Evolution Systems of Measures for Non-autonomous Stochastic Differential Equations with Lévy Noise

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We examine the question of existence and uniqueness of evolution systems of measures for non-autonomous Ornstein-Uhlenbeck-type processes with jumps. In particular, we give examples where we explicitly compute the densities of such families of measures.

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Evolution Systems of Measures for Non-autonomous Stochastic Differential Equations with Lévy Noise

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Dedicated to mom, my first math teacher, and dad, who never let me win and never let me quit.

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Introduction

1.1 Notation

Throughout the thesis we will use the following notation:

- A^T denotes the transpose of a matrix A.
- $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field on $\mathbb{R}^d$.
- $B_r(a) = \{x \in R^d : |x - a| < r\}$ denotes the ball of radius r centered at $a \in \mathbb{R}^d$.
- $B_b(\mathbb{R}^d)$ denotes the space of all bounded Borel functions from $\mathbb{R}^d$ to $\mathbb{R}$.
- $C_b(\mathbb{R}^d)$ denotes the space of all bounded continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$.
- $\mathcal{L}(\mathbb{R}^d)$ denotes the space of all $d \times d$ matrices.
- $\mu * \nu$ denotes the convolution of two Borel probability measures on $\mathbb{R}^d$,

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A - x)\nu(dx),$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$.

- $a \wedge b = \min\{a, b\}$.

- $X \overset{d}{=} Y$: X and Y are equal in distribution.

- $X \sim \mathcal{N}(\mu, R)$: X is Gaussian with mean μ and covariance matrix R.

- $\hat{\mu}$ denote the Fourier transform of the probability measure μ on $\mathbb{R}(d)$,

$$\hat{\mu}(a) = \int_{\mathbb{R}^d} e^{i\langle a,x\rangle} \mu(dx), \quad a \in \mathbb{R}^d.$$

1.2 Introduction

Evolution systems of measures arise in place of invariant measures when looking at time-dependent stochastic differential equations. The main topics of thesis concern existence and uniqueness of evolution systems of measures for time-dependent Ornstein-Uhlenbeck-type stochastic differential equations with Lévy noise and in particular, examples where the noise term is a symmetric stable Lévy process. Unlike mathematical Brownain motion, Lévy processes in general do not have continuous sample paths (although Brownian motion is an example of a Lévy process). Allowing for jumps makes Lévy processes much better for modeling phenomenon with high variability, where quantities can change quickly, and the interest in Lévy processes has grown significantly in the last twenty years or so. Applications of jump processes occur in finance, the physical sciences, and engineering.

In Chapter 2 we review some basic facts and properties of Lévy processes and give some examples, including the stable Lévy processes. In addition to offering more flexibility in modeling, another advantage in working with Lévy processes is that their characteristic functions have an especially useful form, which is given by the Lévy-Khintchine formula (Theorem 2.3.5).

In Chapter 3 we give some background information on the Ornstein-Uhlenbeck process. This process was introduced in 1930 by Leonard Ornstein and George Eugene Uhlenbeck as a model of the velocity of physical Brownian motion, see [6]. We then consider the stochastic initial value problem

$$\begin{aligned} dX(t) &= \lambda(\mu - X(t-))dt + \sigma dZ(t). \\ X(0) &= x, \end{aligned} \tag{1.2.1}$$

where $\lambda > 0, \mu \in \mathbb{R}, \sigma \in \mathbb{R}$, and Z is a 1-dimensional Lévy process. The solution to (1.2.1) is often referred to as the *mean-reverting Ornstein-Uhlebeck process.* In particular, we state some of the well-known properties of the solution when Z is a Brownian motion, including computing its law and its invariant measure. We make a similar investigation when Z is a symmetric stable Lévy process. The important thing, as far as this thesis is concerned, is that the solution to (1.2.1) is *time-homogeneous.* The purpose here is meant to highlight the differences between the autonomous case and the time-dependent case, which is investigated in Chapter 4.

In Chapter 4 we consider a time-dependent, d-dimensional version of (1.2.1),

$$dX(t) = (A(t)X(t-) + f(t))\,dt + B(t)dZ(t)$$
$$X(s) = x, \tag{1.2.2}$$

where $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d), B : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d), f : \mathbb{R} \to \mathbb{R}^d$ are all bounded and continuous, Z is a d-dimensional Lévy process, $x \in \mathbb{R}^d$, and $s \in \mathbb{R}$. Since the solution to (1.2.2) is not time-homogeneous, we cannot expect to be able to find a single invariant measure. Instead we define the notion of an evolution system of measures.

The main result of this thesis is Theorem 4.3.8, where we give conditions in which there exists a unique evolution system of measures. To do this we make a stability assumption on the evolution family, $U(t, s)$, associated to $A(t)$. That is, $U(t, s)$ solves the matrix differential equation

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s),$$
$$U(s,s) = I,$$

where $s, t \in \mathbb{R}$ and I is the identity operator. Our assumption is that

$$||U(t,s)|| \leq Ce^{-\epsilon(t-s)}, \qquad s,t \in \mathbb{R}.$$

Previous work in this area was done by Da Prado and Lunardi in 2007, see [3]. In their paper Da Prado and Lunardi made the assumption that the coefficients A, B, f were T-periodic and that the noise term was a Brownian motion. We do not impose such restrictions in this thesis, although the result here coincides with

Da Prado and Lunardi's result in that particular case. By not assuming periodicity and allowing for jump processes makes our proof much harder. We make extensive use of the Lévy-Khintchine formula and the Lévy-Itô decomposition.

To prove Theorem 4.3.8 we use the theory of probability measures in metric spaces, particularly from Parthasarthy's book, [7]. In [4], Fuhrman and Röckner proved a theorem about existence and uniqueness of an invariant measure for a semigroup arising from an autonomous Ornstein-Uhlenbeck process. We use similar techniques here, however there are some significant differences in our proof than in the autonomous case.

We conclude the thesis with some examples were we compute the densities and characteristic functions of evolutions systems of measures for the solution to a 1-dimensional version of (1.2.2) to which Theorem 4.3.8 applies. We return to the cases where Z is Brownian motion and where Z is a stable process.

Chapter 2

Lévy processes

2.1 Definition of a Lévy process

This chapter consists of a brief introduction to Lévy processes. For a more detailed treatment, see [1].

We fix without further mention a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual hypothesis. That is, $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$ and $\mathcal{F}_0$ is complete. A stochastic process is a collection of random variables $\{X_t(\omega), t \geq 0\}$ taking values in $\mathbb{R}^d$. We will often write the process $X_t(\omega)$ as $X_t, X(t)$, or even just X. A process X is said to be *adapted* if $X(t) \in \mathcal{F}_t$ for each t. A *sample path* of a stochastic process X is a function, $\omega(t)$, from $[0, \infty) \to \mathbb{R}^d$, where

$$\omega(t) := X_t(\omega),$$

for fixed $\omega \in \Omega$.

We now state the definition of a Lévy process.

Definition 2.1.1. A stochastic process $\{X_t, t \geq 0\}$ is a *Lévy process* if

(i) $X(0) = 0$ a.s.

(ii) X has *independent increments*, i.e. for each $n \in \mathbb{N}$, and each $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1} < \infty$ the random variables $(X_{t_{j+1}} - X_{t_j}, 1 \leq j \leq n)$ are indepedent for $1 \leq j \leq n$.

(iii) X has *stationary increments*, i.e. for each $n \in \mathbb{N}$, and each $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1} < \infty$, the random variables $X_{t_{j+1}} - X_{t_j} \stackrel{d}{=} X_{t_{j+1}-t_j} - X(0)$.

(iv) X is *stochastically continuous*, i.e. for all $a > 0$ and $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(|X_t - X_s| > a) = 0.$$

It is well-known (see e.g. [8]) that a Lévy process X has a unique càdlàg version (right-continuous with left limits) which is also a Lévy process. Therefore we assume, with no loss of generality, that every Lévy process is càdlàg.

2.2 Examples of Lévy processes

We now give some examples of Lévy processes.

Example 2.2.1 (Gaussian processes). A *standard Brownian motion* in $\mathbb{R}^d$ is a Lévy process $B(t)$ which satisfies

(i) $B(t) \sim \mathcal{N}(0, tI)$ for each $t \geq 0$,

(ii) B has continuous sample paths a.s.

More generally let R be a nonnegative-definite symmetric $d \times d$ matrix and let σ be a square root of R. The Lévy process

$$B_R(t) := \sigma B(t),$$

(where B is a standard Brownian motion) is called a *Brownian motion with covariance R*.

The process

$$Y(t) = bt + B_R(t)$$

is a *Gaussian Lévy process* with drift $b \in \mathbb{R}^d$ and covariance R.

Gaussian processes are the only Lévy processes with continuous sample paths.

Example 2.2.2 (Poisson process). A *Poisson process* is a Lévy process N taking values in $\mathbb{N} \cup \{0\}$, where for each $t \geq 0$,

$$\mathbb{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

λ is a positive constant, which we call the *intensity* of the process.

Let $T_0 = 0$, and define

$$T_n := \inf\{t \geq 0; N(t) = n\}, n \in \mathbb{N}.$$

It is well known that the T_n are gamma distributed, and are often called the *waiting times of the Poisson process N*.

The *inter-arrival times*, $T_n - T_{n-1}$, are i.i.d. exponentially distributed, with mean $1/\lambda$.

Example 2.2.3 (Compound Poisson process). Let Y_n, $n \in \mathbb{N}$ be a sequence of $\mathbb{R}^d$-valued i.i.d. random variables with law μ_Y and let N be a Poisson process with intensity λ that is independent of all the Y_n. A *compound Poisson process* is defined

$$X(t) = Y_1 + \cdots + Y_{N(t)}$$

for each $t \geq 0$.

