DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **36**, Number **2**, February **2016**

doi:10.3934/dcds.2016.36.577

pp. **577–600**

POLYHARMONIC EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH IN THE WHOLE SPACE $\mathbb{R}^n$

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Abstract. In this paper, we apply the sharp Adams-type inequalities for the Sobolev space $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for any positive real number m less than n, established by Ruf and Sani [46] and Lam and Lu [30, 31], to study polyharmonic equations in $\mathbb{R}^{2m}$. We will consider the polyharmonic equations in $\mathbb{R}^{2m}$ of the form

$$(I-\Delta)^m u = f(x,u) \text{ in } \mathbb{R}^{2m}.$$

We study the existence of the nontrivial solutions when the nonlinear terms have the critical exponential growth in the sense of Adams' inequalities on the entire Euclidean space. Our approach is variational methods such as the Mountain Pass Theorem ([5]) without Palais-Smale condition combining with a version of a result due to Lions ([39, 40]) for the critical growth case. Moreover, using the regularity lifting by contracting operators and regularity lifting by combinations of contracting and shrinking operators developed in [14] and [11], we will prove that our solutions are uniformly bounded and Lipschitz continuous. Finally, using the moving plane method of Gidas, Ni and Nirenberg [22, 23] in integral form developed by Chen, Li and Ou [12] together with the Hardy-Littlewood-Sobolev type inequality instead of the maximum principle, we prove our positive solutions are radially symmetric and monotone decreasing about some point. This appears to be the first work concerning existence of nontrivial nonnegative solutions of the Bessel type polyharmonic equation with exponential growth of the nonlinearity in the whole Euclidean space.

2010 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Dimension theory, Poincaré recurrences, multifractal analysis.

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The first author's research is partly supported by Beijing Municipal Commission of Education for the Supervisor of Excellent Doctoral Dissertation (20131002701), NSFC (11371060) and the Fundamental Research Funds for the Central Universities. The second and third authors' research is partly supported by a US NSF grant DMS-1301595. The third author is also partly supported by a Simons Fellowship (a grant from the Simons Foundation #341447).

1. **Introduction.** We begin with recalling the sharp Adams inequalities in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an arbitrary domain. The well-known Sobolev embedding theorem says that the Sobolev space $W_0^{k,\frac{n}{k}}(\Omega)$, $0 < k < n$, is continuously embedded into $L^q(\Omega)$ for all $\frac{n}{k} \leq q < \infty$. However, we can show by easy examples that $W_0^{k,\frac{n}{k}}(\Omega) \nsubseteq L^\infty(\Omega)$. When Ω is a bounded domain and $k = 1$, it was showed independently by Yudovich [50], Pohozaev [44] and Trudinger [49] that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp\left(|t|^{n/(n-1)}\right) - 1$. Using rearrangement arguments and one-dimensional variational techniques, J. Moser in his 1971 paper [42] found the largest positive real number $\beta_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit $n-$ball, such that if Ω is a domain with finite $n-$measure in Euclidean $n-$space $\mathbb{R}^n$, then there is a constant c_0 depending only on n such that

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\beta\left|u\right|^{\frac{n}{n-1}}\right)dx \leq c_0$$

for any $0 \leq \beta \leq \beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega \left|\nabla u\right|^n dx \leq 1$. Moreover, this constant β_n is sharp in the meaning that if $\beta > \beta_n$, then the above inequality can no longer hold with some c_0 independent of u. The existence of extremal functions was established by Carleson and Chang [9] and subsequent works in [19], [38] in Euclidean spaces, and on compact Riemannian manifolds by Li [35, 36].

Research on finding the sharp constants for higher order Moser's inequality goes back to the work of D. Adams [1]. In this case, Moser's approach encounters many difficulities due to the absence of rearrangement properties for the higher order operators. To overcome this fact, Adams used Riesz potential representation. With the helps of the rearrangement of convolution functions, a result of O'Neil, he can again reduce the problem to one-dimensional case. To state Adams' result, we use the symbol $\nabla^m u$, m is a positive integer, to denote the $m-$th order gradient for $u \in C^m$, the class of $m-$th order differentiable functions:

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{for } m \text{ even} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{for } m \text{ odd} \end{cases}.$$

where ∇ is the usual gradient operator and $\triangle$ is the Laplacian. We use $||\nabla^m u||_p$ to denote the L^p norm $(1 \leq p \leq \infty)$ of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also use $W_0^{k,p}(\Omega)$ to denote the Sobolev space which is a completion of $C_0^\infty(\Omega)$ under the norm of $\left(\sum_{j=0}^{k} ||\nabla^j u||_{L^p(\Omega)}^p\right)^{1/p}$. Then Adams proved the following:

Theorem A. *Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ and $||\nabla^m u||_{L^{\frac{n}{m}}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|u(x)|^{\frac{n}{n-m}})dx \leq C_0$$

for all $\beta \leq \beta(n,m)$ *where*

$$\beta(n,\ m)\ =\begin{cases} \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\ \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is even} \end{cases}.$$

Furthermore, for any $\beta > \beta(n,m)$, *the integral can be made as large as possible.*

Note that $\beta(n,1)$ coincides with Moser's value of β_n and $\beta(2m,m) = 2^{2m}\pi^m\Gamma(m+1)$ for both odd and even m. In 2011, the Adams inequality was extended by Tarsi [48]. More precisely, Tarsi used the Sobolev space with Navier boundary conditions $W_N^{m,\frac{n}{m}}(\Omega)$ which contains the Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$ as a closed subspace. The Moser-Trudinger and Adams inequalities were also extended to compact Riemannian manifolds without boundary Fontana [20].

Since the Moser-Trudinger's inequality and Adams inequality are meaningless when Ω has infinite volume, it is meaningful to study these kinds of inequalities in this case. In the first order case, the Moser-Trudinger type inequality when $|\Omega| = +\infty$ was obtained by B. Ruf [45] in dimension two and Li-Ruf [37] in general dimension. In the case of higher order, Kozono et al. [27] could find a constant $\beta^*_{n,m} \leq \beta(n,\ m)$, with $\beta^*_{2m,m} = \beta(2m,\ m)$, such that if $\beta < \beta^*_{n,m}$ then

$$\sup_{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\|u\|_{m,n}\leq 1}\int_\Omega \phi\left(\beta\,|u|^{\frac{n}{n-m}}\right)dx \leq C_{n,m,\beta}$$

where $C_{n,m,\beta} > 0$ is a constant depending on $\beta,\ n$ and m, while if $\beta > \beta(n,\ m)$, the supremum is infinite. Here they use the following notation:

$$\phi(t) = e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2}\frac{t^j}{j!},$$

$$j_{\frac{n}{m}} = \min\left\{j \in \mathbb{N} : j \geq \frac{n}{m}\right\} \geq \frac{n}{m}$$

and use the norm

$$\|u\|_{m,n} = \left\|(I-\Delta)^{\frac{m}{2}}\,u\right\|_{\frac{n}{m}}$$

which is equivalent to the Sobolev norm

$$\|u\|_{W^{m,\frac{n}{m}}} = \left(\|u\|^{\frac{n}{m}}_{\frac{n}{m}} + \sum_{j=1}^{m}\left\|\nabla^j u\right\|^{\frac{n}{m}}_{\frac{n}{m}}\right)^{\frac{m}{n}}.$$

In particular, if $u \in W_0^{m,\frac{n}{m}}(\Omega)$ or $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, then $\|u\|_{W^{m,\frac{n}{m}}} \leq \|u\|_{m,n}$.

To do this, they followed main steps similar to that of Adams: using the Bessel potentials instead of the Riesz potentials, they apply O'Neil's result on the rearrangement of convolution functions and use techniques of symmetric decreasing rearrangements. But with this approach, the critical case $\beta = \beta(n,\ m)$ cannot be observed. In the recent paper [46], Ruf and Sani studied this limiting case when m is an even integer number. In fact, Ruf and Sani proved the following Adams type inequality using the comparison principle of the polyharmonic operators and symmetrization arguments of the PDEs:

Theorem B. *Let m be an even integer less than n. There exists a constant $C_{m,n} > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^n$*

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega), \|u\|_{m,n} \leq 1} \int_\Omega \phi\left(\beta_0(n,m)|u|^{\frac{n}{n-m}}\right) dx \leq C_{m,n}$$

where

$$\beta_0(n,m) = \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}} .$$

and this inequality is sharp.

Because the result of Ruf and Sani [46] only treats the case when m is even, thus it leaves an open question if Ruf and Sani's theorem still holds when m is odd. Recently, the authors of [30, 31] have established the results of Adams type inequalities on unbounded domains when m is odd. More precisely, the main result of Lam and Lu [31] is as follows:

Theorem C. *Let m be a positive integer less than n and $\tau > 0$. There holds*

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \left\|(\tau I - \Delta)^{\frac{m}{2}} u\right\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta_0(n,m)|u|^{\frac{n}{n-m}}\right) dx < \infty.$$

Moreover, the constant $\beta_0(n,m)$ is sharp.

