A SUBELLIPTIC TAYLOR ISOMORPHISM ON INFINITE-DIMENSIONAL HEISENBERG GROUPS

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Abstract. Let G denote an infinite-dimensional Heisenberg-like group, which is a class of infinite-dimensional step 2 stratified Lie groups. We consider holomorphic functions on G that are square integrable with respect to a heat kernel measure which is formally subelliptic, in the sense that all appropriate finite-dimensional projections are smooth measures. We prove a unitary equivalence between a subclass of these square integrable holomorphic functions and a certain completion of the universal enveloping algebra of the "Cameron-Martin" Lie subalgebra. The isomorphism defining the equivalence is given as a composition of restriction and Taylor maps.

Contents

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1. Introduction

We study spaces of holomorphic functions on infinite-dimensional Heisenberg-like groups based on an abstract Wiener space as constructed in [7]. In particular, we consider holomorphic functions which are square integrable with respect to a subelliptic heat kernel measure and prove a unitary equivalence between a subclass of these functions and a certain completion of the universal enveloping algebra of the Cameron-Martin Lie subalgebra. These results may be viewed as an analogue of the results in [8] for degenerate heat kernel measures, or as an extension of the finite-dimensional results in [10] to a special infinite-dimensional case. Perhaps more particularly, it is an infinite-dimensional extension of [11] in a special case, as the Heisenberg-like groups considered here are nilpotent. There are considerable differences from both cases in techniques, as analytically our setting is very different from the elliptic case in [8], and there are numerous subtle issues when dealing with infinite dimensions versus the finite-dimensional nilpotent case in [11]. In particular, in the infinite-dimensional setting, it is necessary to consider two different norms on the Lie algebra, one which defines the space on which the functions live and one which controls the analysis. This is directly analogous to the abstract Wiener space construction.

1.1. **Background.**

We give a brief (incomplete) background of the development of the Taylor isomorphism to put our results into context. See the papers cited here and their bibliographies for more complete references. Also, the paper [18] gives a very nice discussion and extensive history of the theory.

Let us first recall the classical result. Let $f:\mathbb{C}\to\mathbb{C}$ be a holomorphic function. Then it is well known that f is everywhere determined by the values of its derivatives at the origin and in particular

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k.$$

Moreover, if $d\mu_t(z)=p_t(z)\,dz$ where $p_t(z)=\frac{1}{\pi t}e^{-|z|^2/t}$ is the standard Gaussian density on $\mathbb{C}$, then $\langle z^k,z^\ell\rangle_{L^2(\mu_t)}=\delta_{k\ell}t^k k!$, which implies that

$$\|f\|^2_{L^2(\mu_t)}=\sum_{k=0}^{\infty}\frac{t^k}{k!}|f^{(k)}(0)|^2. \tag{1.1}$$

Thus, one may consider the Taylor expansion as an isometric isomorphism from the space of square integrable, holomorphic functions onto the sequence space of derivatives at 0 endowed with an appropriate norm.

This isomorphism first appeared in the paper of Fock [12] (actually for $\mathbb{C}^n$), but was not made explicit until the work of Segal [25, 26] and Bargmann [2]. Multiple authors contributed to various extensions of this theory, all of which culminated in the paper [9]. In this paper, Driver and Gross considered the case of a connected complex (finite-dimensional) Lie group G with Lie algebra $\mathfrak{g}$. Equip $\mathfrak{g}$ with any inner product, and suppose that $\{V_i\}_{i=1}^n$ is an orthonormal basis of $\mathfrak{g}$. Consider $L=\sum_{i=1}^n\tilde{V}_i^2$, where $\tilde{V}$ is the left invariant vector on G field associated to $V\in\mathfrak{g}$. Then L is an elliptic second order differential operator, and we let $\{g_t\}_{t\ge0}$ denote a Brownian motion on G with generator L. For $t>0$, let $\mathcal{H}L^2(G,\mu_t)$ denote the space of holomorphic functions on G which are square integrable with respect to the heat kernel measure $\mu_t=\mathrm{Law}(g_t)$ on G. Then it was proved in [9] that the analogous Taylor map in this setting is an isometric isomorphism from $\mathcal{H}L^2(G, \mu_t)$ to the space of derivatives at the identity equipped with a norm inspired by the expression in (1.1).

Recently, in [10], Driver, Gross, and Saloff-Coste have further extended this theory to the case of subelliptic (or hypoelliptic) heat kernel measures on a connected complex Lie group. That is, suppose in the previous setting that $\{V_i\}_{i=1}^k \subset \mathfrak{g}$ is not itself a full basis of $\mathfrak{g}$, but does satisfy the Hörmander (or bracket generating) condition

$$\operatorname{span}\{V_i, [V_i, V_j], [V_i, [V_j, V_k]], \ldots\} = \mathfrak{g}.$$

Then due to the classical result of Hörmander [21], it is well known that, for the process $\{g_t\}_{t\geq 0}$ generated by $L = \sum_{i=1}^k \tilde{V}_i^2$, $\mu_t = \mathrm{Law}(g_t)$ is a smooth measure for all $t > 0$. In [10], it is proved that the Taylor map is an isometric isomorphism, this time from $\mathcal{H}L^2(G, \mu_t)$ onto the space of derivatives at the identity with an appropriately modified norm.

There have also been several infinite-dimensional settings in which Taylor isomorphisms have been shown to hold. In particular, in [8] Driver and the first named author proved a Taylor isomorphism theorem for nondegenerate heat kernel measure on the same infinite-dimensional Heisenberg-like groups considered in the present paper. The first named author has proved analogues on the infinite-dimensional complex Hilbert-Schmidt groups [14,15] and for the group of invertible operators in a factor of type II_1 [16]. Also, in [5], Cecil proved an analogue for path groups over stratified nilpotent Lie groups. To our knowledge, the present paper represents the first analogous result for an infinite-dimensional subelliptic setting.

1.2. **Statement of Results.**

1.2.1. *Heisenberg-like groups and subelliptic heat kernel measures.* Let (W, H, μ) be a complex abstract Wiener space and let $\mathbf{C}$ be a finite-dimensional complex inner product space. Let $\mathfrak{g} = W \times \mathbf{C}$ be an infinite-dimensional Heisenberg-like Lie algebra, which is constructed as an infinite-dimensional step 2 nilpotent Lie algebra with Lie bracket satisfying the following condition:

$$[W, W] = \mathbf{C}. \tag{1.2}$$

Let G denote $W \times \mathbf{C}$ thought of as a group with operation

$$g_1 \cdot g_2 = g_1 + g_2 + \frac{1}{2}[g_1, g_2].$$

Then G is a Lie group with Lie algebra $\mathfrak{g}$, and G contains the subgroup $G_{CM} = H \times \mathbf{C}$ which has Lie algebra $\mathfrak{g}_{CM}$. See Section 2.2 for definitions and details.

Now let $\{B_t\}_{t\geq 0}$ be a Brownian motion on W. The solution to the stochastic differential equation

$$dg_t = g_t \circ dB_t \qquad \text{with } g_0 = e \tag{1.3}$$

is a Brownian motion on G, which is given explicitly in Proposition 2.21 and Definition 2.22. For all $t > 0$, let $\nu_t = \mathrm{Law}(g_t)$ denote the heat kernel measure at time t. If W is finite dimensional, then (1.2) implies that $\operatorname{span}\{(\xi_i, 0), [(\xi_i, 0), (\xi_j, 0)]\} = \mathfrak{g}$, where $\{\xi_i\}_{i=1}^{\dim(W)}$ is some orthonormal basis of W, and thus we would have satisfaction of Hörmander's condition implying that ν_t is absolutely continuous with respect to Haar measure on $G = W \times \mathbf{C}$ and its density is a smooth function on

G. If W is infinite-dimensional, then the notion of subellipticity is not so well defined as there is no canonical reference measure. But we say that ν_t is formally subelliptic (or hypoelliptic) in the sense that all appropriate finite-dimensional projections (which will be discussed subsequently) are subelliptic. Similar "definitions" of subellipticity in infinite dimensions have been taken in [1, 13, 23], for example.

Let $\mathrm{Proj}(W)$ denote the collection of finite rank continuous linear maps $P : W \to H$ so that $P|_H$ is orthogonal projection. Further, let $G_P := PW \times \mathbf{C}$ which is a subgroup of G_{CM}. For each $P \in \mathrm{Proj}(W)$, G_P is a finite-dimensional Lie group and Brownian motion on G_P is defined analogously to how it is defined on G. The finite-dimensional heat kernel measures ν_t^P will play an important role in the sequel. In particular, under the assumption that $[PW, PW] = \mathbf{C}$, Hörmander's theorem implies that $d\nu_t^P(x) = p_t^P(x)\,dx$, where p_t^P is a smooth density and dx is finite-dimensional Haar measure.

As has been the case in previous infinite-dimensional contexts [5, 8, 14–16], our results actually take the form of two unitary isomorphisms: the "skeleton" or "restriction" map and the Taylor map on "square integrable holomorphic functions" on G_{CM}.

1.2.2. *The restriction isomorphism theorem.* We must first define the Hilbert spaces involved. Let $\mathcal{H}(G)$ and $\mathcal{H}(G_{CM})$ denote the holomorphic functions on G and G_{CM} respectively. Let $\mathcal{P}$ be the space of holomorphic cylinder polynomials on G. Then Proposition 2.29 implies that $\mathcal{P} \subset L^2(\nu_t)$, and so for $t > 0$ define $\mathcal{H}_t^2(G) := L^2(\nu_t)$-closure of $\mathcal{P}$. For $f \in \mathcal{H}(G_{CM})$, let

$$\|f\|_{\mathcal{H}_t^2(G_{CM})} := \sup_{P \in \mathrm{Proj}(W)} \|f|_{G_P}\|_{L^2(\nu_t^P)}$$

and $\mathcal{H}_t^2(G_{CM}) := \{f \in \mathcal{H}(G_{CM}) : \|f\|_{\mathcal{H}_t^2(G_{CM})} < \infty\}$. It is proved in Proposition 2.30 that as usual $\nu_t(G_{CM}) = 0$; however, $\mathcal{H}_t^2(G_{CM})$ should still be roughly thought of as ν_t-square integrable holomorphic functions on G_{CM}. Having made these definitions, we can state our first theorem.

Theorem 1.1. *For all $t > 0$, there is a map $R_t : \mathcal{H}_t^2(G) \to \mathcal{H}_t^2(G_{CM})$ such that R_t is an isometric isomorphism, $R_t p = p|_{G_{CM}}$ for any $p \in \mathcal{P}$, and*

$$|(R_t f)(g)| \leq \|f\|_{L^2(\nu_t)} e^{d_h(e,g)^2/2t}, \text{ for all } g \in G_{CM},$$

where d_h is the horizontal distance on G_{CM} (see Notation 2.14).

The proof of the pointwise bound and that R_t is actually restriction on $\mathcal{P}$ are in Theorem 4.15. The proof of the isometry and surjectivity are in Theorem 4.16.

1.2.3. *The Taylor isomorphism theorem.* Now let $T(\mathfrak{g}_{CM})$ be the algebraic tensor algebra over $\mathfrak{g}_{CM}$, $T(\mathfrak{g}_{CM})'$ be its algebraic dual, $J = J(\mathfrak{g}_{CM})$ be the two-sided ideal in $T(\mathfrak{g}_{CM})$ generated by

$$\{h \otimes k - k \otimes -[h, k] : h, k \in \mathfrak{g}_{CM}\},$$

and $J^0 = \{\alpha \in T(\mathfrak{g}_{CM})' : \alpha(J) = 0\}$ be the backwards annihilator of J. For $t > 0$, define

$$\|\alpha\|_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{\xi_1, \ldots, \xi_k \in \Gamma} |\langle \alpha, (\xi_1, 0) \otimes \cdots \otimes (\xi_k, 0) \rangle|^2, \tag{1.4}$$

where Γ is an orthonormal basis of H, and let $J_t^0 := \{\alpha \in J^0 : \|\alpha\|_t < \infty\}$. Given $f \in \mathcal{H}(G)$, let $\hat{f}(e)$ denote the element of J^0 defined by $\langle \hat{f}(e), 1\rangle = f(e)$ and

$$\langle \hat{f}(e), h_1 \otimes \cdots \otimes h_n \rangle = \left(\tilde{h}_1 \cdots \tilde{h}_n f\right)(e), \text{ for all } h_1, \ldots, h_k \in \mathfrak{g}_{CM}$$

where $\tilde{h}_i$ is the left invariant vector field on G_{CM} such that $\tilde{h}_i(e) = h_i$. For $f \in \mathcal{H}_t^2(G_{CM})$, let $\mathcal{T}_t f = \hat{f}(e)$.

Theorem 1.2. *For all $t > 0$, the map $\mathcal{T}_t : \mathcal{H}_t^2(G_{CM}) \to J_t^0(\mathfrak{g}_{CM})$ is an isometric isomorphism.*

The isometry in Theorem 1.2 is proved in Proposition 3.13 and the surjectivity is proved in Theorem 3.19. The combination of Theorems 1.1 and 1.2 implies that the mapping $f \mapsto (\mathcal{T}_t \circ R_t) f = \widehat{R_t f}(e)$, where

$$\left\langle \widehat{R_t f}(e), h_1 \otimes \cdots \otimes h_k \right\rangle = \left(\tilde{h}_1 \cdots \tilde{h}_k R_t f\right)(e), \text{ for all } h_1, \ldots, h_k \in \mathfrak{g}_{CM},$$

is a unitary equivalence between $\mathcal{H}_t^2(G)$ and J_t^0.

The organization of the paper is as follows. In Section 2, the definitions of infinite-dimensional Heisenberg-like Lie algebras and groups are revisited. This includes a brief review of complex abstract Wiener spaces in Section 2.1. In Section 2.3 we explore the relationship between linear and left invariant derivatives on G which will later be useful in several limiting arguments. In Section 2.4, we prove that the homogeneous norm and horizontal distance topologies are equivalent. This fact is necessary to make use of the finite-dimensional projection groups introduced in Section 2.5 as approximations to G. In Section 2.6, we define the subelliptic diffusion $\{g_t\}_{t\geq 0}$ and its heat kernel measure ν_t and review various properties that will be necessary for the sequel. Most of these properties follow directly from properties for the nondegenerate heat kernel measures treated in [7] and [8]. Also, in Section 2.7 we review the notion of holomorphic functions in this infinite-dimensional setting.

Section 3 gives the proof of the Taylor isomorphism theorem, including a proof in Section 3.1 that the semi-norm defined in (1.4) is in fact a norm. The proofs in this section are mostly standard.

In Section 4, the restriction map is constructed and we prove its isometry and surjectivity properties. Here the proofs are complicated by several factors, including the use of the horizontal distance and the fact that the norm defining J_t^0 is not the full Hilbert-Schmidt norm as is used in the nondegenerate case. Ultimately, the overall steps here are analogous to those in the nondegenerate setting, but the proofs are necessarily adjusted to account for these complications.

1.3. **Discussion of open questions.** Recall that [8] treated the case of nondegenerate heat kernel measures on the same infinite-dimensional Heisenberg-like groups considered here. One of the main ingredients used there was the quasi-invariance of the heat kernel measure under shifts by elements of the Cameron-Martin subgroup. In particular, this allowed the skeleton or restriction map from $\mathcal{H}_t^2(G)$ to $\mathcal{H}_t^2(G_{CM})$ to be defined via quasi-invariance. At the time of the writing of the present paper, quasi-invariance results for the subelliptic heat kernel measure were unknown. Thus, the construction of the restriction map given here does not rely on quasi-invariance. This construction is analogous to that in [5], which treats the case of nondegenerate heat kernel measures on complex path groups, a case in particular where quasi-invariance results are not known. After the present paper was accepted, a quasi-invariance result for the subelliptic heat kernel measure in this setting was proved in [3]. Thus, it may now be possible to give a different proof of our results including the skeleton map defined via quasi-invariance.

One should also comment that the assumption that $\dim(\mathbf{C}) < \infty$ is necessary at several points. For example, it is used in an essential way for the proof that the homogeneous norm topology is equivalent to that of the horizontal distance. Some readers might be concerned that this restriction on the dimension of the center means that this subelliptic example is in some sense only finitely many steps from being elliptic. This concern would be justified if the Lie bracket is non-trivial on only a finite-dimensional subspace of W, as then the solution to (1.3) is somehow only a finite-dimensional subelliptic diffusion coupled with an infinite-dimensional flat Brownian motion. However, if the Lie bracket is in fact non-trivial on an infinite-dimensional subspace of W, then this does introduce several non-trivial complications, for example, in the proof of equivalence of topologies and more generally in working with the horizontal distance and "projections" of horizontal paths.

Another interesting question is to try to generate holomorphic functions similar to how it was done in [11, Section 4]. Even though one of the techniques of that section, the Fourier-Wigner transform, has been studied in infinite dimensions (for example, [17]), it is still not clear how this question can be approached for infinite-dimensional Heisenberg groups.

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2. Infinite-dimensional complex Heisenberg-like groups

2.1. Complex abstract Wiener spaces.

Let us first briefly recall the definition of a complex abstract Wiener space. We record here only the basic construction and some standard facts that will be useful for the sequel. For more details, see for example Section 2 of [8] and its references.

Suppose that W is a complex separable Banach space and $\mathcal{B}_W$ is the Borel σ-algebra on W. Let W_{Re} denote W thought of as a real Banach space. For $\lambda \in \mathbb{C}$, let $M_\lambda : W \to W$ be the operation of multiplication by λ.

Definition 2.1. A measure μ on $(W, \mathcal{B}_W)$ is called a (mean zero, non-degenerate) *Gaussian measure* provided that its characteristic functional is given by

$$\hat{\mu}(u) := \int_W e^{iu(w)} d\mu(w) = e^{-\frac{1}{2}q(u,u)}, \text{ for all } u \in W_{\mathrm{Re}}^*,$$

where $q = q_\mu : W_{\mathrm{Re}}^* \times W_{\mathrm{Re}}^* \to \mathbb{R}$ is an inner product on W_{Re}^*. If in addition, μ is invariant under multiplication by i, that is, $\mu \circ M_i^{-1} = \mu$, we say that μ is a *complex Gaussian measure* on W.