Example 2.2.4 (α-stable process). A real-valued random variable X has an *α-stable distribution* if, for some $\alpha \in (0, 2]$, its characteristic function is of the form

$$\phi_X(a) = \begin{cases} \exp\left\{i\mu a - \sigma^\alpha |a|^\alpha \left[1 - \beta \operatorname{sgn}(a) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right\} & \alpha \neq 1 \\ \\ \exp\left\{i\mu a - \sigma|a| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(a) \log|a|\right]\right\} & \alpha = 1, \end{cases}$$

where $\sigma > 0, -1 \leq \beta \leq 1, \mu \in \mathbb{R}$. The constant α is called the *index of stability.* If $\alpha = 2$, the random variable is Gaussian and has finite moments of all orders. If $\alpha \in (1, 2)$ it has a finite mean but infinite variance. Finally for $\alpha \in (0, 1]$, an α-stable random variable has infinite mean and variance. For this reason, α-stable random variables are useful for describing phenomenon with large deviations. We can only write the density function in closed form for an α-stable random variable for the following values of α:

- $\alpha = 2$: $X \sim \mathcal{N}(\mu, 2\sigma^2)$, i.e. X has density function

$$d_X(x) = \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-\frac{(x-\mu)^2}{2(2\sigma^2)}}.$$

- $\alpha = 1, \beta = 0$: X has a *Cauchy distribution* with density function

$$d_X(x) = \frac{\sigma}{\pi[(x-\mu)^2 + \sigma^2]}.$$

- $\alpha = 1/2, \beta = 1$: X has a *Lévy distribution* with density function

$$d_X(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left[-\frac{\sigma}{2(x-\mu)}\right], \text{for } x > \mu.$$

A *stable Lévy process* is a Lévy process where for each fixed t, $X(t)$ is a stable random variable. We are particularly interested in symmetric, stable Lévy processes, i.e. stable processes where $\mu = \beta = 0$. Such a process has characteristic function

$$\phi_{X(t)}(a) = \exp\left(-t\sigma^\alpha |a|^\alpha\right).$$

2.3 Important properties of Lévy processes

In this section we state some of the well-known properties of Lévy processes. The two most useful of which are the Lévy-Itô decomposition (Theorem 2.3.1) and the Lévy-Khintchine formula (Theorem 2.3.4). The Lévy-Itô decomposition essentially states that a Lévy process is the sum of a deterministic drift term, a Brownian motion, and a jump part governed by an independent Poisson integral. The Lévy-Khintchine formula gives the characteristic function of a Lévy process in terms of three parameters, (b, R, M), known as the *triple of the process.* This triple uniquely determines the process.

Theorem 2.3.1 (Lévy-Itô decomposition [1]). Let X be a Lévy process. Then there exists a $b \in \mathbb{R}^d$, a Brownian motion B_R with covariance matrix R, and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ such that X has the decomposition

$$X(t) = bt + B_R(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx),$$

for each $t \geq 0$.

The drift term, bt, and Poisson integral that controls the large jumps, $\int_{|x|\geq 1} xN(t, dx)$, are processes of bounded variation on compact sets. The Brownian part, $B_R(t)$, and the compensated Poisson integral, $\int_{|x|\geq 1} x\tilde{N}(t, dx)$, are martingales. Therefore we have the following corollary which makes Lévy processes good candidates for stochastic integrators, see [8].

Corollary 2.3.2. Every Lévy process is a semimartingale.

Definition 2.3.3. A Borel measure M on $\mathbb{R}^d$ is a *Lévy measure* if $M(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) M(dy) < \infty.$$

We now define infinitely divisible probability distributions. Such distributions have an especially useful form for their characteristic functions, given by the Lévy-Khintchine formula.

A random variable X has an *infinitely divisible distribution* if for any $n \in \mathbb{N}$ there exists i.i.d. random variables $Y_1, ..., Y_n$ such that

$$X \stackrel{d}{=} Y_1 + \cdots + Y_n.$$

Theorem 2.3.4 (Lévy-Khintchine formula [1]). Let X be an infinitely divisible random variable. Then there exists a $b \in \mathbb{R}^d$, a non-negative definite symmetric matrix R, and a Lévy measure M such that for all $a \in \mathbb{R}^d$,

$$\mathbb{E}e^{i\langle a,X\rangle} = \exp\left\{ i\langle b, a\rangle - \frac{1}{2}\langle a, Ra\rangle + \int_{\mathbb{R}^d} \left[e^{i\langle a,y\rangle} - 1 - \frac{i\langle a, y\rangle}{1+|y|^2} \right] M(dy) \right\}. \quad (2.3.1)$$

Conversely, any mapping of the form (2.3.1) is the characteristic function of an infinitely divisible distribution.

Theorem 2.3.5. Let $X(t)$ be a Lévy process. Then for each t, X_t is infinitely divisible, and

$$\phi_{X(t)}(a) = e^{-t\eta(a)}, \qquad a \in \mathbb{R},$$

where

$$\eta(a) = -i\langle b, a\rangle + \frac{1}{2}\langle a, Ra\rangle - \int_{\mathbb{R}^d} \left[e^{i\langle a,y\rangle} - 1 - \frac{i\langle a, y\rangle}{1+|y|^2} \right] M(dy),$$

for each $a \in \mathbb{R}^d$. The three-tuple (b, R, M) is called the *triple* or *characteristics* of X, while η is called the *Lévy symbol* of X.

Proof. See Theorem 1.3.3, [1]. ***Qed***

Chapter 3

The Ornstein-Uhlenbeck process and invariant measures

3.1 The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process was first introduced in 1930 by Leonard Ornstein and George Eugene Uhlenbeck as a model of physical Brownian motion [6]. They argued that the force on a particle of mass suspended in a liquid should arise from both a macroscopic frictional force and by random molecular bombardment. Using Newton's second law this can be written formally,

$$m\frac{dv}{dt} = -\lambda m v + m\frac{dB}{dt}.$$

The constant $\lambda > 0$ is related to the viscosity of the liquid, and the formal quantity '$\frac{dB}{dt}$' describes random velocity changes due to molecular bombardment.

This equation can be interpreted as the Itô stochastic differential equation

$$dv(t) = -\lambda v(t) + dB(t), \tag{3.1.1}$$

where B is a Brownian motion.

We now modify (3.1.1) by adding a drift term $\mu \in \mathbb{R}$ (which can be thought of as the velocity of the current in a moving liquid), a scaling of the strength of the random molecular bombardment $\sigma \in \mathbb{R}$, a starting point $x \in \mathbb{R}$, and by replacing the Brownian motion with a Lévy process,

Consider the stochastic initial value problem

$$\begin{aligned} dX(t) &= \lambda(\mu - X(t-))dt + \sigma dZ(t) \\ X(0) &= x. \end{aligned} \tag{3.1.2}$$

We now solve (3.1.2) by multiplying by the integrating factor $e^{\lambda}t$:

$$d(e^{\lambda t}X(t)) = \lambda e^{\lambda t}X(t) + e^{\lambda t}X(t) = e^{\lambda t}\lambda\mu \; dt + e^{\lambda t}\sigma dZ(t).$$

Integrating from 0 to t,

$$e^{\lambda t}X(t) = x + \int_0^t e^{\lambda s}\lambda\mu \; ds + \int_0^t e^{\lambda s}\sigma dZ(s),$$

we obtain the solution to (3.1.2)

$$X_x(t) = e^{-\lambda t}x + \mu(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)}\sigma dZ(s) \tag{3.1.3}$$

The subscript in $X_x(t)$ denotes the starting position. The process $X_x(t)$ is often called the *mean-reverting Ornstein-Uhlenbeck process.*

Example 3.1.1. Let $Z(t)$ be a 1-dimensional Brownian motion. The first and second moments of (3.1.3) are

$$\mathbb{E}X_x(t) = e^{-\lambda t}x + \mu(1 - e^{-\lambda t}),$$

and

$$\begin{aligned}\mathbb{E}\left(X_x(t)\right)^2 &= \left[e^{-\lambda t}x + \mu(1-e^{-\lambda t})\right]^2 \\ &\quad + 2\mathbb{E}\left[e^{-\lambda t}x + \mu(1-e^{-\lambda t})\int_0^t e^{-\lambda(t-s)}\sigma dZ(s)\right] \\ &\quad + \mathbb{E}\left(\int_0^t e^{-\lambda(t-s)}\sigma dZ(s)\right)^2 \\ &= \left[e^{-\lambda t}x + \mu(1-e^{-\lambda t})\right]^2 + \int_0^t e^{-2\lambda(t-s)}\sigma^2 ds \\ &= \left[e^{-\lambda t}x + \mu(1-e^{-\lambda t})\right]^2 + \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t}).\end{aligned}$$

Thus for each fixed t, $X_x(t)$ has a Gaussian distribution with mean

$$\mathbb{E}X_x(t) = e^{-\lambda t}x + \mu(1-e^{-\lambda t}),$$

and variance

$$VarX_x(t) = \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t}).$$

However, it is important to note that $X_x(t)$ is not a Lévy process, as $X_x(t)$ does not have stationary increments.

The following figures are of simulations of sample paths of the mean-reverting Ornstein-Uhlenbeck process done in Matlab.