In fact, in the work [31], by a different approach without using the symmetrization arguments, the authors can set up an Adams type inequality for arbitrary positive number γ on fractional Sobolev spaces $W^{\gamma,\frac{n}{\gamma}}(\mathbb{R}^n)$. Such a symmetrization-free argument has also been carried out on the Heisenberg group where the Pólya-Szegö inequality does not hold [33, 34].

Theorem D. *Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$ and $\tau > 0$. There holds*

$$\sup_{u \in W^{\gamma,p}(\mathbb{R}^n), \left\|(\tau I - \Delta)^{\frac{\gamma}{2}} u\right\|_p \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta_0(n,\gamma)|u|^{p'}\right) dx < \infty$$

where

$$\phi(t) = e^t - \sum_{j=0}^{j_p - 2} \frac{t^j}{j!},$$
$$j_p = \min\{j \in \mathbb{N} : j \geq p\} \geq p.$$

Furthermore this inequality is sharp, i.e., if $\beta_0(n,\gamma)$ is replaced by any $\beta > \beta_0(n,\gamma)$, then the supremum is infinite.

The Moser-Trudinger type inequalities and Adams type inequalities play important roles in geometric analysis and partial differential equations, e.g., in the study of the exponential growth partial differential equations where, roughly speaking, the nonlinearity behaves like $e^{\alpha|u|^{\frac{n}{n-m}}}$ as $|u| \to \infty$. Here we mention Atkinson-Peletier [6], Carlesion-Chang [9], Adimurthi [2], de Figueiredo-Miyagaki-Ruf [16], Lam-Lu [28, 29, 32] and the references therein. In [15], Coti-Zelati and Rabinowitz studied the following periodic nonlinear Schrödinger equation in whole space $\mathbb{R}^n$ with $n \geq 3$

$$-\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^n,\ u \in H^1(\mathbb{R}^n),\ u > 0,$$

where the potential V and the nonlinearity f satisfy some periodicity conditions. Moreover, they required that the nonlinear term has the subcritical (polynomial) growth. They obtained results of existence and multiplicity of homoclinic type solutions by applying the Mountain Pass Theorem together with a sort of Concentration Compactness Principle of Lions. After this initial ground breaking work, there are many papers that utilize this method for both Hamiltonian systems and semilinear elliptic equations. Moreover, the result in [15] was extended and complemented, for example, when there was a small nonperiodic perturbations of V and f with subcritical polynomial growth or when f had critical and supercritical (polynomial) growth (see [3, 41, 43]).

When $n = 2$, if f behaves such as a polynomial, it can be handled quite simply compared to the case $n \geq 3$. So, we may want to look for the maximal growth of the nonlinear term in this case. Thanks to the Moser-Trudinger inequality [42, 49], we know that the maximal growth of the nonlinearity is exponential. There have been many works concerning this situation. For instance, Alves, Do Ó and Miyagaki considered in [4] the critical periodic and asymptotic periodic problem which had the above form in $\mathbb{R}^2$ when the nonlinear term had exponential growth. Do Ó, Medeiros and Severo in [17] investigated the existence, and Lam and Lu in [28] studied the existence and multiplicity of solutions with a positive and blow up potential V and an exponential nonlinearity.

In the case of polyharmonic equation, the situation is quite different. Though there have been substantial works for the polyharmonic equation with the polynomial nonlinearity, it has been absent in the literature on the study of polyharmonic equations with exponential growth of the nonlinear term. Recently, the authors of this paper in [29] investigated the existence of nontrivial solutions to polyharmonic equations on bounded domains with the subcritical or critical growth of nonlinearities in the meaning of Adams inequalities. Moreover, we proved the existence of nontrivial solutions with many different types of conditions: the nonlinear terms may or may not satisfy the Ambrosetti-Rabinowitz condition. However, due to the absence of the Adams type inequalities for the unbounded domains, there have been no work about the polyharmonic equations with exponential growth in the whole space.

Motivated by the above results, we will consider in this paper the critical periodic and asymptotic periodic problem in higher order case, that is,

$$(I - \Delta)^m u = f(x,u) \text{ in } \mathbb{R}^{2m},\ m \in \mathbb{N}. \tag{NP}$$

By the Adams-type inequalities, we consider here the maximal growth on the nonlinear term $f(x,u)$ which allows us to treat problem (NP) variationally in $H^m\left(\mathbb{R}^{2m}\right)$. More precisely, we assume the following growth condition on the nonlinearity $f(x,u)$:

$(f1)$ $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,u) = 0$ for all $(x,u) \in \mathbb{R}^{2m} \times (-\infty, 0]$ and there exists a continuous $1-$periodic function $f_0 : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}$ (i.e. $f_0(x + p, s) = f_0(x,s)$ for all $x \in \mathbb{R}^{2m}$, $p \in \mathbb{Z}^{2m}$, $s \in \mathbb{R}$) such that $f_0(x,u) = 0$ for all $(x,u) \in \mathbb{R}^{2m} \times (-\infty, 0]$ and

(1) $0 \leq f_0(x,s) \leq f(x,s)$ for all $(x,s) \in \mathbb{R}^{2m} \times \mathbb{R}^+$,
(2) for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $s \geq 0$ and $|x| \geq \eta$,

$$|f_0(x,s) - f(x,s)| \leq \varepsilon \exp\left(\beta\left(2m,m\right)|s|^2\right)$$

where $\beta\left(2m,m\right) = 2^{2m}\pi^m\Gamma\left(m+1\right)$.

$(f2)$ f has critical growth at $+\infty$, namely, for all $(x,u)\in\mathbb{R}^{2m}\times\mathbb{R}^{+}$,

$$|f(x,u)|\leq C\exp\left(\beta\left(2m,m\right)|u|^{2}\right).$$

$(f3)$ There exists $\theta>2$ such that for all $x\in\mathbb{R}^{2m}$ and $u>0$,

$$\begin{aligned}0&<\theta F(x,u)\leq uf(x,u)\\ 0&<\theta F_{0}(x,u)\leq uf_{0}(x,u)\end{aligned}$$

where the functions $F_0,\ F$ are the primitives of f_0 and f respectively.

$(f4)$ $f(x,s)=o(s)$ near $s=0$ uniformly with respect to $x\in\mathbb{R}^{2m}$.

$(f5)$ for each fixed $x\in\mathbb{R}^{2m}$, the functions $s\to f_0(x,s)/s$ and $s\to f(x,s)/s$ are increasing;

$(f6)$ there are constants $p>2$ and C_p such that

$$f_0(x,s)\geq C_p s^{p-1},\ \text{ for all }\ (x,s)\in\mathbb{R}^{2m}\times\mathbb{R}^{+},$$

where

$$C_p>\left[\frac{\theta(p-2)}{p\left(\theta-2\right)}\right]^{(p-2)/2}S_p^p,$$

$$S_p=\inf_{u\in H^m(\mathbb{R}^{2m})\setminus\{0\}}\frac{\left[\int\limits_{\mathbb{R}^{2m}}\left|(I-\Delta)^{\frac{m}{2}}u\right|^{2}dx\right]^{1/2}}{\left(\int\limits_{\mathbb{R}^{2m}}|u|^{p}\,dx\right)^{1/p}}.$$

$(f7)$ The nonnegative continuous functions $f(x,s)-f_0(x,s)$ is positive on a set of positive measure.

It is easy to see from the Sobolev embedding that $S_p>0$.

The functional associated to Problem (NP) is $J:H^m\left(\mathbb{R}^{2m}\right)\to\mathbb{R}$

$$J(u)=\frac{1}{2}\int\limits_{\mathbb{R}^{2m}}\left[\sum_{i=0}^{m}\binom{m}{i}\left|\nabla^{i}u\right|^{2}\right]dx-\int\limits_{\mathbb{R}^{2m}}F(x,u)dx$$

where

$$F(x,u)=\int\limits_{0}^{u}f(x,s)ds.$$

Notice that from the Fourier transform, we have

$$\sum_{i=0}^{m}\binom{m}{i}\left\|\nabla^{i}u\right\|_{2}^{2}=\left\|(I-\Delta)^{\frac{m}{2}}u\right\|_{2}^{2}.$$

Thus

$$J(u)=\frac{1}{2}\int\limits_{\mathbb{R}^{2m}}\left|(I-\Delta)^{\frac{m}{2}}u\right|^{2}dx-\int\limits_{\mathbb{R}^{2m}}F(x,u)dx.$$

By the Adams-type inequality, the assumptions on the nonlinearity and standard arguments, we can easily check that J is well-defined. Moreover, J is $C^1\left(H^m\left(\mathbb{R}^{2m}\right),\mathbb{R}\right)$ and

$$DJ(u)v=\int\limits_{\mathbb{R}^{2m}}\left[\sum_{i=0}^{m}\binom{m}{i}\nabla^i u\nabla^i v\right]dx-\int_{\mathbb{R}^{2m}}f(x,u)vdx,\ v\in H^m\left(\mathbb{R}^{2m}\right)$$

Thus, the critical point of J are precisely the weak solutions of problem (NP).

