B_R}|x|^{\nu_1}x\cdot\nabla(u^{1-p_1}(x))dx\\
&\quad=\frac{1}{1-p_1}\int_{\partial B_R}u^{1-p_1}(x)R^{1+\nu_1}d\sigma-\frac{n+\nu_1}{1-p_1}\int_{B_R}|x|^{\nu_1}u^{1-p_1}(x)dx.
\end{aligned}$$

Similarly, one can also derive that

$$\begin{aligned}
&\int_{B_R}|x|^{\nu_2}v^{-p_2}(x)(x\cdot\nabla v(x))dx\\
&\quad=\frac{1}{1-p_2}\int_{\partial B_R}v^{1-p_2}(x)R^{1+\nu_2}d\sigma-\frac{n+\nu_2}{1-p_2}\int_{B_R}|x|^{\nu_2}v^{1-p_2}(x)dx.
\end{aligned}$$

By Lemma 14, we have

$$\int_{\mathbb{R}^n}|x|^{\nu_1}u^{1-p_1}(x)dx<\infty,\quad \int_{\mathbb{R}^n}|x|^{\nu_2}v^{1-p_2}(x)dx<\infty.$$

Then, there exists $R=R_j\to+\infty$ such that

$$R^{1+\nu_1}\int_{\partial B_R}u^{1-p_1}(x)d\sigma\to 0,\quad R^{1+\nu_2}\int_{\partial B_R}v^{1-p_2}(x)d\sigma\to 0.$$

Therefore, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}|x|^{\nu_1}u^{-p_1}(x)(x\cdot\nabla u(x))dx+\int_{\mathbb{R}^n}|x|^{\nu_2}v^{-p_2}(x)(x\cdot\nabla v(x))dx\\
&\quad=-\frac{n+\nu_1}{1-p_1}\int_{\mathbb{R}^n}|x|^{\nu_1}u^{1-p_1}(x)dx-\frac{n+\nu_2}{1-p_2}\int_{\mathbb{R}^n}|x|^{\nu_2}v^{1-p_2}(x)dx.
\end{aligned}\tag{32}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^{\nu_1} u^{-p_1}(x)(x\cdot\nabla u(x))dx \\
&\quad = \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x\cdot(x-y)|x-y|^{\lambda-2}|x|^{\nu_1}|y|^{\nu_2} u^{-p_1}(x)v^{-p_2}(y)dydx \\
&\quad = \frac{\lambda}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x\cdot(x-y)|x-y|^{\lambda-2}|x|^{\nu_1}|y|^{\nu_2} u^{-p_1}(x)v^{-p_2}(y)dydx \\
&\qquad + \frac{\lambda}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} y\cdot(y-x)|x-y|^{\lambda-2}|x|^{\nu_2}|y|^{\nu_1} u^{-p_1}(y)v^{-p_2}(x)dxdy.
\end{aligned}$$

One can also derive

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^{\nu_2} v^{-p_2}(x)(x\cdot\nabla v(x))dx \\
&\quad = \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x\cdot(x-y)|x-y|^{\lambda-2}|x|^{\nu_2}|y|^{\nu_1} v^{-p_2}(x)u^{-p_1}(y)dydx \\
&\quad = \frac{\lambda}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x\cdot(x-y)|x-y|^{\lambda-2}|x|^{\nu_2}|y|^{\nu_1} v^{-p_2}(x)u^{-p_1}(y)dydx \\
&\qquad + \frac{\lambda}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} y\cdot(y-x)|x-y|^{\lambda-2}|x|^{\nu_1}|y|^{\nu_2} v^{-p_2}(y)u^{-p_1}(x)dxdy.
\end{aligned}$$

Applying Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^{\nu_1} u^{-p_1}(x)(x\cdot\nabla u(x))dx + \int_{\mathbb{R}^n} |x|^{\nu_2} v^{-p_2}(x)(x\cdot\nabla v(x))dx \\
&\quad = \frac{\lambda}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda}|x|^{\nu_2}|y|^{\nu_1} v^{-p_2}(x)u^{-p_1}(y)dxdy \\
&\qquad + \frac{\lambda}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda}|x|^{\nu_1}|y|^{\nu_2} v^{-p_2}(y)u^{-p_1}(x)dxdy \\
&\quad = \lambda \int_{\mathbb{R}^n} |x|^{\nu_1} u^{1-p_1}(x)dx \\
&\quad = \lambda \int_{\mathbb{R}^n} |x|^{\nu_2} v^{1-p_2}(x)dx.
\end{aligned}$$

Then by (32), we derive that $\frac{n+\nu_1}{p_1-1} + \frac{n+\nu_2}{p_2-1} = \lambda$.