Intuitively, think of the figures as graphs of the velocity of particles in different fluids. Both fluids have a velocity in a direction opposite to the initial velocity of each of the particles. The fluid in Figure 3.1 has less viscosity and stronger molecular bombardments, which result in greater variations in the motion of the particle. The fluid in Figure 3.2 has much stronger viscosity and the effect of

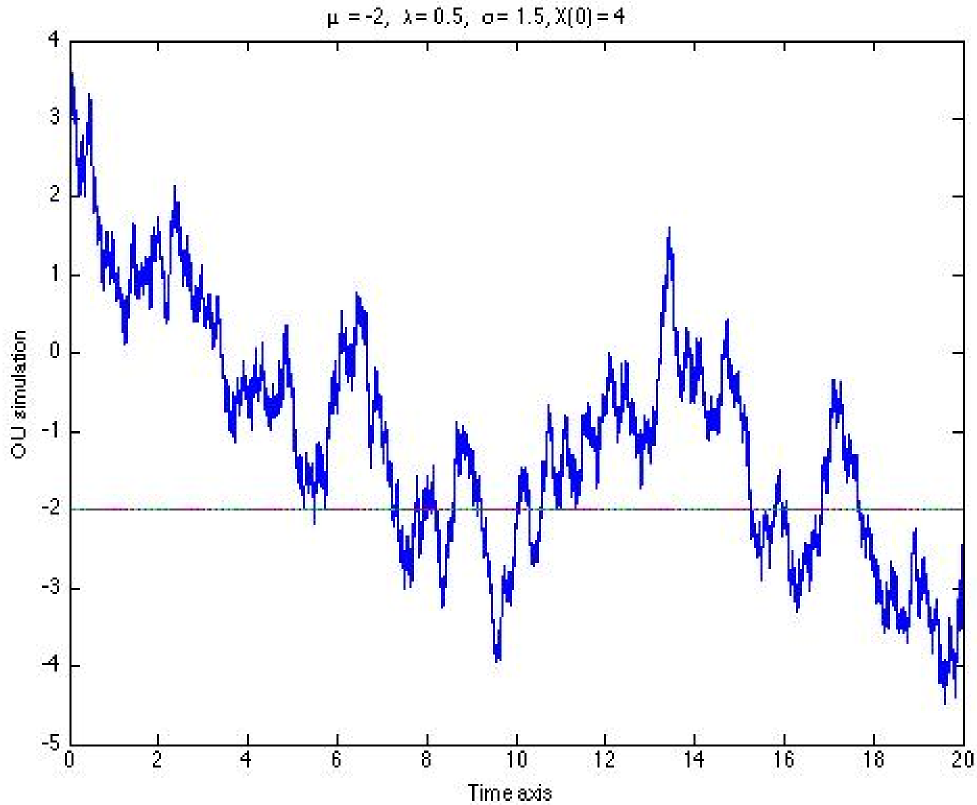

Fig. 3.1: Ornstein-Uhlenbeck sample path with small λ and large σ

molecular bombardments is weaker. This is why there are only slight differences between the velocity of the fluid and the velocity of the particle.

In Figure 3.3, we have the plots of two sample paths of the Ornstein-Uhlenbeck process that differ only in their initial value. This can be thought of as two particles in the same fluid with different initial velocities. Eventually, their initial values don't matter, and it is only the characteristics of the fluid (viscosity, strength of molecular bombardment, and velocity of the fluid) which governs the motion of the particles. Indeed note that the distribution of both particles ap-

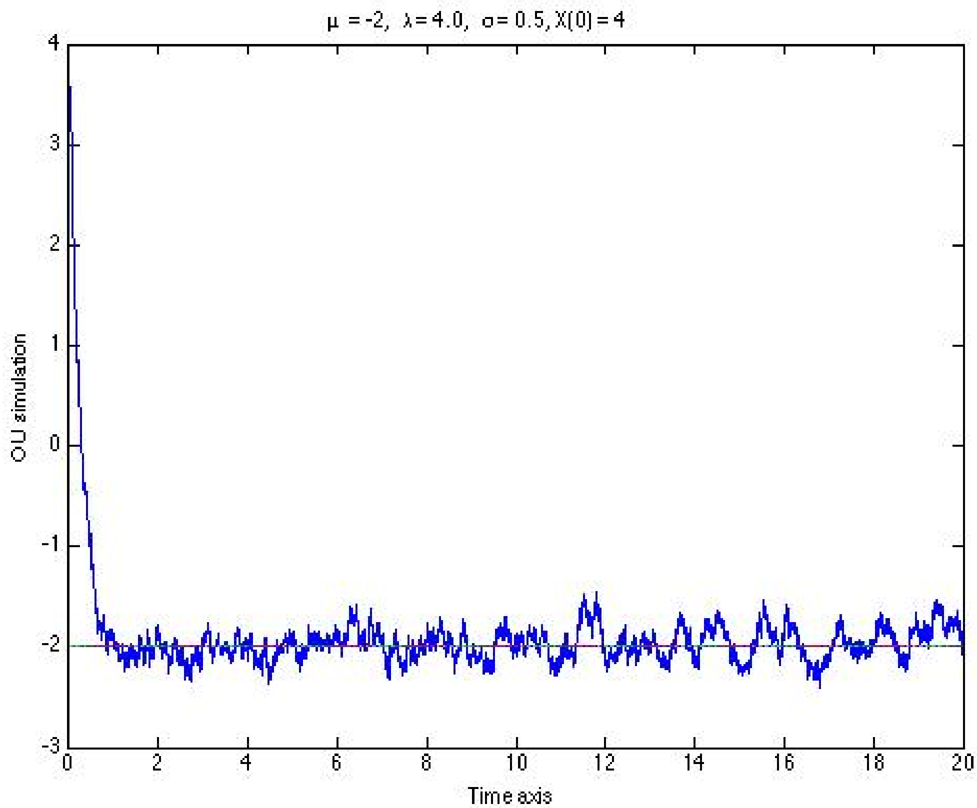

Fig. 3.2: Ornstein-Uhlenbeck sample path with large λ and small σ

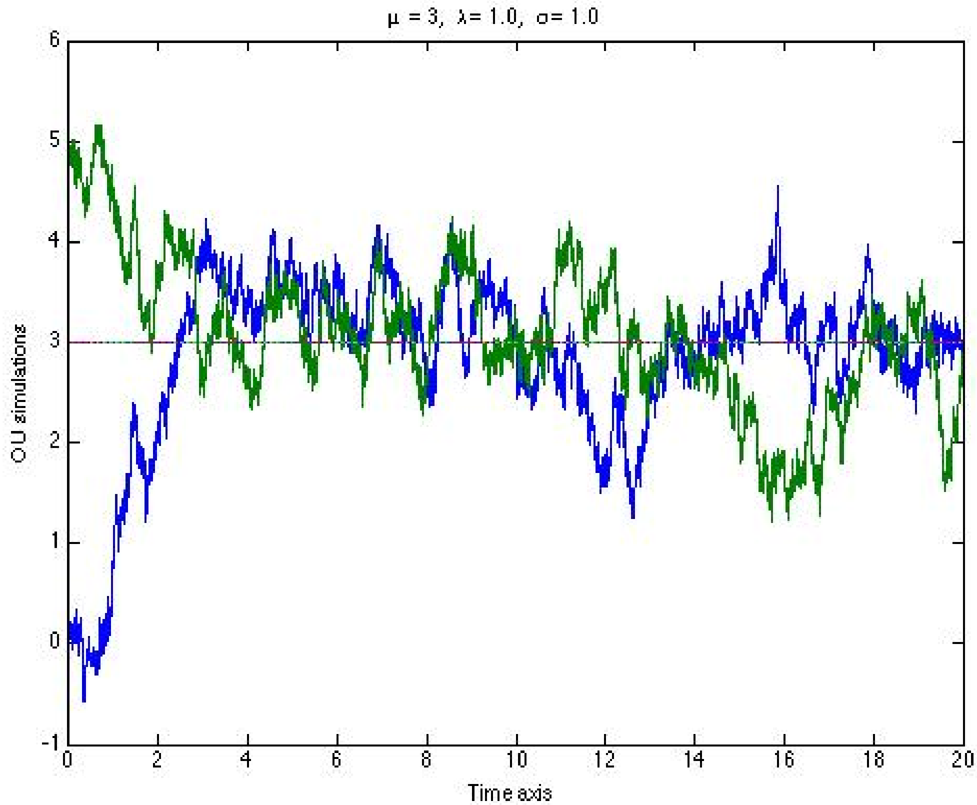

Fig. 3.3: Two Ornstein-Uhlenbeck sample paths that differ only in their initial values

pears to be approaching the same *equilibrium distribution* as time increases. This phenomenon is exactly the notion of an *invariant measure*, which is made precise in Section 3.2.

Before we consider the case where $Z(t)$ in (3.1.3) is a 1-dimensional, symmetric, α-stable process, we will need the following fact about stochastic integrals with respect to symmetric α-stable Lévy processes. A proof can be found in [9].

Proposition 3.1.2. Suppose $Z(t)$ is a 1-dimensional, symmetric, α-stable Lévy process with characteristic function

$$\phi_{Z(t)}(a) = e^{-t\sigma^\alpha |a|^\alpha}$$

Then for each fixed t, the process

$$Y(t) = \int_0^t e^{-\lambda(t-s)} dZ(s)$$

has an α-stable distribution with characteristic function

$$\phi_{Y(t)}(a) = \exp\left\{ -\frac{\sigma^\alpha}{\lambda\alpha}(1 - e^{-\lambda\alpha t})|a|^\alpha \right\}.$$

Example 3.1.3. Let $Z(t)$ be a 1-dimensional, symmetric, α-stable process. Then for each t, (3.1.3) is α-stable with characteristic function

$$\phi_{X_x(t)} = \exp\left\{ ia\left[e^{-\lambda t}x + \mu\left(1 - e^{-\lambda t}\right)\right] - \frac{\sigma^\alpha}{\lambda\alpha}\left(1 - e^{-\lambda\alpha t}\right)|a|^\alpha \right\}. \tag{3.1.4}$$

In the case where $\alpha = 1$, we can write the density function for $X_x(t)$,

$$d_{X_x(t)}(y) = \frac{(\sigma/\lambda)(1 - e^{-\lambda t})}{\pi\left\{ \left[y - (e^{-\lambda t}x + \mu(1 - e^{-\lambda t}))\right]^2 + \left[(\sigma/\lambda)(1 - e^{-\lambda t})\right]^2 \right\}}.$$

3.2 Invariant measures

Let $X(t)$ be a Markov process taking values in $\mathbb{R}^d$. The *transition semigroup of operators* associated with X is the family operators defined

$$P_{s,t}f(x) = \mathbb{E}\left[f(X(t)|X(s) = x\right],$$

for $s \leq t$ and $f \in B_b(\mathbb{R}^d)$.

