ON THE ONSAGER-MACHLUP FUNCTIONAL FOR THE BROWNIAN MOTION ON THE HEISENBERG GROUP

MARCO CARFAGNINI† AND MARIA GORDINA†

Abstract. Given a stochastic process, the problem of finding the corresponding Onsager-Machlup functional is well known in probability theory, and it has been intensively studied over the last fifty years. For a stochastic process taking values in a Riemannian manifold and whose infinitesimal generator is an elliptic operator, the Onsager-Machlup functional is already known. In this paper, we present a way to find the Onsager-Machlup functional associated to the Brownian motion taking values in the Heisenberg group. Its generator is a hypoelliptic operator instead of an elliptic one, and the state space changes from a Riemannian manifold to the Heisenberg group. This space is the simplest non-trivial example of a sub-Riemannian manifold. As the standard geometric and analytic tools from Riemannian geometry are not available, we need to use new techniques.

Contents

1. Introduction 2
2. The Onsager-Machlup functional for an elliptic diffusion 4
3. The setting and the main result 6
3.1. Heisenberg group as Lie group 6
3.2. Heisenberg group as a sub-Riemannian manifold 9
3.3. Main result 13
4. Proof of Theorem 3.13 13
4.1. Reduction of the asymptotics 13
4.2. Girsanov's transformation 16
Appendix A. Girsanov's theorem 21
References 23

1991 *Mathematics Subject Classification.* Primary 58J65; Secondary 60J60, 35R03, 53C17.

Key words and phrases. Diffusion processes, Onsager-Machlup functional, Heisenberg group, hypoelliptic operator.

† Research was supported in part by NSF Grant DMS-1712427.

1. Introduction

The purpose of this paper is to find an Onsager-Machlup functional in a hypoelliptic setting. In general, the Onsager-Machlup functional is used to describe asymptotic behavior of the probability that the paths of a given stochastic process are contained in a small tube around a given smooth curve.

The Onsager-Machlup functional first appeared in [13, 14] where it was used to determine the most probable path of a diffusion process. It can be considered as a probabilistic analogue of the Lagrangian of a dynamical system. For simplicity, let W_t be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\phi \in \mathcal{C}^2([0,1], \mathbb{R})$. Suppose one can find a function L such that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |W_t - \phi(t)| < \varepsilon\right) \sim C(\varepsilon) \exp\left(\int_0^1 L\left(\phi(s), \phi'(s)\right) ds\right), \text{ as } \varepsilon \longrightarrow 0,$$

then $L(\phi(s), \phi'(s))$ is called the *Onsager-Machlup functional*. Here $C(\varepsilon)$ does not depend on the curve ϕ. In particular, minimization of L will yield the most probable deterministic path for W_t.

Note that the problem of finding an Onsager-Machlup functional can be considered for different types of stochastic processes, or smoothness of the curve ϕ, and finally for different tubes around the trajectories. The latter refers to the distance we use on the path space to describe these tubes.

The case of a diffusion process with an elliptic infinitesimal generator is the subject of [4, 6–8, 17], which we will describe briefly. Suppose X_t is an $\mathbb{R}^m$-valued elliptic diffusion which is a solution to a stochastic differential equation. We can endow $\mathbb{R}^m$ with a Riemannian metric using the diffusion coefficient for X_t. Then X_t can be thought of as a manifold-valued process, and one can apply techniques from Riemannian geometry and geometric analysis.

Different norms on the path space and less regular curves have been considered as well, though still in the elliptic setting. We refer to [16] for the case of curves which are not necessarily C^2, and to [12, 18], where convex norms and L^p-norms are considered. We are not be able to refer to all the vast literature on the subject, but we mention some results which are more relevant to the techniques we use in this paper.

In the current paper we are interested in finding the Onsager-Machlup functional if the assumption on the generator being elliptic is relaxed. This has not been addressed so far. A natural setting is to consider diffusions whose generator is a hypoelliptic operator. If Hörmander's condition is satisfied and in addition it induces a sub-Riemannian structure, one can try to use recent geometric techniques such as generalized curvature-dimension inequalities. We are not relying on these geometric methods in our work, though Cameron-Martin type results in [1, 2] might be useful in the future.

Our approach is closer in spirit to [4], which allows to avoid using the PDE techniques such as the Aronson type estimates which are the key ingredient in [18, Lemma 2.1]. Let us explain how a hypoelliptic setting differs from the Riemannian case. For example, the main tools in [4] are the Levi-Civita connection, the corresponding parallel transport and the curvature tensor. While there are recent results dealing with metric connections and corresponding curvature tensors on a class of sub-Riemannian manifolds, these are difficult to use to tackle the Onsager-Machlup functional. In addition, the existing results on Riemannian manifolds relies on smooth exponential map, which allow to construct a local diffeomorphism from the manifold to a Euclidean space. Instead we use the group structure of the Heisenberg group, in particular, the group multiplication instead of a parallel transport on a Riemannian manifold. This aspect is crucial since it is not possible to use tools from geometric analysis as the exponential map or the parallel transport. The main tool in our proof is the Cameron-Martin-Girsanov theorem carefully applied to the two-dimensional Brownian motion driving the hypoelliptic diffusion.

The main result of this paper is Theorem 3.13, which describes the asymptotic behavior of the probability that the paths of the Brownian motion contained in a small tube around a given horizontal curve in the Heisenberg group. Recall that for the Brownian motion on $\mathbb{R}^m$, the Onsager-Machlup functional is given by

$$L(\phi\left(s\right),\phi'\left(s\right)) = -\frac{1}{2}||\phi'\left(s\right)||^2_{\mathbb{R}^m}.$$

Note that Theorem 3.13 describes the Onsager-Machlup functional using the Cameron-Martin norm for horizontal paths of finite energy, which makes it consistent with the Euclidean results.

The case of a diffusion process whose infinitesimal generator is hypoelliptic has been studied recently in [15], which we briefly describe below. Suppose X_t is a two-dimensional diffusion solution to

$$X_t = x_0 + \int_0^t \sigma\left(X_s\right) \circ dW_s + \int_0^t b\left(X_s\right) ds$$

for t in $[0,T]$, where W_t is a one-dimensional Brownian motion. One can associated to X_t a deterministic path $x_t\left(\phi\right)$ as being the solution of the ordinary differential equation

$$x_t\left(\phi\right) = x_0 + \int_0^t \sigma\left(x_s\left(\phi\right)\right)\phi\left(s\right) ds + \int_0^t b\left(x_s\left(\phi\right)\right) ds,$$

where $\phi \in L^2\left([0,T]\right)$ is considered as a control. One of the results in [15] gives upper and lower bounds of the probability that the paths of X_t are contained in a tube around $x_t\left(\phi\right)$. While these bounds are sharper than the ones we use to find the asymptotics in Theorem 3.13, the radius of the tube considered by P. Pigato cannot be arbitrarily small. Therefore these estimates can not be used to find the Onsager-Machlup functional.

Our paper is organized as follows. In Section 2 we recall the definition of the Onsager-Machlup functional and its expression in the case of elliptic diffusions such as diffusions taking values in Riemannian manifolds (see Equation (2.3)). In Section 3 we describe the Heisenberg group $\mathbb{H}$ and the corresponding sub-Laplacian and hypoelliptic Brownian motion. Then in Section 4 we describe the Onsager-Machlup functional for this Brownian motion. Using the left-invariance of the distance we can reduce the problem to finding asymptotics of a different stochastic process around the identity in the group. This allows us to apply the Cameron-Martin-Girsanov Theorem to the two-dimensional Brownian motion driving the hypoelliptic diffusion and then estimate a conditional expectation of exponential martingales.

As we already pointed out, in the known results on the Onsager-Machlup functional in geometric settings, ellipticity was crucial. In the hypoelliptic setting of the Heisenberg group we can overcome this by using the group structure. In the future we hope to develop the techniques used in this paper for a broader class of sub-Riemannian manifolds such as Carnot groups.

2. The Onsager-Machlup functional for an elliptic diffusion

We start by recalling the definition of the Onsager-Machlup functional for a diffusion process whose infinitesimal generator is an elliptic operator. Denote by $C_0\left([0,1];\mathbb{R}^d\right)$ the space of $\mathbb{R}^d$-valued continuous functions vanishing at the origin, that is, being zero at $t=0$. For the purpose of this article we consider the supremum norm $\|\cdot\|$ on this space, though in the future we might have a different choice. For the Borel σ-field $\mathcal{B}$ of $\left(C_0\left([0,1];\mathbb{R}^d\right), ||\cdot||\right)$ we denote by $\mathbb{P}$ the Wiener measure on $\left(C_0\left([0,1];\mathbb{R}^d\right), \mathcal{B}\right)$, and by $B^t=(B_1^t,\dots,B_d^t)$ the Brownian motion on the Wiener space $\left(C_0\left([0,1];\mathbb{R}^d\right), \mathcal{B}, \mathbb{P}\right)$.