Theorem 2.2. *Let μ be a complex Gaussian measure on a complex separable Banach space W. For $1 \le p < \infty$, let*

$$C_p := \int_W \|w\|_W^p \, d\mu(w) < \infty \tag{2.1}$$

For $w \in W$, let

$$\|w\|_H := \sup_{u \in W^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u,u)}},$$

and define the Cameron-Martin subspace $H \subset W$ *by*

$$H := \{h \in W : \|h\|_H < \infty\}.$$

(1) *For all $1 \leq p < \infty$, $C_p < \infty$.*
(2) *H is a dense complex subspace of W.*
(3) *There exists a unique inner product, $\langle \cdot, \cdot \rangle_H$, on H such that $\|h\|_H^2 = \langle h, h \rangle_H$ for all $h \in H$. Moreover, with this inner product H is a separable complex Hilbert space.*
(4) *For any $h \in H$,*

$$\|h\|_W \leq \sqrt{C_2}\|h\|_H. \tag{2.2}$$

Notation 2.3. *The triple (W, H, μ) appearing in Theorem 2.2 will be called a* complex abstract Wiener space*.*

We will also need the following facts about linear maps from W into a complex Hilbert space K. The proof of the next lemma may be found as part of Lemma 2.6 in [8].

Lemma 2.4. *If $\varphi : W \to K$ is a linear map, then*

$$\int_W \|\varphi(w)\|_K^2 \, d\mu(w) = 2\|\varphi\|_{H^* \otimes K}^2.$$

Now suppose that $\rho : W \times W \to K$ is a continuous bilinear map so that

$$\|\rho\|_0 := \sup\{\rho(w, w')\|_K : \|w\|_W = \|w'\|_W = 1\} < \infty.$$

The continuity of ρ and Lemma 2.4 give the following proposition which is analogous to Proposition 3.14 in [7].

Proposition 2.5. *The bilinear form $\rho : H \times H \to K$ is Hilbert-Schmidt; that is, for any orthonormal basis $\{\xi_j\}_{j=1}^\infty$ of H,*

$$\|\rho\|_{HS}^2 := \sum_{j,k=1}^\infty \|\rho(\xi_j, \xi_k)\|_K^2 < \infty$$

(where $\| \cdot \|_{HS}^2$ is independent of basis).

Proof. By Lemma 2.4,

$$\begin{aligned} \|\rho(w, \cdot)\|_{H^* \otimes K}^2 &= \frac{1}{2} \int_W \|\rho(w, w')\|_K^2 \, d\mu(w') \\ &\leq \frac{1}{2} \|\rho\|_0^2 \|w\|_W^2 \int_W \|w'\|_W^2 \, d\mu(w') = \frac{1}{2} C_2 \|\rho\|_0^2 \|w\|_W^2, \end{aligned}$$

where $C_2 < \infty$ is as defined in (2.1). Similarly, viewing $w \mapsto \rho(w, \cdot)$ as a continuous linear map from W to $H^* \otimes K$,

$$\begin{aligned} \|\rho\|_{HS}^2 = \|h \mapsto \rho(h, \cdot)\|_{H^* \otimes (H^* \otimes K)}^2 &= \frac{1}{2} \int_W \|\rho(w, \cdot)\|_{H^* \otimes K}^2 \, d\mu(w) \\ &\leq \frac{1}{4} \int_W C_2 \|\rho\|_0^2 \|w\|_W^2 \, d\mu(w) = \frac{1}{4} C_2^2 \|\rho\|_0^2. \end{aligned}$$

□

2.2. **Infinite-dimensional complex Heisenberg-like groups.** In this section, we revisit the definition of the infinite-dimensional complex Heisenberg-like groups constructed in [8]. Note that since we are interested in subelliptic heat kernel measures on these groups, there are some necessary modifications to the topology. First we set the following notation which will hold for the entirety of this paper.

Notation 2.6. *Let (W, H, μ) be a complex abstract Wiener space. Let $\mathbf{C}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ and $\dim(\mathbf{C}) = N < \infty$. Let $\omega : W \times W \to \mathbf{C}$ be a continuous skew-symmetric bilinear form on W. We will also trivially assume that ω is surjective (otherwise, we just restrict to a linear subspace of $\mathbf{C}$).*

Definition 2.7. Let $\mathfrak{g}$ denote $W \times \mathbf{C}$ when thought of as a Lie algebra with the Lie bracket given by

$$[(X_1, V_1), (X_2, V_2)] := (0, \omega(X_1, X_2)).$$

Let G denote $W \times \mathbf{C}$ when thought of as a group with multiplication given by

$$g_1 g_2 := g_1 + g_2 + \frac{1}{2}[g_1, g_2], \tag{2.3}$$

where g_1 and g_2 are viewed as elements of $\mathfrak{g}$. For $g_i = (w_i, c_i)$, this may be written equivalently as

$$(w_1, c_1) \cdot (w_2, c_2) = \left(w_1 + w_2, c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2) \right). \tag{2.4}$$

We will call G constructed in this way a *Heisenberg-like group*.

It is easy to verify that, given this bracket and multiplication, $\mathfrak{g}$ is indeed a Lie algebra and G is a group. Note that $g^{-1} = -g$ and the identity $e = (0, 0)$.

Notation 2.8. *Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when thought of as a Lie subalgebra of $\mathfrak{g}$, and we will refer to $\mathfrak{g}_{CM}$ as the* Cameron-Martin subalgebra *of $\mathfrak{g}$. Similarly, let G_{CM} denote $H \times \mathbf{C}$ when thought of as a subgroup of G, and we will refer to G_{CM} as the* Cameron-Martin subgroup *of G.*

We will equip $\mathfrak{g} = G$ with the homogeneous norm

$$\|(w, c)\|_{\mathfrak{g}} := \sqrt{\|w\|_W^2 + \|c\|_{\mathbf{C}}},$$

and analogously on $\mathfrak{g}_{CM} = G_{CM}$ we define

$$\|(A, a)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|a\|_{\mathbf{C}}}.$$

Lemma 2.9. *G and G_{CM} are topological groups with respect to the topologies induced by the homogeneous norms.*

Proof. This is proved similarly to [7, Lemma 3.3]. Since $g^{-1} = -g$, the map $g \mapsto g^{-1}$ is continuous in the $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ topologies. Also $(g_1, g_2) \mapsto [g_1, g_2]$ and $(g_1, g_2) \mapsto g_1 + g_2$ are continuous in both the $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ topologies. Thus, it follows from Equation (2.3) that $(g_1, g_2) \mapsto g_1 \cdot g_2$ is continuous as well. ∎

Before proceeding, let us give the basic motivating examples for the construction of these infinite-dimensional Heisenberg-like groups.

Example 2.1 (Finite-dimensional complex Heisenberg group)**.** Let $W = H = \mathbb{C}^n \times \mathbb{C}^n$, $\mathbf{C} = \mathbb{C}$, and

$$\omega((w_1, w_2), (z_1, z_2)) := w_1 \cdot z_2 - w_2 \cdot z_1.$$

Then $G = \mathbb{C}^{2n} \times \mathbb{C}$ equipped with a group operation as defined in (2.4) is a finite-dimensional complex Heisenberg group.

Example 2.2 (Heisenberg group of a symplectic vector space)**.** Let $(K, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and Q be a strictly positive trace class operator on K. For $h, k \in K$, let $\langle h, k\rangle_Q := \langle h, Qk\rangle$ and $\|h\|_Q := \sqrt{\langle h, h\rangle_Q}$, and let $(K_Q, \langle\cdot,\cdot\rangle_Q)$ denote the Hilbert space completion of $(K, \|\cdot\|_Q)$. Further assume that K is equipped with a conjugation $k \mapsto \bar{k}$ which is isometric and commutes with Q. Let $W = K_Q \times K_Q$, $H = K \times K$, and $\omega : W \times W \to \mathbb{C}$ be defined by

$$\omega((w_1, w_2), (z_1, z_2)) = \langle w_1, \bar{z}_2\rangle_Q - \langle w_2, \bar{z}_1\rangle_Q.$$

Then $G = (K_Q \times K_Q) \times \mathbb{C}$ equipped with a group operation as defined in (2.4) is an infinite-dimensional complex Heisenberg-like group.

2.3. **Derivatives on G.** For $g \in G$, let $L_g : G \to G$ and $R_g : G \to G$ denote left and right multiplication by g, respectively. As G is a vector space, to each $g \in G$ we can associate the tangent space T_gG to G at g, which is naturally isomorphic to G.

Notation 2.10 (Linear and group derivatives)**.** *For $f : G \to \mathbb{C}$, $x \in G$, and $h \in \mathfrak{g}$, let*

$$f'(x)h := \partial_h f(x) = \frac{d}{dt}\bigg|_0 f(x + th),$$

whenever this derivative exists. More generally, for $h_1, \ldots, h_n \in \mathfrak{g}$, let

$$f^{(n)}(x)(h_1 \otimes \cdots \otimes h_n) := \partial_{h_1} \cdots \partial_{h_n} f(x).$$

For $v, x \in G$, let $v_x \in T_xG$ denote the tangent vector satisfying $v_x f = f'(x)v$. If $x(t)$ is any smooth curve in G such that $x(0) = x$ and $\dot{x}(0) = v$ (for example, $x(t) = x + tv$), then

$$L_{g*}v_x = \frac{d}{dt}\bigg|_0 g \cdot x(t).$$

In particular, for $x = e$ and $v_e = h \in \mathfrak{g}$, let $\tilde{h}(g) := L_{g*}h$, so that $\tilde{h}$ is the unique left invariant vector field on G such that $\tilde{h}(e) = h$. We view $\tilde{h}$ as a first order differential operator acting on smooth functions by

$$(\tilde{h}f)(g) = \frac{d}{dt}\bigg|_0 f(g \cdot \sigma(t)),$$

where $\sigma(t)$ is a smooth curve in G such that $\sigma(0) = e$ and $\dot{\sigma}(0) = h$ (for example, $\sigma(t) = th$).

The following proposition is Proposition 3.7 of [7] and a special case of Proposition 3.16 of [24]. The proof is a simple computation and is included here for the reader's convenience.

Proposition 2.11. *For $g, x \in G$ and $v_x \in T_xG$,*

$$L_{g*}v_x = v + \frac{1}{2}[g, v],$$

and this expression does not depend on x. In particular, taking $x = e$, $g = (w, c)$, and $v_e = h = (A, a) \in \mathfrak{g}$ gives

$$\tilde{h}(g) = \left(A, a + \frac{1}{2}\omega(w, A)\right).$$

Proof. Let $x(t) = x + tv$. Then

$$L_{g*}v_x = \frac{d}{dt}\bigg|_0 g \cdot x(t) = \frac{d}{dt}\bigg|_0 g + x(t) + \frac{1}{2}[g, x(t)] = v + \frac{1}{2}[g, v].$$

□

In the sequel, it will be useful to have an expression for the left invariant derivatives of a smooth function on G in terms of its linear derivatives. To do this, we first set the following notation.

Notation 2.12. *For $k \in \mathbb{N}$, let*

$$\Lambda^k := \{\text{partitions } \theta \text{ of } \{1, \ldots, k\} : \text{ for all } A \in \theta, \#A \leq 2\}.$$

If $\{i, j\} \in \theta \in \Lambda^k$, we will always assume without loss of generality that $i > j$. For $\ell = 0, \ldots, \lfloor k/2 \rfloor$, let

$$\Lambda^k_\ell := \{\theta \in \Lambda^k : \#\{A \in \theta : \#A = 2\} = \ell\}.$$

Proposition 2.13. *For $g \in G$, $h \in \mathfrak{g}$, and $f : G \to \mathbb{C}$ a smooth function,*

$$\tilde{h}f(g) = f'(g)\tilde{h}(g). \tag{2.5}$$

More generally, for $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in \mathfrak{g}$,

$$\tilde{h}_k \cdots \tilde{h}_1 f(g) = \sum_{j=\lceil k/2 \rceil}^{k} f^{(j)}(g) \left(\sum_{\theta \in \Lambda^k_{k-j}} (h_k, \ldots, h_1)^{\otimes\theta}(g) \right), \tag{2.6}$$

where, for $\theta = \{\{i_1, i_2\}, \ldots, \{i_{2\ell-1}, i_{2\ell}\}, \{i_{2\ell+1}\}, \ldots, \{i_k\}\} \in \Lambda^k_\ell$,

$$(h_k, \ldots, h_1)^{\otimes\theta}(g) := \frac{1}{2^\ell}[h_{i_1}, h_{i_2}] \otimes \cdots \otimes [h_{i_{2\ell-1}}, h_{i_{2\ell}}] \otimes \tilde{h}_{i_{2\ell+1}}(g) \otimes \cdots \otimes \tilde{h}_{i_k}(g).$$

Proof. The first assertion holds by Proposition 2.11 and an application of the chain rule. Equation (2.6) may be then proved by induction. So assume the formula holds for k and consider $k + 1$.

$$\begin{aligned}
\tilde{h}_{k+1}\tilde{h}_k \cdots \tilde{h}_1 f(g) &= \frac{d}{dt}\bigg|_0 \tilde{h}_k \cdots \tilde{h}_1 f(g \cdot th_{k+1}) \\
&= \frac{d}{dt}\bigg|_0 \sum_{j=\lceil k/2 \rceil}^{k} f^{(j)}(g \cdot th_{k+1}) \sum_{\theta \in \Lambda^k_{k-j}} (h_k, \ldots, h_1)^{\otimes\theta}(g \cdot th_{k+1}) \\
&= \sum_{j=\lceil k/2 \rceil}^{k} f^{(j+1)}(g) \sum_{\theta \in \Lambda^k_{k-j}} \tilde{h}_{k+1}(g) \otimes (h_k, \ldots, h_1)^{\otimes\theta}(g) \\
&\quad + \sum_{j=\lceil k/2 \rceil}^{k} f^{(j)}(g) \sum_{\theta \in \Lambda^k_{k-j}} \frac{d}{dt}\bigg|_0 (h_k, \ldots, h_1)^{\otimes\theta}(g \cdot th_{k+1}).
\end{aligned}$$

For $g = (w, c)$, $h = (A, a)$, and $k = (B, b)$,

$$\frac{d}{dt}\bigg|_0 \tilde{h}(g \cdot tk) = \left(A, a + \frac{1}{2}\omega(w + tB, A)\right) = \left(0, \frac{1}{2}\omega(B, A)\right) = \frac{1}{2}[k, h],$$

which is independent of g. (Note that $\widetilde{[k,h]}(g) = [k, h]$.) Thus, for $\theta = \{\{i_1, i_2\}, \ldots, \{i_{2\ell-1}, i_{2\ell}\}, \{i_{2\ell+1}\}, \ldots, \{i_k\}\} \in \Lambda_\ell^k$,

$$\begin{aligned}
&\frac{d}{dt}\bigg|_0 (h_k, \ldots, h_1)^{\otimes\theta}(g \cdot th_{k+1}) \\
&= \frac{d}{dt}\bigg|_0 \frac{1}{2^\ell}\left\{[h_{i_1}, h_{i_2}] \otimes \cdots \otimes [h_{i_{2\ell-1}}, h_{i_{2\ell}}] \otimes \tilde{h}_{i_{2\ell+1}}(g \cdot th_{k+1}) \otimes \cdots \otimes \tilde{h}_{i_k}(g \cdot th_{k+1})\right\} \\
&= \sum_{j=2\ell+1}^{k} \frac{1}{2^{\ell+1}}\bigg([h_{i_1}, h_{i_2}] \otimes \cdots \otimes [h_{i_{2\ell-1}}, h_{i_{2\ell}}] \\
&\qquad \otimes \tilde{h}_{i_{2k+1}}(g) \otimes \cdots \otimes \tilde{h}_{j-1}(g) \otimes [h_{k+1}, h_j] \otimes \tilde{h}_{j+1}(g) \otimes \cdots \otimes \tilde{h}_{i_k}(g)\bigg).
\end{aligned}$$

Rearranging terms and indices gives the desired formula. □

Let us write out (2.6) for the first few n. The expression for $n = 1$ is already given in equation (2.5). For $n = 2$ and $n = 3$, we have

$$\tilde{h}_2\tilde{h}_1 f(g) = f''(g)\left(\tilde{h}_2(g) \otimes \tilde{h}_1(g)\right) + \frac{1}{2}f'(g)[h_2, h_1] \tag{2.7}$$

$$\begin{aligned}
\tilde{h}_3\tilde{h}_2\tilde{h}_1 f(g) &= f'''(g)\left(\tilde{h}_3(g) \otimes \tilde{h}_2(g) \otimes \tilde{h}_1(g)\right) \\
&\quad + \frac{1}{2}f''(g)\left([h_3, h_2] \otimes \tilde{h}_1(g) + [h_3, h_1] \otimes \tilde{h}_2(g) + [h_2, h_1] \otimes \tilde{h}_3(g)\right).
\end{aligned} \tag{2.8}$$

In particular, (2.7) implies that, for $h, k \in \mathfrak{g}$,

$$\left(\tilde{h}\tilde{k} - \tilde{k}\tilde{h}\right) f = \widetilde{[h,k]} f. \tag{2.9}$$

2.4. **Distances on G_{CM}.** We define here the sub-Riemannian distance on G_{CM} and show that the topology induced by this metric is equivalent to the topology induced by the homogeneous norm $\|\cdot\|_{\mathfrak{g}_{CM}}$. Note that in finite dimensions this result is standard and is usually proved via compactness arguments (see for example Chapter 5 of [4]). Of course, these arguments are invalid in infinite dimensions and so we resort to more direct methods of proof. Note that the results in this subsection rely directly on the fact that $N = \dim(\mathbf{C}) < \infty$.