6. The proof of Theorem 7

In this section, we use the stereographic projection to establish an equivalent form of the reversed Stein–Weiss inequality (4) on the sphere $\mathbb{S}^n$ in the case of $q' = p$.

Let $S : x \in \mathbb{R}^n \to \xi \in \mathbb{S}^n \setminus \{(0,0,\dots,-1)\}$ be the inverse of a stereographic projection, defined by

$$\xi_i = \frac{2x_i}{1+|x|^2} \quad \text{for } i = 1,2,\dots,n; \quad \xi_{n+1} = \frac{1-|x|^2}{1+|x|^2}.$$

For x, $y \in \mathbb{R}^n$ and $\xi \in \mathbb{S}^n$, one can refer to [30,31] to obtain

$$|S(x)-S(y)| = \Big(\frac{4|x-y|^2}{(1+|x|^2)(1+|y|^2)}\Big)^{\frac{1}{2}}, \quad d\xi = \Big(\frac{2}{1+|x|^2}\Big)^n dx.$$

For $\lambda,\ \alpha,\ \beta > 0$, $q' = p = \frac{2n}{2n+\lambda+\alpha+\beta}$, $\xi,\ \eta \in \mathbb{S}^n$, $f \in L^{q'}(\mathbb{R}^n)$, and $g \in L^p(\mathbb{R}^n)$, define

$$F(\xi) = \Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}} f(x),$$

$$F_1(\xi) = \Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}} \Big(\frac{1}{1+|x|^2}\Big)^{\frac{\beta}{2}} f(x),$$

$$G(\eta) = \Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}} g(y),$$

$$G_1(\eta) = \Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n+\lambda+\alpha+\beta}{2}} \Big(\frac{1}{1+|y|^2}\Big)^{\frac{\alpha}{2}} g(y),$$

where $x = S^{-1}(\xi), y = S^{-1}(\eta)$. Direct computation leads to

$$\begin{aligned}\int_{\mathbb{S}^n} |F(\xi)|^{q'} d\xi &= \int_{\mathbb{R}^n} |f(x)|^{q'} \Big(\frac{2}{1+|x|^2}\Big)^{-q'\frac{2n+\lambda+\alpha+\beta}{2}} \Big(\frac{2}{1+|x|^2}\Big)^{n} dx \\ &= \int_{\mathbb{R}^n} |f(x)|^{q'} dx\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{S}^n} |G(\eta)|^{p} d\eta &= \int_{\mathbb{R}^n} |g(y)|^{p} \Big(\frac{2}{1+|y|^2}\Big)^{-p\frac{2n+\lambda+\alpha+\beta}{2}} \Big(\frac{2}{1+|y|^2}\Big)^{n} dy \\ &= \int_{\mathbb{R}^n} |g(y)|^{p} dy.\end{aligned}$$

Recall the reversed Stein–Weiss inequality in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^{\alpha}|x-y|^{\lambda} f(x)g(y)|y|^{\beta} dxdy \geq C_{n,\alpha,\beta,p,q'} \|f\|_{L^{q'}} \|g\|_{L^p},$$

where $p,\ q' \in (0,1)$, $\alpha,\ \beta,\ \lambda > 0$ such that $\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2$.

In the case of $q' = p$, we can apply the stereographic projection to obtain

$$\begin{aligned}&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} |\xi - S(0)|^{\alpha}|\xi-\eta|^{\lambda} F_1(\xi)G_1(\eta)|\eta - S(0)|^{\beta} d\xi d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\frac{4|x|^2}{1+|x|^2}\Big)^{\frac{\alpha}{2}} \Big(\frac{4|x-y|^2}{(1+|x|^2)(1+|y|^2)}\Big)^{\frac{\lambda}{2}} \Big(\frac{2}{1+|x|^2}\Big)^{-\frac{\lambda+\alpha+\beta}{2}} f(x) \\ &\qquad \times \Big(\frac{1}{1+|x|^2}\Big)^{\frac{\beta}{2}} \Big(\frac{2}{1+|y|^2}\Big)^{-\frac{\lambda+\alpha+\beta}{2}} g(y) \Big(\frac{1}{1+|y|^2}\Big)^{\frac{\alpha}{2}} \Big(\frac{4|y|^2}{1+|y|^2}\Big)^{\frac{\beta}{2}} dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^{\alpha}|x-y|^{\lambda} f(x)g(y)|y|^{\beta} dxdy \\ &\geq C_{n,\alpha,\beta,p,q'} \Big(\int_{\mathbb{R}^n} |f(x)|^{q'} dx\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^n} |g(y)|^{q'} dy\Big)^{\frac{1}{p}} \\ &= C_{n,\alpha,\beta,p,q'} \Big(\int_{\mathbb{S}^n} |F(\xi)|^{q'} d\xi\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{S}^n} |G(\eta)|^{p} d\eta\Big)^{\frac{1}{p}}.\end{aligned} \tag{33}$$

This accomplishes the proof of Theorem 7.