Let X_t be the diffusion process which is a solution to the stochastic differential equation in $\mathbb{R}^m$

$$dX_t=\sigma(t,X_t)dB_t+b(t,X_t)dt, \quad X_0=x_0, \tag{2.1}$$

where $\sigma=\sigma\left(t,x\right)$ is an $m\times d$ matrix whose entries are smooth functions in t and x, and $b\in\mathbb{R}^m$ is a smooth function in t and x. In what follows σ usually only depends on x, but we need the drift b to depend on both t and x.

In this section we assume that the generator of X_t is elliptic, therefore the inverse of the matrix σ exists. Then we can view $\mathbb{R}^m$ as a Riemannian manifold equipped with the metric $g=(\sigma\sigma^T)^{-1}$, and X_t as a diffusion process on the manifold $M=(\mathbb{R}^m,g)$ with infinitesimal generator $\frac{1}{2}\Delta_M+Z$, where Z is a smooth vector field and Δ_M is the Laplace-Beltrami operator on (M,g). More precisely, we have

$$Z_i(x) := b_i(x) + \frac{1}{2}\sum_{j,k=1}^{m}(\sigma\sigma^T)_{kj}(x)\Gamma^i_{kj},$$

$$\Delta_M f = \sum_{i,j=1}^{m}(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial x_i \partial x_j}f - \sum_{i,j,k=1}^{m}(\sigma\sigma^T)_{ij}\Gamma^k_{ij}\frac{\partial}{\partial x_k}f,$$

where Γ^k_{ij} are the Christoffel symbols corresponding to the Levi-Civita connection on the Riemannian manifold (M, g).

Let us denote by d_M the Riemannian distance on M induced by g. As usual we denote by TM the tangent bundle over M. For the purpose of our work we use the following definition of the Onsager-Machlup functional which can be found in [10, Chapter VI.9, p. 532].

Definition 2.1 (Onsager-Machlup functional)**.** *Consider two curves ϕ and ψ in M starting at $x_0 \in M$. Suppose X_t is a stochastic process with values in M such that $\mathbb{P}(X_0 = x_0) = 1$. For a fixed t we consider two real-valued random variables $d_M(X_t, \phi(t))$ and $d_M(X_t, \psi(t))$. If the limit*

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup\limits_{t\in[0,1]} d_M(X_t, \phi(t)) < \varepsilon\right)}{\mathbb{P}\left(\sup\limits_{t\in[0,1]} d_M(X_t, \psi(t)) < \varepsilon\right)} \tag{2.2}$$

exists and can be written as

$$\exp\left(\int_0^1 L\left(\phi(s), \phi'(s)\right) ds - \int_0^1 L\left(\psi(s), \psi'(s)\right) ds\right)$$

for some functional L defined on the tangent bundle TM, then L is called the Onsager-Machlup functional of X_t.

Y. Takahashi and S. Watanabe showed in [17] that if the infinitesimal generator of X_t is elliptic and the curves ϕ and ψ are C^2-smooth, then the Onsager-Machlup functional is given by

$$L(p, v) = -\frac{1}{2}||Z_p - v||_p^2 - \frac{1}{2}\mathrm{div} Z_p + \frac{1}{12}R(p) \text{ for any } (p, v) \in TM, \tag{2.3}$$

where for every $p \in M$ we denote by $||\cdot||_p$ the Riemannian norm on T_pM and by $R(p)$ the scalar curvature at p. It is clear that while Definition 2.1 does not require elliptic or Riemannian structure, the form of the Onsager-Machlup functional in (2.3) uses them significantly. Our approach is to use an explicit form of the Brownian motion in the Heisenberg group and left-invariance of the distance function.

3. The setting and the main result

3.1. **Heisenberg group as Lie group.** The Heisenberg group $\mathbb{H}$ as a set is $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with the group multiplication given by

$$
\begin{aligned}
&(\mathbf{v}_1, z_1) \cdot (\mathbf{v}_2, z_2) := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}\omega(\mathbf{v}_1, \mathbf{v}_2) \right), \\
&\text{ where } \mathbf{v}_1 = (x_1, y_1), \mathbf{v}_2 = (x_2, y_2) \in \mathbb{R}^2, \\
&\omega : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \\
&\omega(\mathbf{v}_1, \mathbf{v}_2) := x_1 y_2 - x_2 y_1
\end{aligned}
$$

is the standard symplectic form on $\mathbb{R}^2$. The identity in $\mathbb{H}$ is $e = (0, 0, 0)$ and the inverse is given by $(\mathbf{v}, z)^{-1} = (-\mathbf{v}, -z)$.

The Lie algebra of $\mathbb{H}$ can be identified with the space $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with the Lie bracket defined by

$$
[(\mathbf{a}_1, c_1), (\mathbf{a}_2, c_2)] = (0, \omega(\mathbf{a}_1, \mathbf{a}_2)).
$$

The set $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with this Lie algebra structure will be denoted by $\mathfrak{h}$.

Let us now recall some basic notation for Lie groups. Suppose G is a Lie group, then the left and right multiplication by an element $k \in G$ are denoted by

$$
\begin{aligned}
L_k &: G \longrightarrow G, & g &\longmapsto k^{-1}g, \\
R_k &: G \longrightarrow G, & g &\longmapsto gk.
\end{aligned}
$$

Recall that the tangent space T_eG can be identified with the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G, that is, vector fields X on G such that $dL_k \circ X = X \circ L_k$, where dL_k is the differential of L_k. More precisely, if A is a vector in T_eG, then we denote by $\tilde{A} \in \mathfrak{g}$ the (unique) left-invariant vector field such that $\tilde{A}(e) = A$. A left-invariant vector field is determined by its value at the identity, namely, $\tilde{A}(k) = dL_k \circ \tilde{A}(e)$.

Before turning to the Heisenberg group, we recall the connection between left-invariant vector fields on a Lie group G and one-parameter groups and integral curves for left-invariant vector fields.

Notation 3.1 (One-parameter groups and integral curves)**.** *For a left-invariant vector field X on a Lie group G we denote by $\gamma_g^X(t)$, $t \in \mathbb{R}$, the integral curve for X starting at g, that is, $\gamma_g^X(0) = g$ and $\left(\gamma_g^X\right)'(t) = X_{\gamma_g^X(t)}$ for all t. The one-parameter group is then $\gamma^X(t) = \gamma_e^X(t) = e^{tX}$. Often we will write γ_g instead of γ_g^X if there is no ambiguity.*

That is, we have one-to-one correspondence between one-parameter subgroups in G, left-invariant vector fields on G, and tangent vectors at $e \in G$. Note that if X is a left-invariant vector field on G and γ_k is an integral curve for X starting at k, then $L_{g^{-1}}\gamma_k$ is an integral curve for X starting at gk. Indeed,

$$
\begin{aligned}
\frac{d}{dt}\left(L_{g^{-1}}\gamma_g(t)\right) &= d\left(L_{g^{-1}}\right)\circ\frac{d}{dt}\gamma_g(t) \\
&= d\left(L_{g^{-1}}\right)\circ X_{\gamma_k'(t)} = X_{L_{g^{-1}}\gamma_k(t)},
\end{aligned}
$$

and $L_{g^{-1}}\gamma_k(0) = gk$.

The identification of T_eG with $\mathfrak{g}$ can be described explicitly for the Heisenberg group as follows. This proof can be found in [5], and we include it for completeness.

Lemma 3.2 (Proposition 3.7 in [5]). *For every $h = (\mathbf{a}, c) = (a, b, c) \in T_e\mathbb{H}$ and $g = (\mathbf{x}, z) = (x, y, z) \in \mathbb{H}$, then the (unique) left-invariant vector field such that $\tilde{h}(e) = h$ is given by*

$$
\tilde{h}(g) = \left(a, b, c + \frac{1}{2}\omega(\mathbf{x}, \mathbf{a})\right). \tag{3.1}
$$

Proof. Suppose now that $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a one-parameter group for $\tilde{h}$, that is, $\gamma(0) = e = (0, 0, 0)$ and $\gamma'(0) = h$, then by the argument above we can determine the left-invariant vector field by taking $L_{g^{-1}}\gamma$. Note that

$$
L_{g^{-1}}\gamma(t) = \left(\gamma_1(t) - x, \gamma_2(t) - y, \gamma_3(t) - z + \frac{1}{2}\omega\left(\left(\gamma_1(t), \gamma_2(t)\right), \mathbf{x}\right)\right)
$$

and therefore

$$
\begin{aligned}
\tilde{h}(g) &= \left.\frac{d}{dt}\right|_{t=0}\left(L_{g^{-1}}\gamma(t)\right) \\
&= \left(\gamma_1'(0), \gamma_2'(0), \gamma_3'(0) + \frac{1}{2}\omega\left(\left(\gamma_1'(0), \gamma_2'(0)\right), (x, y)\right)\right) \\
&= \left(a, b, c + \frac{1}{2}\omega(\mathbf{x}, \mathbf{a})\right).
\end{aligned}
$$

Finally it is clear that for the vector field defined by (3.1) we have $\tilde{h}(e) = h$. □

As we point out in the next lemma, we can find one-parameter groups in $\mathbb{H}$ explicitly. This means that we can also find integral curves starting at any point in the Heisenberg group.