Notation 2.14. *(Riemannian and horizontal distances)*

(1) *Let C^1_{CM} denote the set of C^1-paths $\sigma : [0, 1] \to G_{CM}$.*

(2) *For $x = (A, a) \in G_{CM}$, let*

$$|x|^2_{\mathfrak{g}_{CM}} := \|A\|^2_H + \|a\|^2_{\mathbf{C}}.$$

The length *of a C^1-path $\sigma : [a, b] \to G_{CM}$ is defined as*

$$\ell(\sigma) := \int_a^b |L_{\sigma^{-1}(s)*}\dot{\sigma}(s)|_{\mathfrak{g}_{CM}}\, ds.$$

(3) *The* Riemannian distance *between $x, y \in G_{CM}$ is defined by*

$$d_{CM}(x, y) := \inf\{\ell(\sigma) : \sigma \in C^1_{CM} \text{ such that } \sigma(0) = x \text{ and } \sigma(1) = y\}.$$

(4) *A* C^1*-path* $\sigma : [a, b] \to G_{CM}$ *is* horizontal *if* $L_{\sigma(t)^{-1}*}\dot{\sigma}(t) \in H \times \{0\}$ *for a.e.* t*. Let* $C^{1,h}_{CM}$ *denote the set of horizontal paths* $\sigma : [0, 1] \to G_{CM}$*.*

(5) *The* horizontal distance *between* $x, y \in G_{CM}$ *is defined by*

$$d_h(x, y) := \inf\{\ell(\sigma) : \sigma \in C^{1,h}_{CM} \text{ such that } \sigma(0) = x \text{ and } \sigma(1) = y\}.$$

Remark 2.15. Note that if $\sigma(t) = (A(t), a(t)) \in C^{1,h}_{CM}$, then

$$L_{\sigma(t)^{-1}*}\dot{\sigma}(t) = \left(\dot{A}(t), \dot{a}(t) - \frac{1}{2}\omega(A(t), \dot{A}(t))\right) \in H \times \{0\}$$

implies that σ must satisfy

$$a(t) = a(0) + \frac{1}{2}\int_0^t \omega(A(s), \dot{A}(s))\, ds,$$

and the length of σ is given by

$$\ell(\sigma) = \int_0^1 |L_{\sigma^{-1}(s)*}\dot{\sigma}(s)|_{\mathfrak{g}_{CM}}\, ds = \int_0^1 \|\dot{A}(s)\|_H\, ds.$$

Proposition 3.10 of [7] gives the following comparison of the $|\cdot|_{\mathfrak{g}_{CM}}$ and Riemannian metrics.

Proposition 2.16. *There exists* $\delta = \delta(\omega) > 0$ *such that, for all* $x, y \in G_{CM}$*,*

$$d_{CM}(x, y) \le \left(1 + \frac{1}{4\delta}|x|_{\mathfrak{g}_{CM}} \wedge |y|_{\mathfrak{g}_{CM}}\right)|y - x|_{\mathfrak{g}_{CM}},$$

and, in particular, $d_{CM}(e, x) \le |x|_{\mathfrak{g}_{CM}}$ *for any* $x \in G_{CM}$*. Also, there exists* $k = k(\omega) < \infty$ *such that, if* $x, y \in G_{CM}$ *satisfy* $d_{CM}(x, y) \le \delta$*, then*

$$|y - x|_{\mathfrak{g}_{CM}} \le k(1 + |x|_{\mathfrak{g}_{CM}} \wedge |y|_{\mathfrak{g}_{CM}})d_{CM}(x, y).$$

Proposition 2.16 implies for example that the topology induced by $|\cdot|_{\mathfrak{g}_{CM}}$ is equivalent to that induced by the Riemannian distance. For the subelliptic case, these are of course not the relevant topologies. However, this result may be used to prove that the homogenous norm on $\mathfrak{g}_{CM}$ and the horizontal distance are comparable at the identity. The following proposition is Theorem C.2 of [7]. We record the proof here for the reader's convenience and to emphasize the dependence of the upper bound constant K_2 on $N = \dim(\mathbf{C})$.

Proposition 2.17. *If* $\{\omega(A, B) : A, B \in H\} = \mathbf{C}$*, then there exist finite constants* $K_1 = K_1(\omega)$ *and* $K_2 = K_2(N, \omega)$ *such that, for all* $(A, a) \in \mathfrak{g}_{CM}$*,*

$$K_1\|(A, a)\|_{\mathfrak{g}_{CM}} \le d_h(e, (A, a)) \le K_2\|(A, a)\|_{\mathfrak{g}_{CM}}.$$

Proof. For any left-invariant metric d on G_{CM} (for example d_{CM} or d_h), we have

$$d(e, xy) \le d(e, x) + d(x, xy) = d(e, x) + d(e, y), \tag{2.10}$$

for all $x, y \in G_{CM}$. Given any horizontal path $\sigma = (w, c)$ joining e to (A, a), we have from Remark 2.15 that

$$\ell(\sigma) = \int_0^1 \|\dot{w}(s)\|_H\, ds \ge \|A\|_H.$$

Taking the infimum over all horizontal paths connecting e to (A, a), it then follows that

$$d_h(e, (A, a)) \ge \|A\|_H.$$

Since the path $\sigma(t) = (tA, 0)$ is horizontal and

$$\|A\|_H = \ell(\sigma) \geq d_h(e, (A,0)) \geq \|A\|_H,$$

it follows that

$$d_h(e, (A,0)) = \|A\|_H \text{ for all } A \in H. \tag{2.11}$$

Given $A, B \in H$, let $\gamma(t) = A\cos 2\pi t + B \sin 2\pi t$ for $0 \leq t \leq 1$, and consider the path

$$\sigma(t) = \left(\gamma(t) - A, \frac{1}{2}\int_0^t \omega(\gamma(s) - A, \dot{\gamma}(s))\, ds\right).$$

Note that σ is a horizontal curve with $L_{\sigma(t)^{-1}_*}\dot{\sigma}(t) = (\dot{\gamma}(t), 0)$, $\sigma(0) = e$, and

$$\sigma(1) = \left(0, \frac{1}{2}\int_0^1 \omega(\gamma(s), \dot{\gamma}(s))\, ds\right) = \left(0, \pi\int_0^1 \omega(A,B)\, ds\right) = (0, \pi\omega(A,B)).$$

Thus, we may conclude that

$$\begin{aligned} d_h(e, (0, \pi\omega(A,B))) \leq \ell(\sigma) &= 2\pi \int_0^1 \|-A\sin 2\pi s + B\cos 2\pi s\|_H\, ds \\ &\leq 2\pi(\|A\|_H + \|B\|_H). \end{aligned} \tag{2.12}$$

Now choose $\{A_\ell, B_\ell\}_{\ell=1}^N \subset H$ such that $\{\pi\omega(A_\ell, B_\ell)\}_{\ell=1}^N$ is a basis for $\mathbf{C}$. Let $\{\varepsilon^\ell\}_{\ell=1}^N$ be the corresponding dual basis. Hence, for any $a \in \mathbf{C}$, we have

$$\begin{aligned} d_h(e, (0,a)) &= d_h\left(e, \prod_{\ell=1}^N (0, \varepsilon^\ell(a)\pi\omega(A_\ell, B_\ell))\right) \\ &\leq \sum_{\ell=1}^N d_h(e, (0, \varepsilon^\ell(a)\pi\omega(A_\ell, B_\ell))) \\ &= \sum_{\ell=1}^N d_h\left(e, \left(0, \pi\omega\left(\operatorname{sgn}(\varepsilon^\ell(a))\sqrt{|\varepsilon^\ell(a)|}A_\ell, \sqrt{|\varepsilon^\ell(a)|}B_\ell\right)\right)\right) \\ &\leq 2\pi \sum_{\ell=1}^N \left(\left\|\sqrt{|\varepsilon^\ell(a)|}A_\ell\right\|_H + \left\|\sqrt{|\varepsilon^\ell(a)|}B_\ell\right\|_H\right), \end{aligned}$$

wherein we have used (2.10) for the first inequality and (2.12) for the second inequality. Then Hölder's inequality implies that

$$d_h(e, (0,a)) \leq 4\pi \sum_{\ell=1}^N \sqrt{|\varepsilon^\ell(a)|} \leq 4\pi C\sqrt{\|a\|_{\mathbf{C}}}, \tag{2.13}$$

for a finite constant $C = C(N, \omega)$. Combining equations (2.10), (2.11), and (2.13) gives,

$$\begin{aligned} d_h(e, (A,a)) &= d_h(e, (A,0)(0,a)) \\ &\leq d_h(e, (A,0)) + d_h(e, (0,a)) \\ &\leq \|A\|_H + C(N,\omega)\sqrt{\|a\|_{\mathbf{C}}} \leq \sqrt{2}\,(1 \wedge C(N,\omega))\,\|(A,a)\|_{\mathfrak{g}_{CM}}, \end{aligned}$$

which completes the proof of the upper bound.

To prove the lower bound, consider first the dilations defined by

$$\varphi_\lambda(w,c) := (\lambda w, \lambda^2 c), \quad \text{ for } \lambda > 0 \text{ and } (w,c) \in \mathfrak{g}_{CM} = G_{CM}.$$

One easily verifies that φ_λ is both a Lie algebra homomorphism on $\mathfrak{g}_{CM}$ and a group homomorphism on G_{CM}. Using the homomorphism property, it follows that, for any C^1-path σ,

$$L_{\varphi_\lambda(\sigma(t))^{-1}*}\frac{d}{dt}\varphi_\lambda(\sigma(t)) = \varphi_\lambda(L_{\sigma(t)^{-1}*}\dot{\sigma}(t)).$$

Consequently, if σ is a horizontal curve, then $\varphi_\lambda\circ\sigma$ is again horizontal and $\ell(\varphi_\lambda\circ\sigma) = \lambda\ell(\sigma)$. Thus, we may conclude that

$$d_h(\varphi_\lambda(x),\varphi_\lambda(y)) = \lambda d_h(x,y), \tag{2.14}$$

for all $x, y \in G_{CM}$.

Now, by the first part of Proposition 2.16, $d_{CM}(e,x) \le |x|_{\mathfrak{g}_{CM}}$, for all $x \in G_{CM}$. Combining this with the second part of the same proposition implies that there exist $\delta > 0$ and $k < \infty$ such that, if $|x|_{\mathfrak{g}_{CM}} \le \delta$, then $|x|_{\mathfrak{g}_{CM}} \le k d_{CM}(x,y)$. So, for arbitrary $x = (A,a) \in G_{CM}$, choose $\lambda = \lambda(x) > 0$ so that

$$\delta^2 = |\varphi_\lambda(x)|^2_{\mathfrak{g}_{CM}} = \lambda^2\|A\|_H^2 + \lambda^4\|a\|^2_{\mathbf{C}};$$

that is, take

$$\lambda^2 = \frac{\sqrt{\|A\|_H^4 + 4\|a\|^2_{\mathbf{C}}\delta^2} - \|A\|_H^2}{2\|a\|^2_{\mathbf{C}}}.$$

Equation (2.14) and Proposition 2.16 then imply that

$$\lambda k d_h(e,x) = k d_h(e,\varphi_\lambda(x)) \ge k d_{CM}(e,\varphi_\lambda(x)) \ge |\varphi_\lambda(x)|_{\mathfrak{g}_{CM}} = \delta$$

Thus,

$$\begin{aligned} d_h(e,x)^2 &\ge \frac{\delta^2}{k^2\lambda^2} = \frac{\delta^2}{k^2}\frac{2\|a\|^2_{\mathbf{C}}}{\sqrt{\|A\|_H^4 + 4\delta^2\|a\|^2_{\mathbf{C}}} - \|A\|_H^2} \\ &= \frac{2\delta^2\|a\|^2_{\mathbf{C}}}{k^2\|A\|^2_H}\frac{1}{\sqrt{1+\frac{4\delta^2\|a\|^2_{\mathbf{C}}}{\|A\|^4_H}}-1}. \end{aligned} \tag{2.15}$$

Since $\sqrt{1+x}-1 \le \min(x/2, \sqrt{x})$, we have

$$\frac{1}{\sqrt{1+x}-1} \ge \max\left(\frac{2}{x},\frac{1}{\sqrt{x}}\right) \ge \frac{1}{x} + \frac{1}{2\sqrt{x}}.$$

Using this estimate with $x = 4\delta^2\|a\|^2_{\mathbf{C}}\|A\|_H^{-4}$ in equation (2.15) shows that

$$d_h(e,x)^2 \ge \frac{2\delta^2\|a\|^2_{\mathbf{C}}}{k^2\|A\|^2_H}\left(\frac{\|A\|^4_H}{4\delta^2\|a\|^2_{\mathbf{C}}} + \frac{\|A\|^2_H}{4\delta\|a\|_{\mathbf{C}}}\right) = \frac{1}{2k^2}(\|A\|^2_H + \delta\|a\|_{\mathbf{C}}),$$

which implies the lower bound. □

Since G_{CM} is stratified, it turns out that comparability of the metrics at e is sufficient to imply the equivalence of their respective topologies.

Proposition 2.18. *The topologies generated by d_h and $\|\cdot\|_{\mathfrak{g}_{CM}}$ are equivalent.*

Proof. Fix $x = (A,a) \in G_{CM}$. First note that, by Proposition 2.17 and the left invariance of the horizontal distance, there exists $K_1 = K_1(\omega) < \infty$ such that, for any $y = (B,b) \in G_{CM}$,

$$\sqrt{\|B-A\|^2_H + \left\|b - a - \frac{1}{2}\omega(A,B)\right\|_{\mathbf{C}}} = \|x^{-1}y\|_{\mathfrak{g}_{CM}} \le K_1 d_h(e,x^{-1}y) = K_1 d_h(x,y).$$

So if $d_h(x,y) < \delta$ for some $\delta > 0$, then

$$\|B-A\|_H \le K_1 d_h(x,y) < K_1\delta,$$

and

$$\begin{aligned}\|b-a\|_{\mathbf{C}} &\le \left\|b-a-\frac{1}{2}\omega(A,B)\right\|_{\mathbf{C}} + \frac{1}{2}\|\omega(A,B)\|_{\mathbf{C}} \\ &\le K_1^2 d_h(x,y)^2 + \frac{1}{2}\|\omega(A,B-A)\|_{\mathbf{C}} \\ &< K_1^2\delta^2 + \frac{1}{2}\|\omega\|_{op}\|A\|_H\|B-A\|_H < K_1^2\delta^2 + \frac{1}{2}\|\omega\|_{op}\|A\|_H\delta,\end{aligned}$$

where

$$\|\omega\|_{op} := \sup\{\|\omega(h,k)\|_{\mathbf{C}} : \|h\|_H = \|k\|_H = 1\} < \infty,$$

by the continuity of ω and (2.2). Thus, given any $R \in (0,1)$, one may clearly choose $c = c(x,\omega)$ sufficiently large (for example, $c = 2(\sqrt{2}K_1 + \frac{1}{2}\|\omega\|_{op}\|A\|)$) so that $d_h(x,y) < \delta = R^2/c$ implies that

$$\begin{aligned}\|y-x\|_{\mathfrak{g}_{CM}} &= \sqrt{\|B-A\|_H^2 + \|b-a\|_{\mathbf{C}}} \\ &< \sqrt{K_1^2\delta^2 + K_1^2\delta^2 + \frac{1}{2}\|\omega\|_{op}\|A\|_H\delta} \\ &= \sqrt{2K_1^2\frac{R^4}{c^2} + \frac{1}{2}\|\omega\|_{op}\|A\|_H\frac{R^2}{c}} < \sqrt{R^2} = R.\end{aligned}$$

Similarly, the left invariance of d_h and Proposition 2.17 imply that there exists $K_2 = K_2(N,\omega) < \infty$ such that

$$d_h(x,y) \le K_2\|x^{-1}y\|_{\mathfrak{g}_{CM}} = K_2\sqrt{\|B-A\|_H^2 + \left\|b-a-\frac{1}{2}\omega(A,B)\right\|_{\mathbf{C}}}.$$

So if we suppose that $\|y-x\|_{\mathfrak{g}_{CM}} = \sqrt{\|B-A\|_H^2 + \|b-a\|_{\mathbf{C}}} < \delta'$, then

$$\begin{aligned}d_h(x,y) &\le K_2\sqrt{\|B-A\|_H^2 + \|b-a\|_{\mathbf{C}} + \frac{1}{2}\|\omega(A,B-A)\|_{\mathbf{C}}} \\ &\le K_2\left(\|y-x\|_{\mathfrak{g}_{CM}} + \sqrt{\frac{1}{2}\|\omega\|_{op}\|A\|_H\|B-A\|_H}\right) \\ &\le K_2\left(\delta' + \sqrt{\frac{1}{2}\|\omega\|_{op}\|A\|_H\delta'}\right).\end{aligned}$$

Again, given any $R \in (0,1)$, one may find $c' = c'(x,N,\omega)$ such that $\|y-x\|_{\mathfrak{g}_{CM}} < \delta' = R^2/c'$ implies that $d_h(x,y) < R$. □

2.5. **Finite-dimensional projection groups.** The finite-dimensional projections of G defined in this section will be important in the sequel. Note that the construction of these projections is quite natural in the sense that they come from the usual projections of the abstract Wiener space; however, the projections defined here are not group homomorphisms, which is a complicating factor in some of the following proofs.

As usual, let (W, H, μ) denote a complex abstract Wiener space. Let $i : H \to W$ be the inclusion map, and $i^* : W^* \to H^*$ be its transpose so that $i^*\ell := \ell \circ i$ for all $\ell \in W^*$. Also, let

$$H_* := \{h \in H : \langle \cdot, h\rangle_H \in \mathrm{Range}(i^*) \subset H^*\}.$$

That is, for $h \in H$, $h \in H_*$ if and only if $\langle \cdot, h\rangle_H \in H^*$ extends to a continuous linear functional on W, which we will continue to denote by $\langle \cdot, h\rangle_H$. Because H is a dense subspace of W, i^* is injective and thus has a dense range. Since $H \ni h \mapsto \langle \cdot, h\rangle_H \in H^*$ is a linear isometric isomorphism, it follows that $H_* \ni h \mapsto \langle \cdot, h\rangle_H \in W^*$ is a linear isomorphism also, and so H_* is a dense subspace of H.

