# Multiplicative Probability Limit Theorems and Their Applications

**Rajeshwari Majumdar**

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**Abstract**

Limit theorems, such as the Central Limit Theorem and the Law of Large Numbers, are fundamental concepts in probability theory. They have been studied extensively in various settings and at widely varying levels of generality over the last three hundred years, culminating in significant results about the asymptotic behavior of sums of random variables (which may be substituted by vectors, matrices, or elements of more abstract spaces). More recently, there has been research on the asymptotic behavior of products of such random elements when the notion of multiplication is defined in the ambient space. We explore some of these "multiplicative limit theorems" and present two new applications of these tools: deriving the Black-Scholes European call option pricing model and understanding the behavior of Lyapunov exponents in the context of products of random matrices.

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# Chapter 1

# Introduction

The Central Limit Theorem is a cornerstone of modern probability theory, with Laplace, Poisson, Cauchy, Lindeberg, and Lévy among the major contributors to its development in the nineteenth and twentieth centuries. The basic form of the Central Limit Theorem, as given in [13], is as follows: the sum of a sufficiently large number of independent and identically distributed random variables with finite mean and variance approximates a normal random variable in distribution. Formally, let $X_1, X_2, \ldots,$ be a sequence of random variables and $S_n = \sum_{k=1}^{n} X_k$. Then, under a variety of different conditions, the distribution function of the appropriately centered and normalized sum $S_n$ converges pointwise to the standard normal distribution function as $n \to \infty$, that is, the centered and normalized $S_n$ converges weakly to $Z$, the standard normal random variable.

Empirically, a normal distribution model has turned out to be appropriate for describing a wide variety of phenomena in "real life." For all we know, that is an illustration of the Central Limit Theorem in action. For many phenomena and processes in real life, how they evolve is determined by a large number of factors, with each making its own, small contribution. That is exactly the concept that finds its mathematical formulation in the statement of the Central Limit Theorem.

Another extremely significant probability limit theorem is the Law of Large Numbers. While in ten tosses of a "fair" coin, we expect 5 heads and 5 tails, it is not out of the ordinary to see 4 heads or 4 tails. On the other hand, if in a thousand tosses of a "fair" coin, we see 400 heads or 400 tails, that will be considered irrefutable evidence against the coin being fair. The phenomenon that the fluctuation in the proportion of heads in a large number of tosses of a fair coin steadily wanes as the number of tosses gets larger and larger (and the proportion gets increasingly closer to 0.5) has been known for a long time. The first rigorous formulation of the result was provided by (Jacob) Bernoulli in the early 18$^{\text{th}}$ century. More than 100 years after Bernoulli, Poisson expanded on Bernoulli's work and coined the term "Law of Large Numbers." Chebyshev, Markov, Kolmogorov, and Khinchin, among others, strengthened both the Weak Law of Large Numbers as well as the Strong Law of Large Numbers by successively weakening the assumptions. Today, a fairly general and standard version of the Law of Large numbers can be found in [5]: Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed random variables. If $\mathbb{E}\left(|X_1|\right) < \infty$, then $\frac{X_1 + \cdots + X_n}{n} \to \mathbb{E}\left(X_1\right)$ almost surely as $n \to \infty$.

The Central Limit Theorem and Law of Large Numbers have found applications in various disciplines, ranging from physics and engineering to economics and finance. We propose two additional applications: (1) deriving the Black-Scholes European call option pricing model without having to use advanced technical machinery such as stochastic differential equations and (2) understanding the behavior of Lyapunov exponents in the context of products of random matrices. Both of these applications concern *products of random variables* rather than the random variables themselves. Thus, we employ multiplicative versions of the traditional, additive limit theorems to handle these applications.

In Chapter 2, we consider an alternative approach to the standard derivation of the Black-Scholes European call option pricing model using the Central Limit Theorem; more specifically, we exploit properties of products of random variables under a “multiplicative” version of the Lindeberg-Feller Central Limit Theorem. The Black-Scholes model was proposed by Fischer Black and Myron Scholes in their 1973 paper entitled “The Pricing of Options and Corporate Liabilities.” They derived a formula to find the value of a “European-style” option in terms of the price of the stock by utilizing techniques from stochastic calculus and partial differential equations [2]. Later in 1973, Robert C. Merton expanded the mathematical ideas underlying the Black-Scholes model in his paper entitled “Theory of Rational Option Pricing” [11]. Since its introduction, the formula has been widely used by option traders to approximate prices and has lead to a variety of new models for pricing derivatives. The standard derivation of the Black-Scholes model, which can be found in [15], uses over 100 pages to arrive at the formula and requires a discussion on geometric Brownian motion.

Our second application of multiplicative analogues of probability limit theorems, presented in Chapter 3, involves the Lyapunov exponent, which measures the exponential growth rate of the operator norm of the partial products of a sequence of independent and identically distributed random matrices. It is difficult to compute this quantity explicitly from the distribution of the matrices as there is no general method. Using analogues to the Central Limit Theorem and Law of Large Numbers for the norm of the partial products of a sequence of such random matrices, we explore new ways of efficiently computing Lyapunov exponent for several random matrix models and numerically estimating corresponding variances.

# Chapter 2

# A Derivation of the Black-Scholes Option Pricing Model

Our derivation of the Black-Scholes European call option pricing model using a Central Limit Theorem approach is inspired by Chapters 17 and 18 of [16]. We give a rigorous mathematical treatment of the results discussed in that text using an elementary approach that is accessible to students who have taken an undergraduate probability course. We first introduce the basic financial concepts underlying the Black-Scholes model.

A **financial instrument** is any asset that can be traded on the market. Consider the following type of instrument: If an event $B$ occurs, the holder of the instrument receives one dollar, and if $B$ does not occur, the holder receives nothing. The value of such an instrument is dependent on the probability that the event occurs. This probability is assessed through a **pricing measure**, denoted by $Q$. A pricing measure can be understood as a way to determine the amount of the underlying asset that one would be willing to pay in order to own a financial instrument. For example, if a financial instrument involves the exchange of one dollar given the event $B$ occurs, and the probability that event $B$ occurs is $Q(B)$, an individual would be willing to risk $Q(B)$ dollars to own the instrument.

A measuring unit for the price of a financial instrument is called a **numeraire**. In the previous example, the dollar would function as a numeraire and the pricing measure would be with respect to dollars. Numeraires have time stamps, so their value corresponds to a specific date. Consider numeraires such as one unit of cash today, or one unit of cash at a future time $t$; the value of that unit of cash today may differ from its value at time $t$. Thus, we specify that a pricing measure is with respect to the numeraire unit cash at time-$t$.

A call (respectively, put) **option** is a contract that gives the option holder the right to buy (respectively, sell) an asset for a certain price $K$, called the **strike price**, during the time period $[0, t]$ (for an American option) or at time $t$ (for a European option), where $t$ is the expiration time of that right (often referred to as just the **expiration time**). We price a **European call option**, which entitles the holder to purchase a unit of the underlying asset at expiration $t$ for strike $K$.

The Black-Scholes model for the price of a European call option is derived under the assumption that there is no arbitrage opportunity surrounding a trade of the option (or the underlying instrument), that is, one cannot expect to generate a risk-free profit by purchasing (or selling) the option (or the underlying instrument). We denote the time when the option is priced as time 0, when the underlying instrument is valued at $X_0$. Recall that the option expires at time $t$ and the strike price for the option is $K$. Suppose the risk-free rate of interest is $r$. If the option is priced for

$C$, then the future-value of it at time $t$, under continuous compounding, is $Ce^{rt}$. With $X_t$ denoting the price of the underlying instrument at time $t$, the payoff of the option is $\max(X_t - K, 0)$. The no arbitrage opportunity on the option trade requires the equation

$$C = e^{-rt}\mathbb{E}\left(\max\left(X_t - K, 0\right)\right) \tag{2.0.1}$$

to hold; similarly, the no arbitrage opportunity on the trade of the underlying instrument requires the equation

$$\mathbb{E}\left(X_t\right) = X_0 e^{rt} \tag{2.0.2}$$

to hold.

The formula used to price the European call option under the **Black-Scholes European option pricing model** is given by

$$C = X_0 N(d_+) - Ke^{-rt}N(d_-), \tag{2.0.3}$$

where $N$ is the standard Normal CDF, that is,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy,$$

$$d_\pm = \frac{1}{\sigma\sqrt{t}} \log\left[e^{rt}X_0/K\right] \pm \frac{1}{2}\sigma\sqrt{t},$$

and $\sigma$ is the volatility of the return on the underlying asset through expiration.

**Example 2.0.1.** *Consider the pricing of a European call option on a stock with a present value of 50 Euros and a strike price of 52 Euros under the following conditions: $r = 4\%$ (per annum), $t = 1$ (year), $\sigma = 0.15$. To calculate the price of this option we use (2.0.3). We first find*

$$d_+ = \frac{\log\left[e^{0.04(1)}50/52\right]}{0.15} + \frac{1}{2}(0.15) = 0.0802$$

*and*

$$d_- = \frac{\log\left[e^{0.04(1)}50/52\right]}{0.15} - \frac{1}{2}(0.15) = -0.0698;$$

*we then have*

$$\begin{aligned} C &= 50N(0.0802) - 52e^{-(0.04)(1)}N(-0.0698) \\ &= 50(.532) - 52(0.96)(0.472) \\ &= 3.04. \end{aligned}$$

*Thus, from the Black-Scholes model, the price of this call option would be 3.04 Euros.*

In Section 2.1, we derive the call option pricing formula assuming the log-normality of the underlying asset price; in Section 2.2, we prove the log-normality. In Section 2.3, we examine a related model of stock price, calculate the expected value and variance of the log-asset price at time $t$, and perform tests of normality.

## 2.1 Pricing the European Call Option

To derive the call option pricing formula in (2.0.3) we first show the following fact regarding normal random variables.

**Lemma 2.1.1.** *For any normal random variable $Y$ with mean $\mu_Y$, standard deviation $\sigma_Y$, and $M > 0$, we have*

$$\mathbb{E}\left(\max\left(e^Y - M, 0\right)\right) = \mathbb{E}\left(e^Y\right) N\left(h_+\right) - MN\left(h_-\right),$$

*where*

$$h_\pm = \left[\log\left(\mathbb{E}\left(e^Y\right) \big/ M\right) \pm \frac{1}{2}\sigma_Y^2\right] \big/ \sigma_Y.$$

*Proof.* For a normal random variable $Y$,

$$\mathbb{E}\left(e^Y\right) = e^{\mu_Y + \frac{\sigma_Y^2}{2}}.$$

As such,

$$h_+ = \frac{\mu_Y + \sigma_Y^2 - \log M}{\sigma_Y}$$

and

$$h_- = \frac{\mu_Y - \log M}{\sigma_Y}.$$

Now note that

$$\mathbb{E}\left(\max\left(e^Y - M, 0\right)\right) = \int_{\log M}^{\infty} e^y \phi_{\mu_Y, \sigma_Y}(y)\, dy - MP\left(Y > \log M\right),$$

where $\phi_{\mu_Y, \sigma_Y}$ is the density of $Y$. Completing the square

$$y - \frac{\left(y - \mu_Y\right)^2}{2\sigma_Y^2} = \mu_Y + \frac{\sigma_Y^2}{2} - \frac{\left(y - \left(\mu_Y + \sigma_Y^2\right)\right)^2}{2\sigma_Y^2},$$

we obtain, using the identity $1 - N(x) = N(-x)$,

$$\int_{\log M}^{\infty} e^y \phi_{\mu_Y, \sigma_Y}(y)\, dy = e^{\mu_Y + \frac{\sigma_Y^2}{2}} N\left(h_+\right).$$

Since

$$P\left(Y > \log M\right) = N\left(h_-\right),$$

the equality follows. □

Recall that under the assumption of no arbitrage, the price of a European call option must equal the expected payoff of the option. Expectation is computed with respect to the pricing measure $Q_t$, corresponding to time-$t$ cash numeraire.