Our first main result in this paper is as follows:

Theorem 1.1. *If f satisfies $(f1)-(f7)$, then Problem (NP) possesses a nontrivial weak solution $u\in H^m\left(\mathbb{R}^{2m}\right)$, which satisfies $u\geq 0$ a.e. in $\mathbb{R}^{2m}$.*

We will find such a nontrivial weak solution of (NP), i.e., find a nontrivial critical point of J by the Mountain-pass Theorem (see e.g., [5], [10]) combining with a version of a result due to Lions for the critical growth case. To do that, we will first work on the periodic problem, and then using the assumptions on the nonlinear term, we prove the existence of such a critical point.

Our next concern is about the regularity and symmetry of the solution u. It can be easily shown that under appropriate decay assumptions of the solutions at infinity, (NP) is equivalent to

$$u=G_{2m}*N_f(u) \tag{I}$$

where $*$ denotes the convolution, G_α is the Bessel kernel and N_f is the Nemitskii operator: $N_f(u)(x)=f(x,u(x))$. We will show that in the case the nonlinearity doesn't depend on x, we can have some regularities for our nonnegative solutions. We will use the new methods developed recently by W. Chen, C. Li and C. Ma [14] to prove the regularity of the solutions to integral equations, namely, the regularity lifting by contracting operators and regularity lifting by combinations of contracting and shrinking operators, which is explicitly explained in [14] and in the monograph of W. Chen and C. Li [11]. These methods are applied for integral equations and system of integral equations associated with Wolff potentials in [14], for integral equations associated with Bessel potentials in [24] and system of integral equations associated with Bessel potentials in [25]. Moreover, instead of the maximum principle, we can use a moving plane method of Gidas, Ni and Nirenberg [22, 23] in the integral form due to Chen, Li and Ou [12] together with the Hardy-Littlewood-Sobolev (HLS) type inequality (see e.g., [12, 13], etc.) to obtain the radial symmetry and monotonicity of our positive solutions.

Our next results are

Theorem 1.2. *If u is a solution of (I) and*

$$g(x)=\left|f\left(x,u(x)\right)\right|\log\left|f\left(x,u(x)\right)\right|\chi\left\{x:\left|f\left(x,u(x)\right)\right|\geq 1\right\}\ \text{is in}\ L^1\left(\mathbb{R}^{2m}\right), \tag{1}$$

then u is uniformly bounded in $\mathbb{R}^{2m}$. Moreover, if f does not depend on x, i.e. $f(x,s)=f(s)$, $f(0)=0$ and satisfies that

$(f8)$ There exist $M\geq 2\left\|u\right\|_\infty$ and $C(M)>0$ such that for every $s,t\in[-M,M]$, $s\neq t$, we have $\left|\frac{f(s)-f(t)}{s-t}\right|\leq C(M)$.

Then u is a Lipschitz continuous function.

Theorem 1.3. *Suppose u is a positive solution of (I), f does not depend on x, i.e. $f(x,s)=f(s)$ and satisfies that*

$(f9)$ *f is nondecreasing and f' is nondecreasing*

$(f10)$ *There exist $q > \beta > 1$ such that $u \in L^q(\mathbb{R}^{2m})$ and $f'(u) \in L^{\frac{q}{\beta-1}}(\mathbb{R}^{2m})$ then u must be radially symmetric and monotonely decrease about some point.*

We notice that if $f(x, u(x))$ is in L^q for some $q > 1$, then we have (1).

The organization of the paper is as follows: In Section 2, we introduce our notations and also some facts about the Bessel potential, Regularity lifting Theorems and the Hardy-Littlewood-Sobolev type inequality. In Section 3, we deal with the periodic problem and prove Theorem 1.1 when the nonlinear term is periodic. In Section 4, we use the results in the periodic case to prove Theorem 1.1 in Section 4. Finally, we prove Theorems 1.2 and 1.3 in Section 5.

2. **Preliminaries.**

In this section, we provide some preliminaries. For $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, we will denote by $\nabla^j u$, $j \in \{1, 2, ..., m\}$, the $j-th$ order gradient of u, namely

$$\nabla^j u = \begin{cases} \triangle^{\frac{j}{2}} u & \text{for } j \text{ even} \\ \nabla \triangle^{\frac{j-1}{2}} u & \text{for } j \text{ odd} \end{cases}.$$

Now, given $\alpha \geq 0$, we define the Bessel kernel $G_\alpha(x)$ and Bessel potentials $B_\alpha(f)$.

Definition 2.1 ((Bessel kernel))**.**

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\,\Gamma(\alpha/2)} \int_0^\infty \exp\left(\frac{-\pi |x|^2}{\delta}\right) \exp\left(\frac{-\delta}{4\pi}\right) \delta^{\frac{\alpha-2m-2}{2}} d\delta$$

Definition 2.2 (Bessel potentials)**.** Given $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^{2m})$, we define the Bessel potential $B_\alpha(f)$ of f as

$$B_\alpha(f) = \begin{cases} G_\alpha * f \text{ if } \alpha > 0, \\ f \text{ if } \alpha = 0. \end{cases}.$$

We give here some basic properties about Bessel kernel and Bessel potentials and also the exponential decay for G_α at infinity, the proper decay at 0. More detailed information can be found in the book of Stein [47].

Lemma 2.3. (i) *For each $\alpha > 0$, $G_\alpha \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} G_\alpha(x)dx = 1$.*

(ii) *$\|B_\alpha(f)\|_p \leq \|f\|_p$, for $1 \leq p \leq \infty$.*

(iii) *Given $\alpha > 0$, the Bessel kernel $G_\alpha(x)$ has the decay estimate as,*

$$G_\alpha(x) \sim \exp\left(-\frac{|x|}{2}\right)$$

when $|x| \geq 2$, and when $|x| \leq 2$,

$$G_\alpha(x) \sim \begin{cases} |x|^{\alpha-n}, & \text{if } 0 < \alpha < n, \\ \log \frac{2}{|x|}, & \text{if } \alpha = n, \\ 1 & \text{if } \alpha > n. \end{cases}$$

It can be noted that with the Bessel potential, we can define the generalized Sobolev space $W^{\alpha,p}(\mathbb{R}^n) = \{G_\alpha * f : f \in L^p(\mathbb{R}^n)\}$. Moreover, when k is a positive integer and $1 < p < \infty$, $W^{\alpha,p}(\mathbb{R}^n)$ becomes the standard Sobolev space. See [47] for more details.

Similarly, for $\tau > 0$, $\alpha \geq 0$, we define the operator $L_{\tau,\alpha}(x)$ by

$$L_{\tau,\alpha}(x) = \tau^{\frac{n-\alpha}{2}} \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \exp\left(\frac{-\pi\tau |x|^2}{\delta}\right) \exp\left(\frac{-\delta}{4\pi}\right) \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta}.$$

We notice that $L_{1,\alpha} = G_\alpha$. Now, by Fourier transform, we can prove the following lemma:

Lemma 2.4. *(i)* $L_{\tau,\alpha} \in L^1(\mathbb{R}^n)$.

(ii) $\widehat{L_{\tau,\alpha}}(x) = \left(\tau + 4\pi^2 |x|^2\right)^{-\frac{\alpha}{2}}$.

(iii) Let $1 < p < \infty$*, and* α *is a positive real number. Then* $u \in W^{\alpha,p}(\mathbb{R}^n)$ *iff* $u = L_{\tau,\alpha} * f$ *for some* $f \in L^p(\mathbb{R}^n)$.

In fact, the properties of the potential $L_{\tau,\alpha}$ are pretty much the same with the properties of the Bessel potential. Also, noticing that from the following identity (see [47]):

$$\frac{|x|^{-n+\alpha}}{\gamma(\alpha)} = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \exp\left(\frac{-\pi |x|^2}{\delta}\right) \delta^{(-n+\alpha)/2} \frac{d\delta}{\delta} \tag{2}$$

where

$$\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma(n/2 - \alpha/2)$$

we have that

$$L_{\tau,\alpha}(x) \leq \frac{|x|^{-n+\alpha}}{\gamma(\alpha)}. \tag{3}$$

Next, we present the regularity lifting by contracting operators. Suppose that V is a topological vector space with two extended norms

$$\|\cdot\|_X, \ \|\cdot\|_Y : V \to [0, \infty].$$

Let $X = \{v \in V : \|v\|_X < \infty\}$ and $Y = \{v \in V : \|v\|_Y < \infty\}$. The operator $T : X \to Y$ is said to be contracting if

$$\|Tf - Tg\|_Y \leq \alpha \|f - g\|_X$$

for all $f, g \in X$ and for some $\alpha < 1$. Also, T is called to be shrinking if

$$\|Tf\|_Y \leq \beta \|f\|_X$$

for all $f \in X$ and for some $\beta < 1$.

Definition 2.5 ($XY-$pair)**.** Suppose that X, Y are two normed subspaces described in the preceding section, X and Y are called an XY-pair if whenever the sequence $\{u_k\} \subset X$ with $u_k \to u$ in X and $\|u_k\|_Y \leq C$ will imply $u \in Y$.