Suppose that $P : H \to H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{\xi_j\}_{j=1}^m$ be an orthonormal basis for PH. Then we may extend P to a (unique) continuous operator from $W \to H$ (still denoted by P) by letting

$$Pw := \sum_{j=1}^{m} \langle w, \xi_j\rangle_H \xi_j \tag{2.16}$$

for all $w \in W$.

Notation 2.19. *Let* $\mathrm{Proj}(W)$ *denote the collection of finite rank projections on* W *such that*

1. $PW \subset H_*$,
2. $P|_H : H \to H$ *is an orthogonal projection (that is,* P *has the form given in equation* (2.16)*), and*
3. PW *is sufficiently large to satisfy Hörmander's condition (that is,* $\{\omega(A, B) : A, B \in PW\} = \mathbf{C}$*).*

For each $P \in \mathrm{Proj}(W)$, we may define $G_P := PW \times \mathbf{C} \subset H_* \times \mathbf{C}$ and a corresponding projection $\pi_P : G \to G_P$

$$\pi_P(w, x) := (Pw, x).$$

We will also let $\mathfrak{g}_P = \mathrm{Lie}(G_P) = PW \times \mathbf{C}$.

For any $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ such that $P_n|_H \uparrow I_H$, we may choose a sequence of complex orthonormal bases Γ_n for each P_nH so that $\Gamma_n \uparrow \Gamma$ a complex orthonormal basis for H. Thus, for the sequel, we will often consider a sequence of projections with respect to a fixed orthonormal basis.

Notation 2.20. *Let* $\{\xi_j\}_{j=1}^\infty \subset H_*$ *be a fixed orthonormal basis of* H*. We will let* P_n *denote the corresponding projections onto* P_nW*, that is,*

$$P_n w = \sum_{j=1}^{n} \langle w, \xi_j\rangle_H \xi_j.$$

Let $G_n = G_{P_n}$*,* $\mathfrak{g}_n = \mathrm{Lie}(G_n)$*, and* $\pi_n = \pi_{P_n} : G \to G_n$*. So* $\{\pi_n\}_{n=1}^\infty$ *is an increasing sequence of projections so that* $\pi_n|_{G_{CM}} \uparrow I|_{G_{CM}}$*. In the sequel, it will also be convenient to let* $\Gamma = \{\eta_j\}_{j=1}^\infty = \{(\xi_j, 0)\}_{j=1}^\infty$ *denote a basis of* $H \times \{0\}$*.*

(It is clear that, in order for $P_n \in \mathrm{Proj}(W)$*, it will be necessary to have a minimal* n *so that* $\mathrm{span}\{\omega(\xi_i, \xi_j) : i, j = 1, \ldots, n\} = \mathbf{C}$*. However, since these projections will be primarily used for large* n *as approximations to* G*, we will ignore this issue in the sequel and always assume we have a large enough projection.)*

2.6. **Brownian motion on G.** Here we define a "subelliptic" Brownian motion $\{g_t\}_{t\geq 0}$ on G and collect various of its properties that are necessary for the sequel. The primary references for this section are [7, 8].

Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on W with variance determined by

$$\mathbb{E}\left[\langle B_s, h\rangle_H \langle B_t, k\rangle_H\right] = \langle h, k\rangle_H \min(s,t),$$

for all $s, t \geq 0$ and $h, k \in H_*$. The following is Proposition 4.1 of [7] and this result implicitly relies on the fact that Proposition 2.5 implies that the bilinear form ω is a Hilbert-Schmidt.

Proposition 2.21. *For $P \in \mathrm{Proj}(W)$, let M_t^P be the continuous L^2-martingale on $\mathbf{C}$ defined by*

$$M_t^P = \int_0^t \omega(PB_s, dPB_s).$$

In particular, if $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ is an increasing sequence of projections as in Notation 2.20 and $M_t^n := M_t^{P_n}$, then there exists an L^2-martingale $\{M_t\}_{t\geq 0}$ in $\mathbf{C}$ such that, for all $p \in [1,\infty)$ and $t > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{\tau\leq t} \|M_\tau^n - M_\tau\|_{\mathbf{C}}^p\right] = 0,$$

and M_t is independent of the sequence of projections.

As M_t is independent of the defining sequence of projections, we will denote the limiting process by

$$M_t = \int_0^t \omega(B_s, dB_s).$$

Definition 2.22. The continuous G-valued process given by

$$g_t = \left(B_t, \frac{1}{2}M_t\right) = \left(B_t, \frac{1}{2}\int_0^t \omega(B_s, dB_s)\right).$$

is a *Brownian motion* on G. For $t > 0$, let $\nu_t = \mathrm{Law}(g_t)$ denote the *heat kernel measure at time t* on G.

Definition 2.23. A function $f : G \to \mathbb{C}$ is a *cylinder function* if it may be written as $f = F \circ \pi_P$, for some $P \in \mathrm{Proj}(W)$ and $F : G_P \to \mathbb{C}$. We say that f is a *smooth (holomorphic) cylinder function* if F is smooth (holomorphic).

Proposition 2.24. *If $f : G \to \mathbb{C}$ is a smooth cylinder function, let*

$$Lf := \sum_{j=1}^\infty \left[\tilde{\eta}_j^2 + \widetilde{i\eta_j}^2\right] f,$$

where $\{\eta_j\}_{j=1}^\infty$ is a basis for $H \times \{0\}$ as in Notation 2.20. Then Lf is well defined, that is, the above sum is convergent and independent of basis. Moreover, $\frac{1}{4}L$ is the generator for $\{g_t\}_{t\geq 0}$, so that

$$f(g_t) - \frac{1}{4}\int_0^t Lf(g_s)\, ds$$

is a local martingale for any smooth cylinder function f.

Proposition 2.21 along with the fact that, for all $p \in [1, \infty)$ and $t > 0$,

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{\tau \leq t} \|B_\tau - P_n B_\tau\|_W^p\right] = 0$$

(see for example Proposition 4.6 of [7]) makes the following proposition clear.

Proposition 2.25. *For $P \in \operatorname{Proj}(W)$, let g_t^P be the continuous process on G_P defined by*

$$g_t^P = \left(PB_t, \frac{1}{2}\int_0^t \omega(PB_s, dPB_s)\right).$$

Then g_t^P is a Brownian motion on G_P . In particular, let $\{P_n\}_{n=1}^\infty \subset \operatorname{Proj}(W)$ be increasing projections as in Notation 2.20 and $g_t^n := g_t^{P_n}$. Then, for all $p \in [1, \infty)$ and $t > 0$,

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{\tau \leq t} \|g_\tau^n - g_\tau\|_{\mathfrak{g}}^p\right] = 0.$$

Notation 2.26. *For all $P \in \operatorname{Proj}(W)$ and $t > 0$, let $\nu_t^P := \operatorname{Law}(g_t^P)$, and for all $n \in \mathbb{N}$ let $\nu_t^n := \operatorname{Law}(g_t^n) = \operatorname{Law}(g_t^{P_n})$.*

For all projections satisfying Hörmander's condition, the Brownian motions on G_P are true subelliptic diffusions in the sense that their laws are absolutely continuous with respect to the finite-dimensional reference measure and their transition kernels are smooth.

Lemma 2.27. *For all $P \in \operatorname{Proj}(W)$ and $t > 0$, we have $\nu_t^P(dx) = p_t^P(e, x)dx$, where dx is the Riemannian volume measure (equal to Haar measure) and $p_t^P(x, y)$ is the heat kernel on G_P.*

Proof. An application of Proposition 2.24 with G replaced by G_P implies that $\nu_t^P = \operatorname{Law}(g_t^P)$ is a weak solution to the heat equation on G_P with generator

$$L^P f := \sum_{j=1}^m \left[\widetilde{(\xi_j, 0)}^2 + \widetilde{(i\xi_j, 0)}^2\right] f$$

for smooth functions $f : G_P \to \mathbb{C}$, where $\{\xi_j\}_{j=1}^m$ is a complex orthonormal basis of PH. The result now follows from the fact that $[PW, PW] = \mathbf{C}$, as this implies $\{(\xi_j, 0), (i\xi_j, 0)\}_{j=1}^m$ satisfies Hörmander's condition, and thus L^P is a hypoelliptic operator [21]. □

The next proposition is a version of Fernique's theorem for the subelliptic heat kernel measures and follows directly from the proof in the elliptic case (see Theorem 4.16 of [7]). In particular, this kind of exponential integrability result is required to have a nontrivial class of holomorphic square integrable functions.

Proposition 2.28 (Subelliptic Fernique's theorem)**.** *There exists $\delta > 0$ such that, for all $\varepsilon \in (0, \delta)$ and $t > 0$,*

$$\sup_{P \in \operatorname{Proj}(W)} \int_{G_P} e^{\varepsilon \|g\|_{\mathfrak{g}}^2 / t} \, d\nu_t^P(g) = \sup_{P \in \operatorname{Proj}(W)} \mathbb{E}\left[e^{\varepsilon \|g_t^P\|_{\mathfrak{g}}^2 / t}\right] < \infty$$

and

$$\int_G e^{\varepsilon \|g\|_{\mathfrak{g}}^2 / t} \, d\nu_t(g) = \mathbb{E}\left[e^{\varepsilon \|g_t\|_{\mathfrak{g}}^2 / t}\right] < \infty.$$

The next proposition follows from Propositions 2.25 and 2.28 and the proof of Proposition 4.12 in [8].

Proposition 2.29. *Let $\delta > 0$ be as in Proposition 2.28, and suppose that $f : G \to \mathbb{C}$ is a continuous function such that, for some $\varepsilon \in (0, \delta)$ and $p \in [1, \infty)$,*

$$|f(g)| \leq Ce^{\varepsilon \|g\|_{\mathfrak{g}}^2/pt},$$

for all $g \in G$. Then $f \in L^p(\nu_t)$, and, for all $h \in G$,

$$\lim_{n\to\infty} \mathbb{E}|f(hg_t^n) - f(hg_t)|^p = 0 \tag{2.17}$$

and

$$\lim_{n\to\infty} \mathbb{E}|f(g_t^n h) - f(g_t h)|^p = 0. \tag{2.18}$$

Finally, we include the following proposition, which states that, as the name suggests, the Cameron-Martin subgroup is a subspace of heat kernel measure 0. The proof is identical to Proposition 4.6 of [8].

Proposition 2.30. *For all $t > 0$, $\nu_t(G_{CM}) = 0$.*

Proof. Let μ_t denote Wiener measure on W with variance t. Then for a bounded measurable function $f : G = W \times \mathbf{C} \to \mathbb{C}$ such that $f(w, x) = f(w)$,

$$\int_G f(w)\, d\nu_t(w, x) = \mathbb{E}[f(B_t)] = \int_W f(w)\, d\mu_t(w).$$

Let $\pi : W \times \mathbf{C} \to W$ be the projection $\pi(w, x) = w$. Then $\pi_*\nu_t = \mu_t$, and thus

$$\nu_t(G_{CM}) = \nu_t(\pi^{-1}(H)) = \pi_*\nu_t(H) = \mu_t(H) = 0.$$

□

2.7. Holomorphic functions on G and G_{CM}.

We recall here the basic facts for holomorphic functions on infinite-dimensional spaces required for the sequel. For complete proofs of any of these results, see Section 5 of [8].

2.7.1. *Holomorphic functions on Banach spaces.*

The material in this subsection is based on the theory in [20]. Let X and Y be two complex Banach spaces, and for $a \in X$ and $\delta > 0$ let

$$B_X(a, \delta) := \{x \in X : \|x - a\|_X < \delta\}$$

be the open ball in X with center a and radius δ. The following is Definition 3.17.2 of Hille and Phillips [20].

Definition 2.31. Let $\mathcal{D}$ be an open subset of X. A function $f : \mathcal{D} \to Y$ is said to be *holomorphic* or *analytic* if the following two conditions hold.

(1) f is locally bounded, namely, for all $a \in \mathcal{D}$ there exists $r_a > 0$ such that

$$M_a := \sup\{\|f(x)\|_Y : x \in B_X(a, r_a)\} < \infty.$$

(2) The function f is complex Gâteaux differentiable on $\mathcal{D}$, that is, for each $a \in \mathcal{D}$ and $h \in X$, the function $\lambda \mapsto f(a + \lambda h)$ is complex differentiable at $\lambda = 0 \in \mathbb{C}$.

Remark 2.32. Holomorphic and analytic will be considered to be synonymous for the purposes of this paper. We will use "holomorphic."

The next proposition gathers together a number of basic properties of holomorphic functions which may be found in [20], see also [19]. One of the key ingredients to all of these results is Hartog's theorem, see [20, Theorem 3.15.1].

Proposition 2.33. *If $f : \mathcal{D} \to Y$ is holomorphic, then there exists a function $f' : \mathcal{D} \to \operatorname{Hom}(X, Y)$, the space of bounded complex linear operators from X to Y, satisfying the following:*

1. *If $a \in \mathcal{D}$, $x \in B_X(a, r_a/2)$, and $h \in B_X(0, r_a/2)$, then*
$$\|f(x+h) - f(x) - f'(x)h\|_Y \leq \frac{4M_a}{r_a(r_a - 2\|h\|_X)}\|h\|_X^2.$$
In particular, f is continuous and Frechét differentiable on $\mathcal{D}$.
2. *The function $f' : \mathcal{D} \to \operatorname{Hom}(X, Y)$ is holomorphic.*

By applying Proposition 2.33 repeatedly, it follows that any holomorphic function $f : \mathcal{D} \to Y$ is Frechét differentiable to all orders and each of the Frechét differentials is again a holomorphic function on $\mathcal{D}$.

2.7.2. *Holomorphic functions on G and G_{CM}.* Now we describe results for holomorphic functions on G and G_{CM}. For the next proposition, take $G_0 = G$ and $\mathfrak{g}_0 = \mathfrak{g}$ or $G_0 = G_{CM}$ and $\mathfrak{g}_0 = \mathfrak{g}_{CM}$. Note that as usual we treat group elements as Lie algebra elements when we write the group multiplication below. This linearization explains why the proof is identical to [8], and why we omit it.

Proposition 2.34. *For each $g \in G_0$, the left translation map $L_g : G_0 \to G_0$ is holomorphic in the $\|\cdot\|_{\mathfrak{g}_0}$-topology. Moreover, a function $f : G_0 \to \mathbb{C}$ defined in a neighborhood of $g \in G_0$ is Gâteaux (Frechét) differentiable at g if and only if $f \circ L_g$ is Gâteaux (Frechét) differentiable at e. If f is Frechét differentiable at g, then*
$$(f \circ L_g)'(e)h = f'(g)\left(h + \frac{1}{2}[g, h]\right).$$
Thus, a function $f : G_0 \to \mathbb{C}$ is holomorphic if and only if f is locally bounded and $h \mapsto f(g \cdot e^h) = f(g \cdot h)$ is Gâteaux (Frechét) differentiable at 0 for all $g \in G_0$. If f is holomorphic and $h \in \mathfrak{g}_0$, then
$$(\tilde{h}f)(g) = \left.\frac{d}{d\lambda}\right|_0 f(g \cdot e^{\lambda h}) = f'(g)\left(h + \frac{1}{2}[g, h]\right)$$
is holomorphic as well.

A simple induction argument using Proposition 2.34 allows us to conclude that $\tilde{h}_1 \dots \tilde{h}_n f \in \mathcal{H}(G_0)$ for all $f \in \mathcal{H}(G_0)$ and $h_1, \dots, h_n \in \mathfrak{g}_0$.

Notation 2.35. *The space of globally defined holomorphic functions on a group U will be denoted by $\mathcal{H}(U)$.*

Finally, we also record the following result, which is completely analogous to Proposition 5.7 and Corollary 5.8 of [8].

Proposition 2.36. *If $f \in \mathcal{H}(G)$ and $h \in \mathfrak{g}$, then $\widetilde{ih}f = i\tilde{h}f$, $\widetilde{ih}\bar{f} = -i\tilde{h}\bar{f}$,*
$$\left(\widetilde{ih}^2 + \tilde{h}^2\right) f = 0, \text{ and}$$
$$\left(\widetilde{ih}^2 + \tilde{h}^2\right) |f|^2 = 4|\tilde{h}f|^2.$$

Thus, for L as in Proposition 2.24 and $f : G \to \mathbb{C}$ a holomorphic cylinder function, $Lf = 0$ and

$$L|f|^2 = \sum_{j=1}^{\infty} |\tilde{\eta}_j f|^2$$

for any $\{\eta_j\}_{j=1}^{\infty}$ a basis of $H \times \{0\}$ as in Notation 2.20.

3. The Taylor isomorphism

Before we define the Taylor map, we must first define the relevant Hilbert spaces. First of these is the noncommutative Fock space, which plays the role of the derivative space of holomorphic functions.

3.1. Noncommutative Fock space.

We set the now standard notation for the noncommutative Fock space, making the appropriate changes in the definition of the norm to accomodate the subelliptic setting.