**Proposition 2.1.2.** *Assume there are no opportunities for arbitrage and the risk-free interest rate is $r$. Consider a European call option on an instrument with expiration $t$ and strike $K$. Let $X_t$ be the time-$t$ price of the underlying instrument, where $X_t = X_0 e^{Y_t}$ and the $Q_t$-induced distribution of $Y_t$ is $\mathcal{N}(\mu_{Y_t}, \sigma^2_{Y_t})$. Then, the discounted (that is, time-0) price of the call option, $C$, is given by*

$$C = X_0 N(d_+) - K e^{-rt} N(d_-), \tag{2.1.1}$$

*where*

$$d_\pm = \frac{1}{\sigma_{Y_t}} \log\left[e^{rt} X_0 / K\right] \pm \frac{1}{2}\sigma_{Y_t}.$$

*Proof.* From the definition of $X_t$,

$$\max\left(X_t - K, 0\right) = X_0 \max\left(e^{Y_t} - \frac{K}{X_0}, 0\right).$$

By Lemma 2.1.1,

$$\mathbb{E}\left(\max\left(e^{Y_t} - \frac{K}{X_0}, 0\right)\right) = \mathbb{E}\left(e^{Y_t}\right) N\left(h_+\right) - \frac{K}{X_0} N\left(h_-\right)$$

with

$$h_\pm = \left[\log\left(\mathbb{E}\left(e^{Y_t}\right)\frac{X_0}{K}\right) \pm \frac{1}{2}\sigma^2_{Y_t}\right] \Big/ \sigma_{Y_t} = d_\pm,$$

where the second equality follows from (2.0.2). The proof follows from (2.0.1). □

## 2.2 Log-Normality of Prices

In the previous section, we derived the Black-Scholes model with the premise that our prices follow a log-normal distribution. In this section, we use the Lindeberg-Feller Central Limit Theorem (as stated in [17]) to prove this premise under suitable assumptions.

**Theorem 2.2.1 (Lindeberg-Feller).** *Suppose for each $n$ and $i = 1, \ldots n$, $X_{ni}$ are independent and have mean $0$. Let $S_n = \sum_{i=1}^n X_{ni}$. Suppose that $\sum_{i=1}^n \mathbb{E}[X^2_{ni}] \to \sigma^2$ for $0 < \sigma^2 < \infty$. Then, the following two conditions are equivalent:*

*(a) $S_n$ converges weakly to a normal random variable with mean $0$ and variance $\sigma^2$, and the triangular array $\{X_{ni}\}$ satisfies the condition that*

$$\lim_{n\to 0} \max_i \mathbb{E}\left(X^2_{ni}\right) = 0.$$

*(b) (Lindeberg Condition) For all $\epsilon > 0$,*

$$\sum_{i=1}^n \mathbb{E}\left[X^2_{ni}; |X_{ni}| > \epsilon\right] \to 0.$$

We make the following three assumptions.

**Assumption 1.** *For each $t$, the random variable $Y_t = \log \frac{X_t}{X_0}$ has finite variance.*

**Assumption 2.** *The process $Y_t$ has stationary and independent increments. That is, the differences $Y_t - Y_s$ are independent for disjoint intervals $[s,t]$; for intervals of equal length, they are i.i.d.*

**Assumption 3.** *For every $\epsilon > 0$, $n\mathbb{E}\left[\left(Y_{t/n} - Y_0\right)^2; \left|Y_{t/n} - Y_0\right| > \epsilon\right] \to 0$ as $n \to \infty$.*

**Theorem 2.2.2.** *Under Assumptions 1, 2, and 3, for every $t > 0$, $Y_t$ is a normal random variable with respect to the pricing measure $Q_0$ with variance $\sigma^2 t$ for some constant $\sigma \geq 0$ and all $t \geq 0$.*

In addition to the Lindeberg-Feller Central Limit Theorem (Theorem 2.2.1), the proof of Theorem 2.2.2 makes use of the following lemma.

**Lemma 2.2.3.** *Suppose $f : [0,\infty) \to [0,\infty)$ satisfies $f(x+y) = f(x) + f(y)$. There exists a constant $C$ such that $f(x) = Cx$ for all $x \geq 0$.*

*Proof.* We first observe that

$$f(0) = f(0+0) = f(0) + f(0) = 2f(0),$$

implying that $f(0) = 0$. We can prove by induction that

$$f(m) = mf(1) \text{ for all } m \geq 1. \tag{2.2.1}$$

Let $p$ and $q$ $(> 1)$ be positive integers with no common factors. By induction again,

$$f\left(m\frac{p}{q}\right) = mf\left(\frac{p}{q}\right) \text{ for all } m \geq 1. \tag{2.2.2}$$

Using (2.2.1) followed by (2.2.2),

$$pf(1) = f(p) = f\left(q\frac{p}{q}\right) = qf\left(\frac{p}{q}\right);$$

hence, for every positive rational number $r$ of the form $p/q$,

$$f(r) = f\left(\frac{p}{q}\right) = \frac{p}{q}f(1) = rf(1). \tag{2.2.3}$$

Thus, we have shown that for every rational number $x$ in $\left[0,\infty\right)$,

$$f(x) = Cx \text{ where } C = f(1). \tag{2.2.4}$$

Note that if $x \leq y$, then

$$f(y) = f(x + y - x) = f(x) + f(y - x) \geq f(x),$$

showing that $f$ is non-decreasing.

We claim that (2.2.4) holds for a positive irrational number $d$. Let $n_0$ be a positive integer such that for all $n \geq n_0$,

$$\frac{1}{n} < d.$$

For every $n \geq n_0$, choose $r_n \in \left(d - \frac{1}{n}, d\right)$ and $s_n \in \left(d, d + \frac{1}{n}\right)$ to be arbitrary rational numbers. Then, by (2.2.4) and the observed monotonicity of $f$,

$$r_n f(1) = f(r_n) \leq f(d) \leq f(s_n) = s_n f(1).$$

Since $r_n \to d$ and $s_n \to d$, by the squeeze theorem we have that $f(d)$ converges to $df(1)$, as needed. □

We are now ready to prove Theorem 2.2.2.

*Proof.* We first show that

$$\mathrm{Var}\left[Y_t\right] = \sigma^2 t. \tag{2.2.5}$$

To that end, note that

$$Y_{t+s} - Y_0 = Y_{t+s} - Y_t + Y_t - Y_0; \tag{2.2.6}$$

by Assumption 2 (independent increments followed by stationary increments),

$$\begin{aligned} \mathrm{Var}\left[Y_{t+s} - Y_0\right] &= \mathrm{Var}\left[Y_{t+s} - Y_t\right] + \mathrm{Var}\left[Y_t - Y_0\right] \\ &= \mathrm{Var}\left[Y_s - Y_0\right] + \mathrm{Var}\left[Y_t - Y_0\right]. \end{aligned} \tag{2.2.7}$$

With $f(u) = \mathrm{Var}\left[Y_u - Y_0\right]$, (2.2.7) reduces to

$$f(t+s) = f(t) + f(s). \tag{2.2.8}$$

Since $f$ is non-negative, by Lemma 2.2.3, $f(t) = tf(1)$, where $f(1) = \mathrm{Var}\left[Y_1 - Y_0\right] = \sigma^2$, thus establishing (2.2.5).

Now, to prove the assertion of the theorem, we show that $Y_t - Y_0$, where $Y_0$ is a deterministic quantity, is normally distributed with variance $\sigma^2 t$ using the Lindeberg-Feller Theorem.

With

$$X_{ni} = Y_{ti/n} - Y_{t(i-1)/n}, \tag{2.2.9}$$

we obtain, by telescopic cancellation,

$$Y_t - Y_0 = \sum_{i=1}^{n} X_{ni}, \tag{2.2.10}$$

where the dependence of $X_{ni}$ on $t$ is suppressed for notational convenience. Since

$$Y_t - Y_0 - \mathbb{E}\left[Y_t - Y_0\right] = \sum_{i=1}^{n} \left[Y_{ti/n} - Y_{t(i-1)/n} - \left(\mathbb{E}\left[Y_{ti/n}\right] - \mathbb{E}\left[Y_{t(i-1)/n}\right]\right)\right],$$

without loss of generality, we can assume that $Y_t - Y_0$ and $X_{ni} = Y_{ti/n} - Y_{t(i-1)/n}$ have mean zero. By stationary increments in Assumption 2, $X_{ni}$ has the same distribution as $Y_{t/n} - Y_0$. Consequently, by (2.2.5),

$$\mathbb{E}\left[X_{ni}^2\right] = \sigma^2 \frac{t}{n}, \tag{2.2.11}$$

implying $\sum_{i=1}^{n} \mathbb{E}\left[X_{ni}^2\right] = \sigma^2 t$. Thus we can apply the Lindeberg-Feller Central Limit Theorem once the Lindeberg condition is satisfied.

Let $\epsilon > 0$. By the consequence of the assumption of stationary increments noted above,

$$\sum_{i=1}^{n} \mathbb{E}\left[X_{ni}^2; |X_{ni}| > \epsilon\right] = n\mathbb{E}\left[\left(Y_{t/n} - Y_0\right)^2; \left|Y_{t/n} - Y_0\right| > \epsilon\right],$$

whence the Lindeberg condition follows from Assumption 3. □

A couple remarks are in order.

**Remark 2.2.4.** *While Assumptions 1 and 2 reflect reasonable properties of the asset price process, it is difficult to interpret Assumption 3. It seems that the only significance of this assumption is its sufficiency for the Lindeberg condition. However, note that the array defined in* (2.2.9)*, by virtue of Equation* (2.2.11)*, satisfies the second part of the first condition in the Lindeberg-Feller Theorem, rendering Assumption 3 necessary for the desired asymptotic normality.*

**Remark 2.2.5.** *We note that Lindeberg's condition is needed, in principle, to avoid jumps in the stochastic process $Y_t$. Without this condition one can obtain the Poisson process as a limit (or more generally a Lévy process), but this is beyond the scope of this thesis. See for example [1, Theorem 28.5].*

## 2.3 Simulations

We now turn our attention to simulating stock prices according to the Black-Scholes model and empirically testing them for log-normality.

First, note that the Black-Scholes stochastic differential equation is written

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \tag{2.3.1}$$

where $X_t$ denotes the price of the underlying instrument, $W_t$ is Brownian motion, $\mu$ is drift, and $\sigma$ is the volatility. The unique process satisfying (2.3.1) is the geometric Brownian motion

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \tag{2.3.2}$$

that is, $X_t$ is log-normal for all $t$.

We will make a discrete approximation to this process. We assume that

$$X_0, X_1, X_2, \ldots, X_t, \ldots$$

satisfy

$$\begin{aligned} X_{t+1} - X_t &= \mu X_t + \sigma X_t \varepsilon_t \\ X_{t+1} &= X_t(1 + \mu + \sigma\varepsilon_t) \\ X_{t+1} &= X_0 \prod_{i=0}^{t} (1 + \mu + \sigma\varepsilon_i) \end{aligned} \tag{2.3.3}$$

for an appropriate choice of i.i.d. random variables $\varepsilon_i$, as this serves as a discrete simulation of geometric Brownian motion. That is, we expect $X_t$ to be "approximately" log-normal for all $t$ as well. Moreover, by (2.3.2), we expect the Gaussian associated to $X_t$ to have parameters approximately $(\mu - \sigma^2/2)t$ and $\sigma^2 t$ respectively. We thus sample simulated stock paths according to (2.3.3).

### 2.3.1 Theoretical Mean and Variance

Let $X_0, X_1, X_2, \ldots, X_t, \ldots$ follow the updating rules in (2.3.3); $X_0$ is deterministic. Further, let $\widetilde{X_t} = e^{-rt}X_t$ and $Y_t = \log \widetilde{X_t}$. We compute the theoretical mean and variance of the random variable $Y_t$ using discrete intervals created by the partition. First note that $Y_t - Y_s$ are i.i.d. if and only if $\log \frac{e^{-rt}X_t}{e^{-rs}X_s}$ are also i.i.d. We can then compute

$$\begin{aligned}
\mathbb{E}[Y_t - Y_0] &= \sum_{i=1}^{t} \mathbb{E}\left[\log\left(e^{-ri}X_i\right) - \log\left(e^{-r(i-1)}X_{(i-1)}\right)\right] \\
&= \sum_{i=1}^{t} \mathbb{E}\left[\log\left(e^{-r}\frac{X_i}{X_{(i-1)}}\right)\right] \\
&= \sum_{i=1}^{t} (-r + \mathbb{E}[\log(1+\mu+\sigma\varepsilon_1)]) \\
&= t(-r + \mathbb{E}[\log(1+\mu+\sigma\varepsilon_1)]), \qquad (2.3.4)
\end{aligned}$$

where the last equality uses the fact that each $\varepsilon_i$ have the same distribution. $Y_0$ is deterministic, so we have $\mathbb{E}[Y_t - Y_0] = \mathbb{E}[Y_t] - Y_0$. Since $Y_0 = \log X_0$, we obtain, rearranging (2.3.4),

$$\mathbb{E}[Y_t] = \log X_0 + t(-r + \mathbb{E}[\log(1+\mu+\sigma\varepsilon_1)]). \qquad (2.3.5)$$

Note that this is a linear equation with respect to the variable $t$.