Lemma 3.3 (Integral curves in the Heisenberg group). *Let $h = (a, b, c) = (\mathbf{a}, c) \in T_e\mathbb{H}$, and let $\tilde{h}$ be the corresponding left-invariant vector field on $\mathbb{H}$. Then for any $g = (x, y, z) = (\mathbf{x}, z) \in \mathbb{H}$ we have*

$$
\gamma_g(t) = \gamma_g^{\tilde{h}}(t) = \left(x + at, y + bt, z + ct + \frac{t}{2}\omega(\mathbf{a}, \mathbf{x})\right) \tag{3.2}
$$

Proof. We start by describing the one-parameter groups first. In this exposition we follow [5, Lemma 3.8]. If we write $\gamma_e(t) = (x(t), y(t), z(t)) = (\mathbf{x}(t), z(t))$, then by (3.1) and definition of an integral curve for the vector field $\tilde{h}$ we have

$$\tilde{h}(\gamma_e(t)) = \left(a, b, c + \frac{1}{2}\omega(\mathbf{x}(t), \mathbf{a})\right) = (x'(t), y'(t), z'(t)). \tag{3.3}$$

The solution to this equation is given by $\gamma_e(t) = (at, bt, ct)$.

Now we can use the argument preceding Lemma 3.2 to find the integral curve γ_g for $\tilde{h}$ starting at $g = (x, y, z) = (\mathbf{x}, z) \in \mathbb{H}$. Namely,

$$\gamma_g(t) = (x, y, z)(at, bt, ct) = \left(x + at, y + bt, z + ct + \frac{t}{2}\omega(\mathbf{a}, \mathbf{x})\right)$$

□

In addition to actions of G on itself by left and right multiplication, we can also consider the action of G on itself by conjugation, namely, for each $k \in G$ we can define the diffeomorphism called the *inner automorphism defined by* k as

$$\operatorname{Inn}_k(g) := k^{-1}gk, g \in G.$$

Note that $\operatorname{Inn}_k(e) = e$, and therefore we have an invertible linear map $(d\operatorname{Inn}_k)_e : T_eG \longrightarrow T_eG$.

Using the identification of the Lie algebra $\mathfrak{g}$ with T_eG we can introduce the adjoint representation of G.

Definition 3.4. *The adjoint representation of a Lie group G is the representation $\operatorname{Ad} : G \longrightarrow \operatorname{Ad}(\mathfrak{g})$ defined by*

$$\operatorname{Ad}_k = (d\operatorname{Inn}_k)_e.$$

For the Heisenberg group Ad can be described explicitly as follows.

Proposition 3.5. *Let $k = (k_1, k_2, k_3) = (\mathbf{k}, k_3)$ and $g = (g_1, g_2, g_3) = (\mathbf{g}, g_3)$ be two elements in $\mathbb{H}$. Then, for every $v = (v_1, v_2, v_3) = (\mathbf{v}, v_3)$ in $T_g\mathbb{H}$, the differentials of the left and right multiplication are given by*

$$\begin{aligned}
& dL_k : T_g\mathbb{H} \longrightarrow T_{k^{-1}g}\mathbb{H}, \\
& dR_k : T_g\mathbb{H} \longrightarrow T_{gk}\mathbb{H}, \\
& dL_k(v) = \left(v_1, v_2, v_3 + \frac{1}{2}\omega(\mathbf{v}, \mathbf{k})\right), \\
& dR_k(v) = \left(v_1, v_2, v_3 + \frac{1}{2}\omega(\mathbf{v}, \mathbf{k})\right). \qquad (3.4)
\end{aligned}$$

If $h=(a,b,c)=(\mathbf{a},c)\in\mathfrak{h}$, then the adjoint representation is given by

$$\begin{aligned} &\mathrm{Ad}_k:\mathfrak{h}\longrightarrow\mathfrak{h},\\ &\mathrm{Ad}_k(h)=(a,b,c+\omega(\mathbf{a},\mathbf{k}))\,. \end{aligned} \tag{3.5}$$

Proof. Suppose γ_g is an integral curve for the vector field $\tilde{h}$ starting at g. Then $\gamma_g'(0)=w$ and therefore

$$\begin{aligned} \mathrm{Ad}_k\left(h\right)&=\left.\frac{d}{dt}\right|_{t=0}\left(\mathrm{Inn}_k\,\gamma_g\left(t\right)\right)\\ &=\left(\gamma_1'(0),\gamma_2'(0),\gamma_3'(0)+\frac{1}{2}\omega\left(\left(\gamma_1'(0),\gamma_2'(0)\right),\mathbf{k}\right)\right)\\ &=(a,b,c+\omega(\mathbf{a},\mathbf{k}))\,. \end{aligned}$$

The rest of the statement can be shown similarly. □

For completeness we give an explicit description of the exponential map $\exp:\mathfrak{h}\longrightarrow\mathbb{H}$. Note that $\mathbb{H}$ is a nilpotent Lie group, therefore the exponential map is a global diffeomorphism.

Recall that the *exponential* for a Lie group is defined as a map $\exp:\mathfrak{g}\longrightarrow G$ defined by $\exp\left(A\right)=\gamma_e^{\tilde{A}}\left(1\right)$ for any $A\in\mathfrak{g}$. Here as before γ_e is the one-parameter group for the left-invariant vector field $\tilde{A}$. Thus for any $h=(a,b,c)\in\mathfrak{h}$ we have

$$\exp(h)=(a,b,c)\,.$$

3.2. **Heisenberg group as a sub-Riemannian manifold.** The Heisenberg group $\mathbb{H}$ is the simplest non-trivial example of a sub-Riemannian manifold. We define X, Y and Z as the unique left-invariant vector fields satisfying $X_e=\partial_x$, $Y_e=\partial_y$ and $Z_e=\partial_z$, that is,

$$\begin{aligned} X&=\partial_x-\frac{1}{2}y\partial_z,\\ Y&=\partial_y+\frac{1}{2}x\partial_z,\\ Z&=\partial_z. \end{aligned}$$

Note that the only non-zero Lie bracket for these left-invariant vector fields is $[X,Y]=Z$, so the vector fields $\{X,Y\}$ satisfy Hörmander's condition. We define the *horizontal distribution* as $\mathcal{H}:=\operatorname{span}\{X,Y\}$ fiberwise, thus making $\mathcal{H}$ a sub-bundle in the tangent bundle $T\mathbb{H}$. To finish the description of the Heisenberg group as a sub-Riemannian manifold we need to equip the horizontal distribution $\mathcal{H}$ with an inner product. For any $p\in\mathbb{H}$ we define the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_p}$ on $\mathcal{H}_p$ so that $\{X\left(p\right),Y\left(p\right)\}$ is an orthonormal (horizontal) frame at any $p\in\mathbb{H}$. Vectors in $\mathcal{H}_p$ will be called *horizontal*, and the corresponding norm will be denoted by $\|\cdot\|_{\mathcal{H}_p}$.

In addition, Hörmander's condition ensures that a natural sub-Laplacian on the Heisenberg group

$$\Delta_{\mathcal{H}} = X^2 + Y^2 \tag{3.6}$$

is a hypoelliptic operator by [9].

We recall now two other important objects in sub-Riemannian geometry, namely, horizontal curves and horizontal gradient. We start by introducing Maurer-Cartan forms on a Lie group G.

Notation 3.6. *By θ^l (θ^r) we will denote the left (right) Maurer–Cartan form on G, i.e. θ is the $\mathfrak{g}$-valued 1-form on $\mathfrak{h}$ defined by*

$$\begin{aligned}\theta^l_k(v) &:= dL_k(v)\\ \theta^r_k(v) &:= dR_{k^{-1}}(v).\end{aligned}$$

for any $g \in G, v \in T_gG$.