Notation 3.1. *Let V be a complex vector space. We will denote the algebraic dual to V by V'. For $k \in \mathbb{N}$, let $V^{\otimes k}$ denote the k-fold algebraic tensor product of V with itself. For any tensors a, b, we write $a \wedge b$ for $a \otimes b - b \otimes a$. Let $T(V)$ denote the algebraic tensor algebra over V, so that $a \in T(V)$ is a finite sum*

$$a = \sum_{k=0}^{n} a_k, \qquad a_k \in V^{\otimes k},$$

where $V^{\otimes 0} = \mathbb{C}$. For $\alpha \in T(V)'$ and $k \in \{0\} \cup \mathbb{N}$, let $\alpha_k := \alpha|_{V^{\otimes k}} \in \left(V^{\otimes k}\right)'$, so that

$$\alpha = \sum_{k=0}^{\infty} \alpha_k, \qquad \alpha_k \in (V^{\otimes k})'.$$

When V is a Lie algebra, let $J(V)$ be the two-sided ideal in $T(V)$ generated by $\{a \wedge b - [a, b] : a, b \in V\}$ and let $J^0(V)$ be the backward annihilator of $J(V)$, that is,

$$J^0(V) = \{\alpha \in T(V)' : \langle \alpha, J(V) \rangle = 0\}.$$

In particular, we will be concerned with the vector spaces $\mathfrak{g}_{CM}$ and $\mathfrak{g}_P = PW \times \mathbf{C}$. We will let $J^0(\mathfrak{g}_{CM}) = J^0$. Now we will define norms on J^0 and $J^0(\mathfrak{g}_P)$.

In order to put a norm on J^0, let $\{\xi_j\}_{j=1}^{\infty} \subset H_*$ be a fixed complex orthonormal basis of H and $\{\eta_j\}_{j=1}^{\infty} = \{(\xi_j, 0)\}_{j=1}^{\infty}$ be a complex basis of $H \times \{0\}$ as in Notation 2.20. For $k \in \{0\} \cup \mathbb{N}$, we define a non-negative sesqui-linear form on $(\mathfrak{g}_{CM}^{\otimes k})'$ by

$$(\alpha, \beta)_k := \sum_{j_1, \ldots, j_k = 1}^{\infty} \langle \alpha, \eta_{j_1} \otimes \cdots \otimes \eta_{j_k} \rangle \overline{\langle \beta, \eta_{j_1} \otimes \cdots \otimes \eta_{j_k} \rangle}, \text{ for all } \alpha, \beta \in (\mathfrak{g}_{CM}^{\otimes k})'.$$

For $\alpha \in (\mathfrak{g}_{CM}^{\otimes k})'$, we will write

$$\|\alpha\|_k^2 := (\alpha, \alpha)_k = \sum_{j_1, \ldots, j_k = 1}^{\infty} |\langle \alpha, \eta_{j_1} \otimes \cdots \otimes \eta_{j_k} \rangle|^2.$$

The following lemma is clear from the definition of $\| \cdot \|_k$.

Lemma 3.2. *Let $\alpha \in (\mathfrak{g}_{CM}^{\otimes k})'$ for some $k \in \mathbb{N}$. Then $\|\alpha\|_k > 0$ if and only if there exist some $\xi_1, \ldots, \xi_k \in H$ such that $\langle \alpha, (\xi_1, 0) \otimes \cdots \otimes (\xi_k, 0) \rangle \neq 0$.*

For any projection $P \in \mathrm{Proj}(W)$, we define an analogous norm for the finite-dimensional Lie algebras $\mathfrak{g}_P = PW \times \mathbf{C}$. Let $\{\xi_j\}_{j=1}^n$ be a complex orthonormal basis for PH, and let $\{\eta_j\}_{j=1}^n = \{(\xi_j, 0)\}_{j=1}^n$. Define the non-negative sesqui-linear form

$$(\alpha, \beta)_P := \sum_{j=1}^n \langle \alpha, \eta_j\rangle \overline{\langle \beta, \eta_j\rangle} \qquad \text{for all } \alpha, \beta \in \mathfrak{g}_P'.$$

This induces a form on $(\mathfrak{g}_P^{\otimes k})'$ determined by

$$(\alpha_1 \otimes \cdots \otimes \alpha_k, \beta_1 \otimes \cdots \otimes \beta_k)_{P,k} := \prod_{\ell=1}^k (\alpha_\ell, \beta_\ell)_P \qquad \text{for all } \alpha_j, \beta_j \in \mathfrak{g}_P'$$

For $\alpha \in (\mathfrak{g}_P^{\otimes k})'$, we will write

$$\|\alpha\|_{P,k}^2 := (\alpha, \alpha)_{P,k} = \sum_{j_1, \ldots, j_k = 1}^n |\langle \alpha, \eta_{j_1} \otimes \cdots \otimes \eta_{j_k}\rangle|^2.$$

One may easily verify that $\|\cdot\|_k$ and $\|\cdot\|_{P,k}$ are independent of the choice of orthonormal basis.

Definition 3.3 (Noncommutative Fock spaces)**.** For $t > 0$ and $\alpha = \sum_{k=1}^\infty \alpha_k \in J^0$, let

$$\|\alpha\|_t^2 := \sum_{k=0}^\infty \frac{t^k}{k!} \|\alpha_k\|_k^2,$$

and

$$J_t^0 := \{\alpha \in J^0 : \|\alpha\|_t < \infty\}.$$

Similarly, for $t > 0$, $P \in \mathrm{Proj}(W)$, and $\alpha \in J^0(\mathfrak{g}_P)$, let

$$\|\alpha\|_{P,t}^2 := \sum_{k=0}^\infty \frac{t^k}{k!} \|\alpha_k\|_{P,k}^2,$$

and

$$J_{P,t}^0 := \{\alpha \in J^0(\mathfrak{g}_P) : \|\alpha\|_{P,t} < \infty\}.$$

For $\{P_n\}_{n=1}^\infty$ an increasing sequence of projections in $\mathrm{Proj}(W)$, let $\|\cdot\|_{n,k} := \|\cdot\|_{P_n,k}$, $\|\alpha\|_{n,t} := \|\alpha\|_{P_n,t}$, $J_{n,t}^0 := J_{P_n,t}^0$.

The functions $\|\cdot\|_t$ and $\|\cdot\|_{P,t}$ are clearly semi-norms on J_t^0 and $J_{P,t}^0$, respectively. It is proved in Theorem 2.7 of [10] that, for any $t > 0$ and $P \in \mathrm{Proj}(W)$, the semi-norm $\|\cdot\|_{P,t}$ is a norm on $J_{P,t}^0$ (using the fact that $[PW, PW] = \mathbf{C}$). In fact, $J_{P,t}^0$ is a Hilbert space when equipped with the inner product

$$\langle \alpha, \beta\rangle_{P,t} := \sum_{k=0}^\infty \frac{t^k}{k!} (\alpha_k, \beta_k)_{P,k} \qquad \text{for all } \alpha, \beta \in J_{P,t}^0.$$

To compare our notation with that used in [10], for each $P \in \mathrm{Proj}(W)$, let

$$K_P := \left\{ \alpha \in \mathfrak{g}_P' : (\alpha, \alpha)_P = \sum_{j=1}^n |\langle \alpha, \eta_j\rangle|^2 = 0 \right\}.$$

Then clearly

$$K_P^0 := \{a \in \mathfrak{g}_P : \langle \alpha, a\rangle = 0 \text{ for all } \alpha \in K_P\} = PH \times \{0\}.$$

If the Lie algebra generated by PH is all of $\mathfrak{g}_P$, then $(\cdot,\cdot)_P$ satisfies Hörmander's condition as defined in Definition 2.6 of [10].

Here we follow the proof in [10] to show that, since Hörmander's condition $[H,H]=\mathbf{C}$ holds, $\|\cdot\|_t$ is a norm on J_t^0. (Indeed, it is shown in [10] that, at least in the finite-dimensional case, $\|\cdot\|_t$ is a norm on J_t^0 if and only if Hörmander condition holds.) First, we need the following lemma.

Lemma 3.4. *There exists an algebra homomorphism* $\Psi : T(\mathfrak{g}_{CM}) \to T(H)$ *such that* $T(\mathfrak{g}_{CM}) = T(H) \oplus \mathrm{Nul}(\Psi)$, *where* $\mathrm{Nul}(\Psi) \subset J(\mathfrak{g}_{CM})$.

Proof. Let $\{\xi_j\}_{j=1}^\infty$ be an orthonormal basis of H. Since $[H,H]=\mathbf{C}$, we may also choose $\{A_\ell,B_\ell\}_{\ell=1}^N \subset H$ such that $\{\omega(A_\ell,B_\ell)\}_{\ell=1}^N$ is a basis of $\mathbf{C}$ with dual basis $\{\varepsilon^\ell\}_{\ell=1}^N$. Define $\psi : \mathfrak{g}_{CM} \to H \oplus H^{\otimes 2}$ for

$$(A,a) = \sum_{j=1}^\infty \langle A,\xi_j\rangle_H(\xi_j,0) + \sum_{\ell=1}^N \varepsilon^\ell(a)(0,\omega(A_\ell,B_\ell)) \in \mathfrak{g}_{CM}$$

by

$$\psi(A,a) := \sum_{j=1}^\infty \langle A,\xi_j\rangle_H(\xi_j,0) + \sum_{\ell=1}^N \varepsilon^\ell(a)(A_\ell\wedge B_\ell,0),$$

where again $u\wedge v = u\otimes v - v\otimes u$ for any $u,v\in H$. Then ψ is a linear operator such that $\psi(A,0)=(A,0)$ for any $A\in H$, and, as

$$(A\wedge B,0)-(0,\omega(A,B)) = (A,0)\wedge(B,0)-(0,\omega(A,B)) \in J(\mathfrak{g}_{CM}),$$

for any $A,B\in H$, we have $\psi h - h \in J(\mathfrak{g}_{CM})$ for all $h\in\mathfrak{g}_{CM}$. One may also show that ψ is bounded as an operator into $T(H)$: for any $x=(A,a)\in G_{CM}$ such that $\|x\|_{\mathfrak{g}_{CM}}^2 = \|A\|_H^2 + \|a\|_{\mathbf{C}} \le 1$,

$$\begin{aligned}
\|\psi(A,a)\|^2_{H\oplus H^{\otimes 2}} &= \|A\|_H^2 + \sum_{j,k=1}^\infty \left|\left\langle \sum_{\ell=1}^N \varepsilon^\ell(a)A_\ell\wedge B_\ell, \xi_j\otimes\xi_k\right\rangle\right|^2 \\
&\le \|A\|_H^2 + \sum_{j,k=1}^\infty \left(\sum_{\ell=1}^N \varepsilon^\ell(a)^2 \sum_{\ell=1}^N |\langle A_\ell\wedge B_\ell,\xi_j\otimes\xi_k\rangle|^2\right) \\
&\le \|A\|_H^2 + C\|a\|_{\mathbf{C}}^2 \le C'(\|A\|_H^2 + \|a\|_{\mathbf{C}}),
\end{aligned}$$

where $C' = C'(N,\omega) < \infty$, and the final inequality follows from the fact that $\|A\|_H^2 + \|a\|_{\mathbf{C}} \le 1$ implies that $\|A\|_H^2 + \|a\|_{\mathbf{C}}^2 \le \|A\|_H^2 + \|a\|_{\mathbf{C}}$.

By the universal property of the tensor algebra, there is a unique extension of ψ to an algebra homomorphism $\Psi : T(\mathfrak{g}_{CM}) \to T(H)$, such that $\Psi 1_{T(\mathfrak{g}_{CM})} = 1_{T(H)}$. Since for $h_1,\ldots,h_n \in \mathfrak{g}_{CM}$

$$\Psi(h_1\otimes\cdots\otimes h_n) = \psi h_1\otimes\cdots\otimes\psi h_n \in (h_1+J(\mathfrak{g}_{CM}))\otimes\cdots\otimes(h_n+J(\mathfrak{g}_{CM}))$$

and $J(\mathfrak{g}_{CM})$ is an ideal, it follows that $\Psi(h_1\otimes\cdots\otimes h_n) - h_1\otimes\cdots\otimes h_n \in J(\mathfrak{g}_{CM})$. ☐

This lemma immediately gives the following.

Theorem 3.5. *Let* $t>0$. *The semi-norm* $\|\cdot\|_t$ *on* J_t^0 *is a norm.*

Proof. Suppose that $\alpha = \sum_{k=0}^\infty \alpha_k \in J^0$ is such that

$$0 = \|\alpha\|_t^2 = \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{i_1,\dots,i_k=1}^\infty |\langle \alpha_k, \eta_{i_1} \otimes \cdots \otimes \eta_{i_k}\rangle|^2.$$

Thus, $\alpha|_{T(H)} = 0$ and, for Ψ as in Lemma 3.4, $\alpha = \alpha \circ \Psi = \alpha|_{T(H)} \circ \Psi = 0$. □

Corollary 3.6. *The space J_t^0 is a Hilbert space equipped with the inner product*

$$\langle \alpha, \beta\rangle_t := \sum_{k=0}^\infty \frac{t^k}{k!} (\alpha_k, \beta_k)_k.$$

3.2. **The Taylor map.** The other relevant space for the Taylor map should be thought of as the ν_t-square integrable holomorphic functions on G_{CM}. For $t > 0$, $f : G_{CM} \to \mathbb{C}$, and $P \in \mathrm{Proj}(W)$, let

$$\|f\|^2_{L^2(\nu_t^P)} := \|f|_{G_P}\|^2_{L^2(\nu_t^P)} = \mathbb{E}|f(g_t^P)|^2,$$

where $\{g_t^P\}_{t\geq 0} \subset G_P \subset G_{CM}$ is a Brownian motion on G_P as in Proposition 2.25.

Definition 3.7. For $t > 0$ and $f \in \mathcal{H}(G_{CM})$, let

$$\|f\|_{\mathcal{H}_t^2(G_{CM})} := \sup_{P\in\mathrm{Proj}(W)} \|f\|_{L^2(\nu_t^P)},$$

and define

$$\mathcal{H}_t^2(G_{CM}) := \{f \in \mathcal{H}(G_{CM}) : \|f\|_{\mathcal{H}_t^2(G_{CM})} < \infty\}.$$

We set one more piece of notation before defining the Taylor map.

Notation 3.8. *Given $f \in \mathcal{H}(G_{CM})$, $g \in G_{CM}$, $k \in \{0\}\cup\mathbb{N}$, let $\hat{f}_k(g) := (D^k f)(g)$ denote the unique element of $(\mathfrak{g}_{CM}^{\otimes k})'$ given by*

$$(D^0 f)(g) = f(g)$$
$$\langle (D^k f)(g), h_1 \otimes \cdots \otimes h_k\rangle = \left(\tilde{h}_1 \cdots \tilde{h}_k f\right)(g)$$

for all $h_1, \dots, h_k \in \mathfrak{g}_{CM}$. Let $\hat{f}(g)$ be the element of $T(\mathfrak{g}_{CM})'$ determined by

$$\langle \hat{f}(g), \beta\rangle = \langle \hat{f}_k(g), \beta\rangle, \qquad \text{for all } \beta \in \mathfrak{g}_{CM}^{\otimes k}.$$

Remark 3.9. As a consequence of equation (2.9), $\hat{f}(g) \in J^0$ for all $f \in \mathcal{H}(G_{CM})$ and $g \in G_{CM}$.

Definition 3.10. For each $t > 0$, the *Taylor map* is the linear map $\mathcal{T}_t : \mathcal{H}_t^2(G_{CM}) \to J_t^0$ defined by $\mathcal{T}_t f = \hat{f}(e)$.

3.3. **Proof of isometry.** We will prove that the Taylor map is an isometry by limiting arguments for the finite-dimensional projections. Let us first recall the finite-dimensional theory.

Notation 3.11. *For any $P \in \mathrm{Proj}(W)$, we set derivative notation for $f \in \mathcal{H}(G_P)$ similarly to how it was done in Notation 3.8. That is, for $g \in G_P$ and $k \in \{0\}\cup\mathbb{N}$, let $\hat{f}_k(g) := (D_P^k f)(g)$ denote the element of $(\mathfrak{g}_P^{\otimes k})'$ given by*

$$\langle (D_P^k f)(g), h_1 \otimes \cdots \otimes h_k\rangle = \left(\tilde{h}_1 \cdots \tilde{h}_k f\right)(g),$$

for all $h_1, \dots, h_k \in \mathfrak{g}_P$, and let $\hat{f}(g)$ be the element of $T(\mathfrak{g}_P)'$ determined by

$$\langle \hat{f}(g), \beta\rangle = \langle \hat{f}_k(g), \beta\rangle, \qquad \text{for all } \beta \in \mathfrak{g}_P^{\otimes k}.$$

Also, let $\mathcal{H}L^2(\nu_t^P) = \mathcal{H}(G_P) \cap L^2(G_P, \nu_t^P)$. *If* $\{P_n\}_{n=1}^\infty$ *is an increasing sequence in* $\mathrm{Proj}(W)$, *let* $\mathcal{H}L^2(\nu_t^n) = \mathcal{H}L^2(\nu_t^{P_n})$. *The* finite-dimensional Taylor map *is the linear map* $f \mapsto \hat{f}(e)$ *from* $\mathcal{H}L^2(\nu_t^P)$ *to* $J_{P,t}^0$, *where the latter is as defined in Definition 3.3*

For each $P \in \mathrm{Proj}(W)$, G_P is a finite-dimensional connected, simply connected complex Lie group. If $[PW, PW] = \mathbf{C}$, then $(\cdot,\cdot)_P$ is a non-negative Hermitian form on $\mathfrak{g}_P'$ satisfying Hörmander's condition. Thus, we have the following theorem.

Theorem 3.12. *Suppose that* $P \in \mathrm{Proj}(W)$ *such that* $[PW, PW] = \mathbf{C}$. *Then the finite-dimensional Taylor map* $f \mapsto \hat{f}(e)$ *is a unitary map from* $\mathcal{H}L^2(\nu_t^P)$ *onto* $J_{P,t}^0$. *Moreover, for any* $t > 0$, $f \in \mathcal{H}L^2(\nu_t^P)$, *and* $g \in G_P$,

$$|f(g)| \leq \|\hat{f}(e)\|_{P,t} e^{d_h^2(e,g)/2t} \tag{3.1}$$

where d_h *is the horizontal distance on* G_P *(defined analogously on* G_P *to the horizontal distance on* G_{CM} *as in Notation 2.14).*

The isometry and surjectivity follow from the finite-dimensional Taylor isomorphism proved in Theorem 6.1 of [10], and the estimate in (3.1) is a consequence of Corollary 5.15 of that same reference. The paper [11] gives an alternate proof of the surjectivity, as each G_P is a nilpotent Lie group. In Section 3.4, we will apply the methods used in [11] to show that the Taylor map is surjective in this infinite-dimensional setting as well. Here we use the finite-dimensional isometries to show that $\mathcal{T}_t$ is an isometry for all $t > 0$ as follows.