Now we can compute the variance. Squaring (2.3.5), we find that

$$\mathbb{E}[Y_t]^2 = (-rt + \log X_0 + t\mathbb{E}[\log(1+\mu+\sigma\varepsilon_i)])^2. \qquad (2.3.6)$$

In order to compute $\mathbb{E}[(Y_t)^2]$, we first observe that

$$Y_t = \log \widetilde{X_t} = \log\left[e^{-rt}X_0 \prod_{i=1}^{t}(1+\mu+\sigma\varepsilon_i)\right] = -rt + \log X_0 + \sum_{i=1}^{t}\log(1+\mu+\sigma\varepsilon_i). \qquad (2.3.7)$$

Then, by squaring the result in (2.3.7) and taking its expectation, we have

$$\begin{aligned}
\mathbb{E}[(Y_t)^2] &= \mathbb{E}\left[\left(-rt + \log X_0 + \sum_{i=1}^{t}\log(1+\mu+\sigma\varepsilon_i)\right)^2\right] \\
&= (rt)^2 + (\log X_0)^2 - 2rt\log X_0 - 2rt\sum_{i=1}^{t}\mathbb{E}\left[\log(1+\mu+\sigma\varepsilon_i)\right] \\
&\quad + 2\log X_0 \sum_{i=1}^{t}\mathbb{E}\left[\log(1+\mu+\sigma\varepsilon_i)\right] + \mathbb{E}\left[\left(\sum_{i=1}^{t}\log(1+\mu+\sigma\varepsilon_i)\right)^2\right]. \qquad (2.3.8)
\end{aligned}$$

The final term on the right-hand side of (2.3.8) can be written as

$$\mathbb{E}\left[\left(\sum_{i=1}^{t}\log(1+\mu+\sigma\varepsilon_i)\right)^2\right] = \sum_{i=1}^{t}\mathbb{E}\left[(\log(1+\mu+\sigma\varepsilon_i))^2\right] + \sum_{i\neq j}^{t}\mathbb{E}\left[\log(1+\mu+\sigma\varepsilon_i)\right]\mathbb{E}\left[\log(1+\mu+\sigma\varepsilon_j))\right].$$

We can thus conclude, using the results of (2.3.6) and (2.3.8),

$$\mathrm{Var}(Y_t) = \mathbb{E}[(Y_t)^2] - \mathbb{E}[Y_t]^2 = \Big(\mathbb{E}[\log^2(1+\mu+\sigma\varepsilon_1)] - \big(\mathbb{E}[\log(1+\mu+\sigma\varepsilon_1)]\big)^2\Big)t. \tag{2.3.9}$$

Note that this is a linear equation with respect to the variable $t$.

**Proposition 2.3.1.** *Let $\mu, \sigma, r, X_0 > 0$. Suppose that*

$$X_t = X_0 \prod_{i=0}^{t-1}(1+\mu+\sigma\epsilon_i),$$

*where $\epsilon_i$ are i.i.d. random variables satisfying $\mathbb{E}\left[\log^2(1+\mu+\sigma\epsilon_1)\right] < \infty$. If $Y_t = \log \tilde{X}_t$, where $\tilde{X}_t = e^{-rt}X_t$, then*

$$\mathbb{E}[Y_t] = \log X_0 + t\left(-r + \mathbb{E}\left[\log(1+\mu+\sigma\epsilon_1)\right]\right)$$

*and*

$$Var(Y_t) = \left(\mathbb{E}\left[\log^2(1+\mu+\sigma\epsilon_1)\right] - \left(\mathbb{E}\left[\log(1+\mu+\sigma\epsilon_1)\right]\right)^2\right)t.$$

We will compute two examples for specific $\varepsilon_i$.

**Example 2.3.1** (Rademacher)**.** *Consider the case when $\varepsilon_i \sim$ Rademacher, taking values 1 and $-1$ with probability $\frac{1}{2}$ each. Using (2.3.5) and computing the expectation with $\varepsilon_i \sim$ Rademacher, we have*

$$\mathbb{E}[Y_t] = \log X_0 + t\left[-r + \frac{1}{2}\log\left((1+\mu)^2 - \sigma^2\right)\right]. \tag{2.3.10}$$

*Now, using (2.3.9) and computing the expectations, we have*

$$Var(Y_t) = t\left(\frac{1}{2}\log^2(1+\mu+\sigma) + \frac{1}{2}\log^2(1+\mu-\sigma) - \left(\frac{1}{2}\log\left((1+\mu)^2-\sigma^2\right)\right)^2\right). \tag{2.3.11}$$

**Example 2.3.2** (Uniform)**.** *Consider the case when $\varepsilon_i \sim \mathrm{Uniform}[-1,1]$. Again, using (2.3.5) and computing the expectation, we have*

$$\mathbb{E}[Y_t] = \log X_0 + t(-r+\lambda_1). \tag{2.3.12}$$

*Similarly, using (2.3.9),*

$$Var(Y_t) = t(\lambda_2 - {\lambda_1}^2). \tag{2.3.13}$$

*We define* $\lambda_1$ *and* $\lambda_2$ *as follows.*

$$\lambda_1 = \frac{1}{2\sigma}\Big[(-\mu+\sigma-1)log(\mu-\sigma+1)+(\mu+\sigma+1)log(\mu+\sigma+1)-2\sigma\Big]$$

*and*

$$\begin{aligned}\lambda_2 = &\frac{1}{2\sigma}\Big[(\mu+\sigma+1)\big((log(\mu+\sigma+1)-2)log(\mu+\sigma+1)+2\big)\\ &-(\mu-\sigma+1)\big((log(\mu-\sigma+1)-2)log(\mu-\sigma+1)+2\big)\Big].\end{aligned}$$

We continue with these two examples in the following section, where we use numerical simulations to examine the properties of $X_t$.

### 2.3.2 Numerical Results

Our goal in this section is to verify the following:

1. The mean of our log-sample paths at time $t$ is approximately equal to $(\mu - \sigma^2/2)t$.
2. The variance of our log-sample paths at time $t$ is approximately equal to $\sigma^2 t$.
3. At each time $t$, $X_t$ is log-normal.

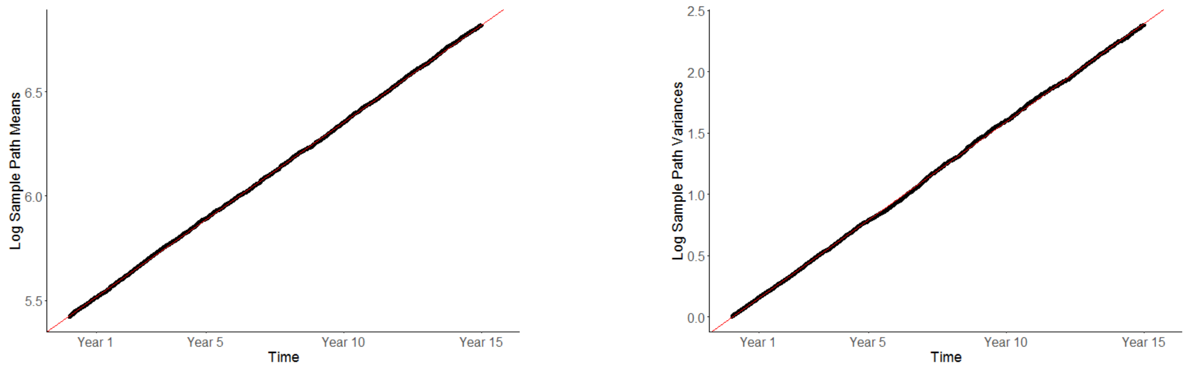
To begin with, we simulate daily data spanning a period of 15 years for a total of 5,475 days (we do not account for leap years); our total number of sample paths is 5,000. Our initial price, $X_0$, is set at 225.93, the position of the Vanguard 500 at the time of simulation. Further, we set the drift, $\mu$, at $\frac{0.1732}{365}$, where 0.1732 is the annual return of the Vanguard 500, and the volatility, $\sigma$, at $\frac{0.114}{\sqrt{30}}$, where 0.114 is the volatility index on the S&P 500. We take $r = 0$ as we are only concerned with the normality of the random variable and discounting does not impact this characteristic. The final component of our simulation is the random term. We run two sets of simulations, one with a Rademacher random term and the other with a Uniform random term over the interval $(-1, 1)$. For each set of simulations, we examine log sample path means and variances, comparing our empirical results with the theoretical calculations done below. We also test $X_t$ for normality at the end of 5 years, 10 years, and 15 years.

Using the aforementioned parameters, we compute the following theoretical means and variances for both sets of simulations (i.e., for both $\varepsilon_i \sim$ Rademacher and $\varepsilon_i \sim$ Uniform) before running the actual market simulations.

$$\begin{aligned}\varepsilon_i \sim \text{Rademacher} \quad & \mathbb{E}[Y_t] = 6.8326 \quad && \text{Slope} = 0.0002580\\ & \text{Var}(Y_t) = 2.370 \quad && \text{Slope} = 0.0004329\end{aligned}$$

$$\begin{aligned}\varepsilon_i \sim \text{Uniform} \quad & \mathbb{E}[Y_t] = 7.6231 \quad && \text{Slope} = 0.0004023\\ & \text{Var}(Y_t) = 0.790 \quad && \text{Slope} = 0.0001443.\end{aligned}$$

In our first set of simulations, the random term is Rademacher. From Figures 2.1a and 2.1b, we note that the slopes given by our model, both simulated and theoretical, closely match the drift and volatility of the geometric Brownian motion as calculated above.




(a) Log Sample Path Means

(b) Log Sample Path Variances

Figure 2.1: Simulated slopes for $\varepsilon_i \sim$ Rademacher.

The histogram in Figure 2.2 shows the log prices at the end of the 15$^{\text{th}}$ year.

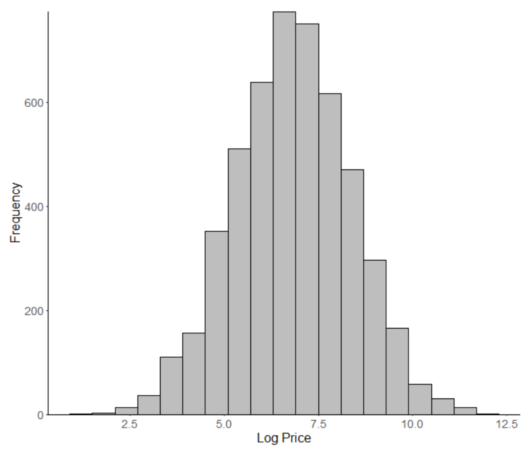
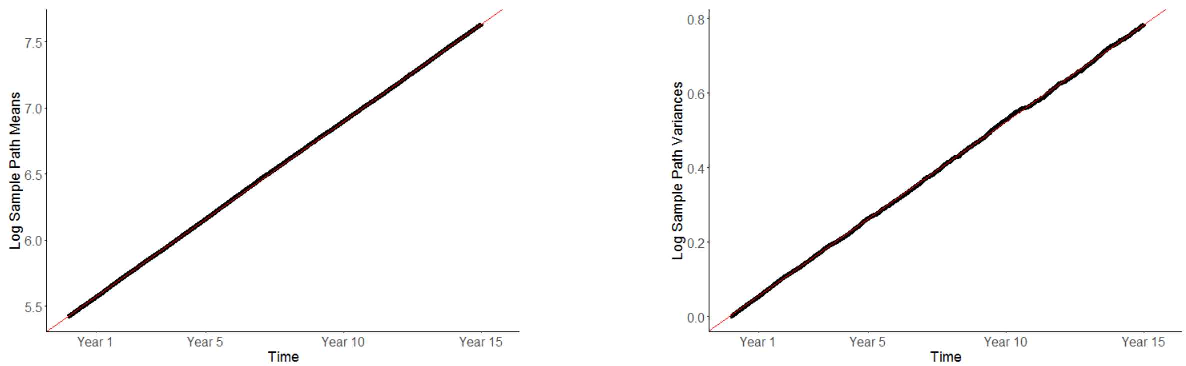



Figure 2.2: Histogram of log prices at Year 15 with $\varepsilon_i \sim$ Rademacher.