If $P_{s,t} = P_{0,t-s}$ for all $s \leq t$, then X is said to be *time-homogeneous*. In this case the action of the semigroup only depends on the length of the time interval and not the actual times involved. Therefore the transition semigroup for a time-homogeneous Markov process is really a one-parameter semigroup, and in this case we write $P_{0,t}$ as P_t.

Remark 3.2.1. The solution to (3.1.2) is a time-homogeneous Markov process, see for e.g. Theorem 6.4.5, [1], which is why we assume the process starts at time 0, and not at some arbitrary time s.

Let $p_t(x, \cdot)$ denote the law of a time-homogeneous Markov process starting at x, i.e.

$$p_t(x, A) = \mathbb{P}\left[X(t) \in A|X(0) = x\right].$$

Definition 3.2.2. A Borel probability measure μ is an *invariant measure* for X if

$$\int_{\mathbb{R}^d} (P_t f)(x)\mu(dx) = \int_{\mathbb{R}^d} f(x)\mu(dx), \tag{3.2.1}$$

for all $t \geq 0$, $f \in B_b(\mathbb{R}^d)$.

Remark 3.2.3. By taking indicator functions and using a standard monotone class argument, it is not hard to show that μ is invariant for X if and only if

$$\int_{\mathbb{R}^d} p_t(x, A)\mu(dx) = \mu(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^d), x \in \mathbb{R}^d$ and $t \geq 0$.

Similarly, for any $f \in B_b(\mathbb{R}^d)$, there exists a sequence in the linear span of the real and imaginary parts of the set of all g_a of the form $g_a(x) = e^{i\langle a,x\rangle}$, for arbitrary $a \in \mathbb{R}^d$, converging to f and bounded by f in the supremum norm. Thus (3.2.1) need only hold for g of the form $g_a(x) = e^{i\langle a,x\rangle}$, $a \in \mathbb{R}^d$. In this case (3.2.1) becomes

$$\int_{\mathbb{R}^d} P_t e^{i\langle a,x\rangle}\mu(dx) = \hat{\mu}(dx),$$

where $\hat{\mu}$ denotes the Fourier transform of μ.

Invariant measures, also commonly called stationary measures, intuitively describe the steady-state or long-term behavior of the process. Indeed if a time-homogeneous Markov process X has a law which approaches a probability measure μ as $t \to \infty$, in the sense that

$$\lim_{t\to\infty} p_t(x, A) = \mu(A), \tag{3.2.2}$$

for all $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$, then μ is an invariant measure for X. Note μ does not depend on the starting point x.

To prove this assertion, fix t, let $A \in \mathcal{B}(\mathbb{R}^d)$, and $x, y \in \mathbb{R}^d$. Using the Chapman-

Kolomogorov equations,

$$\begin{aligned}\int_{\mathbb{R}^d} p_t(x,A)\mu(dx) &= \int_{\mathbb{R}^d} p_t(x,A) \lim_{s\to\infty} p_s(y,dx) \\ &= \lim_{s\to\infty} \int_{\mathbb{R}^d} p_t(x,A) p_s(y,dx) \\ &= \lim_{s\to\infty} p_{s+t}(y,A) = \mu(A).\end{aligned}$$

3.3 Examples of transition semigroups and invariant measures

In this section we compute formulas for the transition semigroups for the mean-reverting Ornstein-Uhlenbeck processes examined in Examples 3.1.1 and 3.1.3. We will also compute invariant measures for these processes and in each case, show that the invariant measure is unique. The results here are well-known and we include them as a comparison with the time-dependent examples in Chapter 4.

Example 3.3.1. First we compute the transition semigroup of operators for (3.1.3) where $Z(t)$ is a Brownian motion, as in Example 3.1.1. This result is referred to as *Mehler's formula*. For $f \in B_b(\mathbb{R})$,

$$\begin{aligned}
P_t f(x) = \mathbb{E} f(X_x(t)) &= \int_{-\infty}^{\infty} f(y) p_t(x, dy) \\
&= \frac{1}{\sqrt{2\pi \left[\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})\right]}} \int_{-\infty}^{\infty} f(y) \exp\left\{ -\frac{[y - (e^{-\lambda t}x + \mu(1-e^{-\lambda t}))]^2}{2\left[\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})\right]} \right\} dy \\
&= \frac{1}{\sqrt{2\pi \left[\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})\right]}} \int_{-\infty}^{\infty} f(e^{-\lambda t}x + z) \exp\left\{ -\frac{[z - \mu(1-e^{-\lambda t})]^2}{2\left[\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})\right]} \right\} dz \\
&= \int_{-\infty}^{\infty} f(e^{-\lambda t}x + z) p_t(0, dz).
\end{aligned}$$

Since $X_x(t) \sim \mathcal{N}\left(e^{-\lambda t}x + \mu(1-e^{-\lambda t}), \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})\right) \to \mathcal{N}\left(\mu, \frac{\sigma^2}{2\lambda}\right)$ as $t \to \infty$, we would expect a Gaussian measure ν, with mean μ and variance $\frac{\sigma^2}{2\lambda}$ to be an invariant measure based on (3.2.2). This is indeed the case. Using Mehler's formula,

$$\begin{aligned}
\int_{-\infty}^{\infty} P_t f(x) \nu(dx) &= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{2\lambda}\right)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(e^{-\lambda t}x + z) p_t(0, dz) \exp\left\{ -\frac{(x-\mu)^2}{2\left(\frac{\sigma^2}{2\lambda}\right)} \right\} dx \\
&= \frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{2\lambda}\right)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(e^{-\lambda t}x + z) \exp\left\{ -\frac{(x-\mu)^2}{2\left(\frac{\sigma^2}{2\lambda}\right)} \right\} dx \; p_t(0, dz). \qquad (3.3.1)
\end{aligned}$$

After a change of variables, (3.3.1) is equal to

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{2\lambda}\right)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y+z) \exp\left\{ -\frac{(e^{\lambda t}y - \mu)^2}{2\left(\frac{\sigma^2}{2\lambda}\right)} \right\} e^{-\lambda t} dy \; p_t(0, dz) \\
&\qquad = \frac{e^{-\lambda t}}{\sqrt{2\pi\left(\frac{\sigma^2}{2\lambda}\right)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y+z) \exp\left\{ -\frac{e^{2\lambda t}(y - e^{-\lambda t}\mu)^2}{2\left(\frac{\sigma^2}{2\lambda}\right)} \right\} dy \; p_t(0, dz) \\
&\qquad = \frac{1}{\sqrt{2\pi\left(\frac{e^{-2\lambda t}\sigma^2}{2\lambda}\right)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y+z) \exp\left\{ -\frac{(y - e^{-\lambda t}\mu)^2}{2\left(\frac{e^{-2\lambda t}\sigma^2}{2\lambda}\right)} \right\} dy \; p_t(0, dz) \\
&\qquad = \int_{-\infty}^{\infty} f(x)(\nu_1 * \nu_2)(dx),
\end{aligned}$$

where $\nu_1 \sim \mathcal{N}\left(e^{-\lambda t}\mu, \frac{e^{-2\lambda t}\sigma^2}{2\lambda}\right)$ and $\nu_2 \sim \mathcal{N}\left(\mu(1-e^{-\lambda t}), \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})\right)$. Thus if we can show $\nu := \nu_1 * \nu_2 \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{2\lambda}\right)$, then μ is an invariant measure.

Taking characteristic functions,

$$
\begin{aligned}
\hat{\nu}(a) &= \hat{\nu}_1(a)\hat{\nu}_2(a) \\
&= \exp\left\{iae^{-\lambda t}\mu - \frac{1}{2}\cdot\frac{e^{-2\lambda t}\sigma^2}{2\lambda}a^2\right\}\exp\left\{ia\mu(1-e^{-\lambda t}) - \frac{1}{2}\cdot\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})a^2\right\} \\
&= \exp\left\{ia\mu - \frac{1}{2}\cdot\frac{\sigma^2}{2\lambda}a^2\right\}.
\end{aligned}
$$

And so $\nu \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{2\lambda}\right)$ as desired.

To show uniqueness, suppose ρ is also an invariant measure. Let $a \in \mathbb{R}$, then taking $f(x) = e^{iax}$,

$$
\begin{aligned}
\hat{\rho}(a) &= \int_{-\infty}^{\infty} e^{iax}\rho(dx) = \int_{-\infty}^{\infty} P_t e^{iax}\rho(dx) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ia(e^{-\lambda t}x+z)}p_t(0,dz)\rho(dx) \\
&= \int_{-\infty}^{\infty} e^{iae^{-\lambda t}x}\rho(dx)\int_{-\infty}^{\infty} e^{iaz}p_t(0,dz) \\
&= \hat{\rho}(e^{-\lambda t}a)\exp\left\{ia\mu(1-e^{-\lambda t}) - \frac{1}{2}\cdot\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})a^2\right\}.
\end{aligned}
$$

Since this holds for all $t \geq 0$, by continuity we have

$$
\begin{aligned}
\hat{\rho}(a) &= \lim_{t\to\infty}\left[\hat{\rho}(e^{-\lambda t}a)\exp\left\{ia\mu(1-e^{-\lambda t}) - \frac{1}{2}\cdot\frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t})a^2\right\}\right] \\
&= \hat{\rho}(0)\exp\left\{ia\mu - \frac{1}{2}\cdot\frac{\sigma^2}{2\lambda}a^2\right\} = \exp\left\{ia\mu - \frac{1}{2}\cdot\frac{\sigma^2}{2\lambda}a^2\right\}.
\end{aligned}
$$

Thus ρ must be $\mathcal{N}\left(\mu, \frac{\sigma^2}{2\lambda}\right)$.