Proposition 3.13. *Let* $f \in \mathcal{H}(G_{CM})$ *and* $t > 0$. *Then*

$$\|\hat{f}(e)\|_t = \|f\|_{\mathcal{H}_t^2(G_{CM})}.$$

Proof. By the finite-dimensional Taylor isomorphism theorem, for all $P \in \mathrm{Proj}(W)$,

$$\|\hat{f}(e)\|_{J_{P,t}^0} = \|f\|_{L^2(\nu_t^P)}.$$

Thus, by definition of $\|\cdot\|_{\mathcal{H}_t^2(G_{CM})}$,

$$\|f\|_{\mathcal{H}_t^2(G_{CM})} = \sup_{P\in\mathrm{Proj}(W)} \|f\|_{L^2(\nu_t^P)} = \sup_{P\in\mathrm{Proj}(W)} \|\hat{f}(e)\|_{J_{P,t}^0}.$$

So showing that

$$\sup_{P\in\mathrm{Proj}(W)} \|\hat{f}(e)\|_{J_{P,t}^0} = \|\hat{f}(e)\|_t$$

completes the proof.

Let $P \in \mathrm{Proj}(W)$ with $\{\xi_j\}_{j=1}^\infty$ an orthonormal basis of H, such that $\{\xi_j\}_{j=1}^n$ is an orthonormal basis of PH. Let $\eta_j = (\xi_j, 0)$. Then

$$\begin{aligned}\|\hat{f}(e)\|_{J_t^0(\mathfrak{g}_P)} &= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^n |\langle \hat{f}(e), \eta_{j_1} \otimes \cdots \otimes \eta_{j_k}\rangle|^2 \\ &\leq \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^\infty |\langle \hat{f}(e), \eta_{j_1} \otimes \cdots \otimes \eta_{j_k}\rangle|^2 = \|\hat{f}(e)\|_t,\end{aligned}$$

and so $\sup_{P\in\mathrm{Proj}(W)} \|\hat{f}(e)\|_{J_t^0(\mathfrak{g}_P)} \leq \|\hat{f}(e)\|_t$. On the other hand, if $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ is an increasing sequence of projections, then

$$\sup_{P\in\mathrm{Proj}(W)} \|\hat{f}(e)\|_{J_t^0(\mathfrak{g}_P)} \geq \lim_{n\to\infty} \|\hat{f}(e)\|_{n,t}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\dots,j_k=1}^{n} |\langle \hat{f}(e), \eta_{j_1} \otimes \cdots \otimes \eta_{j_k} \rangle|^2$$
$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\dots,j_k=1}^{\infty} |\langle \hat{f}(e), \eta_{j_1} \otimes \cdots \otimes \eta_{j_k} \rangle|^2 = \|\hat{f}(e)\|_t.$$

□

The following corollary follows from Propositions 3.13 and 2.29.

Corollary 3.14. *Let $\delta > 0$ be as in Proposition 2.28, and suppose that $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_{CM}} \in \mathcal{H}(G_{CM})$ and, for some $\varepsilon \in (0, \delta)$,*

$$|f(g)| \leq Ce^{\varepsilon \|g\|_{\mathfrak{g}}^2/2t}$$

for all $g \in G$. Then $f|_{G_{CM}} \in \mathcal{H}_t^2(G_{CM})$ and $\widehat{f|_{G_{CM}}}(e) \in J_t^0$.

In particular, Corollary 3.14 implies that, for all $t > 0$, $\mathcal{P}_{CM} \subset \mathcal{H}_t^2(G_{CM})$ and, for any $p \in \mathcal{P}$, $\widehat{p|_{G_{CM}}}(e) \in J_t^0$. Thus, $\mathcal{H}_t^2(G_{CM})$ and J_t^0 are non-trivial spaces.

Corollary 3.15. *The Taylor map $\mathcal{T}_t : \mathcal{H}_t^2(G_{CM}) \to J_t^0$ is injective, and $\|\cdot\|_{\mathcal{H}_t^2(G_{CM})}$ is a norm on $\mathcal{H}_t^2(G_{CM})$ induced by the inner product*

$$\langle u, v \rangle_{\mathcal{H}_t^2(G_{CM})} := \langle \hat{u}(e), \hat{v}(e) \rangle_t, \qquad \text{for all } u, v \in \mathcal{H}_t^2(G_{CM}).$$

Proof. If $\hat{f}(e) = 0$, then Proposition 3.13 implies that $\|f\|_{\mathcal{H}_t^2(G_{CM})} = 0$ and thus $f|_{G_P} = 0$ for all $P \in \mathrm{Proj}(W)$. As f is continuous and $\cup_{P\in\mathrm{Proj}(W)} G_P$ is dense in G_{CM} by Proposition 2.18, it follows that $f \equiv 0$. Thus, $\mathcal{T}_t$ is injective.

Since $\|\cdot\|_t$ is a Hilbert norm, Proposition 3.13 then also implies that $\|\cdot\|_{\mathcal{H}_t^2(G_{CM})}$ is the norm on $\mathcal{H}_t^2(G_{CM})$ given by the above inner product. □

3.4. **A density theorem and proof of surjectivity.** We will now apply the methods used in [11] to show that the Taylor map is surjective. In fact, the infinite-dimensional proof is directly analogous to the finite-dimensional proof presented there, and no special considerations need to be made for the infinite-dimensional case. Similar arguments were used in [5] and [8]. Still, we collect the proofs here for completeness and to stress the dimension independence of the arguments. Additionally, Corollary 3.20 will be critical in the proof of surjectivity of the restriction map in Section 4, and this proof will require some adaptation for the subelliptic construction.

Definition 3.16. A tensor $\alpha = \sum_{k=0}^{\infty} \alpha_k \in T(\mathfrak{g}_{CM})'$ is said to have *finite rank* if $\alpha_k = 0$ for all but finitely many $k \in \mathbb{N}$.

The next lemma is essentially a special case of [10, Lemma 3.5]. See also [5, Theorem 41] and [8, Lemma 7.3].

Lemma 3.17. *The finite rank tensors in J_t^0 are dense in J_t^0.*

Proof. First note that $\mathfrak{g}_{CM} = H \times \mathbf{C}$ is a graded Lie algebra with $[H, H] = \mathbf{C}$, $[H, \mathbf{C}] = 0$, and $[\mathbf{C}, \mathbf{C}] = 0$. Thus, for $\theta \in \mathbb{R}$, we may define the dilations $\varphi_\theta : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ by

$$\varphi_\theta(A, a) := (e^{i\theta} A, e^{2i\theta} a), \qquad \text{for all } (A, a) \in \mathfrak{g}_{CM},$$

and it is straightforward to verify that φ_θ is an automorphism of $\mathfrak{g}_{CM}$. Let $\Phi_\theta : T(\mathfrak{g}_{CM}) \to T(\mathfrak{g}_{CM})$ be the automorphism of the tensor algebra over $\mathfrak{g}_{CM}$ induced by φ_θ, that is,

$$\Phi_\theta := \overbrace{\varphi_\theta \otimes \cdots \otimes \varphi_\theta}^{k \text{ times}} \text{ on } \mathfrak{g}_{CM}^{\otimes k}.$$

Then

$$\begin{aligned}\Phi_\theta(\xi \wedge \xi' - [\xi, \xi']) &= (\varphi_\theta \xi) \wedge (\varphi_\theta \xi') - \varphi_\theta[\xi, \xi'] \\ &= (\varphi_\theta \xi) \wedge (\varphi_\theta \xi') - [\varphi_\theta \xi, \varphi_\theta \xi'].\end{aligned}$$

From this it follows that $\Phi_\theta(J) \subset J$ and therefore if $\alpha \in J^0$, then $\alpha \circ \Phi_\theta \in J^0$. Letting $\{\xi_j\}_{j=1}^\infty$ be an orthonormal basis of H and $\Gamma = \{(\xi_j, 0)\}_{j=1}^\infty$, we have $\varphi_\theta \eta = e^{i\theta}\eta$ for all $\eta \in \Gamma$. Therefore,

$$\begin{aligned}|\langle \alpha \circ \Phi_\theta, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2 &= |\langle \alpha, \varphi_\theta \eta_1 \otimes \cdots \otimes \varphi_\theta \eta_k \rangle|^2 \\ &= |\langle \alpha, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2,\end{aligned}$$

and hence

$$\begin{aligned}\|\alpha \circ \Phi_\theta\|_t^2 &= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{\eta_1, \ldots, \eta_k \in \Gamma} |\langle \alpha \circ \Phi_\theta, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2 \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{\eta_1, \ldots, \eta_k \in \Gamma} |\langle \alpha, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2 = \|\alpha\|_t^2 .\end{aligned}$$

So the map $J_t^0 \ni \alpha \mapsto \alpha \circ \Phi_\theta \in J_t^0$ is unitary. Moreover, since

$$|\langle \alpha, \varphi_\theta \eta_1 \otimes \cdots \otimes \varphi_\theta \eta_k \rangle - \langle \alpha, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2 \leq 2|\langle \alpha, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2,$$

the dominated convergence theorem implies that

$$\begin{aligned}&\lim_{\theta \to 0} \|\alpha \circ \Phi_\theta - \alpha\|_t^2 \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{\eta_1, \ldots, \eta_k \in \Gamma} \lim_{\theta \to 0} |\langle \alpha, \varphi_\theta \eta_1 \otimes \cdots \otimes \varphi_\theta \eta_k \rangle - \langle \alpha, \eta_1 \otimes \cdots \otimes \eta_k \rangle|^2 \\ &= 0,\end{aligned} \tag{3.2}$$

and $\alpha \mapsto \alpha \circ \Phi_\theta$ is continuous. (Notice that $\Phi_\theta \circ \Phi_\alpha = \Phi_{\theta+\alpha}$, so it suffices to check continuity at $\theta = 0$.)

Now, for any $n \in \mathbb{N}$, let

$$F_n(\theta) = \frac{1}{2\pi n} \sum_{j=0}^{n-1} \sum_{\ell=-j}^{j} e^{i\ell\theta} = \frac{1}{2\pi n} \frac{\sin^2(j\theta/2)}{\sin^2(\theta/2)}$$

denote Fejer's kernel [27, p. 143]. Then one may show the following: $\int_{-\pi}^{\pi} F_n(\theta) d\theta = 1$ for all $n \in \mathbb{N}$;

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} F_n(\theta) u(\theta) d\theta = u(0),$$

for all continuous functions $u : [-\pi, \pi] \to \mathbb{C}$; and

$$\int_{-\pi}^{\pi} F_n(\theta) e^{im\theta}\, d\theta = 0$$

whenever $m > n$. Given $\alpha \in J_t^0$, we let

$$\alpha(n) := \int_{-\pi}^{\pi} \alpha \circ \Phi_\theta F_n(\theta)\, d\theta.$$

If $\beta = h_1 \otimes \cdots \otimes h_m \in \mathfrak{g}_{CM}^{\otimes m}$, then there exist $\beta_\ell \in \mathfrak{g}_{CM}^{\otimes m}$ such that

$$\Phi_\theta \beta = \sum_{\ell=m}^{2m} e^{i\ell\theta} \beta_\ell.$$

So, if $m > n$,

$$\langle \alpha(n), \beta \rangle = \int_{-\pi}^{\pi} \langle \alpha, \Phi_\theta \beta \rangle\, F_n(\theta)\, d\theta = \sum_{\ell=m}^{2m} \langle \alpha, \beta_\ell \rangle \int_{-\pi}^{\pi} e^{i\ell\theta} F_n(\theta)\, d\theta = 0,$$

from which it follows that $\alpha(n)_m \equiv 0$ for all $m > n$. Thus $\alpha(n)$ is a finite rank tensor for all $n \in \mathbb{N}$, and (3.2) implies that

$$\begin{aligned}\limsup_{n\to\infty} \|\alpha - \alpha(n)\|_t^2 &= \limsup_{n\to\infty} \left\| \int_{-\pi}^{\pi} [\alpha - \alpha \circ \Phi_\theta] F_n(\theta)\, d\theta \right\|_t \\ &\leq \limsup_{n\to\infty} \int_{-\pi}^{\pi} \|\alpha - \alpha \circ \Phi_\theta\|_t F_n(\theta)\, d\theta = 0.\end{aligned}$$

□

The surjectivity of the Taylor map may now be proved by finding a preimage in $\mathcal{H}_t^2(G_{CM})$ under $\mathcal{T}_t$ for any finite rank tensor in J_t^0. The following lemma is a special case of Proposition 5.1 in [6] and motivates our construction of the inverse of the Taylor map. This version of the result may also be found in Lemma 6.9 of [8].

Lemma 3.18. *For every $f \in \mathcal{H}(G_{CM})$ and $g \in G_{CM}$,*

$$f(g) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \hat{f}_k(e), g^{\otimes k} \rangle,$$

where by convention $g^{\otimes 0} = 1 \in \mathbb{C}$ and the above sum is absolutely convergent.

Proof. The function $u(z) := f(zg)$ is a holomorphic function of $z \in \mathbb{C}$. Therefore,

$$f(g) = u(1) = \sum_{k=0}^{\infty} \frac{1}{k!} u^{(k)}(0),$$

and the sum is absolutely convergent. In fact, for all $r > 0$, there exists $C(r) < \infty$ such that $\frac{1}{k!}|u^{(k)}(0)| \leq C(r) r^{-k}$ for all $k \in \mathbb{N}$. Finally, note that

$$\begin{aligned}u^{(k)}(0) &= \frac{d^k}{dt^k}\Big|_{t=0} u(t) = \frac{d^k}{dt^k}\Big|_{t=0} f(tg) \\ &= \frac{d^k}{dt^k}\Big|_{t=0} f(e^{tg}) = (\tilde{g}^k f)(e) = \langle \hat{f}_k(e), g^{\otimes k} \rangle.\end{aligned}$$

□

The following proof of the surjectivity of the Taylor map is directly analogous to the proof of Lemma 3.6 in [11].

Theorem 3.19. *The Taylor map $\mathcal{T}_t : \mathcal{H}_t^2(G_{CM}) \to J_t^0$ is surjective.*

Proof. Consider first α a finite rank tensor in J_t^0. By Lemma 3.18, if $f = \mathcal{T}_t^{-1}\alpha$ exists, then it must be given by

$$f_\alpha(g) := \sum_{k=0}^\infty \frac{1}{k!}\langle \alpha_k, g^{\otimes k}\rangle,$$

for all $g \in G_{CM}$. This is a finite sum since α is of finite rank, and thus f_α is a finite sum of continuous complex multilinear forms in $g \in G_{CM}$. Thus, f_α is holomorphic, and, in particular, for any $h \in \mathfrak{g}_{CM}$,

$$\langle \hat{f}_\alpha(e), h^{\otimes k}\rangle = \frac{d^k}{dt^k}\bigg|_{t=0} f_\alpha(th) = \frac{d^k}{dt^k}\bigg|_{t=0} \sum_{n=0}^\infty \frac{1}{n!}\langle \alpha_n, (th)^{\otimes n}\rangle = \langle \alpha_k, h^{\otimes k}\rangle.$$

So $\hat{f}_\alpha(e) = \alpha$ on $\operatorname{span}\{h^{\otimes k} : h \in \mathfrak{g}_{CM}, k \in \{0\} \cup \mathbb{N}\} = \{\text{symmetric } \mathbb{R}\text{-tensors}\} =: \mathcal{S}$. By the Poincaré-Birkhoff-Witt theorem (see [28, Lemma 3.3.3] or [22, Corollary E]), $T(\mathfrak{g}_{CM}) = \mathcal{S} \oplus J$, and, since $\hat{f}_\alpha(e) - \alpha$ annihilates J, this implies that $\hat{f}_\alpha(e) = \alpha$ on $T(\mathfrak{g}_{CM})$.

Thus, for every finite rank tensor $\alpha \in J_t^0$, the function f_α is holomorphic and $\hat{f}_\alpha(e) = \alpha$, and so Proposition 3.13 implies that $f_\alpha \in \mathcal{H}_t^2(G_{CM})$. Hence, the image of $f \mapsto \hat{f}(e)$ is dense in J_t^0, which suffices to prove surjectivity. □

The following is an immediate consequence of Lemma 3.17 and Theorem 3.19.

Corollary 3.20. *The vector space,*

$$\mathcal{H}_{t,\mathrm{fin}}^2(G_{CM}) := \left\{ f \in \mathcal{H}_t^2(G_{CM}) : \hat{f}(e) \in J_t^0 \text{ is finite rank}\right\}$$

is a dense subspace of $\mathcal{H}_t^2(G_{CM})$.

4. The restriction map

In this section, we construct the "skeleton" or "restriction" map between a class of square integrable holomorphic functions on G and $\mathcal{H}_t^2(G_{CM})$, and we prove that this map is an isometric isomorphism. Before proceeding, we must first define the appropriate class of holomorphic functions on G we wish to deal with.

Recall from Definition 2.23 that a function $f : G \to \mathbb{C}$ is a *cylinder function* if $f = F \circ \pi_P$ for some $P \in \operatorname{Proj}(W)$ and $F : G_P \to \mathbb{C}$. We say that f is a *holomorphic cylinder polynomial* if F is a holomorphic polynomial on G_P. The space of holomorphic cylinder polynomials will be denoted by $\mathcal{P}$. Propositions 2.28 and 2.29 imply that $\mathcal{P} \subset L^p(\nu_t)$ for all $p \in [1, \infty)$, so we may make the following definition.

Definition 4.1. For $t > 0$, let $\mathcal{H}_t^2(G)$ denote the $L^2(\nu_t)$-closure of $\mathcal{P}$.