Clearly, the data is normally distributed; this is further verified by the Shapiro-Wilk Normality Test results below:

| Year | 5 | 10 | 15 |
|---|---|---|---|
| Shapiro-Wilk $p$-value | 0.276 | 0.441 | 0.498 |

The second set of simulations features a Uniform random term. As in the case with the Rademacher random term, the simulated and theoretical means and variances given by our model are quite close (see Figure 2.3), although they are closer in the Rademacher model. It should be noted that Black-Scholes simulations generally tend to use Rademacher as the distribution of choice, since one may think of Brownian motion as the scaling limit of a simple random walk.




(a) Log Sample Path Means (b) Log Sample Path Variances

Figure 2.3: Simulated slopes for $\varepsilon_i \sim$ Uniform.

Our log-prices are normally distributed at the end of five, ten, and fifteen years as evidenced by the results of the Shapiro-Wilk Normality Test in the table below (and in Figure 2.4, for the 15$^{\text{th}}$ year).

| Year | 5 | 10 | 15 |
|---|---|---|---|
| Shapiro-Wilk $p$-value | 0.305 | 0.920 | 0.941 |

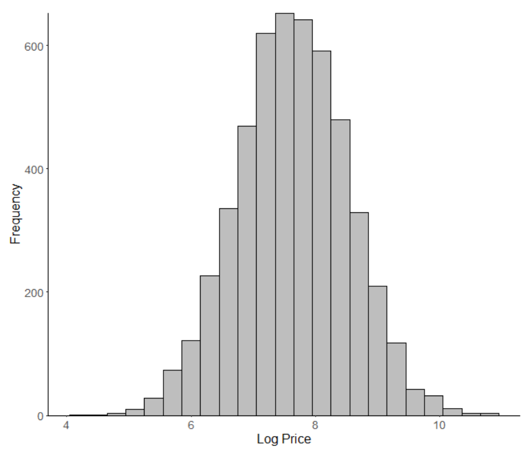



Figure 2.4: Histogram of log prices at Year 15 with $\varepsilon_i \sim$ Uniform.

# Chapter 3

# Lyapunov Exponents and Products of Random Matrices

Let $\{Y_i\}_{i\geq 1}$ be a sequence of i.i.d. random matrices equipped with measure $\mu$. Further, let $S_n = \prod_{i=1}^{n} Y_i$. We assume that $\mathbb{E}(\log^+ \|Y_1\|) < \infty$. The *Lyapunov exponent* $\lambda$ associated with $\mu$ is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(\log \|S_n\|\right), \tag{3.0.1}$$

with $\lambda \in \mathbb{R} \cup \{-\infty\}$. In this chapter, we investigate the behavior of the Lyapunov exponent as the common distribution of the sequence of random matrices varies over a wide class.

The Lyapunov exponent gives a measure of the exponential growth rate of the matrix norm. Since all finite-dimensional norms are equivalent, $\lambda$ is unique regardless of which norm $\|\cdot\|$ we use. At the same time, $\lambda$ depends on $\mu$, which we usually take to be fixed in any discussion of $\lambda$ and so omit from our notation. Occasionally, when we are considering $\lambda$ over a family of distributions parametrized by some variable, we will write $\lambda$ as a function of that variable.

The Lyapunov exponent is known explicitly for very few random matrix models as there is no closed-form formula that can be applied universally. While there are examples in the literature where explicit expressions have been obtained for some matrices under certain conditions, such as in [3, 4, 9], it is often not possible to compute directly.

There have also been multiple forays into approximating the Lyapunov exponent for models where it cannot be calculated explicitly. In [18], the author finds that random Fibonacci sequences can be used to better understand the behavior of $\lambda$ and employs new computational methods to find exact values for a given $\lambda$. Further, in [14], $\lambda$ is expressed in terms of associated complex functions and a more general algorithm to numerically approximate $\lambda$ is proposed. In [10], the authors construct explicit invariant measures for a family of random matrix models where one entry follows a Gamma distribution in order to calculate the corresponding Lyapunov exponents.

In Section 3.2, we provide analytic upper and lower bounds on the Lyapunov exponent associated with the product of random matrices where one entry is a Bernoulli random variable with probability $p$ equal to $\frac{1}{2}$. We find that these bounds converge to the Lyapunov exponent. Interestingly, these bounds are related to Fibonacci sequences.

We further develop new tools to analytically and numerically compute the Lyapunov exponents associated with various "parameter models" as defined in Section 3.3. For each of these models, we also examine the variance associated with a multiplicative Central Limit Theorem for products of random matrices under different priors on the model parameters. Compared to the calculation of the Lyapunov exponent, there have been relatively few attempts to explicitly compute or numerically approximate this variance. Our work aims to bridge this gap in the context of the parameter models that we consider. These models are all based on the products of matrices that have one element distributed as a random variable – in Section 3.4, this element is a standard Cauchy random variable multiplied by a real-valued parameter $\xi$; in Section 3.5, a Bernoulli random variable with probability $p$ equal to $\frac{1}{2}$ multiplied by a real-valued parameter $\xi$; in Section 3.6, a Bernoulli random variable with the probability $p$ as the parameter; and in Section 3.7, a Rademacher random variable multiplied by a real-valued parameter $\xi$.

In Section 3.1, we discuss some well-known results regarding $\lambda$ that we make use of in our subsequent analysis of the aforementioned random matrix models.

## 3.1 Preliminaries

In what follows, we introduce notational conventions and terminology and recall well-known results regarding the Lyapunov exponent. Let $\mathbb{P}^1(\mathbb{R})$ denote the one-dimensional projective space. Recall that $\mathbb{P}^1(\mathbb{R})$ denotes the set of directions in $\mathbb{R}^2$. To describe $\mathbb{P}^1(\mathbb{R})$ let us define the following equivalence relation $\sim$ on $\mathbb{R}^2\setminus\{\mathbf{0}\}$. We say the vectors $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \in \mathbb{R}^2\setminus\{\mathbf{0}\}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2\setminus\{\mathbf{0}\}$ are equivalent, denoted $\mathbf{x} \sim \tilde{\mathbf{x}}$, if there exists a nonzero real number $c$ such that $\mathbf{x} = c\tilde{\mathbf{x}}$. We define $\bar{\mathbf{x}}$ to be the equivalence class of a vector $\mathbf{x}$. We can thus define $\mathbb{P}^1(\mathbb{R})$ as the set of all equivalence classes $\bar{\mathbf{x}}$. We can also define a bijective map $\phi : \mathbb{P}^1(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ by

$$\phi(\bar{\mathbf{x}}) = \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \infty & \text{if } x_2 = 0. \end{cases}$$

With a slight abuse of notation, we can identify $\mathbb{P}^1(\mathbb{R})$ with $\mathbb{R} \cup \{\infty\}$

Let $G$ be a topological semigroup acting on $X$ and let $\mu, \nu$ be measures on $G, X$ respectively. Define $\mu * \nu$ to be the measure satisfying

$$\int_X f(x)\,\mathrm{d}(\mu * \nu)(x) = \int_X \int_G f(g \cdot x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x)$$

for all integrable $f$. If $\mu * \nu = \nu$, then we say that $\nu$ is $\mu$-invariant. Furthermore, we say that $G$ is strongly irreducible if there is no finite family of one-dimensional vector subspaces $\{X_1, \ldots, X_n\}$ such that $g(X_1 \cup \cdots \cup X_n) = X_1 \cup \cdots \cup X_n$ for all $g \in G$. For our purposes, we are interested in the action of $\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ and $x \in \mathbb{P}^1(\mathbb{R})$, we define

$$A \cdot x = \frac{ax + b}{cx + d}.$$

The following result by Furstenberg and Kesten in [6] gives an important analogue to the Law of Large Numbers.

**Theorem 3.1.1.** *Let* $\{Y_i\}_{i\geq 1}$ *be i.i.d.* $\mathrm{GL}(d,\mathbb{R})$*-valued random matrices and* $S_n = \prod_{i=1}^n Y_i$. *If* $\mathbb{E}(\log^+\|Y_1\|) < \infty$*, then a.s.*

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|S_n\|.$$

Suppose $\mu$ is a measure on the group $\mathrm{GL}(d,\mathbb{R})$ and that the matrices $\{Y_i\}_{i\geq 1}$ are distributed according to $\mu$. In [7], Furstenberg and Kifer give an expression for $\lambda$ in terms of the measure $\mu$ and the $\mu$-invariant measures $\nu$ on $\mathbb{P}^1(\mathbb{R})$. The following result is given in [7, Theorem 2.2],

**Theorem 3.1.2.** *Let* $\mu$ *be a measure on the group* $\mathrm{GL}(d,\mathbb{R})$*. Let* $\{Y_i\}_{i\geq 1}$ *be i.i.d. random matrices distributed according to* $\mu$ *and* $S_n = \prod_{i=1}^n Y_i$*. If* $\mathbb{E}(\log^+\|Y_1\| + \log^+\|Y_1^{-1}\|) < \infty$*, then the Lyapunov exponent is given by*

$$\lambda = \sup_{\nu} \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(d,\mathbb{R})} \log \frac{|M\bar{x}|}{|\bar{x}|} \, \mathrm{d}\mu(M) \, \mathrm{d}\nu(x),$$

*where the supremum is taken over all probability measures* $\nu$ *that are* $\mu$*-invariant on* $\mathbb{P}^1(\mathbb{R})$*.*

If $\nu$ is the unique invariant distribution on $\mathbb{P}^1(\mathbb{R})$, then Theorem 3.1.2 shows that the Lyapunov exponent can be written as

$$\lambda = \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(d,\mathbb{R})} \log \frac{|M\bar{x}|}{|\bar{x}|} \, \mathrm{d}\mu(M) \, \mathrm{d}\nu(x).$$

Given a measure $\mu$ on $\mathrm{GL}(d,\mathbb{R})$, Furstenberg and Bougerol state sufficient conditions for the existence of a unique $\mu$-invariant measure $\nu$ on $\mathbb{P}^1(\mathbb{R})$. The following theorem can be found in [3, Theorem 4.1].

**Theorem 3.1.3.** *Let* $\mu$ *be a measure on the group* $\mathrm{GL}(2,\mathbb{R})$ *and* $\{Y_i\}_{i\geq 1}$ *be i.i.d. random matrices distributed according to* $\mu$*. Further, let* $S_n = \prod_{i=1}^n Y_i$*. Suppose* $G_\mu$ *is the smallest closed subgroup containing the support of* $\mu$*. If*

*(i)* $|\det Y_1| = 1$ *a.s.*

*(ii)* $\mathbb{E}(\log^+\|Y_1\|) < \infty$

*(iii)* $G_\mu$ *is not compact and strongly irreducible*

*then there exists a* $\mu$*-invariant distribution* $\nu$ *on* $\mathbb{P}^1(\mathbb{R})$*,* $\lambda > 0$ *a.s. In particular, a.s.*

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|S_n \bar{x}\| = \lim_{n\to\infty} \frac{1}{n} \log \|S_n\|$$

*for any* $x \in \mathbb{P}^1(\mathbb{R})$*, and*

$$\lambda = \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(2,\mathbb{R})} \log \frac{|M\bar{x}|}{|\bar{x}|} \, \mathrm{d}\mu(M) \, \mathrm{d}\nu(x).$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $\mathrm{GL}(2,\mathbb{R})$-valued random matrix and $X$ be a $\mathbb{P}^1(\mathbb{R})$-valued random variable with distributions $\mu$ and $\nu$ respectively. In this chapter, we only study matrices $A$ with $a$ random and all other entries constant. Let us suppose that $\nu$ is the unique $\mu$-invariant distribution on $\mathbb{P}^1(\mathbb{R})$. Following [10, pp. 3421] we see that

$$\begin{aligned}
\lambda &= \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(2,\mathbb{R})} \log \frac{|A\bar{x}|}{|\bar{x}|} \,\mathrm{d}\mu(A)\,\mathrm{d}\nu(x) \\
&= \frac{1}{2} \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(2,\mathbb{R})} \log \frac{(ax+b)^2 + (cx+d)^2}{x^2+1} \,\mathrm{d}\mu(A)\,\mathrm{d}\nu(x) \\
&= \frac{1}{2} \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(2,\mathbb{R})} \log \left[ \left( \frac{ax+b}{cx+d} \right)^2 + 1 \right] + \log \left( \frac{|cx+d|}{x^2+1} \right) \mathrm{d}\mu(A)\,\mathrm{d}\nu(x) \\
&= \frac{1}{2} \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(2,\mathbb{R})} \log\bigl(|A\cdot x|^2+1\bigr) + \log\left[(cx+d)^2\right] - \log|x^2+1| \,\mathrm{d}\mu(A)\,\mathrm{d}\nu(x) \\
&= \int_{\mathbb{P}^1(\mathbb{R})} \log|cx+d| \,\mathrm{d}\nu(x) + \frac{1}{2} \int_{\mathbb{P}^1(\mathbb{R})} \int_{\mathrm{GL}(2,\mathbb{R})} \log\bigl(|A\cdot x|^2+1\bigr) \,\mathrm{d}\mu(A)\,\mathrm{d}\nu(x) \\
&\qquad - \frac{1}{2} \int_{\mathbb{P}^1(\mathbb{R})} \log|x^2+1| \,\mathrm{d}\nu(x) \\
&= \int_{\mathbb{P}^1(\mathbb{R})} \log|cx+d| \,\mathrm{d}\nu(x),
\end{aligned}$$

where in the second-to-last step we apply the definition of $\mu$-invariance to the second term. Thus, if $X$ has distribution $\nu$, then

$$\lambda = \mathbb{E}\left[\log|cX+d|\right]. \tag{3.1.1}$$

Moreover, from the definition of $\mu$-invariance, we can also conclude that $A \cdot x$ has the same distribution as $x$, which we write $A \cdot x \sim x$. Thus, the random variable $X$ with law given by a unique $\mu$-invariant distribution $\nu$ on $\mathbb{P}^1(\mathbb{R})$ must satisfy

$$\frac{aX+b}{cX+d} \sim X. \tag{3.1.2}$$

We make use of this distributional identity for the $\mu$-invariant distribution in later sections.