We now recall the following theorems which can be found in [11].

Theorem 2.6 (Regularity lifting by contracting operators)**.** *Let* T *be a contracting operator from* X *to itself and from* Y *to itself, and assume that* X, Y *are both complete. If* $f \in X$*, and there exists* $g \in X \cap Y$ *such that* $f = Tf + g$ *in* X*, then* $f \in X \cap Y$.

Theorem 2.7 (Regularity lifting by combinations of contracting and shrinking operators)**.** *Let X and Y be an XY-pair, and assume that are both complete. Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of X and Y respectively, and T be a operator, which is contracting from $\mathcal{X}$ to X and shrinking from $\mathcal{Y}$ to Y. Define $Sw = Tw + g$ for some $g \in \mathcal{X} \cap \mathcal{Y}$. Then there exists a unique solution u of the equation $w = Sw$ in $\mathcal{X}$, and $u \in Y$.*

Notice that in this paper, we will choose $X = L^{\infty}\left(\mathbb{R}^{2m}\right)$ and $Y = \Lambda^{1}\left(\mathbb{R}^{2m}\right)$, the space of Lipschitz continuous functions.

Theorem 2.8 (HLS type inequality)**.** *Let $q > \beta > 1$. If $f \in L^{\frac{q}{\beta}}\left(\mathbb{R}^{2m}\right)$, then $B_{2m}(f) \in L^{q}\left(\mathbb{R}^{2m}\right)$. Moreover, we have the estimate*

$$\left\|B_{2m}(f)\right\|_{L^{q}(\mathbb{R}^{2m})} \leq C\left(\beta, q, m\right)\|f\|_{L^{\frac{q}{\beta}}(\mathbb{R}^{2m})}.$$

Let $u : B_R \to \mathbb{R}$ be a measurable function. The distribution function of u is defined by

$$\mu_u(t) = \left|\{x \in B_R |\, |u(x)| > t\}\right| \;\; \forall t \geq 0.$$

The decreasing rearrangement of u is defined by

$$u^{*}(s) = \inf\left\{t \geq 0 : \mu_u(t) < s\right\} \;\; \forall s \in \left[0, |B_R|\right],$$

and the spherically symmetric decreasing rearrangment of u by

$$u^{\#}(x) = u^{*}\left(\sigma_n |x|^{n}\right) \;\; \forall x \in B_R.$$

We have that $u^{\#}$ is the unique nonnegative integrable function which is radially symmetric, nonincreasing and has the same distribution function as $|u|$.

Now, when $n = 2m$, we denote

$$\|u\|_p = \left(\int\limits_{\mathbb{R}^n} |u|^p \, dx\right)^{1/p},$$

$$\|u\| = \left\|(I - \Delta)^{\frac{m}{2}} u\right\|_2 .$$

It is easy to see that the norm $\|u\|$ is equivalent to the Sobolev norm

$$\|u\|_{H^m} = \left(\|u\|_2^2 + \sum_{j=1}^{m}\left\|\nabla^{j} u\right\|_2^2\right)^{1/2}$$

and in particular, if $u \in H^{m}\left(\mathbb{R}^{2m}\right)$ then

$$\|u\|_{H^m} \leq \|u\| .$$

We have the following version of the Adams-type inequality:

Lemma 2.9. *If $u \in H^{m}\left(\mathbb{R}^{2m}\right)$ such that $\|u\| \leq M$ with M sufficiently small ($2\beta M^2 \leq \beta\left(2m, m\right)$) and $q > 2$, we have*

$$\int\limits_{\mathbb{R}^{2m}} \left[\exp\left(\beta |u|^2\right) - 1\right] |u|^q \, dx \leq C\left(\beta\right)\|u\|^q$$

Proof. We will use the following elementary result: For $\beta > 0$, then for all $r \geq 1$ and $s \in \mathbb{R}$, we have

$$\left(e^{\beta s^2} - 1\right)^r \leq e^{\beta r s^2} - 1.$$

First, let $u \in H^m\left(\mathbb{R}^{2m}\right)$ be such that $\|u\| \leq M$, then by the Holder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^{2m}} \left[\exp\left(\beta |u|^2\right) - 1\right] |u|^q \, dx &\leq \left(\int_{\mathbb{R}^{2m}} \left[\exp\left(\beta |u|^2\right) - 1\right]^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^{2m}} |u|^{2q}\right)^{1/2} dx \\ &\leq \left(\int_{\mathbb{R}^{2m}} \left[\exp\left(2\beta \|u\|^2 \left|\frac{u}{\|u\|}\right|^2\right) - 1\right] dx\right)^{1/2} \|u\|_{2q}^q .\end{aligned}$$

By the Sobolev embedding, we have the desired result. □

3. The periodic problem: Theorem 1.1 when the nonlinear term is periodic.

In this section, we study the existence of solutions for the following periodic critical problem:

$$\begin{cases} (I - \Delta)^m u = f_0(x, u) \text{ in } \mathbb{R}^{2m} \\ u \geq 0 \end{cases} \tag{P}$$

Notice that the solutions of (P) are the critical points of the associated functional on $H^m\left(\mathbb{R}^{2m}\right)$ given by

$$J_0(u) = \frac{1}{2} \int_{\mathbb{R}^{2m}} \left|(I - \Delta)^{\frac{m}{2}} u\right|^2 dx - \int_{\mathbb{R}^{2m}} F_0(x, u) dx$$

where

$$F_0(x, u) = \int_0^u f_0(x, s) ds.$$

From $(f1)$, $(f2)$ and $(f4)$, given $\varepsilon > 0$ there exist positive constants C_ε and β such that we obtain for all $(x, u) \in \mathbb{R}^{2m} \times \mathbb{R}$,

$$|F_0(x, u)| \leq \varepsilon \frac{u^2}{2} + C_\varepsilon \left[\exp\left(\beta |u|^2\right) - 1\right]$$

Thus we have by the Adams-type inequalities: $F_0(x, u) \in L^1\left(\mathbb{R}^{2m}\right)$ for all $u \in H^m\left(\mathbb{R}^{2m}\right)$. Therefore, the functional J_0 is well-defined. Moreover, by standard arguments, we see that J_0 is a C^1 functional on $H^m\left(\mathbb{R}^{2m}\right)$ with

$$DJ_0(u)v = \int_{\mathbb{R}^{2m}} \left[\sum_{i=0}^{m} \binom{m}{i} \nabla^i u \nabla^i v\right] dx - \int_{\mathbb{R}^{2m}} f_0(x, u) v dx, \ \forall v \in H^m\left(\mathbb{R}^{2m}\right).$$

3.1. Mountain pass geometry.

Lemma 3.1. *For any $u \in H^m\left(\mathbb{R}^{2m}\right) \setminus \{0\}$ with compact support and $u \geq 0$, we have $J_0(tu) \to -\infty$ as $t \to \infty$.*

Proof. Let $u \in H^m\left(\mathbb{R}^{2m}\right) \setminus \{0\}$ with compact support and $u \geq 0$. By our assumptions, there are positive constants c, d such that

$$F_0(x, u) \geq c u^\theta - d, \ \forall x \in \operatorname{supp}(u), \ \forall s \in [0, \infty).$$

Thus

$$J_0(tu) \geq \frac{t^2}{2}\|u\|^2 - ct^\theta \int_{\mathbb{R}^{2m}} u^\theta dx + d\,|\mathrm{supp}(u)|\,,$$

which implies that $J_0(tu) \to -\infty$ as $t \to \infty$ since $\theta > 2$. □

Lemma 3.2. *There exist $\alpha,\ \rho > 0$ such that $J_0(u) \geq \alpha$ if $\|u\| \geq \rho$.*

Proof. By our assumptions, we have

$$F_0(x,u) \leq \varepsilon |u|^2 + C\left[\exp\left(\beta |u|^2\right) - 1\right]|u|^q$$

for all $(x,u) \in \mathbb{R}^{2m} \times \mathbb{R}$. By Lemma 2.3 and the Sobolev embedding, we get when $\|u\|$ is sufficiently small:

$$J_0(u) \geq \frac{\|u\|^2}{2} - C\varepsilon \|u\|^2 - C\|u\|^q\,.$$

Thus since $q > 2$, we can choose $\alpha,\ \rho > 0$ such that $J_0(u) \geq \alpha$ if $\|u\| \geq \rho$. □

Now, in view of the above two lemmas, we can apply a version of Ambrosetti-Rabinowitz mountain-pass theorem without a compactness condition such as the one of Palais-Smale or Cerami, to get a Palais-Smale sequence of the functional J_0, that is, $(u_k) \in H^m\left(\mathbb{R}^{2m}\right)$ such that

$$J_0\left(u_k\right) \to c_0 \text{ and } DJ_0\left(u_k\right) \to 0, \text{ as } k \to \infty,$$

where the mountain-pass level c_0 is characterized by

$$\begin{aligned} c_0 &= \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} J_0\left(\gamma\left(t\right)\right), \\ \Gamma &= \left\{\gamma \in C\left([0,1], H^m\left(\mathbb{R}^{2m}\right)\right) : J_0\left(\gamma\left(0\right)\right) = 0 \text{ and } J_0\left(\gamma\left(1\right)\right) \leq 0\right\}. \end{aligned}$$