Remark 4.2. Let $\mathcal{A}$ denote the class of holomorphic cylinder functions on G. As remarked in [8], it is natural to expect that $\mathcal{H}_t^2(G)$ coincides with the closure of $\mathcal{A} \cap L^2(\nu_t)$ in $L^2(\nu_t)$, however, this is currently not known even in much simpler settings. But in a sense $\mathcal{H}_t^2(G)$ is the appropriate space to consider, as the polynomials should constitute a dense subset of the square integrable holomorphic functions, when one can make sense of polynomials.

In Section 4.1, we show that the restriction of holomorphic cylinder polynomials to G_{CM} constitutes a dense subspace of $\mathcal{H}_t^2(G_{CM})$, and with this result in hand, in Section 4.2 we construct the restriction map as a linear map on $\mathcal{H}_t^2(G)$.

4.1. **Another density theorem.** Techniques similar to those used in this section were used in [8], as well as in Cecil [5] to prove an analogous result for path groups over stratified Lie groups.

Theorem 4.3. *For all* $t > 0$,

$$\mathcal{P}_{CM} := \{p|_{G_{CM}} : p \in \mathcal{P}\}$$

is a dense subspace of $\mathcal{H}_t^2(G_{CM})$.

This result is analogous to Theorem 7.1 of [8], and as done in that paper, Theorem 4.3 will be proved by showing that $\mathcal{P}_{CM}$ is dense in yet another dense subspace of $\mathcal{H}_t^2(G_{CM})$. In particular, Corollary 3.20 implies that it suffices to show that any element of $\mathcal{H}_{t,\mathrm{fin}}^2(G_{CM})$ may be approximated by elements of $\mathcal{P}_{CM}$. However, the fact that in our case J_t^0 is defined not using the full Hilbert-Schmidt norm complicates some limiting arguments that appear in [8].

Again we recall Notation 2.20: let $\{\xi_j\}_{j=1}^\infty \subset H_*$ be a complex orthonormal basis of H and let $\{\eta_j\}_{j=1}^\infty = \{(\xi_j, 0)\}_{j=1}^\infty$. Define $P_n \in \mathrm{Proj}(W)$ by

$$P_n w = \sum_{j=1}^n \langle w, \xi_j\rangle_H \xi_j \text{ for all } w \in W,$$

and $\pi_n : G \to G_n = P_n W \times \mathbf{C}$ defined by $\pi_n(w, c) = (P_n w, c)$.

We will show that for all $f \in \mathcal{H}_{t,\mathrm{fin}}(G_{CM})$, $f \circ \pi_n \in \mathcal{P}$ and $f \circ \pi_n|_{G_{CM}} \to f$ in $\mathcal{H}_t^2(G_{CM})$. The proof of this statement is complicated by the fact that, for general ω and $P \in \mathrm{Proj}(W)$, $\pi_P : G \to G_P \subset G_{CM}$ is not a group homomorphism. In fact, for $g = (w, c)$ and $g' = (w', c')$,

$$\pi_P(gg') - \pi_P g \cdot \pi_P g' = \Gamma_P(w, w')$$

where

$$\Gamma_P(w, w') := \frac{1}{2}(0, \omega(w, w') - \omega(Pw, Pw')) = \frac{1}{2}\left([g, g'] - [\pi_P g, \pi_P g']\right). \tag{4.1}$$

So unless ω is "supported" on the range of P, π_P is not a group homomorphism. Note that the case where ω *is* supported on a finite-dimensional space is exactly the trivial case where L is "finitely many steps from being elliptic," and the proof of several of the other results included here would be greatly simplified.

The proof of the following proposition is similar to Proposition 2.13 and is left to the reader.

Proposition 4.4. *For any* $P \in \mathrm{Proj}(W)$, $g = (w, c) \in G$, $h_i = (A_i, a_i) \in \mathfrak{g}$, *and* $f : G \to \mathbb{C}$ *a smooth function,*

$$\tilde{h}_n \cdots \tilde{h}_1(f \circ \pi_P)(g) = \sum_{k=\lceil n/2\rceil}^n f^{(k)}(\pi_P g) \sum_{\theta \in \Lambda_{n-k}^n} (h_n, \dots, h_1)_P^{\otimes\theta}(g), \tag{4.2}$$

where, for $\theta = \{\{i_1, i_2\}, \dots, \{i_{2k-1}, i_{2k}\}, \{i_{2k+1}\}, \dots, \{i_n\}\} \in \Lambda_k^n$ *a partition of* $\{1, \cdots, n\}$ *as defined in Notation 2.12,*

$$(h_n, \dots, h_1)_P^{\otimes\theta}(g) := [h_{i_1}, h_{i_2}] \otimes \cdots \otimes [h_{i_{2k-1}}, h_{i_{2k}}] \otimes h_{i_{2k+1}}^P(g) \otimes \cdots \otimes h_{i_n}^P(g),$$

with

$$h^P(g) := \left(PA, a + \frac{1}{2}\omega(w, A)\right).$$

Again as we did for Proposition 2.13, let us write out (4.2) for the first few n:

$$\begin{aligned}\tilde{h}_1(f\circ\pi)(g) &= f'(\pi g)h_1^P(g)\\ \tilde{h}_2\tilde{h}_1(f\circ\pi)(g) &= f''(\pi g)\left(h_2^P(g)\otimes h_1^P(g)\right)+f'(\pi g)[h_2,h_1]\\ \tilde{h}_3\tilde{h}_2\tilde{h}_1(f\circ\pi)(g) &= f'''(\pi g)\left(h_3^P(g)\otimes h_2^P(g)\otimes h_1^P(g)\right)\\ &\quad+\frac{1}{2}f''(\pi g)\left([h_3,h_2]\otimes h_1^P(g)+[h_3,h_1]\otimes h_2^P(g)+[h_2,h_1]\otimes h_3^P(g)\right)\end{aligned}$$

In particular, when $g=e$ and $h_i=(A_i,0)$, we have $h_i^P(e)=(PA_i,0)=\pi h_i$, and the above formulae become

$$\tilde{h}_1(f\circ\pi)(e)=f'(e)\pi h_1 \tag{4.3}$$

$$\tilde{h}_2\tilde{h}_1(f\circ\pi)(e)=f''(e)(\pi h_2\otimes\pi h_1)+f'(e)\frac{1}{2}[h_2,h_1] \tag{4.4}$$

$$\begin{aligned}\tilde{h}_3\tilde{h}_2\tilde{h}_1(f\circ\pi)(e)&=f'''(e)(\pi h_3\otimes\pi h_2\otimes\pi h_1)\\ &\quad+\frac{1}{2}f''(e)\big([h_3,h_2]\otimes\pi h_1+\pi h_2\otimes[h_3,h_1]+\pi h_3\otimes[h_2,h_1]\big).\end{aligned} \tag{4.5}$$

Now using Propositions 2.13 and 4.4 we may prove the following.

Proposition 4.5. *Fix $k\in\mathbb{N}$ and suppose that $f\in\mathcal{H}(G_{CM})$ satisfies $\|\hat{f}_k(e)\|_k<\infty$. Then*

$$\lim_{n\to\infty}\left\|\hat{f}_k(e)-\left(\widehat{f\circ\pi_n}\right)_k(e)\right\|_k=0.$$

Proof. We will write out the first few cases for small k before proving the convergence for arbitrary k. Consider first $k=1$. Then Propositions 2.13 and 4.4, (more particularly, equations (2.5) and (4.3)) imply that

$$\begin{aligned}\|\hat{f}_1(e)-(\widehat{f\circ\pi})_1(e)\|_1^2&=\sum_{j=1}^\infty|\tilde{\eta}_jf(e)-\tilde{\eta}_j(f\circ\pi)(e)|^2\\ &=\sum_{j=1}^\infty|f'(e)\eta_j-f'(e)\pi\eta_j|^2=\sum_{j=n+1}^\infty|f'(e)\eta_j|^2\to0\end{aligned}$$

as $n\to\infty$, since by hypothesis

$$\|\hat{f}_1(e)\|_1^2=\sum_{j=1}^\infty|\tilde{\eta}_jf(e)|^2=\sum_{j=1}^\infty|f'(e)\eta_j|^2<\infty.$$

Now, for $k=2$, equations (2.7) and (4.4)) give

$$\begin{aligned}\|\hat{f}_2(e)-(\widehat{f\circ\pi})_2(e)\|_2^2&=\sum_{j_1,j_2=1}^\infty|\tilde{\eta}_{j_2}\tilde{\eta}_{j_1}f(e)-\tilde{\eta}_{j_2}\tilde{\eta}_{j_1}(f\circ\pi)(e)|^2\\ &=\sum_{j_1,j_2=1}^\infty\Bigg|\left\{f''(e)(\eta_{j_1}\otimes\eta_{j_2})+\frac{1}{2}f'(e)[\eta_{j_1},\eta_{j_2}]\right\}\\ &\qquad-\left\{f''(e)(\pi\eta_{j_1}\otimes\pi\eta_{j_2})+\frac{1}{2}f'(e)[\eta_{j_1},\eta_{j_2}]\right\}\Bigg|^2\\ &=\sum_{j_1,j_2=1}^\infty|f''(e)(\eta_{j_1}\otimes\eta_{j_2}-\pi\eta_{j_1}\otimes\pi\eta_{j_2})|^2\end{aligned}$$

$$\leq \sum_{j_1=1}^{\infty}\sum_{j_2=n+1}^{\infty}\left|f''(e)(\eta_{j_1}\otimes\eta_{j_2})+\frac{1}{2}f'(e)[\eta_{j_1},\eta_{j_2}]\right|^2$$
$$+\sum_{j_1=n+1}^{\infty}\sum_{j_2=1}^{\infty}\left|f''(e)(\eta_{j_1}\otimes\eta_{j_2})+\frac{1}{2}f'(e)[\eta_{j_1},\eta_{j_2}]\right|^2$$
$$+\frac{1}{2}\sum_{j_1=1}^{\infty}\sum_{j_2=n+1}^{\infty}|f'(e)[\eta_{j_1},\eta_{j_2}]|^2+\frac{1}{2}\sum_{j_1=n+1}^{\infty}\sum_{j_2=1}^{\infty}|f'(e)[\eta_{j_1},\eta_{j_2}]|^2\to 0,$$

as $n\to\infty$, since

$$\|\hat{f}_2(e)\|_2^2=\sum_{j_1,j_2=1}^{\infty}\left|f''(e)(\eta_{j_1}\otimes\eta_{j_2})+\frac{1}{2}f'(e)[\eta_{j_1},\eta_{j_2}]\right|^2<\infty,$$

by hypothesis, and

$$\sum_{j_1,j_2=1}^{\infty}|f'(e)[\eta_{j_1},\eta_{j_2}]|^2\leq|f'(e)|^2\sum_{j_1,j_2=1}^{\infty}\|\omega(\xi_{j_1},\xi_{j_2})\|_{\mathbf{C}}^2=|f'(e)|^2\|\omega\|_{HS}^2<\infty,$$

by Proposition 2.33 which states that $f'(e)$ is a bounded operator on G_{CM} and Proposition 2.5 which implies that ω is Hilbert-Schmidt.

For $k=3$, equations (2.8) and (4.5)) give

$$\|\hat{f}_3(e)-(\widehat{f\circ\pi})_3(e)\|_2^2=\sum_{j_1,j_2,j_3=1}^{\infty}|\tilde{\eta}_{j_3}\tilde{\eta}_{j_2}\tilde{\eta}_{j_1}f(e)-\tilde{\eta}_{j_3}\tilde{\eta}_{j_2}\tilde{\eta}_{j_1}(f\circ\pi)(e)|^2$$
$$=\sum_{j_1,j_2,j_3=1}^{\infty}\Big|f'''(e)(\eta_{j_3}\otimes\eta_{j_2}\otimes\eta_{j_1}-\pi\eta_{j_3}\otimes\pi\eta_{j_2}\otimes\pi\eta_{j_1})$$
$$+\frac{1}{2}f''(e)([\eta_{j_3},\eta_{j_2}]\otimes\eta_{j_1}+[\eta_{j_3},\eta_{j_1}]\otimes\eta_{j_2}+[\eta_{j_2},\eta_{j_1}]\otimes\eta_{j_3}$$
$$-[\eta_{j_3},\eta_{j_2}]\otimes\pi\eta_{j_1}-[\eta_{j_3},\eta_{j_1}]\otimes\pi\eta_{j_2}-[\eta_{j_2},\eta_{j_1}]\otimes\pi\eta_{j_3})\Big|^2$$
$$\leq\sum_{\ell=1}^{3}\sum_{j_\ell=n+1}^{\infty}\sum_{\substack{j_i=1\\ i\neq\ell}}^{\infty}\Big|f'''(e)(\eta_{j_3}\otimes\eta_{j_2}\otimes\eta_{j_1})$$
$$+\frac{1}{2}f''(e)([\eta_{j_3},\eta_{j_2}]\otimes\eta_{j_1}+[\eta_{j_3},\eta_{j_1}]\otimes\eta_{j_2}+[\eta_{j_2},\eta_{j_1}]\otimes\eta_{j_3})\Big|^2\to 0$$

as $n\to\infty$, since

$$\|\hat{f}_3(e)\|_2^2=\sum_{j_1,j_2,j_3=1}^{\infty}\Big|f'''(e)(\eta_{j_3}\otimes\eta_{j_2}\otimes\eta_{j_1})$$
$$+\frac{1}{2}f''(e)([\eta_{j_3},\eta_{j_2}]\otimes\eta_{j_1}+[\eta_{j_3},\eta_{j_1}]\otimes\eta_{j_2}+[\eta_{j_2},\eta_{j_1}]\otimes\eta_{j_3})\Big|^2<\infty,$$

again by hypothesis.

More generally, using equations (2.6) and (4.2) with $g = e$ and $\eta_j = (\xi_j, 0)$ for k odd shows that

$$\|\hat{f}_k(e) - (\widehat{f\circ\pi})_k(e)\|_k^2 \leq \sum_{\ell=1}^{k} \sum_{j_\ell=n+1}^{\infty} \sum_{\substack{j_i=1 \\ i\neq\ell}}^{\infty} \left|\langle \hat{f}_k(e), \eta_{j_k}\otimes\cdots\otimes\eta_{j_1}\rangle\right|^2 \to 0$$

as $n\to\infty$. Similarly, for k even,

$$\|\hat{f}_k(e) - (\widehat{f\circ\pi})_k(e)\|_k^2 \leq \sum_{\ell=1}^{k} \sum_{j_\ell=n+1}^{\infty} \sum_{\substack{j_i=1 \\ i\neq\ell}}^{\infty} \left\{ \left|\langle \hat{f}_k(e), \eta_{j_k}\otimes\cdots\otimes\eta_{j_1}\rangle\right|^2 \right.$$
$$\left. + \frac{1}{2}\sum_{\theta\in\Lambda^k_{k/2}} \left| f^{(k/2)}(e)(\eta_{j_k},\dots,\eta_{j_1})^{\otimes\theta}\right|^2 \right\} \to 0,$$

as $n\to\infty$, since for $\theta = \{\{i_1,i_2\},\dots,\{i_{k-1},i_k\}\}\in\Lambda^k_{k/2}$, we have

$$(\eta_{j_k},\dots,\eta_{j_1})^{\otimes\theta} = [\eta_{j_{i_1}},\eta_{j_{i_2}}]\otimes\cdots\otimes[\eta_{j_{i_{k-1}}},\eta_{j_{i_k}}],$$

which implies that

$$\sum_{j_1,\dots,j_k=1}^{\infty} \left| f^{(k/2)}(e)(\eta_{j_k},\dots,\eta_{j_1})^{\otimes\theta}\right|^2$$
$$\leq \left| f^{(k/2)}(e)\right|^2 \sum_{j_1,\dots,j_k=1}^{\infty} \|[\eta_{j_{i_1}},\eta_{j_{i_2}}]\|^2_{\mathbf{C}}\cdots\|[\eta_{j_{i_{k-1}}},\eta_{j_{i_k}}]\|^2_{\mathbf{C}}$$
$$= \left| f^{(k/2)}(e)\right|^2 \|\omega\|^n_{HS} < \infty,$$

again by Propositions 2.33 and 2.5. □

The following proposition completes the proof of Theorem 4.3.

Proposition 4.6. *If $f\in\mathcal{H}^2_{t,\mathrm{fin}}(G_{CM})$ as defined in Corollary 3.20, then $f\circ\pi_n\in\mathcal{P}$ for all $n\in\mathbb{N}$ and $f\circ\pi_n|_{G_{CM}}\to f$ in $\mathcal{H}^2_t(G_{CM})$.*

Proof. Suppose $m\in\mathbb{N}$ is chosen so that $\hat{f}_k(e) = 0$ if $k > m$. Comparing equations (2.6) and (4.2), one may determine that, for $h_1,\dots,h_k\in\mathfrak{g}_{CM}$,

$$\left\langle \left(\widehat{f\circ\pi_n}\right)(e), h_k\otimes\cdots\otimes h_1\right\rangle = \left\langle \hat{f}(e), \kappa^n_k(h_k,\dots,h_1)\right\rangle, \tag{4.6}$$

where κ^n_k is defined as follows: for $h_i = (A_i, a_i)$,

$$\kappa^n_k(h_k,\dots,h_1) := \sum_{j=\lfloor k/2\rfloor}^{k} \sum_{\theta\in\Lambda^k_{k-j}} \Gamma^{\otimes\theta}_{P_n}(h_k,\cdots,h_1),$$

where, for $\theta = \{\{i_1,i_2\},\dots,\{i_{2\ell-1},i_{2\ell}\},\{i_{2\ell+1}\},\dots,\{i_k\}\}\in\Lambda^k_\ell$,

$$\Gamma^{\otimes\theta}_{P_n}(h_k,\dots,h_1) := \Gamma_{P_n}(A_{i_1},A_{i_2})\otimes\cdots\otimes\Gamma_{P_n}(A_{i_{2\ell-1}},A_{i_{2\ell}})\otimes\pi h_{i_{2\ell+1}}\otimes\cdots\otimes\pi h_{i_k},$$

and $\Gamma_P(A_i,A_j) = \frac{1}{2}([h_i,h_j] - [\pi h_i,\pi h_j])$ as in equation (4.1). Alternatively, one may consult Section 7.2 of [8] for a direct derivation of κ^n_k and equation (4.6) (in this reference, our $\kappa^n_k(h_k,\dots,h_1)$ is just $\kappa_k(e)$).