The following result by Le Page stated in [12] gives us an analogue to the standard, additive Central Limit Theorem.

**Theorem 3.1.4.** *Define $\ell$ as the function $\ell(M) = \max\{\log^+\|M\|, \log^+\|M^{-1}\|\}$. Let $\mu$ be a measure on the group $\mathrm{GL}(2,\mathbb{R})$ and $\{Y_i\}_{i\geq 1}$ be i.i.d. random matrices distributed according to $\mu$. Further, let $S_n = \prod_{i=1}^n Y_i$. Suppose that $G_\mu$ is the smallest closed subgroup containing the support of $\mu$. If*

*(i)* $\int_{\mathrm{GL}(2,\mathbb{R})} \exp(t\ell(M))\,\mathrm{d}\mu(M) < \infty$ *for some* $t > 0$

*(ii)* $G_\mu$ *is strongly irreducible*

*(iii)* $\{|\det M|^{-1/d} M : M \in G_\mu\}$ *is not contained in a compact subgroup of* $\mathrm{GL}(2,\mathbb{R})$

*then, for any $x \in \mathbb{R}^d \setminus \{0\}$, there exists a Gaussian random variable $Z$ with mean 0 and variance $\sigma^2$ such that*

$$\frac{1}{\sqrt{n}}\left(\log\|S_n\bar{x}\| - n\lambda\right) \to Z \text{ and } \frac{1}{\sqrt{n}}\left(\log\|S_n\| - n\lambda\right) \to Z.$$

In subsequent sections, we present numerical approximations for the value of $\sigma^2$ for several matrix models. There have been very few attempts to find the value of $\sigma^2$ explicitly.

## 3.2 Bernoulli$\left(\frac{1}{2}\right)$ Model

The first matrix model that we study is based on a Bernoulli $\left(\frac{1}{2}\right)$ random variable. Recall that a random variable $\epsilon \sim$ Bernoulli $\left(\frac{1}{2}\right)$ if $\mathbb{P}(\epsilon = 1) = \frac{1}{2}$ and $\mathbb{P}(\epsilon = 0) = \frac{1}{2}$. In this section we consider a measure $\mu$ on GL$(2,\mathbb{R})$ given by

$$Y_i = \begin{pmatrix} \epsilon_i & 1 \\ 1 & 0 \end{pmatrix}, \ \epsilon_i \sim \text{Bernoulli}\left(\frac{1}{2}\right). \tag{3.2.1}$$

Suppose there exists a unique $\mu$-invariant distribution $\nu$ related to the products of random matrices of the form $Y_i$. By (3.1.2), $\nu$ must have the same distribution of a random variable $X$ that satisfies the following distributional identity

$$X \sim \frac{1}{X} + \epsilon, \tag{3.2.2}$$

where $\epsilon \sim$ Bernoulli $\left(\frac{1}{2}\right)$ and is independent of $X$. Clearly, $X$ must be a positive random variable. In fact, $X$ has full support in $(0,\infty)$. By the formula given in (3.1.1), the Lyapunov exponent related to $Y_i$ in (3.2.1) is of the form

$$\lambda = \mathbb{E}\left[\log X\right].$$

For a detailed study on this random variable and the Markov chain related to it we refer the reader to [8]. In particular, the author proves that the Markov chain related to the system $Y_i$ has a unique invariant distribution $\nu$ that satisfies (3.2.2) in [8, Theorem 5.2]. The author also gives a formula for the distribution function in terms of continued fraction expansions.

In order to compute $\lambda$, we will study properties of the random variable $X$ on $(0,\infty)$ that satisfies (3.2.2). We begin by establishing some identities related to $\mathbb{E}[\log X]$.

**Proposition 3.2.1.** *If $X$ is the random variable satisfying (3.2.2) then*

$$\mathbb{E}\left[\log X\right] = \frac{1}{6}\mathbb{E}\left[\log\left(2X+1\right)\right].$$

*Proof.* Let $X$ be a random variable with distribution $\nu$ as in (3.2.2). Then,

$$\begin{aligned}
\mathbb{E}\left[\log X\right] &= \mathbb{E}\left[\log\left(\frac{1}{X} + \epsilon\right)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\log\left(\frac{1}{X}\right)\right] + \frac{1}{2}\mathbb{E}\left[\log\left(\frac{1}{X}+1\right)\right] \\
&= -\frac{1}{2}\mathbb{E}\left[\log X\right] + \frac{1}{2}\mathbb{E}\left[\log\left(\frac{1+X}{X}\right)\right] \\
&= -\mathbb{E}\left[\log X\right] + \frac{1}{2}\mathbb{E}\left[\log\left(1+X\right)\right].
\end{aligned} \tag{3.2.3}$$

Adding $\mathbb{E}[\log X]$ to both sides of (3.2.3) and dividing by 2,

$$\mathbb{E}[\log X] = \frac{1}{4}\mathbb{E}[\log(1+X)]. \tag{3.2.4}$$

Continuing in a similar fashion with (3.2.4), we obtain

$$\begin{aligned}
\mathbb{E}[\log X] &= \frac{1}{4}\mathbb{E}\left[\log\left(1+\frac{1}{X}+\epsilon\right)\right] \\
&= \frac{1}{8}\mathbb{E}\left[\log\left(1+\frac{1}{X}\right)\right] + \frac{1}{8}\mathbb{E}\left[\log\left(2+\frac{1}{X}\right)\right] \\
&= \frac{1}{8}\mathbb{E}\left[\log\left(\frac{X+1}{X}\right)\right] + \frac{1}{8}\mathbb{E}\left[\log\left(\frac{2X+1}{X}\right)\right] \\
&= \frac{1}{8}\mathbb{E}[\log(X+1)] + \frac{1}{8}\mathbb{E}[\log(2X+1)] - \frac{1}{4}\mathbb{E}[\log X] \\
&= \frac{1}{4}\mathbb{E}[\log X] + \frac{1}{8}\mathbb{E}[\log(2X+1)]. \qquad (3.2.5)
\end{aligned}$$

Subtracting $\frac{1}{4}\mathbb{E}[\log X]$ from both sides of (3.2.5) and dividing by $\frac{3}{4}$ yields

$$\mathbb{E}[\log X] = \frac{1}{6}\mathbb{E}[\log(2X+1)],$$

thus completing the proof. □

The proof of Proposition 3.2.1 helps us generalize several other identities. We can prove a string of these identities in a similar fashion as follows.

$$\begin{aligned}
\mathbb{E}[\log X] &= \frac{1}{6}\mathbb{E}[\log(2X+1)] \\
&= \frac{1}{14}\mathbb{E}[\log(3X+2)(X+2)] \\
&= \frac{1}{32}\mathbb{E}[\log(5X+3)(3X+1)(2X+3)(2X+1)] \\
&= \frac{1}{72}\mathbb{E}[\log(8X+5)(4X+3)(5X+2)(3X+2)(3X+5)(X+3)(3X+2)(X+2)] \\
&\qquad (3.2.6) \\
&\ \vdots
\end{aligned}$$

The string of equalities above is obtained by iteratively exploiting the distributional equivalence of $X$ and $\frac{1}{X}+\epsilon$, the independence of $X$ and $\epsilon$, and elementary logarithmic identities. An interesting pattern emerges.

At the first step of the iteration, we are looking at the expected value of the log of one affine function of $X$ that is obtained by taking the inner product of the vector $(2,1)$ and the vector $(X,1)$. As we move to the second step of the iteration, we encounter the expectaton of the log of the product of two affine functions of $X$. The first one is obtained by taking the inner product of $(3,2)$ and $(X,1)$, while the second is obtained by taking the inner product of $(1,2)$ and $(X,1)$. At the third step, we encounter the expected value of the log of the product of four $(= 2^{3-1})$ affine functions of $X$; these are obtained by respectively taking the inner product of $(X, 1)$ with the vectors $(5, 3)$, $(3, 1)$, $(2, 3)$, and $(2, 1)$.

In what follows, we represent the vectors generating the aforesaid affine functions of $X$ via inner products with $(X, 1)$, which we call "coefficient pairs", in an array where the row number corresponding to the $n^{\text{th}}$ step of the iteration is $n - 1$. The first five rows of the array are shown below. We use the symbol $\mapsto$ to map the collection of coefficient pairs to the real number representing the product of the sum of entries in each coefficient pair in the row; we make extensive use of these quantities later on.

$$\begin{array}{ll} n = 0 & (2,1) \mapsto 3 \\ n = 1 & (3,2)\,(1,2) \mapsto 5 \cdot 3 = 15 \\ n = 2 & (5,3)\,(3,1)\,(2,3)\,(2,1) \mapsto 8 \cdot 4 \cdot 5 \cdot 3 = 480 \\ n = 3 & (8,5)\,(4,3)\,(5,2)\,(3,2)\,(3,5)\,(1,3)\,(2,1)\,(2,3) \mapsto 13 \cdot 7 \cdot 7 \cdot 5 \cdot 8 \cdot 4 \cdot 3 \cdot 5 = 1528800 \\ n = 4 & (13,8)\,(7,4)\,(7,5)\,(5,3)\,(8,3)\,(4,1)\,(3,2)\,(5,2) \\ & (5,8)\,(3,4)\,(2,5)\,(2,3)\,(5,3)\,(3,1)\,(1,2)\,(3,2) \\ & \mapsto 59668697090000 \\ \cdots & \cdots, \end{array} \tag{3.2.7}$$

Now, for the $k^{\text{th}}$ coefficient pair in the $n^{\text{th}}$ row, let $a_n^k$ denote the first element and $b_n^k$ the second. To illustrate this notational convention, consider the example $\frac{1}{14}\mathbb{E}\left[\log\left(3X+2\right)\left(X+2\right)\right]$. This is in row $n = 1$, so we would refer to the 3 in $(3X + 2)$ as $a_1^1$ and the 2 as $b_1^1$. Similarly, in $(X + 2)$, the coefficient $X$, 1, would be labeled $a_1^2$ and the 2 would be labeled $b_1^2$. In terms of $a_n^k$ and $b_n^k$, the expression is $\frac{1}{14}\mathbb{E}\left[\log\left(a_1^1 X + b_1^1\right)\left(a_1^2 X + b_1^2\right)\right]$.

We define the following multilevel recursion.