Example 3.3.2. Now we consider a jump case. As in Example 3.1.3, let $Z(t)$ be a 1-dimensional, symmetric, α-stable Lévy process. Similar to the computation of the transition semigroup in Example 3.3.1, we have the following version of Mehler's formula, generalized to the case where Z is symmetric, α-stable,

$$P_t f(x) = \int_{-\infty}^{\infty} f(e^{-\lambda t}x + z) p_t(0, dz). \tag{3.3.2}$$

In the case where $\alpha = 1$, (3.3.2) can be written as an integral with respect to a Cauchy density,

$$P_t f(x) = \int_{-\infty}^{\infty} f\left(e^{-\lambda t}x + z\right) \frac{(\sigma/\lambda)(1 - e^{-\lambda t})}{\pi\left[(z - (\mu(1 - e^{-\lambda t})))^2 + ((\sigma/\lambda)(1 - e^{-\lambda t}))^2\right]} dz.$$

Based on (3.1.4), by letting $t \to \infty$, we expect the measure ν with characteristic function

$$\hat{\nu}(a) = \exp\left\{ia\mu - \frac{\sigma^\alpha}{\lambda\alpha}|a|^\alpha\right\} \tag{3.3.3}$$

to be an invariant measure for $X_x(t)$. In light of Remark 3.2.3 we take f to be of the form $f(x) = e^{iax}$ for some $a \in \mathbb{R}$.

$$\begin{aligned}
\int_{-\infty}^{\infty} P_t\{\exp(iax)\}\nu(dx) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[ia(e^{-\lambda t}x + z)] p_t(0, dz)\nu(dx) \\
&= \int_{-\infty}^{\infty} \exp(iae^{-\lambda t}x)\nu(dx) \int_{-\infty}^{\infty} \exp(iaz) p_t(0, dz) \\
&= \exp\left\{iae^{-\lambda t}\mu - \frac{\sigma^\alpha}{\lambda\alpha} e^{-\lambda\alpha t}|a|^\alpha\right\} \exp\left\{ia\mu\left(1 - e^{-\lambda t}\right) - \frac{\sigma^\alpha}{\lambda\alpha}\left(1 - e^{-\lambda\alpha t}\right)|a|^\alpha\right\} \\
&= \exp\left\{ia\mu - \frac{\sigma^\alpha}{\lambda\alpha}|a|^\alpha\right\} = \hat{\nu}(a).
\end{aligned}$$

In the case where $\alpha = 1$, this measure has density function

$$d_\nu(x) = \frac{\sigma/\lambda}{\pi[(x - \mu)^2 + (\sigma/\lambda)^2]}.$$

A proof similar to that in Example 3.3.1 shows that this measure is the unique invariant measure.

Chapter 4

Time-dependence and evolution systems of measures

4.1 Introduction

In this chapter we consider an Ornstein-Uhlenbeck-type stochastic differential equation where the coefficients depend on time. We will assume that they are bounded and continuous. In this case, there is a unique mild solution to the initial value problem. However, because the coefficients depend on time, the solution is not time-homogeneous. Therefore its associated transition semigroup is a two-parameter semigroup, so we cannot expect to have an invariant measure. Instead we define what is called an *evolution system of measures*, $\{\nu_t, t \in \mathbb{R}\}$. We will see that this notion is a natural generalization of an invariant measure.

In the main result of the thesis, Theorem 4.3.8 in Section 4.3, we prove the existence and uniqueness of an evolution system of measures under certain conditions on the coefficients in the stochastic differential equation. In 2007 Da Prato and Lunardi proved existence and uniqueness of an evolution system where the noise was Brownian motion and the coefficients were T-periodic [3]. We improve upon this result in two significant ways. First we allow the noise to be an arbitrary Lévy process. In the Brownian noise case, the solution is again a Gaussian process. However an Ornstein-Uhlenbeck-type process where the noise is a Lévy process offers more flexibility because it allows one to model phenomenon with heavy tails, as Gaussian tails decay exponentially. As in Chapter 3, of particular interest to us is when the noise term is a symmetric α-stable process. The other significant improvement is that we do not require the coefficients to be periodic. Removing this restriction makes proving the theorem much harder. In [4], Fuhrman and Röckner proved a theorem on the existence and uniqueness of an invariant measure (the coefficients were not time-dependent) with Lévy noise in a Hilbert space. We have used some of the techniques there, adapted to the time-dependent case, to prove Theorem 4.3.8.

4.2 Time-dependence

In this section we first explore properties of the solution to the stochastic initial value problem

$$\begin{aligned} dX(t) &= (A(t)X(t-) + f(t))\,dt + B(t)dZ(t) \\ X(s) &= x, \end{aligned} \tag{4.2.1}$$

where $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d), B : \mathbb{R} \to \mathcal{L}(\mathbb{R}^d), f : \mathbb{R} \to \mathbb{R}^d$ are all bounded and continuous, $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and Z is a d-dimensional Lévy process with Lévy symbol η and triple $[b, R, M]$.

Remark 4.2.1. In previous chapters we have considered Lévy processes to be defined for times $t \geq 0$. To define $\{Z(t), t < 0\}$, take $Z(t)$ to be an independent copy of $-Z(-t-)$. The resulting process satisfies all the conditions of a Lévy process and has cadlag paths.

Definition 4.2.2. The *evolution operator* in $\mathbb{R}^d$ associated with $A(t)$ is the solution of the matrix-valued equation

$$\begin{aligned} \frac{\partial U(t,s)}{\partial t} &= A(t)U(t,s), \\ U(s,s) &= I, \end{aligned} \tag{4.2.2}$$

$s, t \in \mathbb{R}$ and I is the identity operator.

It satisfies the properties

$$\begin{aligned} U(t,s)U(s,r) &= U(t,r), \quad s,r,t \in \mathbb{R}, \\ U(s,r)^T U(t,s)^T &= U(t,r)^T, \quad s,r,t \in \mathbb{R}, \\ \frac{\partial U(t,s)}{\partial s} &= -U(t,s)A(s), \quad s,r \in \mathbb{R}. \end{aligned}$$

Assumption 1. We make the following exponential stability assumption on $U(t,s)$. There exists a $C \geq 1$ and $\epsilon > 0$ so that

$$||U(t,s)|| \leq Ce^{-\epsilon(t-s)}, \quad s,t \in \mathbb{R}.$$

There is no clear way to replace Assumption 1 with an assumption on $A(t)$ itself. For example, even if the eigenvalues of $A(t)$ are negative and bounded away from zero uniformly for all t, Assumption 1 need not hold. See [2] Example 3.5, p. 61.

Definition 4.2.3. We called the process

$$X(t) = X_{s,x}(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)dZ(r) \tag{4.2.3}$$

a *mild solution* of (4.2.1).

Remark 4.2.4. (4.2.3) is also the unique *strong solution* to (4.2.1). A strong solution to (4.2.1) is an adapted process which satisfying the integral equation

$$X(t) = x + \int_s^t (A(r)X(r-) + f(r))dr + \int_s^t B(r)dZ(r)dr \quad a.s.$$

In the next two propositions we see that for each fixed $s < t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, the process $X_{s,x}(t)$ is an infinitely divisible random variable. In Proposition 4.2.5 we compute the characteristic function of this process. The property that Lévy process have independent and stationary increments is important here. In Proposition 4.2.6, we utilize the Lévy-Khintchine formula to compute the triple of $X_{s,x}(t)$.

Proposition 4.2.5. The characteristic function of the process

$$Y(t) = \int_s^t U(t,r)B(r)dZ(r)$$

is of the form

$$\phi_{Y(t)}(a) = \exp\left[-\int_s^t \eta(B(r)^T U(t,r)^T a)dr\right].$$

Proof. Fix $-\infty < s \leq t < \infty$. Let $P_n = \{s = r_0^{(n)} \leq r_1^{(n)} \leq \cdots \leq r_{m(n)}^{(n)} = t\}$ be a sequence of partitions such that $||P_n|| \to 0$ as $n \to \infty$.

By the construction of the Itô stochastic integral,

$$\begin{aligned}\phi_{Y(t)}(a) &= \mathbb{E}\exp\left(i\langle a, Y(t)\rangle\right)\\ &= \mathbb{E}\exp\left(i\langle a, \int_s^t U(t,r)B(r)dZ(r)\rangle\right)\\ &= \mathbb{E}\exp\left(i\left\langle a, \lim_{n\to\infty}\sum_{j=1}^{m(n)} U(t,r_j^{(n)})B(r_j^{(n)})(Z(r_{j+1}^{(n)}) - Z(r_j^{(n)}))\right\rangle\right).\end{aligned}$$

Next we take the limit out of the expectation using the Dominated Convergence theorem,

$$\begin{aligned}\phi_{X_{s,t}^0}(a) &= \lim_{n\to\infty}\mathbb{E}\exp\left(i\left\langle a, \sum_{j=1}^{m(n)} U(t,r_j^{(n)})B(r_j^{(n)})(Z(r_{j+1}^{(n)}) - Z(r_j^{(n)}))\right\rangle\right)\\ &= \lim_{n\to\infty}\mathbb{E}\prod_{j=1}^{m(n)}\exp\left(i\left\langle a, U(t,r_j^{(n)})B(r_j^{(n)})(Z(r_{j+1}^{(n)}) - Z(r_j^{(n)}))\right\rangle\right).\end{aligned}$$

In the next several steps we use the fact that Z has independent and stationary increments.