Lemma 3.3. *We have $c_0 \in \left[\alpha, (\theta-2)/2\theta\right)$. Moreover, the Palais-Smale $(PS)_{c_0}$ sequence (u_k) is bounded and its weak limit u_0 satisfies $DJ_0\left(u_0\right) = 0$.*

Proof. It's clear that $c_0 \geq \alpha$. Now, we fix a positive function $v_p \in H^m\left(\mathbb{R}^{2m}\right)$ such that

$$S_p \approx \frac{\left[\int_{\mathbb{R}^{2m}} \left|(I-\Delta)^{\frac{m}{2}} v_p\right|^2 dx\right]^{1/2}}{\left(\int_{\mathbb{R}^{2m}} |v_p|^p\, dx\right)^{1/p}}.$$

Then,

$$\begin{aligned} c_0 &\leq \max_{t\geq 0} J_0\left(tv_p\right) \\ &\leq \max_{t\geq 0}\left[\frac{t^2}{2}\int_{\mathbb{R}^{2m}} \left|(I-\Delta)^{\frac{m}{2}} v_p\right|^2 dx - t^p \frac{C_p}{p}\int_{\mathbb{R}^{2m}} |v_p|^p\, dx\right] \\ &\approx \frac{p-2}{2p}\frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}} \\ &< \frac{\theta-2}{2\theta}. \end{aligned}$$

Using the well-known Ambrosetti-Rabinowitz condition, we can check easily that (u_k) is bounded. Hence, WLOG, we can suppose that

$$\begin{gathered} u_k \rightharpoonup u_0 \text{ in } H^m\left(\mathbb{R}^{2m}\right) \\ u_k(x) \to u_0(x) \text{ almost everywhere in } \mathbb{R}^{2m} \\ u_k \to u_0 \text{ in } L^s_{loc}\left(\mathbb{R}^{2m}\right) \text{ for all } s \geq 1. \end{gathered}$$

Now, noting that

$$\begin{aligned} c_0 &= \lim_{k\to\infty} J_0\left(u_k\right) \\ &= \lim_{k\to\infty}\left[J_0\left(u_k\right) - \frac{1}{\theta} DJ_0\left(u_k\right)u_k\right] \\ &\geq \frac{\theta-2}{2\theta}\limsup_{k\to\infty}\left\|u_k\right\|^2 \end{aligned}$$

we can deduce

$$\limsup_{k\to\infty}\left\|u_k\right\|^2 = m \leq \frac{2\theta c_0}{\theta-2} < 1.$$

By Theorem C, we could find two real numbers $\gamma, q > 1$ sufficiently close to 1 such that the sequence

$$h_k(x) = \exp\left(\beta\left(2m,m\right)\gamma u_k^2(x)\right) - 1$$

belongs to $L^q\left(\mathbb{R}^{2m}\right)$ and a universal constant $C > 0$ such that $\left\|h_k\right\|_q \leq C$ for all $k \in \mathbb{N}$. Combining all these together, we have

$$\int_{\mathbb{R}^{2m}} f_0(x,u_k)v dx \to \int_{\mathbb{R}^{2m}} f_0(x,u_0)v dx,\ v \in H^m\left(\mathbb{R}^{2m}\right).$$

From the above result, it's easy to see that $DJ_0\left(u_0\right) = 0$. □

Lemma 3.4. *Let $(u_k) \in H^m\left(\mathbb{R}^{2m}\right)$ be a sequence with $u_k \rightharpoonup 0$ and*

$$\limsup_{k\to\infty}\left\|u_k\right\|^2 \leq m < 1.$$

If there exists $R > 0$ such that

$$\liminf_{k\to\infty}\sup_{y\in\mathbb{R}^{2m}}\int_{B_R(y)}\left|u_k\right|^2 dx = 0,$$

we then have

$$\int_{\mathbb{R}^{2m}} F_0(x,u_k)dx,\ \int_{\mathbb{R}^{2m}} f_0(x,u_k)u_k dx \to 0 \text{ as } k \to \infty.$$

Proof. We first recall the Lemma 8.4 in [26]: Let $1 < p \leq \infty$, $1 \leq q < \infty$ (if $p < 2m$, we assume more that $q \neq p^* = \frac{2mp}{2m-p}$). Assume (u_k) is bounded in $L^q\left(\mathbb{R}^{2m}\right)$ such that $(|\nabla u_k|)$ is bounded in $L^p\left(\mathbb{R}^{2m}\right)$. If there exists $R > 0$ such that

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^{2m}}\int_{B_R(y)}\left|u_k\right|^q dx = 0,$$

then $u_k \to 0$ in $L^r\left(\mathbb{R}^{2m}\right)$ for $\min\left(q,p^*\right) < r < \max\left(q,p^*\right)$ (we denote that $p^* = \infty$ if $p \geq 2m$).

Now with $p = 2m$ and $q = 2$, we get

$$u_k \to 0 \text{ in } L^r\left(\mathbb{R}^{2m}\right) \text{ for all } r > 2.$$

(Note that since $u_k \in H^m\left(\mathbb{R}^{2m}\right)$, we have $\nabla u_k \in H^{m-1}\left(\mathbb{R}^{2m}\right)$ and thanks to the Sobolev imbedding $H^{m-1}\left(\mathbb{R}^{2m}\right) \hookrightarrow L^{2m}\left(\mathbb{R}^{2m}\right)$, we have $|\nabla u_k|$ is bounded in $L^{2m}\left(\mathbb{R}^{2m}\right)$ since (u_k) is bounded in $H^m\left(\mathbb{R}^{2m}\right)$).

From the above limit and the Adams-type inequalities (Theorem C), we can have that for $\kappa > 1$ sufficiently close to 1 :

$$\int\limits_{\mathbb{R}^{2m}} \left(\exp\left(\beta\left(2m,m\right)\kappa u_k^2(x)\right) - 1\right) dx \leq C.$$

By our assumptions on the nonlinear term, given $\varepsilon > 0$ we could find positive constants C_ε and $q, \gamma > 1$ sufficiently close to 1 such that

$$\begin{aligned}
&\left|\int\limits_{\mathbb{R}^{2m}} f_0(x,u_k)u_k dx\right| \\
&\leq \varepsilon \int\limits_{\mathbb{R}^{2m}} |u_k|^2 dx + C_\varepsilon \int\limits_{\mathbb{R}^{2m}} |u_k| \left[\exp\left(\beta\left(2m,m\right)\gamma u_k^2(x)\right) - 1\right] dx \\
&\leq C\left(\int\limits_{\mathbb{R}^{2m}} |u_k|^{q'} dx\right)^{1/q'} \left(\int\limits_{\mathbb{R}^{2m}} \left[\exp\left(\beta\left(2m,m\right)\gamma u_k^2(x)\right) - 1\right]^q dx\right)^{1/q} + \varepsilon C \\
&\leq C\left(\int\limits_{\mathbb{R}^{2m}} |u_k|^{q'} dx\right)^{1/q'} \left(\int\limits_{\mathbb{R}^{2m}} \left[\exp\left(\beta\left(2m,m\right)q\gamma u_k^2(x)\right) - 1\right] dx\right)^{1/q} + \varepsilon C \\
&= C\left\|u_k\right\|_{q'} + \varepsilon C
\end{aligned}$$

where $q' = q/(q-1)$. Hence

$$\int\limits_{\mathbb{R}^{2m}} F_0(x,u_k)dx, \quad \int\limits_{\mathbb{R}^{2m}} f_0(x,u_k)u_k dx \to 0 \text{ as } k \to \infty.$$

$\square$

3.2. **The existence of nontrivial solutions.** In this section, we mainly prove the existence of nontrivial solutions for Problem (P). In view of Lemma 3.3, we see that the weak limit u_0 of the $(PS)_{c_0}$ is a weak solution of Problem (P). So if we can show $u_0 \neq 0$, the problem is then completed. Indeed, if $u_0 = 0$, then we have the following result:

Claim 3.1. *There is a sequence $(z_k) \subset \mathbb{R}^{2m}$, and $R,\ A > 0$ such that*

$$\liminf_{k\to\infty} \int_{B_R(z_k)} |u_k|^2 dx > A.$$

Proof. If not, then by Lemma 3.4, we have

$$\int\limits_{\mathbb{R}^{2m}} F(x,u_k)dx, \quad \int\limits_{\mathbb{R}^{2m}} f(x,u_k)u_k dx \to 0 \text{ as } k \to \infty.$$

which implies that $u_k \to 0$ in $H^m\left(\mathbb{R}^{2m}\right)$. As a consequence, $c_0 = 0$ which is impossible. $\square$

Next, without loss of generality, we may assume that $(z_k) \subset \mathbb{Z}^{2m}$. Now, letting $\widetilde{u}_k(x) = u_k(x - z_k)$, since $f_0,\ F_0$ are 1−periodic functions, by a direct calculation, we get

$$\begin{aligned} \|u_k\| &= \|\widetilde{u}_k\| \\ J_0(u_k) &= J_0(\widetilde{u}_k) \\ DJ_0(\widetilde{u}_k) &\to 0. \end{aligned}$$

Then we can find a $\widetilde{u}_0$ in $H^m\left(\mathbb{R}^{2m}\right)$ such that $\widetilde{u}_k \rightharpoonup \widetilde{u}_0$ weakly in $H^m\left(\mathbb{R}^{2m}\right)$. It's not hard to see that $DJ_0(\widetilde{u}_0) = 0$. Finally, we notice that by taking a subsequence and R sufficiently large, we can get

$$A^{1/2} \leq \|\widetilde{u}_k\|_{L^2(B_R(0))} \leq \|\widetilde{u}_0\|_{L^2(B_R(0))} + \|\widetilde{u}_k - \widetilde{u}_0\|_{L^2(B_R(0))}.$$

By the compact embedding $H^m\left(\mathbb{R}^{2m}\right) \hookrightarrow L^2_{loc}\left(\mathbb{R}^{2m}\right)$ we have that $\widetilde{u}_0$ is nontrivial.