**Definition 3.2.2.** *Set $a_0^1 = 2$ and $b_0^1 = 1$. For any given $n \in \mathbb{N}$, define*

$$\begin{aligned} \left(a_{n+1}^k, b_{n+1}^k\right) &:= \left(a_n^k + b_n^k, a_n^k\right), \quad &\text{for } k = 1, \ldots, 2^n, \\ \left(a_{n+1}^k, b_{n+1}^k\right) &:= \left(b_n^{k-2^n}, a_n^{k-2^n}\right), \quad &\text{for } k = 2^n + 1, \ldots, 2^{n+1}. \end{aligned}$$

We recall that a "Fibonacci-like sequence" $f_0, f_1, f_2 \ldots$ is a sequence determined by the initial values $f_0, f_1$ such that

$$f_{n+1} = f_n + f_{n-1}$$

for all $n \in \mathbb{N}$. When $f_0 = f_1 = 1$, we recover the standard Fibonacci sequence. Fibonacci-like sequences can be given by a closed formula. Let $f_n(f_0, f_1)$ represent the $n$th term in the sequence given initial values $f_0, f_1$. If

$$\phi_1 = \frac{1 + \sqrt{5}}{2} \text{ and } \phi_2 = \frac{1 - \sqrt{5}}{2},$$

then

$$f_n(f_0, f_1) = \frac{f_1 - f_0\phi_2}{\sqrt{5}}\left(\phi_1\right)^n + \frac{f_0\phi_1 - f_1}{\sqrt{5}}\left(\phi_2\right)^n. \tag{3.2.8}$$

We observe several patterns in (3.2.7). Note that the $n$th row is made up of $2^n$ pairs. By definition, given a fixed $k \in \{1, \ldots, 2^{n-1}\}$, we have

$$a_{n+1}^k = a_n^k + b_n^k = a_n^k + a_{n-1}^k.$$

Thus, for each $k$, the sequence $a_n^k$ will be a Fibonacci-like sequence on $n$. Given a fixed $k$, the sequence $b_n^k$ will also be a Fibonacci-like sequence in $n$. Also note that by definition, the second half of the $n$th row is simply the $(n-1)^{\text{th}}$ row where the elements within the coefficient pairs are flipped. We use these patterns to help establish bounds on the Lyapunov exponent. In order to find suitable estimates, we first need to establish some preliminary results. These involve proving the string of equalities on the Lyapunov exponent given in (3.2.6). We also need to some to prove some elementary inequalities on the logarithm of the polynomials given inside the expectation in (3.2.6).

First, we extend the identities given in (3.2.6) for all $n$.

**Proposition 3.2.3.** *If $X$ is the random variable satisfying (3.2.2), then*

$$\mathbb{E}\left[\log X\right] = \frac{1}{(n+6)2^n}\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k X + b_n^k\right)\right)\right], \tag{3.2.9}$$

*for all $n \in \mathbb{N} \cup \{0\}$.*

*Proof.* We begin with $n = 0$. By Proposition 3.2.1,

$$\begin{aligned}\mathbb{E}\left[\log X\right] &= \frac{1}{(0+6)2^0}\mathbb{E}\left[\log\left(a_0^1 X + b_0^1\right)\right] \\ &= \frac{1}{6}\mathbb{E}\left[\log\left(2X+1\right)\right].\end{aligned}$$

Now suppose (3.2.9) holds for $n$. We shall prove (3.2.9) holds for $n+1$. Note that

$$\begin{aligned}\mathbb{E}\left[\log X\right] &= \frac{1}{(n+6)2^n}\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k X + b_n^k\right)\right)\right] \\ &= \frac{1}{(n+6)2^n}\left(\frac{1}{2}\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k\left(\frac{1}{X}+1\right)+b_n^k\right)\right)\right] + \frac{1}{2}\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k\left(\frac{1}{X}\right)+b_n^k\right)\right)\right]\right) \\ &= \frac{1}{(n+6)2^{n+1}}\left(\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(\frac{a_n^k}{X}+a_n^k+b_n^k\right)\right)\right] + \mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(\frac{a_n^k}{X}+b_n^k\right)\right)\right]\right) \\ &= \frac{1}{(n+6)2^{n+1}}\left(\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(\frac{a_n^k+\left(a_n^k+b_n^k\right)X}{X}\right)\right)\right] + \mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(\frac{a_n^k+b_n^k X}{X}\right)\right)\right]\right) \\ &= \frac{1}{(n+6)2^{n+1}}\left(\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(\left(a_n^k+\left(a_n^k+b_n^k\right)X\right)\left(a_n^k+b_n^k X\right)\right)\right)\right]\right) - \frac{2^{n+1}\mathbb{E}\left[\log X\right]}{(n+6)2^{n+1}} \\ &= \frac{1}{(n+6)2^{n+1}}\left(\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(\left(b_{n+1}^k+\left(a_{n+1}^k\right)X\right)\left(a_n^k+b_n^k X\right)\right)\right)\right]\right) - \frac{\mathbb{E}\left[\log X\right]}{(n+6)}.\end{aligned} \tag{3.2.10}$$

After adding the last term in (3.2.10) to both sides and then dividing by the coefficient in front of $\mathbb{E}[\log X]$, we have

$$\mathbb{E}[\log X] = \frac{1}{((n+1)+6)2^{n+1}} \left( \mathbb{E}\left[ \log \left( \prod_{k=1}^{2^{n+1}} \left( a_{n+1}^k X + b_{n+1}^k \right) \right) \right] \right).$$

Here, the product simplifies because the $\left(a_n^k + b_n^k X\right)$ term will show up in the second half of the $(n+1)^{\text{th}}$ row (since the coefficients are flipped from the $n^{\text{th}}$ row). This is taken into account by the new product from $k = 1$ to $2^{n+1}$ which will multiply all of the $2^{n+1}$ sums that appear in the $n+1$ row. Since the formula holds for $n+1$, by induction, it holds for all $n \geq 0$. □

We now prove the elementary inequalities needed on the polynomials in (3.2.9).

**Lemma 3.2.4.** *For $x > 1$,*

$$\log \left( \prod_{k=1}^{2^n} \left( a_n^k x + b_n^k \right) \right) < \log \left( x^{2^n} \prod_{k=1}^{2^n} \left( a_n^k + b_n^k \right) \right).$$

*Proof.* We let $f(x) = \log \left( \prod_{k=1}^{2^n} \left( a_n^k x + b_n^k \right) \right)$ and $g(x) = \log \left( x^{2^n} \prod_{k=1}^{2^n} \left( a_n^k + b_n^k \right) \right)$. Note that $f(1) = g(1)$. Taking the derivative of both of these functions, we obtain

$$f'(x) = \sum_{k=1}^{2^n} \frac{a_n^k}{a_n^k x + b_n^k}$$

and

$$g'(x) = \frac{2^n x^{2^n - 1}}{x^{2n}} = \frac{2^n}{x}.$$

Note that for $x > 1$,

$$\sum_{k=1}^{2^n} \frac{a_n^k}{a_n^k x + b_n^k} < \sum_{k=1}^{2^n} \frac{a_n^k}{a_n^k x} = \sum_{k=1}^{2^n} \frac{1}{x} = \frac{2^n}{x}.$$

This shows that $f'(x) < g'(x)$ when $x > 1$. Since we also have that $f(1) = g(1)$, it must be that $f(x) < g(x)$ for all $x > 1$. □

**Lemma 3.2.5.** *For $x < 1$,*

$$\log \left( \prod_{k=1}^{2^n} \left( a_n^k x + b_n^k \right) \right) > \log \left( x^{2^n} \prod_{k=1}^{2^n} \left( a_n^k + b_n^k \right) \right).$$

*Proof.* Note that when $x < 1$, we have $a_n^k x + b_n^k > \left( a_n^k + b_n^k \right) x$. Taking products and the log of both sides gives us the desired result. □

Using Lemmas 3.2.4 and 3.2.5, we can prove that the Lyapunov exponent is bounded by terms dependent only on $n$. First, we define the following quantity.

**Definition 3.2.6.** *For a fixed $n$, let $c_n$ be the product of the sums of coefficient pairs in the $n^{th}$ row of (3.2.7). That is,*

$$c_n = \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right) = c_{n-1} \prod_{k=1}^{2^{n-1}} \left(a_n^k + b_n^k\right) = \prod_{k=1}^{2^n} a_{n+1}^k.$$

Recall that $c_0, \ldots, c_4$ are displayed in (3.2.7).

**Theorem 3.2.7.** *The Lyapunov exponent $\lambda = \mathbb{E}[\log X]$ associated with the random matrices $Y_i$ in (3.2.1) can be estimated by*

$$p_n \leq \mathbb{E}[\log X] \leq q_n, \tag{3.2.11}$$

*where*

$$p_n = \frac{\log c_n}{(n+7)\, 2^n} \quad \text{and} \quad q_n = \frac{\log c_n}{(n+4)\, 2^n}. \tag{3.2.12}$$

*Proof.* Recall that by Proposition 3.2.3, we have that

$$\mathbb{E}[\log X] = \frac{1}{(n+6)2^n} \left( \mathbb{E}\left[ \log \left( \prod_{k=1}^{2^n} \left(a_n^k X + b_n^k\right) \right) \right] \right).$$

Using this and Lemma 3.2.4, we can write

$$\begin{aligned}
\mathbb{E}[\log X] &= \frac{1}{(n+6)2^n} \left( \mathbb{E}\left[ \log \left( \prod_{k=1}^{2^n} \left(a_n^k X + b_n^k\right) \right) \cdot \mathbf{1}_{(X<1)} \right] + \mathbb{E}\left[ \log \left( \prod_{k=1}^{2^n} \left(a_n^k X + b_n^k\right) \right) \cdot \mathbf{1}_{(X>1)} \right] \right) \\
&\leq \frac{1}{(n+6)2^n} \left( \mathbb{E}\left[ \log \left( \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right) \right) \cdot \mathbf{1}_{(X<1)} \right] + \mathbb{E}\left[ \log \left( X^{2^n} \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right) \right) \cdot \mathbf{1}_{(X>1)} \right] \right).
\end{aligned} \tag{3.2.13}$$

Further,

$$\begin{aligned}
&\mathbb{E}[\log X] \\
\leq& \frac{1}{(n+6)2^n} \left( \log \left( \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right) \right) \cdot \mathbf{1}_{(X<1)} + \mathbb{E}\left[ \log \left(X^{2^n}\right) \cdot \mathbf{1}_{(X>1)} \right] + \log \left( \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right) \right) \cdot \mathbf{1}_{(X>1)} \right) \\
=& \frac{1}{(n+6)2^n} \left( \log \left( \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right) \right) \right) + \frac{2^n}{(n+6)2^n} \mathbb{E}\left[ \log (X) \cdot \mathbf{1}_{(X>1)} \right] \\
=& \frac{\log c_n}{(n+6)2^n} + \frac{1}{(n+6)} \left(2\mathbb{E}[\log X]\right).
\end{aligned} \tag{3.2.14}$$

To obtain (3.2.14), we used the fact that $c_n = \prod_{k=1}^{2^n} \left(a_n^k + b_n^k\right)$ and $\mathbb{E}\left[\log (X) \cdot \mathbf{1}_{(X>1)}\right] = 2\mathbb{E}[\log X]$, which is shown in (3.6.3) by Proposition 3.6.1. Subtracting the last term in (3.2.14) from both sides and dividing by the resulting constant that multiplies $\mathbb{E}[\log X]$, we have

$$\mathbb{E}[\log X] \leq \frac{\log c_n}{(n+4)\, 2^n}.$$

This result gives us the upper bound.

On the other hand, using Lemma 3.2.5,

$$\mathbb{E}[\log X] = \frac{1}{(n+6)2^n}\left(\mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k X + b_n^k\right)\right)\cdot \mathbf{1}_{(X<1)}\right] + \mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k X + b_n^k\right)\right)\cdot \mathbf{1}_{(X>1)}\right]\right)$$
$$\geq \frac{1}{(n+6)2^n}\left(\mathbb{E}\left[\log\left(X^{2^n}\prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right)\right)\cdot \mathbf{1}_{(X<1)}\right] + \mathbb{E}\left[\log\left(\prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right)\right)\cdot \mathbf{1}_{(X>1)}\right]\right) \tag{3.2.15}$$

Further,

$$\begin{aligned}
&\mathbb{E}[\log X] \\
\geq& \frac{1}{(n+6)2^n}\left(\log\left(\prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right)\right)\cdot \mathbf{1}_{(X<1)} + \mathbb{E}\left[\log\left(X^{2^n}\right)\cdot \mathbf{1}_{(X<1)}\right] + \log\left(\prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right)\right)\cdot \mathbf{1}_{(X>1)}\right) \\
=& \frac{1}{(n+6)2^n}\left(\log\left(\prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right)\right)\right) + \frac{2^n}{(n+6)2^n}\mathbb{E}\left[\log(X)\cdot \mathbf{1}_{(X<1)}\right] \\
=& \frac{\log c_n}{(n+6)2^n} - \frac{1}{(n+6)}\mathbb{E}[\log(X)]. 
\end{aligned} \tag{3.2.16}$$

To obtain (3.2.16), we used the fact that $c_n = \prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right)$ and $\mathbb{E}[\log(X)\cdot \mathbf{1}_{X<1}] = -\mathbb{E}[\log X]$, which is shown in (3.6.4) by Proposition 3.6.1. Adding the last term in (3.2.16) to both sides and dividing by the resulting constant that multiplies $\mathbb{E}[\log X]$, we have

$$\frac{\log c_n}{(n+7)2^n} \leq \mathbb{E}[\log(X)].$$

This gives us the lower bound. □

We now show that these bounds converge to the Lyapunov exponent.