$$\begin{aligned}\phi_{X_{s,t}^0}(a) &= \lim_{n\to\infty}\prod_{j=1}^{m(n)}\mathbb{E}\exp\left(i\left\langle a, U(t,r_j^{(n)})B(r_j^{(n)})(Z(r_{j+1}^{(n)}) - Z(r_j^{n}))\right\rangle\right)\\ &= \lim_{n\to\infty}\prod_{j=1}^{m(n)}\mathbb{E}\exp\left(i\left\langle B(r_j^{(n)})^T U(t,r_j^{(n)})^T a, (Z(r_{j+1}^{(n)}) - Z(r_j^{(n)}))\right\rangle\right)\\ &= \lim_{n\to\infty}\prod_{j=1}^{m(n)}\mathbb{E}\exp\left(i\left\langle B(r_j^{(n)})^T U(t,r_j^{(n)})^T a, Z(r_{j+1}^{(n)} - r_j^{(n)})\right\rangle\right).\end{aligned}$$

Finally we use Theorem 2.3.5, continuity of the exponential function, and the definition of the integral to finish the proof.

$$\begin{aligned}
\phi_{X^0_{s,t}}(a) &= \lim_{n\to\infty} \prod_{j=1}^{m(n)} \exp\left(-(r^{(n)}_{j+1} - r^{(n)}_j)\eta(B(r^{(n)}_j)^T U(t, r^{(n)}_j)^T a)\right) \\
&= \lim_{n\to\infty} \exp\left(-\sum_{j=1}^{m(n)} \eta(B(r^{(n)}_j)^T U(t, r^{(n)}_j)^T a)(r^{(n)}_{j+1} - r^{(n)}_j)\right) \\
&= \exp\left(-\lim_{n\to\infty}\sum_{j=1}^{m(n)} \eta(B(r^{(n)}_j)^T U(t, r^{(n)}_j)^T a)(r^{(n)}_{j+1} - r^{(n)}_j)\right) \\
&= \exp\left(-\int_s^t \eta(B(r)^T U(t,r)^T a)dr\right).
\end{aligned}$$

Qed

Proposition 4.2.6. For each $-\infty < s \le t < \infty, x \in R^d$, the process $X_{s,x}(t)$ is infinitely divisible with the triple

$$(U(t,s)x + b_{s,t}, R_{s,t}, M_{s,t}),$$

where

$$\begin{aligned}
b_{s,t} &= \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)b\ dr \\
&\quad + \int_s^t \int_{\mathbb{R}^d} U(t,r)B(r)y\left(\frac{1}{1+|U(t,r)B(r)y|^2} - \frac{1}{1+|y|^2}\right) M(dy)dr, \\
R_{s,t} &= \int_s^t U(t,r)B(r)RB(r)^T U(t,r)^T dr, \\
M_{s,t}(A) &= \int_s^t M(B(r)^{-1}U(t,r)^{-1}(A))dr.
\end{aligned}$$

Proof. Using Proposition 4.2.5,

$$\begin{aligned}\phi_{X_{s,x}(t)}(a) =& \mathbb{E}\exp\left[i\langle a, X_{s,x}(t)\rangle\right]\\ =& \mathbb{E}\exp\left[i\left\langle a, U(t,s)x + \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)dZ(r)\right\rangle\right]\\ =& \exp\left[i\left\langle a, U(t,s)x + \int_s^t U(t,r)f(r)dr\right\rangle\right]\times\\ & \exp\left[-\int_s^t \eta(B(r)^T U(t,r)^T a)dr\right].\end{aligned}$$

Now by Theorem 2.3.5,

$$\begin{aligned}\phi_{X_{s,x}(t)}(a) = \exp\Bigg\{ & i\left\langle a, U(t,s)x + \int_s^t U(t,r)f(r)dr\right\rangle + \\ & + \int_s^t \Bigg(i\left\langle b, B(r)^T U(t,r)^T a\right\rangle \\ & - \frac{1}{2}\left\langle B(r)^T U(t,r)^T a, RB(r)^T U(t,r)^T a\right\rangle \\ & + \int_{\mathbb{R}^d}\left[e^{i\left\langle B(r)^T U(t,r)^T a, y\right\rangle} - 1 - \frac{i\left\langle B(r)^T U(t,r)^T a, y\right\rangle}{1+|y|^2}\right] M(dy)\Bigg) dr\Bigg\}.\end{aligned}$$

Next we rearrange some terms, and add and subtract $\frac{i\langle a, U(t,r)B(r)y\rangle}{1+|U(t,r)B(r)y|^2}$ to obtain

$$\begin{aligned}\phi_{X_{s,x}(t)}(a) = \exp\Bigg\{ & i\left\langle a, U(t,s)x + \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)b\ dr\right\rangle \\ & - \frac{1}{2}\left\langle a, \int_s^t U(t,r)B(r)RB(r)^T U(t,r)^T dr\ a\right\rangle \\ & + \int_s^t \int_{\mathbb{R}^d}\Bigg[e^{i\langle a, U(t,r)B(r)y\rangle} - 1 - \frac{i\langle a, U(t,r)B(r)y\rangle}{1+|U(t,r)B(r)y|^2} \\ & + \frac{i\langle a, U(t,r)B(r)y\rangle}{1+|U(t,r)B(r)y|^2} - \frac{i\langle a, U(t,r)B(r)y\rangle}{1+|y|^2}\Bigg] M(dy)dr\Bigg\}.\end{aligned}$$

After more rearranging and a change of variables ϕ takes the desired form,

$$\begin{aligned}\phi_{X_{s,x}(t)}(a) = \exp\Bigg\{ & i\left\langle a, U(t,s)x + \int_s^t U(t,r)f(r)dr + \int_s^t U(t,r)B(r)b\ dr\right\rangle \\ & + i\left\langle a, \int_s^t \int_{\mathbb{R}^d} U(t,r)B(r)y\left(\frac{1}{1+|U(t,r)B(r)y|^2} - \frac{1}{1+|y|^2}\right)M(dy)dr\right\rangle \\ & - \frac{1}{2}\left\langle a, \int_s^t U(t,r)B(r)RB(r)^T U(t,r)^T dr\ a\right\rangle \\ & + \int_s^t \int_{\mathbb{R}^d}\left[e^{i\langle a,z\rangle} - 1 - \frac{i\langle a,z\rangle}{1+|z|^2} M(B(r)^{-1}U(t,r)^{-1}dz)dr\right\}.\end{aligned}$$

Now we show $R_{s,t}$ is non-negative definite, symmetric, and bounded, and that $M_{s,t}$ is a Lévy measure.

Let $y \in \mathbb{R}^d$ and $s, t \in \mathbb{R}$.

$$\langle y, R_{s,t}y\rangle = \int_s^t \left\langle B(r)^T U(t,r)^T y, RB(r)^T U(t,r)^T y\right\rangle dr \geq 0,$$

since R is non-negative definite.

Furthermore since R is symmetric,

$$\begin{aligned}\langle y, R_{s,t}y\rangle &= \int_s^t \left\langle y, U(t,r)B(r)RB(r)^T U(t,r)^T y\right\rangle dr \\ &= \int_s^t \left\langle U(t,r)B(r)RB(r)^T U(t,r)^T y, y\right\rangle dr = \langle R_{s,t}y, y\rangle.\end{aligned}$$

Let $|y| \leq 1$. Since B is bounded, let C_B be such that $C_B \geq 1$ and $||B(t)|| \leq C_B$

for all $t \in \mathbb{R}$. By Assumption 1 we have,

$$\begin{aligned} |R_{s,t}y| &= \left| \int_s^t U(t,r)B(r)RB(r)^T U(t,r)^T dr \ y \right| \\ &\leq \int_s^t \left| U(t,r)B(r)RB(r)^T U(t,r)^T \ y \right| \ dr \\ &\leq C^2 C_B^2 ||R|| \int_s^t e^{-2\epsilon(t-r)} dr = \frac{C^2 C_B^2 ||R||}{2\epsilon} \left(1 - e^{-2\epsilon(t-s)}\right). \end{aligned}$$

Thus for all $-\infty < s \leq t < \infty$,

$$||R_{s,t}|| = \sup_{|y|\leq 1} |R_{s,t}y| \leq \frac{C^2 C_B^2 ||R||}{2\epsilon} < \infty.$$

Since M is Lévy measure, set

$$K_1 := \int_{\{|y|\leq 1\}} |y|^2 M(dy),$$

and

$$K_2 := M\left(\left\{|y| > \frac{1}{CC_B}\right\}\right).$$

Then for $-\infty < s \leq t < \infty$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) M_{s,t}(dy) = \int_s^t \int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) M(B(r)^{-1} U(t,r)^{-1} dy) dr \\ &= \int_s^t \int_{\mathbb{R}^d} \left(1 \wedge |U(t,r)B(r)z|^2\right) M(dz) dr \leq \int_s^t \int_{\mathbb{R}^d} \left(1 \wedge C^2 C_b^2 |z|^2\right) M(dz) dr \\ &= \int_s^t \left[\int_{\left\{|z|\leq \frac{1}{CC_B}\right\}} \left(1 \wedge C^2 C_b^2 |z|^2\right) M(dz) dr + \int_{\left\{|z|> \frac{1}{CC_B}\right\}} \left(1 \wedge C^2 C_b^2 |z|^2\right) M(dz) \right] dr \\ &= \int_s^t \int_{\left\{|z|\leq \frac{1}{CC_B}\right\}} C^2 C_B^2 |y|^2 M(dy) dr + \int_s^t \int_{\left\{|z|> \frac{1}{CC_B}\right\}} M(dy) dr \\ &\leq \left(C^2 C_B^2 K_1 + K_2\right)(t-s) < \infty. \end{aligned}$$

Qed

Note that as $s \to -\infty$, $M_{s,t}$ is an increasing family of measures. Similarly $R_{s,t}$ is an increasing family of nonnegative symmetric matrices. We define

$$R_{-\infty,t} := \int_{-\infty}^{t} U(t,r)B(r)RB(r)^T U(t,r)^T dr,$$

and

$$M_{-\infty,t}(A) := \sup_{s<t} M_{s,t}(A) = \int_{-\infty}^{t} M(B(r)^{-1}U(t,r)^{-1}(A))dr,$$

$A \in \mathcal{B}(\mathbb{R}^d)$.