4. The nonperiodic problem-The proof of Theorem 1.1.

In this section, we study the existence of nontrivial solutions of Problem (NP). Again, the solutions of Problem (NP) are the critical points of the associated C^1 functional J. Now, as in the above section, we may check that the functional energy J has the geometry of the mountain-pass theorem. Also, we can find a bounded Palais-Smale sequence (v_k) in $H^m\left(\mathbb{R}^{2m}\right)$ such that

$$J\left(v_k\right) \to c_1 \text{ and } DJ\left(v_k\right) \to 0, \text{ as } k \to \infty.$$

Furthermore, we also have $c_1 \in (\kappa, (\theta - 2)/2\theta]$ for positive constant κ and $v_k \rightharpoonup v_0$ in $H^m\left(\mathbb{R}^{2m}\right)$ and that v_0 is a critical point of functional J.

We also get that

$$\limsup_{k\to\infty} \|v_k\|^2 \leq m' < 1$$

which again implies that for $\gamma > 1$ sufficiently close to 1, we have

$$\int\limits_{\mathbb{R}^{2m}} \left(\exp\left(\beta\left(2m, m\right)\gamma v_k^2(x)\right) - 1\right) dx \leq C$$

for some universal constant $C > 0$. Now, it's sufficient to prove that v_0 is nontrivial.

4.1. Proof of Theorem 1.1.

Claim 4.1. *v_0 is nontrivial.*

Proof. We will prove it by contradiction. Suppose, on the contrary, that v_0 is trivial.

First, we will prove that

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^{2m}} |F_0(x, v_k) - F(x, v_k)|\, dx = \lim_{k\to\infty} \int\limits_{\mathbb{R}^{2m}} |f_0(x, v_k)v_k - f(x, v_k)v_k|\, dx = 0.$$

Indeed, given $\varepsilon > 0$, there exists $\eta > 0$ such that by the Adams-type inequalities and Sobolev embeddings:

$$\int\limits_{|x|\geq\eta} |f_0(x, v_k)v_k - f(x, v_k)v_k|\, dx$$

$$\leq \varepsilon \int\limits_{|x|\geq \eta} \left|\left(\exp\left(\beta\left(2m,m\right)\gamma v_k^2(x)\right)-1\right)v_k\right|$$
$$\leq \varepsilon \left(\int\limits_{\mathbb{R}^{2m}} \left|\left(\exp\left(\beta\left(2m,m\right)\gamma v_k^2(x)\right)-1\right)\right|^q dx\right)^{1/q} \left(\int\limits_{\mathbb{R}^{2m}} |v_k|^{q'} dx\right)^{1/q'}$$
$$\leq C\varepsilon.$$

On the other hand, by the compact embedding $H^m\left(\mathbb{R}^{2m}\right) \hookrightarrow L^r_{loc}\left(\mathbb{R}^{2m}\right)$, $r \geq 1$:

$$\int\limits_{|x|\leq \eta} |f_0(x,v_k)v_k - f(x,v_k)v_k|\, dx$$
$$\leq \left(\int\limits_{\mathbb{R}^{2m}} \left|\left(\exp\left(\beta\left(2m,m\right)\gamma v_k^2(x)\right)-1\right)\right|^q dx\right)^{1/q} \left(\int\limits_{|x|\leq\eta} |v_k|^{q'} dx\right)^{1/q'}$$
$$+\varepsilon \int\limits_{\mathbb{R}^4} |v_k|^2 .$$

Combining these two inequalities, we have

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^{2m}} |F_0(x,v_k) - F(x,v_k)|\, dx = \lim_{k\to\infty} \int\limits_{\mathbb{R}^{2m}} |f_0(x,v_k)v_k - f(x,v_k)v_k|\, dx = 0.$$

From this equation, we get

$$|J_0\left(v_k\right) - J\left(v_k\right)| \to 0$$
$$\|DJ_0\left(v_k\right) - DJ\left(v_k\right)\| \to 0.$$

which implies

$$J_0\left(v_k\right) \to c_1$$
$$DJ_0\left(v_k\right) \to 0.$$

As in the previous section, there is a sequence $(z_k) \subset \mathbb{Z}^{2m}$, and $R,\ A > 0$ such that

$$\liminf_{k\to\infty} \int_{B_R(z_k)} |v_k|^2\, dx > A.$$

Now, letting $\widetilde{v}_k(x) = v_k(x - z_k)$, since $f_0,\ F_0$ are $1-$periodic functions, by a routine calculation, we get

$$\|v_k\| = \|\widetilde{v}_k\|$$
$$J_0(v_k) = J_0(\widetilde{v}_k)$$
$$DJ_0(\widetilde{v}_k) \to 0.$$

Then we can find a $\widetilde{v}_0$ in $H^m\left(\mathbb{R}^{2m}\right)$ such that $\widetilde{v}_k \rightharpoonup \widetilde{v}_0$ weakly in $H^m\left(\mathbb{R}^{2m}\right)$ and $DJ_0(\widetilde{v}_0) = 0$.

Next, by Fatou's lemma we have:

$$
\begin{aligned}
J_0(\widetilde{v}_0) &= J_0(\widetilde{v}_0) - \frac{1}{2} D J_0(\widetilde{v}_0)\widetilde{v}_0 \\
&= \frac{1}{2} \int\limits_{\mathbb{R}^{2m}} [f_0(x, \widetilde{v}_0)\widetilde{v}_0 - 2F_0(x, \widetilde{v}_0)] \\
&\le \liminf_{k\to\infty} \frac{1}{2} \int\limits_{\mathbb{R}^{2m}} [f_0(x, \widetilde{v}_k)\widetilde{v}_k - 2F(x, \widetilde{v}_k)] \\
&= \lim_{k\to\infty} \left[J_0(\widetilde{v}_k) - \frac{1}{2} D J_0(\widetilde{v}_k)\widetilde{v}_k \right] = c_1.
\end{aligned}
$$

Similarly as in the previous section, we have that $\widetilde{v}_0 \neq 0$ and

$$
c_1 \ge J_0(\widetilde{v}_0) = \max_{t\ge 0} J_0(t\widetilde{v}_0) \ge c_0
$$

On the other hand, by assumptions $(f1)$ and $(f7)$:

$$
c_1 \le \max_{t\ge 0} J(tu_0) = J(t_1 u_0) < J_0(t_1 u_0) \le \max_{t\ge 0} J_0(tu_0) = J(u_0) = c_0
$$

and we get a contradiction. Therefore, v_0 is nontrivial.

Since $f(x, u) = 0$ for all $(x, u) \in \mathbb{R}^{2m} \times (-\infty, 0]$, from standard arguments, it's easy to see that this weak solution is nonnegative. □

5. **Proof of Theorems 1.2 and 1.3.**

5.1. **Proof of Theorem 1.2.** By the assumption, we have

$$
g(x) = |f(x, u(x))| \log |f(x, u(x))| \chi \{x : |f(x, u(x))| \ge 1\} \text{ is in } L^1\left(\mathbb{R}^{2m}\right).
$$

Since u is a solution of (I), we have

$$
\begin{aligned}
|u(x)| &= |G_{2m} * N_f(u)(x)| \\
&= \left| \int\limits_{\mathbb{R}^{2m}} G_{2m}(x-y) f(y, u(y)) dy \right|.
\end{aligned}
$$

Using the following inequality

$$
st \le \left(e^t - 1\right) + s \log s \text{ for all } t \ge 0 \text{ and } s \ge 1,
$$

we can have the following estimate:

$$
\begin{aligned}
|u(x)| &\le \int\limits_{\mathbb{R}^{2m}} |G_{2m}(x-y)| \, |f(y, u(y))| \, dy \\
&\le \int\limits_{\mathbb{R}^{2m}} \left(e^{|G_{2m}(x-y)|} - 1 \right) dy + \int\limits_{\mathbb{R}^{2m}} |g(y)| \, dy + \int\limits_{\mathbb{R}^{2m}} |G_{2m}(x-y)|
\end{aligned}
$$

By our assumption, the second term is bounded. Since G_{2m} has exponential decay at infinity, and from the properties of the Bessel kernel that

$$
\int\limits_{|x|\le 2} |G_{2m}(x)| \, dx \sim \int\limits_{|x|\le 2} \left| \log \frac{2}{|x|} \right| dx < \infty,
$$

the third term is bounded. So now, we just need to take care of the first one. In fact, we have

$$\int_{|y|\leq 2}\left(e^{|G_{2m}(y)|}-1\right)dy \sim \int_{|y|\leq 2}\left(e^{\left|\log\frac{2}{|y|}\right|}-1\right)dy$$
$$\sim \int_{|y|\leq 2}\frac{2}{|y|}$$

and

$$\int_{|y|\geq 2}\left(e^{|G_{2m}(y)|}-1\right)dy \sim \int_{|y|\geq 2}\left(e^{e^{-\frac{|y|}{2}}}-1\right)dy$$
$$\sim \int_{|y|\geq 2} e^{-\frac{|y|}{2}}dy.$$

Therefore, u is uniformly bounded in $\mathbb{R}^{2m}$.