Note that in general

$$\theta^r_k(v) = \mathrm{Ad}_{k^{-1}}\left(\theta^l_k(v)\right). \tag{3.7}$$

For the Heisenberg group Maurer-Cartan forms can be written explicitly by using relations 3.4 as follows

$$\begin{aligned}\theta^l_k(v) &= \left(v_1, v_2, v_3 + \frac{1}{2}\omega(\mathbf{v},\mathbf{k})\right),\\ \theta^r_k(v) &= \left(v_1, v_2, v_3 - \frac{1}{2}\omega(\mathbf{v},\mathbf{k})\right).\end{aligned} \tag{3.8}$$

Notation 3.7. *A curve $\gamma(t) = (x(t), y(t), z(t))$ in $\mathbb{H}$ will be denoted by $(\mathbf{x}(t), z(t))$, and its corresponding tangent vector $\gamma'(t)$ in $T\mathbb{H}_{\gamma(t)}$ will be denoted by*

$$\gamma'(t) = \left(x'(t), y'(t), z'(t)\right) = \left(\boldsymbol{x}'(t), z'(t)\right).$$

Definition 3.8. *An absolutely continuous path $t \longmapsto \gamma(t) \in \mathbb{H}, t \in [0,1]$ is said to be horizontal if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all t, that is, the tangent vector to $\gamma(t)$ at every point $\gamma(t)$ is horizontal. Equivalently we can say that γ is horizontal if the (left) Maurer-Cartan form $c(t) := \theta^l_{\gamma(t)}(\gamma'(t)) \in \mathcal{H}_e$ for a.e. t.*

Note that for $\gamma(t) = (\mathbf{x}(t), z(t))$ we have

$$\begin{aligned}c_\gamma(t) := c(t) &= \theta^l_{\gamma(t)}\left(\gamma'(t)\right)\\ &= \left(\mathbf{x}'(t), z'(t) - \frac{1}{2}\omega(\mathbf{x}(t), \mathbf{x}'(t))\right),\end{aligned} \tag{3.9}$$

where we used Proposition 3.5. Equation (3.9) can be used to characterize horizontal curves in terms of the components as follows. The curve γ is horizontal if and only if

$$z'(t) - \frac{1}{2}\omega(\mathbf{x}(t), \mathbf{x}'(t))) = 0. \tag{3.10}$$

Definition 3.9. *We say that a horizontal curve $t \longmapsto \gamma(t) \in \mathbb{H}$, $t \in [0,1]$ has finite energy if*

$$\|\gamma\|^2_{H(\mathbb{H})} := \int_0^1 |c_\gamma(s)|^2_{\mathcal{H}_e} ds = \int_0^1 |\theta_{\gamma(s)}\left(\gamma'(s)\right)|^2_{\mathcal{H}_e} ds < \infty. \tag{3.11}$$

We denote by $H(\mathbb{H})$ the Cameron-Martin space of finite energy horizontal curves starting at the identity

$$\begin{aligned} H(\mathbb{H}) := \{&\gamma : [0,1] \longrightarrow \mathbb{H}, \|\gamma\|_{H(\mathbb{H})} < \infty, \gamma(0) = e, \\ &\gamma \textit{ is absolutely continuous and horizontal}\}. \end{aligned}$$

The inner product corresponding to the norm $\|\cdot\|_{H(\mathbb{H})}$ is denoted by $\langle \cdot, \cdot \rangle_{H(\mathbb{H})}$.

Note that the Heisenberg group as a sub-Riemannian manifold comes with a natural left-invariant distance which we will use to define the Onsager-Machlup functional.

Definition 3.10. *For any $g_1, g_2 \in \mathbb{H}$ the Carnot-Carathéodory distance is defined as*

$$\begin{aligned} d_{cc}(g_1, g_2) := \inf \Bigg\{ &\int_0^1 |c_\gamma(s)|^2_{\mathcal{H}_e}, \\ &\gamma : [0,1] \longrightarrow \mathbb{H}, \gamma(0) = g_1, \gamma(1) = g_2, \gamma \textit{ is horizontal} \Bigg\}. \end{aligned}$$

Another consequence of Hörmander's condition for left-invariant vector fields X, Y and Z is that we can apply the Chow–Rashevskii theorem. As a result, given two points in $\mathbb{H}$ there exists a horizontal curve connecting them, and therefore the Carnot-Carathéodory distance is finite on $\mathbb{H}$.

In addition to the Carnot-Carathéodory distance on the Heisenberg group, we will use the following homogeneous distance

$$\rho(g_1, g_2) := \left(\|\mathbf{x}_1 - \mathbf{x}_2\|^2_{\mathbb{R}^2} + |z_1 - z_2 + \omega(\mathbf{x}_1, \mathbf{x}_2)|\right)^{\frac{1}{2}},$$

which is equivalent to the Carnot-Carathéodory distance, that is, there exist two positive constants c and C such that

$$c\rho(g_1, g_2) \leqslant d_{cc}(g_1, g_2) \leqslant C\rho(g_1, g_2) \tag{3.12}$$

for all $g_1, g_2 \in \mathbb{H}$.

Finally, we need to describe a hypoelliptic Brownian motion with values in $\mathbb{H}$. This is a stochastic process whose generator is the sub-Laplacian $\frac{1}{2}\Delta_{\mathcal{H}}$ defined by Equation (3.6). We give several equivalent descriptions of this process.

Notation 3.11. Throughout the paper we use the following notation. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. We denote the expectation under $\mathbb{P}$ by $\mathbb{E}$.

By a standard Brownian motion $\{B_t\}_{t\geqslant 0}$ we mean a continuous adapted $\mathbb{R}$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that for all $0 \leqslant s \leqslant t$, we have that $B_t - B_s$ is independent of $\mathcal{F}_s$ and has a normal distribution with mean 0 and the variance $t-s$.

Definition 3.12. *Let $W_t = (W_1(t), W_2(t), 0)$ be an $\mathfrak{h}$-valued stochastic process, where $\boldsymbol{W}_t := (W_1(t), W_2(t))$ is a standard two-dimensional Brownian motion. A hypoelliptic Brownian motion $g_t = (g_1(t), g_2(t), g_3(t))$ on $\mathbb{H}$ is the continuous $\mathbb{H}$-valued process defined by*

$$g_t := \left(\boldsymbol{W}_t, \frac{1}{2} \int_0^t \omega\left(\boldsymbol{W}_s, d\boldsymbol{W}_s\right) \right). \tag{3.13}$$

Note that we used the Itô integral in the definition rather than the Startonovich integral. However, these two integrals are equal since the symplectic form ω is skew-symmetric, and therefore Lévy's stochastic area functional is the same for both integrals.

One can also write a stochastic differential equation for $g_t = (x_t, y_t, z_t)$, $g_0 = (0,0,0) = e \in \mathbb{H}$. The first form is the standard stochastic differential equation for a Lie group-valued Brownian motion, namely,

$$\begin{aligned} &\theta^l_{g_t}\left(dg_t\right) = dL_{g_t}\left(dg_t\right) = dW_t, \\ &g_0 = e. \end{aligned}$$

Equation (3.13) gives an explicit solution to this stochastic differential equation.

To write it in the form similar to Equation (2.1) we view the process g_t as taking values in $\mathbb{R}^3$. Then

$$dg_t = \sigma\left(g_t\right) dW_t,$$

where σ a 3×3-matrix defined by

$$\sigma(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{pmatrix} \quad \text{for all } (x,y,z) \in \mathbb{R}^3. \tag{3.14}$$

In the setting of the Heisenberg group finding an Onsager-Machlup functional will lead us to understanding of the asymptotics of the following probability

$$\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t, \phi\left(t\right)) < \varepsilon \right),$$

for a horizontal curve ϕ. Equivalently by Equation (3.12) we have

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t,\phi(t))<\varepsilon\right)=\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,1]} \rho(g_t,\phi(t))<\varepsilon\right). \tag{3.15}$$

3.3. **Main result.** Now we have all the ingredients needed to state the main result of this paper. This describes the Onsager-Machlup functional for the hypoelliptic diffusion g_t in spirit of Definition 2.1.