By definition, $\kappa_k^n(h_k,\dots,h_1) \in \bigoplus_{j=\lceil k/2\rceil}^k \mathfrak{g}_{CM}^{\otimes j}$ and so (4.6) implies that $\left\langle (\widehat{f\circ\pi_n})(e), h_k\otimes\cdots\otimes h_1\right\rangle = 0$ when $k \geq 2m+2$. Therefore, $f\circ\pi_n$ restricted to $G_n = P_nH\times\mathbf{C}$ is a holomorphic polynomial, and, since $f\circ\pi_n = (f\circ\pi_n)|_{G_n}\circ\pi_n$, it follows that $f\circ\pi_n\in\mathcal{P}$.

Moreover,

$$\lim_{n\to\infty}\left\|\hat{f}(e) - \left(\widehat{f\circ\pi_n}\right)(e)\right\|_t^2 = \lim_{n\to\infty}\sum_{k=0}^{2m+2}\frac{t^k}{k!}\left\|\hat{f}_k(e) - \left(\widehat{f\circ\pi_n}\right)_k(e)\right\|_k^2 = 0,$$

since Proposition 4.5 implies that $\lim_{n\to\infty}\left\|\hat{f}_k(e) - \left(\widehat{f\circ\pi_n}\right)_k(e)\right\|_k = 0$ for each k. Thus, by Proposition 3.13,

$$\lim_{n\to\infty}\|f - f\circ\pi_n\|_{\mathcal{H}_t^2(G_{CM})} = \lim_{n\to\infty}\left\|\hat{f}(e) - \left(\widehat{f\circ\pi_n}\right)(e)\right\|_t = 0.$$

□

4.2. **Construction and proof of restriction isomorphism.** Before we construct the restriction map, we require some preliminary estimates. Again, we let $\{\eta_j\}_{j=1}^\infty = \{(\xi_j,0)\}_{j=1}^\infty \subset H_*\times\{0\}$, $\{P_n\}_{n=1}^\infty\subset \operatorname{Proj}(W)$, and $\pi_n : G\to G_n$ be as in Notation 2.20. Also, for $f: G\to\mathbb{C}$ or $f: G_{CM}\to\mathbb{C}$, let

$$\|f\|_{L^2(\nu_t^n)}^2 := \|f|_{G_n}\|_{L^2(\nu_t^n)}^2 = \mathbb{E}|f(g_t^n)|^2,$$

where $\{g_t^n\}_{t\geq 0}\subset G_n\subset G_{CM}\subset G$ is a Brownian motion on G_n as in Proposition 2.25.

First we show that these norms are increasing in n (for sufficiently large n). A similar result was proved in [14, Lemma 4.1].

Lemma 4.7. *Suppose $f: G\to\mathbb{C}$ is a continuous function such that $f|_{G_n}\in\mathcal{H}(G_n)$ for all $n\in\mathbb{N}$. Then $\|f\|_{L^2(\nu_t^n)}\leq\|f\|_{L^2(\nu_t^{n+1})}$ for all large enough $n\in\mathbb{N}$.*

Proof. For each $n\in\mathbb{N}$, let $D_n = D_{P_n}^k$ where $D_{P_n}^k$ is as defined in Notation 3.11. By the Taylor isomorphism for subelliptic heat kernels on finite dimension Lie groups stated in Theorem 3.12,

$$\|f\|_{L^2(\nu_t^n)} = \|\hat{f}(e)\|_{n,t},$$

where we recall that

$$\|\hat{f}(e)\|_{n,t}^2 = \sum_{k=0}^\infty\frac{t^k}{k!}\|(D_n^k f(e)\|_{n,k}^2,$$

for all n sufficiently large that $[P_nW, P_nW] = \mathbf{C}$. Observing that, for each such $n\in\mathbb{N}$ and $k\in\{0\}\cup\mathbb{N}$,

$$\begin{aligned}\|(D_n^k f)(e)\|_{n,k}^2 &= \sum_{j_1,\dots,j_k=1}^n|\langle(D_n^k f)(e),\eta_{j_1}\otimes\cdots\otimes\eta_{j_k}\rangle|^2\\ &= \sum_{j_1,\dots,j_k=1}^n|\tilde{\eta}_{j_1}\cdots\tilde{\eta}_{j_k}f(e)|^2 \leq \sum_{j_1,\dots,j_k=1}^{n+1}|\tilde{\eta}_{j_1}\cdots\tilde{\eta}_{j_k}f(e)|^2\\ &= \sum_{j_1,\dots,j_k=1}^{n+1}|\langle(D_{n+1}^k f)(e),\eta_{j_1}\otimes\cdots\otimes\eta_{j_k}\rangle|^2 = \|(D_{n+1}^k f)(e)\|_{n+1,k}^2,\end{aligned}$$

completes the proof. □

Lemma 4.8. *For any continuous function $f : G \to \mathbb{C}$ such that $f|_{G_{CM}} \in \mathcal{H}(G_{CM})$,*

$$\|f\|_{L^2(\nu_t)} \leq \|f|_{G_{CM}}\|_{\mathcal{H}_t^2(G_{CM})}.$$

Proof. First, note that, if $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ such that $P_n|_H \uparrow I_H$, then Proposition 2.25 implies that (passing to a subsequence if necessary) $g_t^n \to g_t$ almost surely. Thus,

$$\|f\|_{L^2(\nu_t)} \leq \sup_n \|f\|_{L^2(\nu_t^n)} \leq \|f|_{G_{CM}}\|_{\mathcal{H}_t^2(G_{CM})},$$

where the first inequality holds by Fatou's lemma and the second by the definition of $\|\cdot\|_{\mathcal{H}_t^2(G_{CM})}$. □

Remark 4.9. Of course this lemma holds for any $p \in [1, \infty)$, for $\mathcal{H}_t^p(G)$ defined analogously to $\mathcal{H}_t^2(G)$ in Definition 4.1.

Corollary 4.10. *Let $\delta > 0$ be as in Proposition 2.28, and suppose that $f : G \to \mathbb{C}$ is a continuous function such that, for some $\varepsilon \in (0, \delta)$,*

$$|f(g)| \leq Ce^{\varepsilon\|g\|_{\mathfrak{g}}^2/2t},$$

for all $g \in G$. Then

$$\|f\|_{L^2(\nu_t^n)} \uparrow \|f\|_{L^2(\nu_t)}.$$

(In particular, this implies that $\|f\|_{L^2(\nu_t^P)} \leq \|f\|_{L^2(\nu_t)}$ for any $P \in \mathrm{Proj(W)}$.) Also, if $f|_{G_{CM}} \in \mathcal{H}(G_{CM})$, then

$$\|f\|_{L^2(\nu_t)} = \|f|_{G_{CM}}\|_{\mathcal{H}_t^2(G_{CM})}. \tag{4.7}$$

Proof. First, Lemma 4.7 implies that $\{\|f\|_{L^2(\nu_t^n)}\}_{n=1}^\infty$ is an increasing sequence. Proposition 2.29 implies that $f \in L^2(\nu_t)$, and taking $h = e$ in equation (2.17) or equation (2.18) shows that the sequence must be increasing to $\|f\|_{L^2(\nu_t)}$. This combined with Lemma 4.8 gives (4.7). □

Lemma 4.11. *Suppose $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_n} \in \mathcal{H}L^2(\nu_t^n)$ for all $n \in \mathbb{N}$. Then, for all $g \in G_{CM}$,*

$$|f(g)| \leq \|f\|_{L^2(\nu_t)} e^{d_h(e,g)^2/2t}.$$

Proof. Let $g = (w, c) \in G_m$, and consider an arbitrary horizontal path $\sigma : [0, 1] \to G_{CM}$ such that $\sigma(0) = e$ and $\sigma(1) = g$. Recall that, by Remark 2.15, σ must have the form

$$\sigma(t) = \left(A(t), \frac{1}{2}\int_0^t \omega(A(s), \dot{A}(s))\, ds\right).$$

For $n \geq m$, consider the "projected" horizontal paths $\sigma_n : [0, 1] \to G_n$ given by

$$\sigma_n(t) = (A_n(t), a_n(t)) := \left(P_nA(t), \frac{1}{2}\int_0^t \omega(P_nA(s), P_n\dot{A}(s))\, ds\right).$$

Note that $A_n(1) = P_nA(1) = P_nw = w$, and let

$$\varepsilon_n := c - a_n(1) = c - \frac{1}{2}\int_0^1 \omega(P_nA(s), P_n\dot{A}(s))\, ds \in \mathbf{C}.$$

Then, for d_n the horizontal distance in G_n,

$$\begin{aligned} d_n(e, g) &= d_n(e, (w, c)) = d_n(e, (w, a_n(1) + \varepsilon_n)) = d_n(e, (w, a_n(1)) \cdot (0, \varepsilon_n)) \\ &\leq d_n(e, (w, a_n(1))) + d_n(e, (0, \varepsilon_n)) \\ &\leq \ell(\sigma_n) + C\sqrt{\|\varepsilon_n\|_{\mathbf{C}}}, \qquad (4.8) \end{aligned}$$

where the first inequality holds by (2.10) and the second inequality holds by (2.13), with constant $C = C(N, \omega)$. Note that (2.12) technically gives only a bound for d_h on G_{CM}; however, it is clear from the proof of this bound that one may find a constant C so that (2.13) holds for all sufficiently large n with the constant C not depending on n.

Now consider a continuous function $f : G \to \mathbb{C}$ such that $f|_{G_n} \in \mathcal{H}L^2(\nu_t^n)$ for all $n \in \mathbb{N}$. For $n \geq m$, $g \in G_m \subset G_n$. Then, for n sufficiently large that $[P_nW, P_nW] = \mathbf{C}$, Theorem 3.12 (in particular (3.1)), Corollary 4.10, and (4.8) imply that

$$|f(g)| \leq \|f\|_{L^2(\nu_t^n)} e^{d_n(e,g)^2/2t} \leq \|f\|_{L^2(\nu_t)} e^{(\ell(\sigma_n)+C\sqrt{\|\varepsilon_n\|_{\mathbf{C}}})^2/2t}. \tag{4.9}$$

One may then show via dominated convergence that

$$\lim_{n\to\infty} \ell(\sigma_n) = \lim_{n\to\infty} \int_0^1 \|P_n\dot{A}(s)\|\, ds = \int_0^1 \|\dot{A}(s)\|\, ds = \ell(\sigma),$$

and that

$$\lim_{n\to\infty} \|\varepsilon_n\|_{\mathbf{C}} = \lim_{n\to\infty} \left\| \frac{1}{2}\int_0^1 \omega(A(s), \dot{A}(s)) - \omega(P_nA(s), P_n\dot{A}(s))\, ds \right\|_{\mathbf{C}} = 0.$$

Thus, passing to the limit in (4.9) as $n \to \infty$ gives

$$|f(g)| \leq \|f\|_{L^2(\nu_t)} e^{\ell(\sigma)^2/2t},$$

and taking the infimum over all horizontal paths σ such that $\sigma(0) = e$ and $\sigma(1) = g$ completes the proof for all $g \in \cup_P G_P$. Since both sides of the inequality are continuous in $g \in G_{CM}$ and $\cup_P G_P$ is dense in G_{CM} by Proposition 2.18, this is sufficient to prove the bound for all $g \in G_{CM}$. □

Notation 4.12. *For $g \in G_{CM}$, define the linear map $R_g : \mathcal{P} \to \mathbb{C}$ by*

$$R_g f := f(g).$$

Proposition 4.13. *For all $g \in G_{CM}$, R_g can be extended uniquely to a continuous linear functional on all of $\mathcal{H}_t^2(G)$ satisfying*

$$|R_g f| \leq \|f\|_{L^2(\nu_t)} e^{d_h(e,g)^2/2t}. \tag{4.10}$$

Proof. Lemma 4.11 implies that (4.10) holds for $f \in \mathcal{P}$ and $g \in G_{CM}$. Thus, $\|R_g\|_{op} \leq e^{d_h(e,g)^2/2t}$ as an operator on $\mathcal{P} \subset L^2(\nu_t)$, and R_g is continuous and defined on a dense subset of $\mathcal{H}_t^2(G)$. Thus, there exists a unique extension of R_g to $\mathcal{H}_t^2(G)$ (still denoted by R_g) so that (4.10) is satisfied for all $f \in \mathcal{H}_t^2(G)$. To define R_g for an arbitrary $f \in \mathcal{H}_t^2(G)$, let $\{f_j\}_{j=1}^\infty \subset \mathcal{P}$ such that $f_j \to f$ in $L^2(\nu_t)$ and define $R_g f := \lim_{j\to\infty} R_g f_j$. □

Remark 4.14. The estimate in (4.10) implies that, if $f_j \to f$ in $L^2(\nu_t)$, then, for any $g \in G_{CM}$, $R_g f_j \to R_g f$ and the convergence is locally uniform.

Theorem 4.15. *There exists a linear map $R : \mathcal{H}_t^2(G) \to \mathcal{H}(G_{CM})$ with the following properties:*

1. *For any $f \in \mathcal{P}$, $Rf = f|_{G_{CM}}$.*
2. *For $g \in G_{CM}$, $|(Rf)(g)| \leq \|f\|_{L^2(\nu_t)} e^{d_h^2(e,g)/2t}$.*

Proof. Given $f \in \mathcal{H}_t^2(G)$, we define Rf by $(Rf)(g) := R_g f$ for all $g \in G_{CM}$. Items (1) and (2) are satisfied by definition of R_g and Proposition 4.13.

To see that $Rf \in \mathcal{H}(G_{CM})$, first consider $f \in \mathcal{P}$. Then $f = F \circ \pi_P$ for some $P \in \mathrm{Proj}(W)$ and polynomial $F \in \mathcal{H}(G_P)$. By Proposition 2.34, $h \mapsto f(g \cdot e^h)$ is Frechét differentiable at $h = 0$ and this derivative is continuous with respect to g.

For general $f \in \mathcal{H}_t^2(G)$, fix $g \in G_{CM}$ and choose $\{f_j\}_{j=1}^{\infty} \subset \mathcal{P}$ such that $f_j \to f$ in $L^2(\nu_t)$. Then

$$|(Rf_j)(g) - (Rf)(g)| = |R_g(f_j - f)| \leq \|f_j - f\|_{L^2(\nu_t)} e^{d_h^2(e,g)/2t},$$

and so Rf is the pointwise limit of $Rf_j = f_j|_{G_{CM}} \in \mathcal{H}(G_{CM})$ with the limit being uniform over any bounded subset of g's contained in G_{CM}. By Theorem 3.18.1 of [20], this is sufficient to imply that $Rf \in \mathcal{H}(G_{CM})$. □

Theorem 4.16. *The map $R : \mathcal{H}_t^2(G) \to \mathcal{H}_t^2(G_{CM})$ is unitary.*

Proof. Given $f \in \mathcal{P}$, Corollary 4.10 implies that $\|Rf\|_{\mathcal{H}_t^2(G_{CM})} = \|f\|_{L^2(\nu_t)}$. Therefore, $R|_{\mathcal{P}}$ extends to an isometry, still denoted by R, from $\mathcal{H}_t^2(G)$ to $\mathcal{H}_t^2(G_{CM})$ such that $R(\mathcal{P}) = \mathcal{P}_{CM}$. Since R is isometric and $\mathcal{P}_{CM}$ is dense in $\mathcal{H}_t^2(G_{CM})$ by Theorem 4.3, it follows that R is surjective. □

Corollary 4.17. *Suppose $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_{CM}} \in \mathcal{H}_t^2(G_{CM})$. Then $f \in \mathcal{H}_t^2(G)$ and $\|f\|_{L^2(\nu_t)} = \|f|_{G_{CM}}\|_{\mathcal{H}_t^2(G_{CM})}$.*

Proof. By Theorem 4.16, there exists $u \in \mathcal{H}_t^2(G)$ such that $Ru = f|_{G_{CM}}$. Let $p_n \in \mathcal{P}$ be chosen so that $p_n \to u$ in $L^2(\nu_t)$. Then $p_n|_{G_{CM}} = Rp_n \to Ru = f|_{G_{CM}}$ in $\mathcal{H}_t^2(G_{CM})$, and, by Lemma 4.8,

$$\|f - p_n\|_{L^2(\nu_t)} \leq \|(f - p_n)|_{G_{CM}}\|_{\mathcal{H}_t^2(G_{CM})}.$$

Thus, $p_n \to f$ in $L^2(\nu_t)$, and since $p_n \to u$ in $L^2(\nu_t)$ also, it must be that $f = u \in \mathcal{H}_t^2(G)$. □

Corollary 4.17 along with Corollary 4.10 immediately give the following. In particular, this result states that, under the assumptions of Corollary 4.10, $f \in \mathcal{H}_t^2(G)$.

Corollary 4.18. *Let $\delta > 0$ be as in Proposition 2.28, and suppose that $f : G \to \mathbb{C}$ is a continuous function such that $f|_{G_{CM}} \in \mathcal{H}(G_{CM})$ and, for some $\varepsilon \in (0, \delta)$,*

$$|f(g)| \leq Ce^{\varepsilon \|g\|_{\mathfrak{g}}/2t},$$

for all $g \in G$. Then $f \in \mathcal{H}_t^2(G)$ and $\|f\|_{L^2(\nu_t)} = \|f|_{G_{CM}}\|_{\mathcal{H}_t^2(G_{CM})}$.

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