4.3 Evolution systems of measures

As in Chapter 3 we recall the transition semigroup of operators associated with X

$$P_{s,t}f(x) = \mathbb{E}[f(X(t)|X(s) = x] = \mathbb{E}[f(X_{s,x}(t)],$$

for $-\infty < s \leq t < \infty$ and $f \in B_b(\mathbb{R}^d)$.

The *transition probabilities* of X are defined

$$p_{s,t}(x, A) = \mathbb{P}(X(t) \in A|X(s) = x),$$

for $s < t, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)$.

Unlike in the autonomous case, the solution to (4.2.1) is not time-homogeneous, and so we cannot expect to have a single invariant measure. Instead we look for a family of probability measures, $\{\nu_t, t \in \mathbb{R}\}$, called an *evolution family (or evolution system) of measures*.

Definition 4.3.1. A family of Borel probability measures, $\{\nu_t\}_{t\in\mathbb{R}}$ is an *evolution family of measures* for the semigroup of operators, $P_{s,t}$, if

$$\int_{\mathbb{R}^d} (P_{s,t}f)(x)\nu_s(dx) = \int_{\mathbb{R}^d} f(x)\nu_t(dx), \tag{4.3.1}$$

for all $-\infty < s \le t < \infty$, $f \in B_b(\mathbb{R}^d)$.

Remark 4.3.2. Similar to Remark 3.2.3, (4.3.1) need only hold for indicator functions or for f of the form $f(x) = \exp(i\langle a, x\rangle)$, $a \in \mathbb{R}^d$.

In the case of $f(x) = \exp(i\langle a, x\rangle)$, using Proposition 4.2.5, (4.3.1) becomes

$$\hat{\nu}_s\left(U(t,s)^T a\right)\exp\left(i\left\langle a, \int_s^t U(t,r)f(r)dr\right\rangle\right)\times \\ \exp\left\{-\int_s^t \eta(B(r)^T U(t,r)^T a)dr\right\} = \hat{\nu}_t(a),$$

where $\hat{\nu}$ denotes the characteristic function of ν.

In the case of $f = \mathbb{1}_A$, $A \in \mathcal{B}(\mathbb{R}^d)$, (4.3.1) becomes

$$\nu_t(A) = p_{s,t}(0,\cdot) * (\nu_s \circ U(t,s)^{-1}\cdot)(A). \tag{4.3.2}$$

Lemma 4.3.3. Suppose Assumption 1 holds. If $\{\nu_s\}_{s\in\mathbb{R}}$ is an evolution system of measures such that there exists an integer N_0 where the sub-collection $\{\nu_s\}_{s<N_0}$ is uniformly tight, then $\nu_s \circ U(t,s)^{-1} \to \delta_0$ weakly as $s \to -\infty$ for each fixed t.

Proof. Fix t and let $f \in C_b(\mathbb{R}^d)$ and choose M so that $|f| \le M$. Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that if $|x| < \delta$, then $|f(x) - f(0)| < \epsilon/2$.

Using the tightness assumption, choose R so that

$$\nu_s(\bar{B}_R(0)) > 1 - \frac{\epsilon}{4M}$$

for $s < N_0$.

By Assumption 1, $U(t,s)x \to 0$ for all $x \in \mathbb{R}^d$ as $s \to -\infty$ Thus $||U(t,s)|| \to 0$ as $s \to -\infty$. Choose $N < N_0$, such that if $s < N$, then $||U(t,s)|| \leq \dfrac{\delta}{R}$.

Then for $s < N$ we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} f(x)\nu_s(U(t,s)^{-1}dx) - \int_{\mathbb{R}^d} f(x)\delta_0(dx) \right| \qquad (4.3.3) \\ &= \left| \int_{\mathbb{R}^d} f(U(t,s)x)\nu_s(dx) - \int_{\mathbb{R}^d} f(x)\delta_0(dx) \right| \\ &= \left| \int_{\mathbb{R}^d} f(U(t,s)x)\nu_s dx - f(0) \right| \end{aligned}$$

Since ν_s is a probability measure, (4.3.3) is equal to

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} [f(U(t,s)x) - f(0)]\, \nu_s(dx) \right| \\ &\leq \left| \int_{\bar{B}_R(0)} [f(U(t,s)x) - f(0)]\, \nu_s(dx) \right| \\ &+ \left| \int_{|x|>R} [f(U(t,s)x) - f(0)]\, \nu_s(dx) \right| \\ &< \epsilon/2 + 2M \cdot \epsilon/4M = \epsilon. \end{aligned}$$

Qed

The next several lemmas are from [7], and are needed in the proof of Theorem 4.3.8. We will also need the notion of *shift-relative compactness.*

Definition 4.3.4. A set of Borel probability measures, $\mathcal{H}$ is said to be *shift-relatively compact* if, for every sequence $\mu_n \in \mathcal{H}$, there is a sequence ν_n such that ν_n is a right (or left) translate of mu_n, and ν_n has a convergent subsequence.

Lemma 4.3.5 (Parthasarathy, Theorem III.2.2)**.** Let X be a complete separable metric group and let $\{\lambda_n\}, \{\mu_n\}, \{\nu_n\}$ be three sequences of measures on X such that $\lambda_n = \mu_n * \nu_n, n = 1, 2, ...$If the sequence $\{\lambda_n\}$ is relatively compact then the sequences $\{\mu_n\}$ and $\{\nu_n\}$ are right- and left-shift compact, respectively.

Proof. See p. 59, [7]. ***Qed***

Theorem 4.3.6 (Parthasarathy, Theorem VI.5.3)**.** In order that a sequence μ_n of infinitely divisible distributions with triples $\mu_n = [x_n, R_n, M_n]$ be relatively compact it is necessary and sufficient that the following hold:

(i) $\{M_n\}$ restricted to to the compliment of any neighborhood of the origin is weakly relatively compact.

(ii) $\{S_n\}$ defined by (4.3.4) is compact.

(iii) x_n is compact in X.

Proof. See p. 187, [7]. ***Qed***

Theorem 4.3.7 (Parthasarathy, Theroem III.2.1)**.** Let X be a complete separable metric group and let $\{\lambda_n\}, \{\mu_n\}, \{\nu_n\}$ be three sequences of measures on X such that $\lambda_n = \mu_n * \nu_n$ for each n. If the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are relatively compact then so is the sequence $\{\nu_n\}$.

Proof. See p. 58, [7]. ***Qed***

Now we are ready to prove the main result of the thesis.

Theorem 4.3.8. If there exists an evolution system of measures for $X_{s,x}(t)$ then the following conditions hold:

(i) For any $t \in \mathbb{R}, \sup_{s<t} \mathrm{tr} R_{s,t} < \infty$,

(ii) For any $t \in \mathbb{R}, \int_{-\infty}^{t} \int_{\mathbb{R}^d} (1 \wedge |U(t,r)B(r)y|^2) M(dy)dr < \infty$.

If in addition,

(iii) for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^d, U(t,s)x \to 0$ as $s \to -\infty$, and there exists an N such that the collection $\{\nu_t\}_{t<N}$ is uniformly tight,

then ν_t is unique and there exists $b_{-\infty,t} := \lim_{s \to -\infty} b_{s,t}$.

Conversely if (i) and (ii) hold and $\lim_{s \to -\infty} b_{s,t}$ exists then for each $t \in \mathbb{R}$, $M_{-\infty,t}$ is a Lévy measure and the evolution system of measures, ν_t, is given by

$$\nu_t = [b_{-\infty,t}, R_{-\infty,t}, M_{-\infty,t}].$$

Proof. We prove the converse first. Suppose (i), (ii) hold and the limit (iii) exists. Fix $t \in \mathbb{R}$. Using (ii),

$$\begin{aligned}
\int_{\mathbb{R}^d} (1 \wedge |y|^2) M_{-\infty,t}(dy) &= \int_{\mathbb{R}^d} (1 \wedge |y|^2) \int_{-\infty}^{t} M(B(r)^{-1}U(t,r)^{-1}(dy))dr \\
&= \int_{-\infty}^{t} \int_{\mathbb{R}^d} (1 \wedge |y|^2) M(B(r)^{-1}U(t,r)^{-1}(dy))dr \\
&= \int_{-\infty}^{t} \int_{\mathbb{R}^d} (1 \wedge |B(r)U(t,r)y|^2) M(dy)dr < \infty
\end{aligned}$$

shows that $M_{-\infty,t}$ is a Lévy measure.

From the computation of the Lévy triple of $X_{s,x}(t)$ in Proposition 4.2.6, it follows that

$$\hat{\nu}_t(a) = \exp\left(i\left\langle a, \int_{-\infty}^{t} U(t,r)f(r)dr\right\rangle\right)\exp\left\{-\int_{-\infty}^{t}\eta(B(r)^T U(t,r)^T a)dr\right\}.$$

Then using Remark 4.3.2 once again,

$$\begin{aligned}
\hat{\nu}_s\left(U(t,s)^T a\right)&\exp\left(i\left\langle a, \int_{s}^{t} U(t,r)f(r)dr\right\rangle\right)\exp\left\{-\int_{s}^{t}\eta(B(r)^T U(t,r)^T a)dr\right\} \\
&= \exp\left(i\left\langle U(t,s)^T a, \int_{-\infty}^{s} U(s,r)f(r)dr\right\rangle\right)\times \\
&\quad \exp\left\{-\int_{-\infty}^{s}\eta(B(r)^T U(s,r)^T U(t,s)^T a)dr\right\}\times \\
&\quad \exp\left(i\left\langle a, \int_{s}^{t} U(t,r)f(r)dr\right\rangle\right)\exp\left\{-\int_{s}^{t}\eta(B(r)^T U(t,r)^T a)dr\right\} \\
&= \exp\left(i\left\langle a, \int_{-\infty}^{t} U(t,r)f(r)dr\right\rangle\right)\times \\
&\quad \exp\left\{-\int_{-\infty}^{t}\eta(B(r)^T U(t,r)^T a)dr\right\} = \hat{\nu}_t(a)
\end{aligned}$$

shows that ν_t is an evolution system of measures.