Theorem 3.13. *Let $(\mathbb{H},\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ be the Heisenberg group, g_t be the hypoelliptic Brownian motion on $\mathbb{H}$ starting at the identity $e\in\mathbb{H}$, and d_{cc} be the Carnot-Carathéodory distance. Then for any $\phi,\psi\in H(\mathbb{H})$ we have*

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t,\phi(t))<\varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t,\psi(t))<\varepsilon\right)} = \exp\left(-\frac{1}{2}\|\phi\|^2_{H(\mathbb{H})}+\frac{1}{2}\|\psi\|^2_{H(\mathbb{H})}\right). \tag{3.16}$$

In particular, this implies that the Onsager-Machlup functional L for the hypoelliptic Brownian motion on $\mathbb{H}$ is given as a functional on the horizontal distribution $\mathcal{H}$ as follows

$$L(p,v)=-\frac{1}{2}\|v\|^2_{\mathcal{H}_p} \text{ for any } v\in\mathcal{H}_p, p\in\mathbb{H}.$$

Note that the form of the Onsager-Machlup functional follows easily from definition of the Cameron-Martin norm $\|\cdot\|_{H(\mathbb{H})}$ and the form of the Maurer-Cartan form for horizontal curves as given by (3.9), namely,

$$L(\phi(s),\phi'(s))=-\frac{1}{2}|c_\phi(s)|^2_{\mathcal{H}_e}=-\frac{1}{2}\left(\phi_1'(s)^2+\phi_2'(s)^2\right),$$

for almost all $s\in[0,1]$.

4. Proof of Theorem 3.13

We will divide the proof of Theorem 3.13 into two steps.

4.1. **Reduction of the asymptotics.** The aim of this step is the reduction of the asymptotics in (3.16) for the hypoelliptic Brownian motion g_t to an asymptotics of a different stochastic process being close to the constant curve $\boldsymbol{e}(t)\equiv e\in\mathbb{H}$. This is what Equation (4.8) represents, where process u_t^ϕ is defined by (4.7).

Let us define the process

$$u_t^\xi := \xi(t)^{-1} g_t, \tag{4.1}$$

where ξ is an absolutely continuous curve starting at the identity. For now we do not assume that ξ is horizontal. We can write the process u_t^ξ as follows

$$
\begin{aligned}
u_t^\xi &= \left(-\boldsymbol{\xi}\left(t\right), -\xi_3\left(t\right)\right) \cdot \left(\boldsymbol{W}_t, \frac{1}{2}\int_0^t \omega\left(\boldsymbol{W}_s, d\boldsymbol{W}_s\right)\right) \\
&= \left(\boldsymbol{W}_t - \boldsymbol{\xi}\left(t\right), \frac{1}{2}\int_0^t \omega\left(\boldsymbol{W}_s, d\boldsymbol{W}_s\right) - \xi_3\left(t\right) - \frac{1}{2}\omega\left(\boldsymbol{\xi}\left(t\right), \boldsymbol{W}_t\right)\right),
\end{aligned} \tag{4.2}
$$

where we used (3.13). Then by (3.15) and left-invariance of both distances we see that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t, \xi\left(t\right)) < \varepsilon\right) \\
&= \lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(u_t^\xi, e) < \varepsilon\right) \\
&= \lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,1]} \rho(u_t^\xi, e) < \varepsilon\right).
\end{aligned} \tag{4.3}
$$

That is, the probability of g_t being close to ξ in the Carnot-Carathéodory distance is the same as the probability of the process u_t^ξ being close to the constant curve $\boldsymbol{e}$ with respect to the homogeneous norm ρ.

Equation (4.2) can be used to see that the process u_t^ξ satisfies the following stochastic differential equation

$$
\begin{aligned}
du_t^\xi = \big(&d\boldsymbol{W}_t - \boldsymbol{\xi}'\left(t\right)dt, \\
&\frac{1}{2}\omega\left(\boldsymbol{W}_t - \boldsymbol{\xi}\left(t\right), d\boldsymbol{W}_t\right) - \left(\xi_3'\left(t\right) + \frac{1}{2}\omega\left(\boldsymbol{\xi}'\left(t\right), \boldsymbol{W}_t\right)\right)dt\bigg).
\end{aligned} \tag{4.4}
$$

The next statement describes the Maurer-Cartan form induced by the process u_t^ξ.

Proposition 4.1. *Let ξ be an absolutely continuous curve starting at the identity e in $\mathbb{H}$. Then the process u_t^ξ, defined by (4.1), satisfies the equation*

$$
\begin{aligned}
&\int_0^t \theta^l_{u_s^\xi}\left(du_s^\xi\right) = \left(W_t - \xi\left(t\right)\right) - \left(\mathbf{0}, \int_0^t \omega(\boldsymbol{\xi}'\left(s\right), \boldsymbol{W}_s - \frac{1}{2}\boldsymbol{\xi}\left(s\right))\right)ds \\
&= \left(\boldsymbol{W}_t - \boldsymbol{\xi}\left(t\right), -\xi_3\left(t\right) - \int_0^t \omega(\boldsymbol{\xi}'\left(s\right), \boldsymbol{W}_s - \frac{1}{2}\boldsymbol{\xi}\left(s\right))ds\right), \\
&u_0^\xi = e.
\end{aligned} \tag{4.5}
$$

Proof. We use the definition of the Maurer-Cartan form (3.8), Equation (3.9) and the fact that ξ is absolutely continuous to see that almost everywhere

$$
\begin{aligned}
\theta^l_{u^\xi_t}\left(du^\xi_t\right) &= \theta^l_{g_t}\left(dg_t\right) - \mathrm{Ad}_{g_t}\,\theta^r_{\xi(t)}\left(\xi'\left(t\right)\right)dt \\
&= dW_t - \mathrm{Ad}_{g_t}\,\theta^r_{\xi(t)}\left(\xi'\left(t\right)\right)dt \\
&= dW_t - \mathrm{Ad}_{g_t}\left(\boldsymbol{\xi}'\left(t\right), \xi_3'\left(t\right) + \frac{1}{2}\omega(\boldsymbol{\xi}\left(t\right), \boldsymbol{\xi}'\left(t\right))\right)dt.
\end{aligned}
$$

Now we can use an explicit form of the Brownian motion g_t given by (3.13) and Equation (3.5) to conclude that

$$
\begin{aligned}
&\theta^l_{u^\xi_t}\left(du^\xi_t\right) = \\
&= dW_t - \left(\boldsymbol{\xi}'\left(t\right), \xi_3'\left(t\right) + \frac{1}{2}\omega(\boldsymbol{\xi}\left(t\right), \boldsymbol{\xi}'\left(t\right)) + \omega(\boldsymbol{\xi}'\left(t\right), \boldsymbol{W}_t)\right)dt \\
&= dW_t - \left(\boldsymbol{\xi}'\left(t\right), \xi_3'\left(t\right) + \omega(\boldsymbol{\xi}'\left(t\right), \boldsymbol{W}_t - \frac{1}{2}\boldsymbol{\xi}\left(t\right))\right)dt
\end{aligned}
$$

which completes the proof. □

Equation (4.4) and Equation (4.5) have a simpler form if the curve ξ is assumed to be horizontal. More precisely we have the following corollary.

Corollary 4.2. *Suppose $\phi \in H\left(\mathbb{H}\right)$. Then the process u^ϕ_t defined by* (4.1), *satisfies the following equation*

$$
\begin{aligned}
&\mathbf{u}^\phi_t = \boldsymbol{W}_t - \boldsymbol{\phi}\left(t\right), \\
&u^\phi_3(t) = \frac{1}{2}\int_0^t \omega\left(\boldsymbol{W}_s - \boldsymbol{\phi}\left(s\right), d\boldsymbol{W}_s\right) + \frac{1}{2}\int_0^t \omega\left(\boldsymbol{W}_s - \boldsymbol{\phi}\left(s\right), \boldsymbol{\phi}'\left(s\right)\right)dt. \\
&u^\phi_0 = e.
\end{aligned}
\tag{4.6}
$$

Similarly Equation (4.5) *becomes*

$$
\begin{aligned}
&\int_0^t \theta^l_{u^\phi_s}\left(du^\phi_s\right) = \left(\boldsymbol{W}_t - \boldsymbol{\phi}\left(t\right), \int_0^t \omega\left(\boldsymbol{W}_s - \boldsymbol{\phi}\left(s\right), \boldsymbol{\phi}'\left(s\right)\right)ds\right), \\
&u^\phi_0 = e,
\end{aligned}
\tag{4.7}
$$

Proof. Indeed, if ϕ is horizontal, then Equation (3.10) gives that $\phi_3' = \frac{1}{2}\omega\left(\boldsymbol{\phi}, \boldsymbol{\phi}'\right)$ and therefore

$$
\begin{aligned}
&\theta^l_{u^\phi_t}\left(du^\phi_t\right) \\
&= \left(d\boldsymbol{W}_t - \boldsymbol{\phi}'\left(t\right)dt, \omega\left(\boldsymbol{W}_t, \boldsymbol{\phi}'\left(t\right)\right)dt - \left(\phi_3'\left(t\right) + \frac{1}{2}\omega\left(\boldsymbol{\phi}\left(t\right), \boldsymbol{\phi}'\left(t\right)\right)\right)dt\right) \\
&= \left(d\boldsymbol{W}_t - \boldsymbol{\phi}'\left(t\right)dt, \omega\left(\boldsymbol{W}_t, \boldsymbol{\phi}'\left(t\right)\right)dt - \omega\left(\boldsymbol{\phi}\left(t\right), \boldsymbol{\phi}'\left(t\right)\right)dt\right) \\
&= \left(d\boldsymbol{W}_t - \boldsymbol{\phi}'\left(t\right)dt, \omega\left(\boldsymbol{W}_t - \boldsymbol{\phi}\left(t\right), \boldsymbol{\phi}'\left(t\right)\right)dt\right)
\end{aligned}
$$

almost everywhere.