We now prove that u is Lipschitz continuous. Let

$$\mathcal{X}=\left\{v\in L^{\infty}(\mathbb{R}^{2m}):\|v\|_{\infty}\leq 2\|u\|_{\infty}\right\},$$
$$\mathcal{Y}=\left\{v\in \Lambda^{1}(\mathbb{R}^{2m}):\|v\|_{\infty}\leq 2\|u\|_{\infty}\right\}.$$

We notice that,

$$\begin{aligned}u(x)&=G_{2m}*f(u(x))\\&=\int_{\mathbb{R}^{2m}}G_{2m}(x-y)f(u(y))dy\\&=\int_{0}^{m}\int_{\partial B_r(x)}G_{2m}(x-y)f(u(y))dydr+\int_{m}^{\infty}\int_{\partial B_r(x)}G_{2m}(x-y)f(u(y))dydr.\end{aligned}$$

Now, define

$$T_m w(x)=\int_{0}^{m}\int_{\partial B_r(x)}G_{2m}(x-y)f(w(y))dydr,$$
$$g(x)=\int_{m}^{\infty}\int_{\partial B_r(x)}G_{2m}(x-y)f(u(y))dydr$$
$$S_m w=T_m w+g.$$

Claim 5.1. *T_m is contracting from $\mathcal{X}$ into $L^{\infty}(\mathbb{R}^{2m})$.*

Proof. Let $\phi,\ \varphi\in\mathcal{X}$ and $x\in\mathbb{R}^{2m}$. We have

$$\begin{aligned}|T_m\phi(x)-T_m\varphi(x)|&\leq\int_{0}^{m}\int_{\partial B_r(x)}G_{2m}(x-y)\left|f(\phi(y))-f(\varphi(y))\right|dydr\\&=\int_{0}^{m}\int_{\partial B_r(x)}G_{2m}(x-y)\left|\phi(y)-\varphi(y)\right|\frac{|f(\phi(y))-f(\varphi(y))|}{|\phi(y)-\varphi(y)|}dydr\end{aligned}$$

$$\leq C(M)\|\phi-\varphi\|_\infty \int_0^m \int_{\partial B_r(0)} G_{2m}(y)dydr.$$

Here we used the assumption $(f8)$.

Next, note that with $m \leq 2$ we have $|y| \leq 2$ and $G_{2m}(y) \sim \log \frac{2}{|y|}$:

$$\int_0^m \int_{\partial B_r(0)} G_{2m}(y)dydr \sim m^{2m} \log \frac{2}{m}.$$

Choosing m sufficiently small such that $C(M)\, m^{2m} \log \frac{2}{m} \leq \frac{1}{4}$, we have the Claim. □

Claim 5.2. *T_m is shrinking from $\mathcal{Y}$ into $\Lambda^1(\mathbb{R}^{2m})$.*

Proof. Let $\phi \in \mathcal{Y}$ and $x, z \in \mathbb{R}^{2m}$ we have

$$\begin{aligned}
&|T_m\phi(x) - T_m\phi(z)| \\
&\leq \int_0^m \left| \int_{\partial B_r(x)} G_{2m}(x-y)f(\phi(y))dy - \int_{\partial B_r(z)} G_{2m}(z-y)f(\phi(y))dy \right| dr \\
&\leq \int_0^m \int_{\partial B_r(x)} G_{2m}(x-y)\left|f(\phi(y)) - f(\phi(y+z-x))\right| dydr \\
&\leq \int_0^m \int_{\partial B_r(x)} G_{2m}(x-y)\left|\phi(y) - \phi(y+z-x)\right| \frac{|f(\phi(y)) - f(\phi(y+z-x))|}{|\phi(y) - \phi(y+z-x)|} dydr \\
&\leq C(M) \int_0^m \int_{\partial B_r(x)} G_{2m}(x-y)\left|\phi(y) - \phi(y+z-x)\right| dydr \\
&\leq C(M)\|\phi\|_{\Lambda^1} |z-x| \int_0^m \int_{\partial B_r(0)} G_{2m}(y)dydr.
\end{aligned}$$

Similarly as in Claim 5.1, we can choose m sufficiently small such that

$$\sup_{x\neq z} \frac{|T_m\phi(x) - T_m\phi(z)|}{|z-x|} \leq \frac{1}{4}\|\phi\|_{\Lambda^1}.$$

Moreover, from Claim 5.1, we have

$$\|T_m\phi\|_\infty \leq \frac{1}{4}\|\phi\|_\infty \leq \frac{1}{4}\|\phi\|_{\Lambda^1},$$

which implies

$$\begin{aligned}
\|T_m\phi\|_{\Lambda^1} &= \|T_m\phi\|_\infty + \sup_{x\neq z} \frac{|T_m\phi(x) - T_m\phi(z)|}{|z-x|} \\
&\leq \frac{1}{2}\|\phi\|_{\Lambda^1}.
\end{aligned}$$

□

Claim 5.3. $g \in \mathcal{X} \cap \mathcal{Y}$

Proof. First, it's clear that $g \in \mathcal{X}$ since

$$\begin{aligned}
\|g\|_\infty &= \left\| \int_m^\infty \int_{\partial B_r(x)} G_{2m}(x-y) f(u(y)) dy dr \right\|_\infty \\
&\leq \left\| \int_0^\infty \int_{\partial B_r(x)} G_{2m}(x-y) f(u(y)) dy dr \right\|_\infty \\
&= \|G_{2m} * f(u(x))\|_\infty \\
&= \|u\|_\infty .
\end{aligned}$$

Now, let $x, z \in \mathbb{R}^{2m}$. Note that $G_{2m}(x)$ is bounded for $|x| \geq m$, we have that

$$\begin{aligned}
&|g(x) - g(z)| \\
&= \left| \int_m^\infty \int_{\partial B_r(x)} G_{2m}(x-y) f(u(y)) dy dr - \int_m^\infty \int_{\partial B_r(z)} G_{2m}(z-y) f(u(y)) dy dr \right| \\
&\leq C \left| \int_m^\infty \int_{\partial B_r(x)} f(u(y)) dy dr - \int_m^\infty \int_{\partial B_r(z)} f(u(y)) dy dr \right| \\
&\leq C \left| \int_{\mathbb{R}^{2m} \setminus B_m(x)} f(u(y)) dy dr - \int_{\mathbb{R}^{2m} \setminus B_m(z)} f(u(y)) dy dr \right| \\
&\leq C M C(M) m^{2m-1} |x - z| .
\end{aligned}$$

Notice that in the above last inequality, we use the assumption $(f8)$: since $|u(y)| \leq M$, $|f(u(y))| \leq MC(M)$ and the fact that $|B_m(x) \bigtriangleup B_m(z)| \sim m^{2m-1} |x - z|$ where $B_m(x) \bigtriangleup B_m(z) = (B_m(x) \setminus B_m(z)) \cup (B_m(z) \setminus B_m(x))$. So $g \in \mathcal{X} \cap \mathcal{Y}$. $\square$

Claim 5.4. $S_m : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X} \cap \mathcal{Y}$.

Proof. Given $\phi \in \mathcal{X} \cap \mathcal{Y}$, then by Claims 5.1 and 5.2, it's clear that $T_m \phi \in L^\infty(\mathbb{R}^{2m}) \cap \Lambda^1(\mathbb{R}^{2m})$. Since $g \in \mathcal{X} \cap \mathcal{Y}$ by Claim 5.3, we can conclude that $S_m \phi = T_m \phi + g \in L^\infty(\mathbb{R}^{2m}) \cap \Lambda^1(\mathbb{R}^{2m})$. Finally, from Claims 5.1 and 5.3 we have

$$\begin{aligned}
\|S_m \phi\|_\infty &\leq \|T_m \phi\|_\infty + \|g\|_\infty \\
&\leq \frac{1}{4} \|u\|_\infty + \|u\|_\infty \\
&= \frac{5}{4} \|u\|_\infty ,
\end{aligned}$$

which thus implies $S_m \phi \in \mathcal{X} \cap \mathcal{Y}$. $\square$

From these four claims and the Regularity lifting by combinations of contracting and shrinking operators (Theorem 2.2), we can conclude that u is a Lipschitz continuous function.