7. The proof of Theorem 9

In this section, we also give an equivalent form of Stein–Weiss inequality (4) on the sphere $\mathbb{S}^n$ in the case of $q'=p$ as we did in Section 6.

Recall the classical Stein–Weiss inequality in the whole space $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x|^{-\alpha}|x-y|^{-\lambda}f(x)g(y)|y|^{-\beta}dxdy \le C_{n,\alpha,\beta,p,q'}\|f\|_{L^{q'}}\|g\|_{L^p}, \tag{34}$$

where p, q', α, β, and λ satisfy the following conditions:

$$\frac{1}{q'}+\frac{1}{p}+\frac{\alpha+\beta+\lambda}{n}=2,\quad \frac{1}{q'}+\frac{1}{p}\ge 1,$$

$$\alpha+\beta\ge 0,\quad \alpha<\frac{n}{q},\quad \beta<\frac{n}{p'},\quad 0<\lambda<n.$$

In the case of $q'=p=\frac{2n}{2n-\lambda-\alpha-\beta}$, for ξ, $\eta\in\mathbb{S}^n$, $f\in L^{q'}(\mathbb{R}^n)$, and $g\in L^p(\mathbb{R}^n)$, define

$$H(\xi)=\Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}f(x),$$

$$H_1(\xi)=\Big(\frac{2}{1+|x|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}\Big(\frac{1}{1+|x|^2}\Big)^{-\frac{\beta}{2}}f(x),$$

$$T(\eta)=\Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}g(y),$$

$$T_1(\eta)=\Big(\frac{2}{1+|y|^2}\Big)^{-\frac{2n-\lambda-\alpha-\beta}{2}}\Big(\frac{1}{1+|y|^2}\Big)^{-\frac{\alpha}{2}}g(y),$$

where $x=S^{-1}(\xi), y=S^{-1}(\eta)$. Direct computation leads to

$$\begin{aligned}\int_{\mathbb{S}^n}|H(\xi)|^{q'}d\xi &= \int_{\mathbb{R}^n}|f(x)|^{q'}\Big(\frac{2}{1+|x|^2}\Big)^{-q'\frac{2n-\lambda-\alpha-\beta}{2}}\Big(\frac{2}{1+|x|^2}\Big)^{n}dx\\ &= \int_{\mathbb{R}^n}|f(x)|^{q'}dx\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{S}^n}|T(\eta)|^{p}d\eta &= \int_{\mathbb{R}^n}|g(y)|^{p}\Big(\frac{2}{1+|y|^2}\Big)^{-p\frac{2n-\lambda-\alpha-\beta}{2}}\Big(\frac{2}{1+|y|^2}\Big)^{n}dy\\ &= \int_{\mathbb{R}^n}|g(y)|^{p}dy.\end{aligned}$$

This together with Stein–Weiss inequality (34) and stereographic projection yields

(35)

$$\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}|\xi-S(0)|^{-\alpha}|\xi-\eta|^{-\lambda}H_1(\xi)T_1(\eta)|\eta-S(0)|^{-\beta}d\xi d\eta\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big(\frac{4|x|^2}{1+|x|^2}\Big)^{-\frac{\alpha}{2}}\Big(\frac{4|x-y|^2}{(1+|x|^2)(1+|y|^2)}\Big)^{-\frac{\lambda}{2}}\Big(\frac{2}{1+|x|^2}\Big)^{\frac{\lambda+\alpha+\beta}{2}}f(x)\\
&\qquad\times\Big(\frac{1}{1+|x|^2}\Big)^{-\frac{\beta}{2}}\Big(\frac{2}{1+|y|^2}\Big)^{\frac{\lambda+\alpha+\beta}{2}}g(y)\Big(\frac{1}{1+|y|^2}\Big)^{-\frac{\alpha}{2}}\Big(\frac{4|y|^2}{1+|y|^2}\Big)^{-\frac{\beta}{2}}dxdy\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x|^{-\alpha}|x-y|^{-\lambda}f(x)g(y)|y|^{-\beta}dxdy\\
&\le C_{n,\alpha,\beta,p,q'}\Big(\int_{\mathbb{R}^n}|f(x)|^{q'}dx\Big)^{\frac{1}{q'}}\Big(\int_{\mathbb{R}^n}|g(y)|^{p}dy\Big)^{\frac{1}{p}}\\
&=C_{n,\alpha,\beta,p,q'}\Big(\int_{\mathbb{S}^n}|H(\xi)|^{q'}d\xi\Big)^{\frac{1}{q'}}\Big(\int_{\mathbb{S}^n}|T(\eta)|^{p}d\eta\Big)^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof of Theorem 9.

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