∎

Hence the asymptotics in Equation 3.16 reduces to

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t,\phi\left(t\right))<\varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} d_{cc}(g_t,\psi\left(t\right))<\varepsilon\right)}=\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho(u_t^{\phi},e)<\varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} \rho(u_t^{\psi},e)<\varepsilon\right)}, \tag{4.8}$$

where u_t^{ψ} is defined by (4.7) with ϕ replaced by ψ.

4.2. **Girsanov's transformation.** Before proceeding to the next step in our proof of Theorem 3.13, we recall several basic definitions such as the Doléans-Dade stochastic exponential. We refer to Appendix A for more details concerning the stochastic exponential and its connection with Girsanov's Theorem.

Notation 4.3. *For a continuous real-valued semimartingale X_t starting at 0 we denote by*

$$\mathcal{E}_t^X:=\exp\left(X_t-\frac{1}{2}\langle X\rangle_t\right),$$

the Doléans-Dade stochastic exponential of X_t, where $\langle X\rangle_t$ is the quadratic variation process for X_t.

Notation 4.4. *For any $\gamma\in H\left(\mathbb{H}\right)$ we denote by X_t^{γ} the real-valued stochastic process*

$$X_t^{\gamma}:=\int_0^t\langle c_{\gamma}\left(s\right),dW_s\rangle_{\mathcal{H}_e}, \tag{4.9}$$

where the Maurer-Cartan form c_{γ} is defined by (3.9), and $\mathcal{E}_t^{\gamma}:=\mathcal{E}\left(X^{\gamma}\right)$ is the Doléans-Dade stochastic exponential of X_t^{γ}.

Note that in this case the quadratic variation process for X_t^{γ} is given by

$$\langle X^{\gamma}\rangle_t=\int_0^t|c_{\gamma}(s)|^2_{\mathcal{H}_e}ds=\|\gamma\|^2_{H(\mathbb{H})}.$$

Remark 4.5. *Before proceeding with the proof, we would like to compare our hypoelliptic setting with the elliptic one. In the elliptic (Riemannian) setting, one can find a process y_t such that the asymptotics in (4.8) can be written as*

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}\left[\mathcal{E}_1^{\phi}\,|\mathbb{1}_{A_\varepsilon}\right]}{\mathbb{E}\left[\mathcal{E}_1^{\psi}\,|\mathbb{1}_{A_\varepsilon}\right]},$$

where $\mathcal{E}_t^{\phi}$ and $\mathcal{E}_t^{\psi}$ are stochastic exponentials and

$$A_\varepsilon := \left\{ \sup_{t\in[0,1]} \rho(y_t, e) < \varepsilon \right\}.$$

Note that the conditional expectation of two stochastic exponentials for ϕ and ψ are conditioned on the same event A_ε which depends on the Riemannian metric.

This is the approach used in [4], where to find an $\mathbb{R}^m$-valued process y_t M. Capitaine uses fundamentally the invertibility and differentiability of the Riemannian metric $g = (\sigma\sigma^T)^{-1}$. This is clearly not possible in the sub-Riemannian case. The main tool then is Girsanov's transformation on [4, p. 319] using the stochastic exponential $\mathcal{E}_t^A$, where

$$A(t,x) = \sigma^{-1}(t,x)\left(b(t,x) - \gamma(t,x)\right)$$

for γ defined in terms of the derivatives of the metric g.

Our main goal now is to evaluate the limit in (4.8), which will be accomplished in Proposition 4.9. We will use the Cameron-Martin-Girsanov Theorem (Theorem A.2).

Let $\gamma \in H(\mathbb{H})$ and define a probability measure $\mathbb{Q}^\gamma$ on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ as in Remark A.1. Then the Cameron-Martin-Girsanov Theorem (Theorem A.2) ensures that

$$\boldsymbol{W}_t^\gamma = (W_1(t) - \gamma_1(t), W_2(t) - \gamma_2(t))$$

is a standard two-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^\gamma)$. Then we can write (4.7) in terms of $\boldsymbol{W}_t^\gamma$ to see that u_t^ϕ satisfies the following equation

$$\begin{aligned} &\int_0^t \theta^l_{u_s^\phi}(du_s^\phi) \\ &= \left(\boldsymbol{W}_t^\gamma + \boldsymbol{\gamma}(t) - \boldsymbol{\phi}(t), \int_0^t \omega\left(\boldsymbol{W}_s^\gamma - \boldsymbol{\phi}(s) + \boldsymbol{\gamma}(s), \boldsymbol{\phi}'(s)\right) ds \right), \\ &u_0^\phi = e. \end{aligned} \tag{4.10}$$

The expression in (4.8) involves both ϕ and ψ, so we would like to incorporate both in the application of Girsanov's transformation. By looking at Equation (4.7) and Equation (4.10), a natural choice for γ would be $\gamma(t) := \phi(t)\psi(t)^{-1}$. But the space $H(\mathbb{H})$ is not closed under pointwise multiplication or taking inverses. Indeed, it is clear from Equation (3.10) that the pointwise inverse and product of horizontal curves is not horizontal in general. Therefore we consider a horizontal curve γ such that

$$c_\gamma(t) = c_{\phi\cdot\psi^{-1}}(t)\big|_{\mathcal{H}_e}.$$

For example, one can take

$$\gamma(t) := \left(\boldsymbol{\phi}(t) - \boldsymbol{\psi}(t), \frac{1}{2} \int_0^1 \omega\left(\boldsymbol{\phi}(s) - \boldsymbol{\psi}(s), \boldsymbol{\phi}'(s) - \boldsymbol{\psi}'(s)\right) ds \right). \tag{4.11}$$

Note that if both ϕ and ψ have finite energy, then the horizontal curve γ also has finite energy. For this choice of γ Equation (4.10) becomes

$$\begin{aligned} \int_0^t \theta^l_{u^\phi_s}(du^\phi_s) &= \left(\boldsymbol{W}^\gamma - \boldsymbol{\psi}(t), \int_0^t \omega\left(\boldsymbol{W}^\gamma_s - \boldsymbol{\psi}(s), \boldsymbol{\phi}'(s)\right) ds \right) \\ &= \left(\boldsymbol{W}^\gamma_t - \boldsymbol{\psi}(t), \int_0^t \omega\left(\boldsymbol{W}^\gamma_s - \boldsymbol{\psi}(s), \boldsymbol{\psi}'(s)\right) ds \right) \\ &\quad + \int_0^t \left(0, 0, \omega\left(\boldsymbol{W}^\gamma_s - \boldsymbol{\psi}(s), \boldsymbol{\gamma}'(s)\right)\right) ds. \end{aligned} \tag{4.12}$$

Remark 4.6. *Note that one might want to evaluate the limit in* (4.8) *by finding a measure $\mathbb{Q}^\gamma$ for $\gamma \in H(\mathbb{H})$ in such a way that the law of u^ϕ_t under $\mathbb{Q}^\gamma$ is the same as the one of u^ψ_t under $\mathbb{P}$. Then*

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u^\phi_t, e\right) < \varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u^\psi_t, e\right) < \varepsilon\right)} = \lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u^\phi_t, e\right) < \varepsilon\right)}{\mathbb{Q}^\gamma\left(\sup_{t\in[0,1]} \rho\left(u^\phi_t, e\right) < \varepsilon\right)}$$

$$= \lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u^\phi_t, e\right) < \varepsilon\right)}{\mathbb{E}\left[\mathcal{E}^\gamma_1 \mathbb{1}_{\left\{\sup_{t\in[0,1]} \rho\left(u^\phi_t, e\right) < \varepsilon\right\}}\right]} = \lim_{\varepsilon \to 0} \frac{1}{\mathbb{E}\left[\mathcal{E}^\gamma_1 | \mathbb{1}_{\left\{\sup_{t\in[0,1]} \rho\left(u^\phi_t, e\right) < \varepsilon\right\}}\right]},$$

so that one is left with estimating a conditional expectation of an exponential martingale.