Suppose now that an evolution system of measures, ν_t, exists. Fix t, then using Remark 4.3.2, for $s < t$,

$$\nu_t = p_{s,t}(0,\cdot) * (\nu_s \circ U(t,s)^{-1}) = \delta_{b_{s,t}} * [0, R_{s,t}, 0] * [0, 0, M_{s,t}] * (\nu_s \circ U(t,s)^{-1}),$$

where δ_y is the Dirac measure.

Set $s = -n$. Then by Lemma 4.3.5, the sequence $\delta_{b_{-n,t}} * [0, R_{-n,t}, 0] * [0, 0, M_{-n,t}]$ is shift relatively compact. This means that there is a sequence $y_n \in \mathbb{R}^d$ (depending on t) such that

$$\delta_{y_n} * \delta_{b_{-n,t}} * [0, R_{-n,t}, 0] * [0, 0, M_{-n,t}] = [y_n + b_{-n,t}, R_{-n,t}, M_{-n,t}]$$

is weakly relatively compact.

Let $S_n : \mathbb{R}^d \to \mathbb{R}^d$ be a sequence of operators defined by

$$\langle S_n y, y\rangle = \langle R_{-n,t} y, y\rangle + \int_{|x|\leq 1} \langle x, y\rangle^2 dM_{-n,t}(x) \tag{4.3.4}$$

By Theorem 4.3.6, the following hold:

(a) $\{M_{-n,t}\}$ restricted to the compliment of any neighborhood of the origin is weakly relatively compact,

(b) $\{S_n\}$ is compact,

(c) $y_n + b_{-n,t}$ is compact in $\mathbb{R}^d$.

Since we are in finite dimensions, part (b) simply means that $\sup_n \mathrm{tr} S_n < \infty$ (see Definition VI.2.4 of [7]). Part (a) implies

$$M_{-\infty,t}(\{|x| \geq 1\}) = \sup_n M_{-n,t}(\{|x| \geq 1\}) < \infty.$$

By (b) we have

$$\begin{aligned} \mathrm{tr} R_{-\infty,t} + \int_{|x|\leq 1} |x|^2 M_{-\infty,t}(dx) &= \sup_n \left(\mathrm{tr} R_{-\infty,t} + \int_{|x|\leq 1} |x|^2 M_{-n,t}(dx) \right) \\ &= \sup_n \mathrm{tr} S_n < \infty. \end{aligned}$$

Thus $M_{-\infty,t}$ is a Lévy measure for each t, and (i) and (ii) hold by Lemma 3.4 of [4].

Now suppose in addition that $U(t,s)x \to 0$ as $s \to -\infty$ and there exists an N such that the collection $\{\nu_t\}_{t<N}$ is uniformly tight. Then by Lemma 4.3.3,

$\nu_s \circ U(t,s)^{-1} \to \delta_0$ weakly as $s \to -\infty$. By Lemma 3.4 [4], $[0, R_{s,t}, 0] \to [0, R_{-\infty,t}, 0]$ and $[0,0,M_{s,t}] \to [0,0,M_{-\infty,t}]$ weakly as $s \to -\infty$. Thus by the weak continuity of convolution we conclude

$$[0, R_{s,t}, 0] * [0,0,M_{s,t}] * (\nu_s \circ U(t,s)^{-1}) \to [0, R_{-\infty,t}, 0] * [0,0,M_{-\infty,t}].$$

Let s_n be a sequence decreasing to $-\infty$. Then

$$\nu_t = \delta_{b_{s,t}} * [0, R_{s,t}, 0] * [0,0,M_{s,t}] * (\nu_s \circ U(t,s)^{-1}),$$

and by Theorem 4.3.7, the collection $\{\delta_{s_n,t}\}_{n\in\mathbb{N}}$ is weakly relatively compact. Thus there is a probability measure σ_t and a subsequence n_k such that $\delta_{s_{n_k},t} \to \sigma$ weakly. Letting $k \to \infty$,

$$\nu_t = \sigma_t * [0, R_{-\infty,t}, 0] * [0,0,M_{-\infty,t}].$$

Taking Fourier transforms of both sides we have

$$\hat{\sigma}_t = \hat{\nu}_t(\widehat{[0, R_{-\infty,t}, 0]} \cdot \widehat{[0,0,M_{-\infty,t}]})^{-1},$$

We see that σ_t does not depend on the subsequence, and so $\delta_{b_{s_n,t}}$ converges weakly. This implies that $b_{-\infty,t} := \lim_{n\to\infty} b_{s_n,t}$ exists. Since s_n is arbitrary, we have $b_{-\infty,t} = \lim_{s\to-\infty} b_{s,t}$,

Thus we have shown that

$$\nu_t = \delta_{b_{-\infty,t}} * [0, R_{-\infty,t}, 0] * [0,0,M_{-\infty,t}]$$

is uniquely determined.

Qed

4.4 Examples of evolution systems measures

In this section we give some examples where we explicitly compute the characteristic functions and densities of evolution systems of measures for the time-dependent 1-dimensional Ornstein-Uhlenbeck process to which Theorem 4.3.8 applies. As in Chapter 3, the noises we consider are a Brownian motion and an symmetric α-stable process.

Consider a 1-dimensional version of (4.2.1),

$$\begin{aligned} dX(t) &= \lambda(t)\left[\mu(t) - X(t-)\right]dt + \sigma(t)dZ(t), \\ X(s) &= x, \end{aligned} \tag{4.4.1}$$

where λ, μ, σ are bounded and continuous on $\mathbb{R}$. In addition we require $\lambda(t) \geq \epsilon > 0$ for all $t \in \mathbb{R}$. This is essentially a time-dependent version of the mean-reverting Ornstein-Uhlenbeck process given by (3.1.2). In one dimension the evolution operator has the form $U(t,s) = e^{-\int_s^t \lambda(r)dr}$. The positivity condition on λ implies Assumption 1 is satisfied.

We write the solution to (4.4.1),

$$\begin{aligned} X(t) = X_{s,x}(t) =& e^{-\int_s^t \lambda(u)du}x + \int_s^t e^{-\int_r^t \lambda(u)du}\lambda(r)\mu(r)dr \\ &+ \int_s^t e^{-\int_r^t \lambda(u)du}\sigma(r)dZ(r). \end{aligned} \tag{4.4.2}$$

The transition semigroup associated with X is computed in a manner similar to

Examples 3.3.1 and 3.3.2,

$$P_{s,t}f(x) = \mathbb{E}f(X_{s,x}(t)) = \int_{-\infty}^{\infty} f\left(e^{-\int_s^t \lambda(r)dr}x + y\right) p_{s,t}(0, dy).$$

Example 4.4.1. Let $Z(t)$ be 1-dimensional Brownian motion. Then for each $t > s$,

$$X_{s,x}(t) \sim \mathcal{N}\left(e^{-\int_s^t \lambda(u)du}x + \int_s^t e^{-\int_r^t \lambda(u)du}\lambda(r)\mu(r)dr, \int_s^t e^{-2\int_r^t \lambda(u)du}\sigma(r)^2 dr\right).$$

By Theorem 4.3.8, the collection of Gaussian measures, $\{\nu_t, t \in \mathbb{R}\}$, given by

$$\nu_t \sim \mathcal{N}\left(\int_{-\infty}^t e^{-\int_r^t \lambda(u)du}\lambda(r)\mu(r)dr, \int_{-\infty}^t e^{-2\int_r^t \lambda(u)du}\sigma(r)^2 dr\right),$$

is the unique evolution system of measures for X.

Example 4.4.2. Now we consider the case where $Z(t)$ is symmetric, α-stable for $0 < \alpha < 2$. Then for each $t > s$, the law of $X_{s,x}(t)$ is α-stable and has characteristic function

$$\phi_{X_{s,x}(t)}(a) = \exp\Bigg\{i\left(e^{-\int_s^t \lambda(u)du}x + \int_s^t e^{-\int_r^t \lambda(u)du}\lambda(r)\mu(r)dr\right)a - \int_s^t e^{-\alpha\int_r^t \lambda(u)du}[\sigma(r)]^\alpha dr\ |a|^\alpha\Bigg\}.$$

The collection of measures, $\{\nu_t, t \in \mathbb{R}\}$, with characteristic functions

$$\hat{\nu}_t = \exp\Bigg\{i\int_{-\infty}^t e^{-\int_r^t \lambda(u)du}\lambda(r)\mu(r)dr\ a - \int_{-\infty}^t e^{-\alpha\int_r^t \lambda(u)du}[\sigma(r)]^\alpha dr\ |a|^\alpha\Bigg\},$$

is the unique evolution system of measures for X. In the case where $\alpha = 1$, we can write the densities of the ν_t,

$$f_{\nu_t}(y) = \frac{a(t)}{\pi\left[(y - b(t))^2 + (a(t))^2\right]},$$

where

$$a(t) = \int_{-\infty}^{t} e^{-\int_r^t \lambda(u)du} \sigma(r) dr,$$

and

$$b(t) = \int_{-\infty}^{t} e^{-\int_r^t \lambda(u)du} \lambda(r)\mu(r) dr.$$

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