**Theorem 3.2.8.** *If $p_n, q_n$ are the sequences defined in (3.2.12), then*

$$p_n, q_n \to \mathbb{E}[\log X].$$

*Proof.* It follows from the earlier observations that $c_n \leq (F_{n+3})^{2^n}$ where $F_n$ is the usual Fibonacci sequence. Also note that

$$\begin{aligned}
\lim_{m\to\infty}\frac{\log\left((F_{m+3})^{2^m}\right)}{(m+4)2^m} &= \lim_{m\to\infty}\frac{\log(F_{m+3})}{(m+4)} \\
&= \lim_{m\to\infty}\frac{1}{m+4}\log\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{m+4}}{\sqrt{5}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^{m+4}}{\sqrt{5}}\right) \\
&= \log(\phi_1).
\end{aligned}$$

Thus, we have

$$\begin{aligned} |q_n - p_n| &= \frac{\log c_n}{(n+4)\,2^n} - \frac{\log c_n}{(n+7)\,2^n} \\ &= \frac{3\log c_n}{(n+7)\,(n+4)\,2^n} \\ &\leq \frac{3}{n+7} \times \frac{\log\left((F_{n+3})^{2^n}\right)}{(n+4)\,2^n} = 0. \end{aligned} \tag{3.2.17}$$

Using (3.2.11) and (3.2.17), we reach the desired result. □

Unfortunately, there is no obvious recursion among the $c_n$ values. In order to compute $c_n$ using its definition, we must consider $2^n$ coefficient pairs. To do so beyond $n = 25$ exceeds our computing power. With $n = 25$, we have the upper bound from (3.2.11) to be 0.225397 and the lower bound to be 0.204266.

## 3.3 Overview of Parameter Models

We consider a parameter model to be one where an element of the random matrix that the model is based on is either a random variable whose parameter we vary or a random variable that is multiplied by a real-valued parameter.

More specifically, the random matrices $\{Y_i\}_{i\geq 1}$ in the first type of parameter model are of the form

$$Y_i = \begin{pmatrix} \epsilon_i & b \\ c & d \end{pmatrix},$$

where $\epsilon_i \sim \text{Bernoulli}\,(p)$ with the parameter $p \in (0,1)$; we investigate the properties of this model in Section 3.6.

In the second type of parameter model, the random matrices $\{Y_i\}_{i\geq 1}$ are of the form

$$Y_i = \begin{pmatrix} \xi\epsilon_i & b \\ c & d \end{pmatrix},$$

where $\epsilon_i$ is distributed as Cauchy $(0,1)$ in Section 3.4, Bernoulli $\left(\frac{1}{2}\right)$ in Section 3.5, and Rademacher in Section 3.7. In all models, $\xi \in \mathbb{R}$.

For the first type, we study the behavior of the following two objects:

1. $\lambda\,(p)$

2. the variance of
$$L_p = \frac{\sum_{i=1}^n \log \|S_i x\| - n\lambda\,(p)}{\sqrt{n}}.$$

For the second type, we study

1. $\lambda\,(\xi)$

2. the variance of
$$L_\xi = \frac{\sum_{i=1}^{n} \log \|S_i x\| - n\lambda(\xi)}{\sqrt{n}}.$$

If $\lambda(\xi)$ or $\lambda(p)$ does not have an explicit form, we approximate it using the method outlined in Section 3.3.1. We simulate $\mathrm{Var}(L_\xi)$ and $\mathrm{Var}(L_p)$ using the method outlined in Section 3.3.2.

### 3.3.1 Approximating $\lambda$

We first define $U_1, U_2, \ldots U_N$ as

$$\begin{aligned} U_1 &= \log \left\| Y_1 \frac{x}{\|x\|} \right\| \\ U_2 &= \log \left\| Y_2 \frac{Y_1 x}{\|Y_1 x\|} \right\| \\ &\vdots \\ U_n &= \log \left\| Y_n \frac{Y_{n-1} \ldots Y_1 x}{\|Y_{n-1} \ldots Y_1 x\|} \right\|. \end{aligned}$$

Since
$$U_n = \log \|Y_n Y_{n-1} \ldots Y_1 x\| - \log \|Y_{n-1} \ldots Y_1 x\|, \tag{3.3.1}$$
we have
$$\begin{aligned} \log \|Y_n Y_{n-1} \ldots Y_1 x\| &= U_n + \log \|Y_{n-1} \ldots Y_1 x\| \\ &= U_n + U_{n-1} + \cdots + U_1 + \log \|x\|. \end{aligned} \tag{3.3.2}$$

Further,
$$\log \frac{\|S_n x\|}{\|x\|} = \sum_{i=0}^{n-1} \log \frac{\|S_{i+1} x\|}{\|S_i x\|}, \tag{3.3.3}$$

which implies
$$\frac{\log \|S_n x\|}{n} = \frac{\sum_{i=0}^{n} \log \frac{\|S_{i+1} x\|}{\|S_i x\|} + \log \|x\|}{n}. \tag{3.3.4}$$
By Furstenberg and Kesten's analogue to the Law of Large Numbers (Theorem 3.1.1), we can approximate $\lambda$ for a large $n$ using the right-hand side of (3.3.4).

### 3.3.2 Variance Simulation Method

Motivated by Theorem 3.1.4, we obtain estimates for the variance associated with a multiplicative Central Limit Theorem for products of random matrices with the following simulation method:

1. Choose an interval $[a, b]$ as the range of $\xi$. Divide this interval into sub-intervals of length $k$. Let $\xi$ be of the form $a + jk$ for $j = 0, 1, \ldots, \frac{b-a}{k} - 1$.

2. Choose a unit vector $x$.
3. For each $\xi$, simulate
$$L_\xi = \frac{\sum_{i=1}^{n} \log \|S_i x\| - n\lambda(\xi)}{\sqrt{n}},$$
and store the result in a data vector of length $\frac{b-a}{k}$.
4. Repeat Step 3 an $m$ number of times to obtain a $m \times \frac{b-a}{k}$ matrix, where the $j^{\text{th}}$ column contains all of the $L_\xi$ corresponding to $\xi_j$.
5. Calculate the variance of each column of the matrix.

To obtain estimates for $\operatorname{Var}(L_p)$, we use the same procedure with
$$L_p = \frac{\sum_{i=1}^{n} \log \|S_i x\| - n\lambda(p)}{\sqrt{n}},$$
in Step 3.

Note that in all of our simulations, we set $x = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in Step 2.

## 3.4 $\xi \cdot$ Cauchy Parameter Model

The first parameter model we consider is based on the standard Cauchy distribution (that is, Cauchy with location $x_0 = 0$ and scale $\gamma = 1$). The probability density function of a Cauchy random variable $X$ with location $x_0$ and scale $\gamma$ is
$$f(x) = \frac{1}{\pi\gamma\left[1 + \left(\frac{x-x_0}{\gamma}\right)^2\right]}$$
for $-\infty < x < \infty$.

In this section, we consider a measure $\mu_\xi$ on $\mathrm{GL}(2, \mathbb{R})$ given by
$$Y_i = \begin{pmatrix} \xi\epsilon_i & -1 \\ 1 & 0 \end{pmatrix}, \ \epsilon_i \sim \text{Cauchy}\,(0, 1)\,, \ \xi \in \mathbb{R}. \tag{3.4.1}$$
It can be seen that $\mu_\xi$ satisfies Theorem 3.1.3 (see [3, pp. 36]). By (3.1.2), the unique $\mu_\xi$-invariant distribution $\nu_\xi$ must have the same distribution of the random variable $X_\xi$ that satisfies the distributional identity
$$X_\xi \sim -\frac{1}{X_\xi} + \xi\epsilon, \tag{3.4.2}$$
where $\epsilon \sim \text{Cauchy}\,(0, 1)$ is independent of $X_\xi$. Following [3, pp. 33], we have an explicit formula for the Lyapunov exponent in terms of the parameter $\xi$.

**Proposition 3.4.1.** *The Lyapunov exponent $\lambda(\xi)$ related to the random matrices $Y_i$ given in (3.4.1) is of the form*
$$\lambda(\xi) = \log\left(\frac{\xi + \sqrt{\xi^2 + 4}}{2}\right). \tag{3.4.3}$$

*Proof.* We want to compute $\lambda(\xi) = \mathbb{E}[\log|X_\xi|]$, where $X_\xi$ is a random variable satisfying (3.4.2). First, we show that the random variable $X_\xi \sim \text{Cauchy}\,(0, a)$ satisfies the distributional identity

$$X_\xi \sim -\frac{1}{X_\xi} + \xi\epsilon, \tag{3.4.4}$$

where $\epsilon$ is a standard Cauchy distribution independent of $X_\xi$.

From the transformation properties of the Cauchy distribution, we know that the right-hand side of (3.4.4) has a Cauchy distribution;

$$-\frac{1}{X_\xi} + \xi\epsilon \sim \text{Cauchy}\left(0, \frac{1}{a} + |\xi|\, a\right).$$

Since the left-hand side of (3.4.4) also has a Cauchy distribution, it suffices to find which parameter $a$ (3.4.4) holds for. Taking the characteristic function of each side (recall that $\varphi_X(\lambda) = e^{-a|\lambda|}$) and using the independence of $X_\xi$ and $\epsilon$,

$$e^{-a|\lambda|} = e^{\frac{-1}{a}|\lambda|}e^{-\xi|\lambda|}.$$

Clearly, $a = \frac{1}{a} + \xi$, which implies that $a^2 - \xi a - 1 = 0$. Hence,

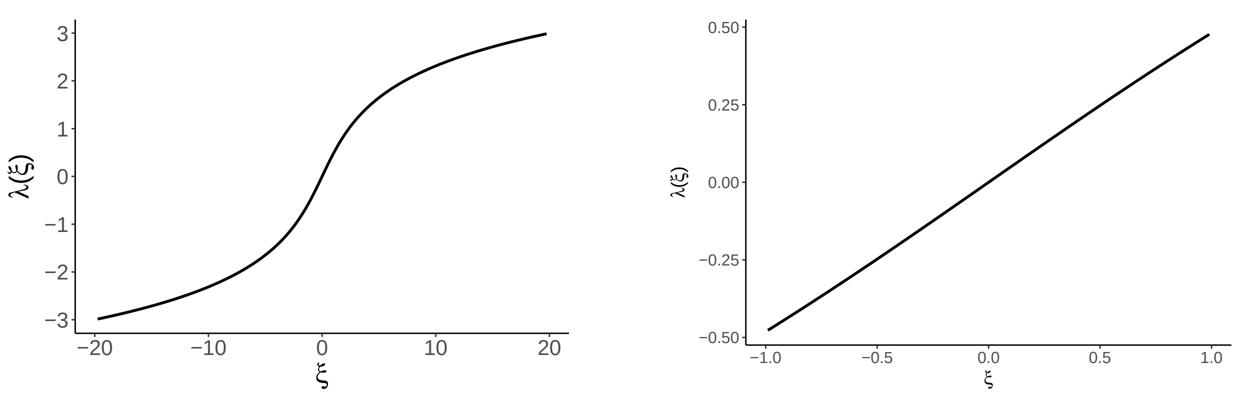
$$a = \frac{\xi + \sqrt{\xi^2 + 4}}{2}.$$

A simple computation thus shows that

$$\begin{aligned}\lambda(\xi) = \int \log|x| \frac{1}{\pi a(1 + \frac{x^2}{a^2})} dx &= \frac{1}{\pi a}(\pi a \ \log|a|) \\ &= \log|a|,\end{aligned}$$

completing the proof. □

Figure 3.1a shows the graph of $\lambda(\xi)$ for $\xi \in [-20, 20]$; in Figure 3.1b, we plot $\lambda(\xi)$ for $\xi \in [-1, 1]$.