The simplest situation for which this holds is if $\phi(t) = \psi(t)$ for all t in $[0,1]$, since Equation (4.12) *becomes*

$$\int_0^t \theta^l_{u^\phi_s}(du^\phi_s) = \left(\boldsymbol{W}^\gamma - \boldsymbol{\psi}(t), \int_0^t \omega\left(\boldsymbol{W}^\gamma_s - \boldsymbol{\psi}(s), \boldsymbol{\psi}'(s)\right) ds \right).$$

Motivated by Equation (4.12) we define a process z_t in $\mathbb{H}$ by

$$\int_0^t \theta^l_{z_s}(dz_s) = \left(\boldsymbol{W}_t - \boldsymbol{\psi}(t), \int_0^t \omega\left(\boldsymbol{W}_s - \boldsymbol{\psi}(s), \boldsymbol{\phi}'(s)\right) ds \right). \tag{4.13}$$

Then comparing Equation (4.13) and Equation (4.12) we see that the law of z_t under $\mathbb{P}$ is the same as the law of u^ϕ_t under $\mathbb{Q}^\gamma$.

Additionally, Equation (4.13) and Equation (4.7) imply that for every fixed t in $[0,1]$ the following identity holds

$$\boldsymbol{z}_t = \boldsymbol{u}_t^{\psi}, \tag{4.14}$$
$$z_3(t) = u_3^{\psi}(t) + \int_0^t \omega\left(\boldsymbol{u}_s^{\psi}, \boldsymbol{\gamma}'(s)\right) ds.$$

As a consequence, we have the following lemma.

Lemma 4.7. *Let z_t and u_t^{ψ} be the processes defined by* (4.13) *and* (4.7) *respectively, and consider the following limits*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\psi}, e\right) < \varepsilon\right), \tag{4.15}$$

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(z_t, e\right) < \varepsilon\right). \tag{4.16}$$

Then the limit (4.15) *exists if and only if the limit* (4.16) *does. Moreover, if one of them exists, they are equal.*

Proof. Assume that $\sup_{t\in[0,1]} \rho\left(u_t^{\psi}, e\right) < \varepsilon$, then

$$\sup_{t\in[0,1]} \left|u_i^{\psi}(t)\right| < \varepsilon, i = 1, 2, \tag{4.17}$$
$$\sup_{t\in[0,1]} \left|u_3^{\psi}(t)\right| < \varepsilon^2.$$

Since $\mathbf{z}_t = \mathbf{u}_t^{\psi}$, to prove the statements we only need to check that in this case $z_3(t)$ is small. Observe that

$$|\omega\left(\mathbf{v}_1, \mathbf{v}_2\right)| \leqslant \max\left\{x_1, y_1\right\}\left(|x_2| + |y_2|\right),$$

which with (4.17) gives that

$$|\omega\left(\mathbf{u}_s^{\psi}, \boldsymbol{\gamma}'(s)\right)| < \varepsilon \sum_{i=1}^{2} \left|\gamma_i'(s)\right| = \varepsilon \|c_\gamma(s)\|_1 = \varepsilon C_\gamma.$$

Here we denote

$$C_\gamma := \int_0^1 \|c_\gamma(s)\|_1 ds = \int_0^1 \sum_{i=1}^{2} \left|\gamma_i'(s)\right| ds$$

for γ defined by (4.11). Therefore

$$|z_3(t)| < \left|u_3^{\psi}(t)\right| + \varepsilon \int_0^1 \|c_\gamma(s)\|_1 ds < \varepsilon^2 + \varepsilon C_\gamma.$$

Combining everything together we get

$$\rho^2\left(z_t, e\right) < 3\varepsilon^2 + \varepsilon C_\gamma$$

for all $t \in [0,1]$.

Similarly if $\rho\left(z_t, e\right) < \varepsilon$ we use the fact that we can write

$$u_3^{\psi}(t) = z_3 - \int_0^t \omega\left(\mathbf{z}_t, \boldsymbol{\gamma}'(s)\right) ds$$

to see that

$$\rho^2\left(u_t^{\psi}(t), e\right) < 3\varepsilon^2 + \varepsilon C_{\gamma}.$$

□

Before proceeding to the proof of Proposition 4.9, we need the following lemma whose proof can be found in [10, pp. 536-537].

Lemma 4.8 (pp. 536-537 in [10])**.** *Let $I_1, \ldots, I_n$ be n random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{A_\varepsilon\}_{0<\varepsilon<1}$ be a family of events in $\mathcal{F}$ and $a_1, \ldots, a_n$ be n numbers. If for every real number c and every $1 \leqslant i \leqslant n$*

$$\limsup_{\varepsilon \to 0} \mathbb{E}\left[\exp(c\, I_i)\, | A_\varepsilon\right] \leqslant \exp(c\, a_i),$$

then

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\exp\left(\sum_{i=1}^n I_i\right) | A_\varepsilon\right] = \exp\left(\sum_{i=1}^n a_i\right).$$

Proposition 4.9. *Let $\gamma \in H(\mathbb{H})$ defined by* (4.11)*, then*

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\psi}, e\right) < \varepsilon\right)} = \exp\left(\frac{1}{2}\|\gamma\|^2_{H(\mathbb{H})} - \langle \gamma, \phi\rangle_{H(\mathbb{H})}\right).$$

Proof. From the definition of z_t, $\mathbb{Q}^{\gamma}$ and by Lemma 4.7 we have

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\psi}, e\right) < \varepsilon\right)} = \lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right)}{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(z_t, e\right) < \varepsilon\right)}$$

$$= \lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right)}{\mathbb{Q}^{\gamma}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right)} = \lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right)}{\mathbb{E}\left[\mathcal{E}_1^{\gamma} \mathbb{1}_{\left\{\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right\}}\right]}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\mathbb{E}\left[\mathcal{E}_1^{\gamma} | \mathbb{1}_{\left\{\sup_{t\in[0,1]} \rho\left(u_t^{\phi}, e\right) < \varepsilon\right\}}\right]},$$

where

$$\mathcal{E}_1^{\gamma} = \exp\left(\int_0^1 \langle c_\gamma(s), dW(s)\rangle_{\mathcal{H}_e} - \frac{1}{2}\int_0^1 |c_\gamma(s)|^2_{\mathcal{H}_e} ds\right).$$

In order to evaluate this limit we apply Lemma 4.8. Note that we can use (4.7) to write

$$\int_0^t \theta^l_{g_s}(dg_s) = \int_0^t \theta^l_{u_s^\phi}\left(du_s^\phi\right)\Big|_{\mathcal{H}_e} ds + \int_0^t c_\phi(s)\, ds,$$

where the projection to the horizontal subspace $\mathcal{H}_e$ is

$$\int_0^t \theta^l_{u_s^\phi}\left(du_s^\phi\right)\Big|_{\mathcal{H}_e} = \left(\boldsymbol{W}(t) - \boldsymbol{\phi}(t), 0\right).$$

Thus the following identity holds

$$\begin{aligned}
&\int_0^1 \langle c_\gamma(s), dW(s)\rangle_{\mathcal{H}_e} = \int_0^1 \langle c_\gamma(s), \theta^l_{g_t}(dg_t)\rangle_{\mathcal{H}_e} \\
&= \int_0^1 \langle c_\gamma(s), \theta^l_{u_s^\phi}(du_s^\phi)\rangle_{\mathcal{H}_e} + \int_0^1 \langle c_\gamma(s), c_\phi(s)\rangle_{\mathcal{H}_e} ds \\
&= \int_0^1 \langle c_\gamma(s), \theta^l_{u_t^\phi}(du_s^\phi)\rangle_{\mathcal{H}_e} + \langle \gamma, \phi\rangle_{H(\mathbb{H})}.
\end{aligned}$$

Therefore for any $c \in \mathbb{R}$

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \mathbb{E}\left[\exp\left(c\int_0^1 \langle c_\gamma(s), dW(s)\rangle_{\mathcal{H}_e}\right)\Big| \mathbb{1}_{\left\{\sup_{t\in[0,1]}\rho\left(u_t^\phi, e\right)<\varepsilon\right\}}\right] \\
&\leqslant \exp\left(c\langle\gamma,\phi\rangle_{H(\mathbb{H})}\right).
\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\mathcal{E}_1^\gamma \Big| \mathbb{1}_{\left\{\sup_{t\in[0,1]}\rho\left(u_t^\phi, e\right)<\varepsilon\right\}}\right] = \exp\left(\langle\gamma,\phi\rangle_{H(\mathbb{H})} - \frac{1}{2}\|\gamma\|^2_{H(\mathbb{H})}\right)$$

which completes the proof. □

In particular, (4.11) implies that the Onsager-Machlup functional L can be explicitly expressed as follow

$$\frac{1}{2}\|\gamma\|^2_{H(\mathbb{H})} - \langle\gamma,\phi\rangle_{H(\mathbb{H})} = -\frac{1}{2}\|\phi\|^2_{H(\mathbb{H})} + \frac{1}{2}\|\psi\|^2_{H(\mathbb{H})}$$

$$= \int_0^1 L\left(\phi(s), \phi'(s)\right) ds - \int_0^1 L\left(\psi(s), \psi'(s)\right) ds.$$