5.2. **Proof of Theorem 1.3.** For a given real number λ, we define

$$\begin{aligned}
\Sigma_\lambda &= \{x = (x_1, ..., x_{2m}) \mid x_1 \geq \lambda\}, \\
\Sigma_\lambda^c &= \{x = (x_1, ..., x_{2m}) \mid x_1 < \lambda\}, \\
x_\lambda &= (2\lambda - x_1, ..., x_{2m}), \\
u_\lambda(x) &= u(x_\lambda), \\
\Sigma_\lambda^- &= \{x \mid x \in \Sigma_\lambda,\ u(x) < u_\lambda(x)\}.
\end{aligned}$$

Lemma 5.1. *For any solution $u(x)$ of (I), we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} [G_{2m}(x-y) - G_{2m}(x_\lambda - y)]\,[f(u(y)) - f(u_\lambda(y))]\,dy.$$

Proof. We have

$$\begin{aligned}
u(x) &= \int_{\Sigma_\lambda} G_{2m}(x-y) f(u(y))dy + \int_{\Sigma_\lambda^c} G_{2m}(x-y) f(u(y))dy \\
&= \int_{\Sigma_\lambda} G_{2m}(x-y) f(u(y))dy + \int_{\Sigma_\lambda} G_{2m}(x-y_\lambda) f(u(y_\lambda))dy \\
&= \int_{\Sigma_\lambda} G_{2m}(x-y) f(u(y))dy + \int_{\Sigma_\lambda} G_{2m}(x_\lambda-y) f(u_\lambda(y))dy
\end{aligned}$$

where we have used the fact that $|x - y_\lambda| = |x_\lambda - y|$.

Similarly, we have

$$u(x_\lambda) = \int_{\Sigma_\lambda} G_{2m}(x_\lambda - y) f(u(y))dy + \int_{\Sigma_\lambda} G_{2m}(x-y) f(u_\lambda(y))dy.$$

Thus

$$\begin{aligned}
u(x) - u_\lambda(x) &= \int_{\Sigma_\lambda} G_{2m}(x-y) f(u(y))dy + \int_{\Sigma_\lambda} G_{2m}(x_\lambda-y) f(u_\lambda(y))dy \\
&\quad - \int_{\Sigma_\lambda} G_{2m}(x_\lambda-y) f(u(y))dy - \int_{\Sigma_\lambda} G_{2m}(x-y) f(u_\lambda(y))dy \\
&= \int_{\Sigma_\lambda} [G_{2m}(x-y) - G_{2m}(x_\lambda - y)]\,[f(u(y)) - f(u_\lambda(y))]\,dy.
\end{aligned}$$

□

Lemma 5.2. *For $\lambda << 0$, Σ_λ^- must be measure zero.*

Proof. First, noting that for any $x,\ y \in \Sigma_\lambda$, we have $|x-y| \leq |x_\lambda - y|$. Also, for $\lambda < 0$, we have $|x| \leq |x_\lambda|$ for any $x \in \Sigma_\lambda$. Now, for any $x \in \Sigma_\lambda^-$,

$$\begin{aligned}
u_\lambda(x) - u(x) &= -\int_{\Sigma_\lambda} [G_{2m}(x-y) - G_{2m}(x_\lambda - y)]\,[f(u(y)) - f(u_\lambda(y))]\,dy \\
&= -\int_{\Sigma_\lambda^-} [G_{2m}(x-y) - G_{2m}(x_\lambda - y)]\,[f(u(y)) - f(u_\lambda(y))]\,dy \\
&\quad - \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} [G_{2m}(x-y) - G_{2m}(x_\lambda - y)]\,[f(u(y)) - f(u_\lambda(y))]\,dy \\
&\leq -\int_{\Sigma_\lambda^-} [G_{2m}(x-y) - G_{2m}(x_\lambda - y)]\,[f(u(y)) - f(u_\lambda(y))]\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Sigma_\lambda^-} G_{2m}(x_\lambda - y)\left[f(u(y)) - f(u_\lambda(y))\right] dy \\
&- \int_{\Sigma_\lambda^-} G_{2m}(x - y)\left[f(u(y)) - f(u_\lambda(y))\right] dy \\
&\le \int_{\Sigma_\lambda^-} G_{2m}(x - y)\left[f(u_\lambda(y)) - f(u(y))\right] dy \\
&\le \int_{\Sigma_\lambda^-} G_{2m}(x - y)\left[u_\lambda(y) - u(y)\right] f'\left(u_\lambda(y)\right) dy
\end{aligned}
$$

Thus using the HLS type inequality (Theorem 2.3), we have:

$$
\begin{aligned}
\|u_\lambda - u\|_{L^q\left(\Sigma_\lambda^-\right)} &\le \left\|B_{2m}\left(f'\left(u_\lambda\right)\left(u_\lambda - u\right)\right)\right\|_{L^q\left(\Sigma_\lambda^-\right)} \\
&\le C\left\|f'\left(u_\lambda\right)\left(u_\lambda - u\right)\right\|_{L^{\frac{q}{\beta}}\left(\Sigma_\lambda^-\right)} \\
&\le C\left\|f'\left(u_\lambda\right)\right\|_{L^{\frac{q}{\beta-1}}\left(\Sigma_\lambda^-\right)} \|u_\lambda - u\|_{L^q\left(\Sigma_\lambda^-\right)} \\
&\le C\left\|f'\left(u\right)\right\|_{L^{\frac{q}{\beta-1}}\left(\Sigma_\lambda^c\right)} \|u_\lambda - u\|_{L^q\left(\Sigma_\lambda^-\right)}
\end{aligned}
$$

Now we choose $M >> 0$ such that for $\lambda \le -M : C\left\|f'\left(u\right)\right\|_{L^{\frac{q}{\beta-1}}\left(\Sigma_\lambda^c\right)} < 1$. Then it's clear that Σ_λ^- must be measure zero. $\square$

Now, we choose $M >> 0$ such that for $\lambda \le -M$,

$$
u(x) \ge u_\lambda(x),\ \forall x \in \Sigma_\lambda. \tag{4}
$$

First, we start moving the planes continuously from $\lambda \le -M$ to the right as long as (4) holds. If at a $\lambda_0 < 0$, we have $u(x) \ge u_{\lambda_0}(x)\ \forall x \in \Sigma_{\lambda_0}$ and $m\left(\{x \in \Sigma_{\lambda_0} : u(x) > u_{\lambda_0}(x)\}\right) > 0$, then by Lemma 5.1, $u(x) > u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Thus $\overline{\Sigma_{\lambda_0}^-} = \{x \in \Sigma_{\lambda_0} : u(x) \le u_{\lambda_0}(x)\}$ has measure zero. Moreover, $\lim\limits_{\lambda \to \lambda_0} \Sigma_\lambda^- \subset \overline{\Sigma_{\lambda_0}^-}$. Let $\left(\Sigma_\lambda^-\right)^*$ be the reflection of Σ_λ^- about the planes $x_1 = \lambda$. Similarly as in Lemma 5.2, we have

$$
\|u_\lambda - u\|_{L^q\left(\Sigma_\lambda^-\right)} \le C\left\|f'\left(u\right)\right\|_{L^{\frac{q}{\beta-1}}\left(\left(\Sigma_\lambda^-\right)^*\right)} \|u_\lambda - u\|_{L^q\left(\Sigma_\lambda^-\right)}.
$$

It's clear that we can choose an $\varepsilon \gtrapprox 0$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$, we have $C\left\|f'\left(u\right)\right\|_{L^{\frac{q}{\beta-1}}\left(\left(\Sigma_\lambda^-\right)^*\right)} < 1$ which implies $\|u_\lambda - u\|_{L^q\left(\Sigma_\lambda^-\right)} = 0$. Hence Σ_λ^- must be measure zero and the planes can be moved further to the right.

Now, the planes either stops at some $\lambda < 0$ or can be moved until $\lambda = 0$. In the former case, we have that $u(x) = u_\lambda(x)\ \forall x \in \Sigma_\lambda$. In the latter case, we have $u(x_1, ..., x_{2m}) \ge u(-x_1, ..., x_{2m})\ \forall x \in \mathbb{R}^{2m},\ x_1 \ge 0$. However, we can move the planes from the right with the same argument to get that $u(x_1, ..., x_{2m}) \le u(-x_1, ..., x_{2m})\ \forall x \in \mathbb{R}^{2m},\ x_1 \ge 0$. Thus $u(x_1, ..., x_{2m}) = u(-x_1, ..., x_{2m})\ \forall x \in \mathbb{R}^{2m}$. Since we can choose any direction to start the process, we can conclude that u is radial symmetric and monotone decreases about some point in $\mathbb{R}^{2m}$.

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Received for publication August 2014.

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