Appendix A. Girsanov's theorem

Let $\gamma \in H(\mathbb{H})$, that is, an absolutely continuous finite energy horizontal curve starting at the identity. Consider the process X_t^γ defined by (4.9), and the corresponding the Doléans-Dade stochastic exponential of X_t^γ, that is, $\mathcal{E}_t^\gamma = \mathcal{E}(X^\gamma)$. First observe that the process $\mathcal{E}_t^\gamma$ is a martingale with respect to the measure $\mathbb{P}$ since γ has finite energy and therefore

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle X^\gamma\rangle_t\right)\right] = \exp\left(\frac{1}{2}\int_0^t |c_\gamma(s)|^2_{\mathcal{H}_e} ds\right) < \infty.$$

Thus X_t^γ satisfies Novikov's condition and therefore $\mathcal{E}_t^\gamma$ is a martingale with respect to the measure $\mathbb{P}$.

Let $\mathbb{Q}_t^\gamma$ be a measure on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{0\leqslant s\leqslant t}\right)$ which is absolutely continuous with respect to $\mathbb{P}$ with the Radon-Nikodym derivative given by

$$\frac{d\mathbb{Q}_t^\gamma}{d\mathbb{P}} = \mathcal{E}_t^\gamma,$$

that is,

$$\mathbb{Q}_t^\gamma(A) = \int_A \mathcal{E}_t^\gamma d\mathbb{P} = \mathbb{E}\left[\mathcal{E}_t^\gamma \mathbb{1}_A\right].$$

Remark A.1. *Denote $\mathbb{Q}^\gamma := \mathbb{Q}_1^\gamma$. Note that since $\mathcal{E}_t^\gamma$ is a martingale, we have*

$$\mathbb{Q}^\gamma(A) = \mathbb{Q}_t^\gamma(A) \tag{A.1}$$

for all $t \in [0,1]$ and $A \in \mathcal{F}_t$. Indeed,

$$\mathbb{Q}_t^\gamma(A) = \mathbb{E}\left[\mathcal{E}_t^\gamma \mathbb{1}_A\right] = \mathbb{E}\left[\mathbb{E}\left[\mathcal{E}_1^\gamma | \mathcal{F}_t\right] \mathbb{1}_A\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathcal{E}_1^\gamma \mathbb{1}_A | \mathcal{F}_t\right]\right] = \mathbb{E}\left[\mathcal{E}_1^\gamma \mathbb{1}_A\right] = \mathbb{Q}^\gamma(A).$$

Suppose now that $\boldsymbol{W}_t = (W_1(t), W_2(t))$ is a two-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We want to describe a two-dimensional Brownian motion under $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^\gamma)$. Let us consider the process

$$\boldsymbol{W}_t^\gamma := \left(W_1(t) - \langle W_1, X^\gamma\rangle_t, W_2(t) - \langle W_2, X^\gamma\rangle_t\right), \tag{A.2}$$

where $\langle X, Y\rangle_t$ denotes the quadratic covariation process for two processes X_t and Y_t. Note that $\boldsymbol{W}_t^\gamma$ can be written as

$$\boldsymbol{W}_t^\gamma = \left(W_1(t) - \gamma_1(t), W_2(t) - \gamma_2(t)\right).$$

Indeed,

$$\langle W_1, X^\gamma\rangle_t = \frac{1}{2}\left[\langle W_1 + X^\gamma\rangle_t - \langle W_1\rangle_t - \langle X^\gamma\rangle(t)\right] =$$
$$\frac{1}{2}\left[\int_0^t \left(\gamma_1'(s)+1\right)^2 + \gamma_2'(s) ds - t - \sum_{i=1}^2 \int_0^t \gamma_i'(s) ds\right] =$$
$$\frac{1}{2}\left[2\int_0^t \gamma_1'(s) ds\right] = \gamma_1(t).$$

Similarly we have that $\langle W_2, X^\gamma\rangle_t = \gamma_2(t)$. Thus we can use the classical Cameron-Martin-Girsanov Theorem which we include below for completeness, the proof can be found in [11, Theorem 5.1]. Note that for deterministic γ this was proved in [3].

Theorem A.2 (Cameron-Martin-Girsanov Theorem)**.** *Assume that* $\boldsymbol{W}_t$ *is a two-dimensional Brownian motion on* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$*. For any* $\gamma \in H(\mathbb{H})$ *we denote by* $\mathbb{Q}^\gamma$ *the probability measure defined by* (A.1)*. Then the process* $\boldsymbol{W}_t^\gamma$ *in* (A.2)*, is a Brownian motion on* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^\gamma)$ *for* $t \in [0, 1]$*.*

Acknowledgement. *The second author would like to thank Ofer Zeitouni for suggesting this problem during his visit to the University of Connecticut in Fall of 2018.*

References

[1] Fabrice Baudoin, Qi Feng, and Maria Gordina, *Integration by parts and quasi-invariance for the horizontal wiener measure on foliated compact manifolds*, Journal of Functional Analysis **277** (2019), no. 5, 1362 – 1422.

[2] Fabrice Baudoin, Maria Gordina, and Tai Melcher, *Quasi-invariance for heat kernel measures on sub-Riemannian infinite-dimensional Heisenberg groups*, Trans. Amer. Math. Soc. **365** (2013), no. 8, 4313–4350. MR 3055697

[3] R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. of Math. (2) **45** (1944), 386–396. MR MR0010346 (6,5f)

[4] Mireille Capitaine, *On the Onsager-Machlup functional for elliptic diffusion processes*, Séminaire de Probabilités, XXXIV, Lecture Notes in Math., vol. 1729, Springer, Berlin, 2000, pp. 313–328. MR 1768071

[5] Bruce K. Driver and Maria Gordina, *Heat kernel analysis on infinite-dimensional Heisenberg groups*, J. Funct. Anal. **255** (2008), no. 9, 2395–2461. MR MR2473262

[6] Takahiko Fujita and Shinichi Kotani, *The Onsager-Machlup function for diffusion processes*, J. Math. Kyoto Univ. **22** (1982/83), no. 1, 115–130. MR 648558

[7] K. Hara and Y. Takahashi, *Stochastic analysis in a tubular neighborhood or Onsager-Machlup functions revisited*, arXiv e-prints (2016), XXX.

[8] Keisuke Hara and Yoichiro Takahashi, *Lagrangian for pinned diffusion process*, Itô's stochastic calculus and probability theory, Springer, Tokyo, 1996, pp. 117–128. MR 1439520

[9] Lars Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171. MR 0222474 (36 #5526)

[10] Nobuyuki Ikeda and Shinzo Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam, 1989. MR MR1011252 (90m:60069)

[11] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR 1121940 (92h:60127)

[12] Terry Lyons and Ofer Zeitouni, *Conditional exponential moments for iterated Wiener integrals*, Ann. Probab. **27** (1999), no. 4, 1738–1749. MR 1742886

[13] S. Machlup and L. Onsager, *Fluctuations and irreversible process. II. Systems with kinetic energy*, Physical Rev. (2) **91** (1953), 1512–1515. MR 0057766

[14] ———, *Fluctuations and irreversible processes*, Physical Rev. (2) **91** (1953), 1505–1512. MR 0057765

[15] Paolo Pigato, *Tube estimates for diffusion processes under a weak Hörmander condition*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 1, 299–342. MR 3765891

[16] L. A. Shepp and O. Zeitouni, *Exponential estimates for convex norms and some applications*, Barcelona Seminar on Stochastic Analysis (St. Feliu de Guíxols, 1991), Progr. Probab., vol. 32, Birkhäuser, Basel, 1993, pp. 203–215. MR 1265050

[17] Y. Takahashi and S. Watanabe, *The probability functionals (Onsager-Machlup functions) of diffusion processes*, Stochastic integrals (Proc. Sympos., Univ. Durham,

Durham, 1980), Lecture Notes in Math., vol. 851, Springer, Berlin, 1981, pp. 433–463. MR 620998

[18] Ofer Zeitouni, *On the Onsager-Machlup functional of diffusion processes around non-C^2-curves*, Ann. Probab. **17** (1989), no. 3, 1037–1054. MR 1009443

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.

Email address: marco.carfagnini@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.

Email address: maria.gordina@uconn.edu