(a) (b)

Figure 3.1: $\xi$ vs. $\lambda(\xi)$

Following our variance simulation method of Section 3.3.2, we specify the set of $\xi$ to use in our parameter models by choosing an interval $[a, b]$ as the range of $\xi$, dividing this interval into sub-intervals of length $k$, and writing $\xi$ in the form $a + jk$ for $j = 0, 1, \ldots, \frac{b-a}{k} - 1$. Figure 3.2 illustrates the results for $\xi \in [-20, 20]$, i.e., with $a = -20$, $b = 20$, and $k = 0.25$, for a total of $\frac{b-a}{k} = \frac{40}{0.25} = 160$ points in the interval. This is the same set of $\xi$ used to produce Figure 3.1a, that is, we use the set of $\lambda(\xi)$ shown in Figure 3.1a in our simulation of $\text{Var}(L_\xi)$ in Figure 3.2.

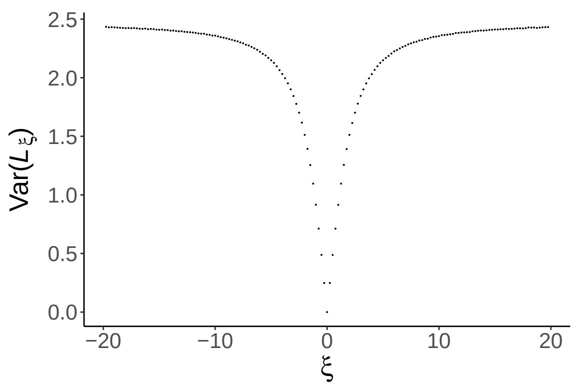



Figure 3.2: $k = 0.25$, $m = 5\,000\,000$, $n = 1000$

In Figure 3.3, we superimpose a curve on the points shown in Figure 3.2. We do this for all $\lambda$ and variance figures that follow, unless noted otherwise.

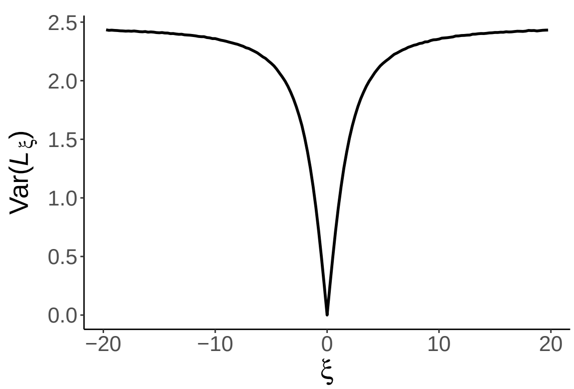



Figure 3.3: $k = 0.25$, $m = 5\,000\,000$, $n = 1000$

Clearly, $\text{Var}(L_\xi)$ is equal to 0 when $\xi = 0$. Further, while a slight perturbation of $\xi$ on either side of zero produces a very steep change in the value of the variance, the rate of change gradually flattens out as $\xi$ moves away from zero.

Next, in Figure 3.4, we plot $\text{Var}(L_\xi)$ for $\xi \in [-1, 1]$ with much finer intervals of length 0.01.

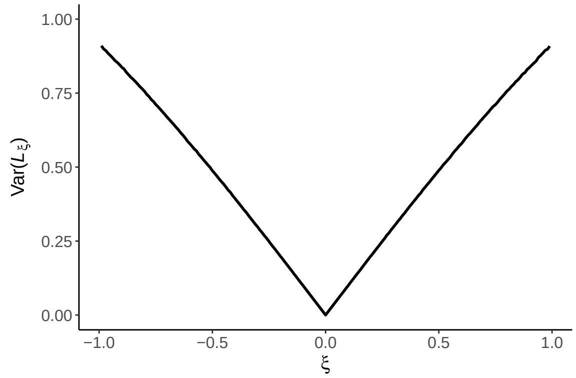



Figure 3.4: $k = 0.01$, $m = 1\,000\,000$, $n = 1000$

The two halves of the graph (i.e., between $-1$ and 0 and between 0 and 1) appear, at first glance, to be linear. While there are some small jumps on either side, especially as $\xi$ moves away from zero, the mean squared error is equal to 0.0001 (for each half), so these deviations are indeed negligible.

## 3.5 $\xi \cdot$ Bernoulli$\left(\frac{1}{2}\right)$ Parameter Model

Our second parameter model is based on the Bernoulli $\left(\frac{1}{2}\right)$ model of Section 3. The random matrices are of the form

$$Y_i = \begin{pmatrix} \xi\epsilon_i & 1 \\ 1 & 0 \end{pmatrix}, \ \epsilon_i \sim \text{Bernoulli}\left(\frac{1}{2}\right), \ \xi \in \mathbb{R}. \tag{3.5.1}$$

In this model, we do not have an explicit expression for $\lambda(\xi)$. Thus, in order to study the asymptotics of $\lambda(\xi)$, we build on our approximation method from Section 3.3.1 and simulate it as follows.

1. Choose an interval $[a, b]$ as the range of $\xi$. Divide this interval into sub-intervals of length $k$. Let $\xi$ be of the form $a + jk$ for $j = 0, 1, \ldots, \frac{b-a}{k} - 1$.

2. For each $\xi_j$,

   (a) simulate
   $$x_{j,i} = \frac{1}{x_{j,i-1}} + \xi_j \epsilon_{j,i},$$
   with $i = 0, 1, \ldots, N$, $\epsilon_{j,i} \sim$ Bernoulli $\left(\frac{1}{2}\right)$ and $x_{j,0} = 1$;

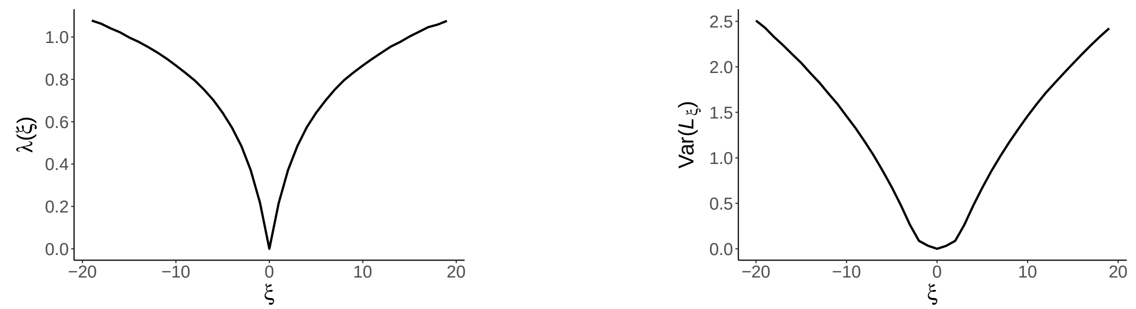
   (b) repeat Step(2)(a) an $M$ number of times to get a set of values
   $$z_j = \left\{x_{j,N}^{(1)}, x_{j,N}^{(2)}, \ldots x_{j,N}^{(M)}\right\}.$$

3. Calculate
   $$\lambda(\xi_j) = \mathbb{E} \log |z_j| \,.$$

For our simulations of $\lambda(\xi)$ and $\mathrm{Var}(L_\xi)$, we use two sets of values for $\xi$:

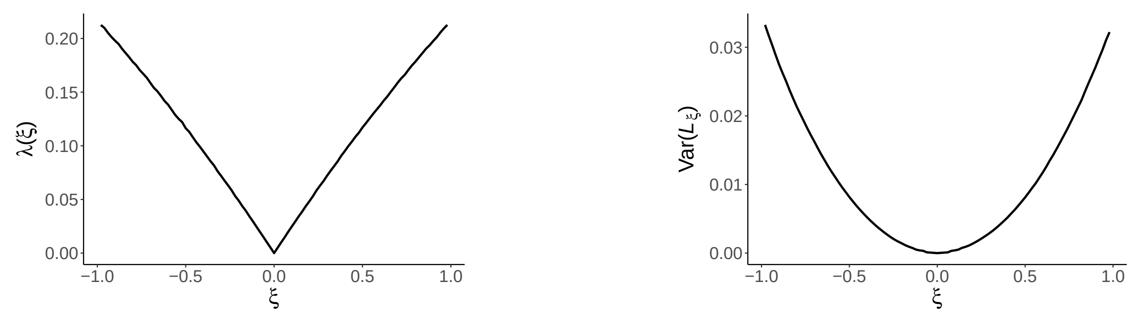
- $a = -20$, $b = 20$, $k = 1$ (Figure 3.5)
- $a = -1$, $b = 1$, $k = 0.02$. (Figure 3.6)

Both sets' $\lambda(\xi)$ are simulated with $M = 1\,000\,000$ and $N = 1000$, and variance with $m = 1\,000\,000$ and $n = 1000$.




(a) (b)

Figure 3.5




(a) (b)

Figure 3.6

## 3.6 Bernoulli $(p)$ Parameter Model

In this section, we consider a random matrix model where the random entry follows a Bernoulli $(p)$ distribution and the parameter of interest is $p$. Let $\mu_p$ be a measure on $\mathrm{GL}(2, \mathbb{R})$ given by

$$Y_i = \begin{pmatrix} \epsilon_i & 1 \\ 1 & 0 \end{pmatrix}, \ \epsilon_i \sim \mathrm{Bernoulli}\,(p)\,, \ 0 \le p \le 1. \tag{3.6.1}$$

Recall that in [8, Theorem 5.2] the author proves the existence of a unique $\mu_p$-invariant distribution $\nu_p$. By (3.1.2), $\nu_p$ must have the same distribution of the random variable $X_p$ that satisfies the following distributional identity

$$X_p \sim \frac{1}{X_p} + \epsilon_i, \tag{3.6.2}$$

where $\epsilon \sim \text{Bernoulli}\,(p)$ is independent of $X_p$.

Let $\lambda(p)$ be the Lyapunov exponent related to the matrices of the form $Y_i$ in (3.6.1). Much like with the Bernoulli $\left(\frac{1}{2}\right)$ model, we do not have an expression for the Lyapunov exponent. Despite this, we can make some observations about its behavior depending on the parameter $p$. As pointed out in [8, Theorem 5.2] we have that

$$X_p \geq 0$$

with support in $(0, \infty)$.

To study the asymptotics of $\lambda(p)$ near zero, we begin with inequalities related to $\mathbb{E}\left[\log\left(X_p\right)\right]$.

**Proposition 3.6.1.** *If $X_p$ is the random variable that satisfies (3.6.2), then we have the following identities:*

$$\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right] = \frac{1}{p}\mathbb{E}\left[\log\left(X_p\right)\right], \tag{3.6.3}$$

$$\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p<1)}\right] = \frac{(p-1)}{p}\mathbb{E}\left[\log\left(X_p\right)\right], \tag{3.6.4}$$

*and*

$$\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p<1)}\right] = (p-1)\,\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right]. \tag{3.6.5}$$

*Proof.* We first prove (3.6.5) by noting that

$$\begin{aligned}
\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p<1)}\right] &= p\mathbb{E}\left[\log\left(\frac{1}{X_p}+1\right)\cdot \mathbf{1}_{\left(\frac{1}{X_p}+1<1\right)}\right] + (1-p)\,\mathbb{E}\left[\log\left(\frac{1}{X_p}\right)\cdot \mathbf{1}_{\left(\frac{1}{X_p}<1\right)}\right] \\
&= 0 + (1-p)\,\mathbb{E}\left[\log\left(\frac{1}{X_p}\right)\cdot \mathbf{1}_{(X_p>1)}\right] \\
&= (p-1)\,\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right].
\end{aligned}$$

To prove (3.6.3), observe that

$$\begin{aligned}
\mathbb{E}\left[\log\left(X_p\right)\right] &= \mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right] + \mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p<1)}\right] \\
&= \mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right] + (p-1)\,\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right] \\
&= p\mathbb{E}\left[\log\left(X_p\right)\cdot \mathbf{1}_{(X_p>1)}\right].
\end{aligned}$$

Combining (3.6.5) and (3.6.3), we can show (3.6.4). □

**Proposition 3.6.2.** *If $X_p$ is the random variable that satisfies (3.6.2), then*

$$\mathbb{E}\left[\log\left(X_p\right)\right] = \frac{p}{2}\mathbb{E}\left[\log\left(X_p+1\right)\right]. \tag{3.6.6}$$

*Proof.* The proof is analogous to the proof of Proposition 3.2.1. □

We can now use these results to establish bounds on the Lyapunov exponent as a function of $p$. We begin with an upper estimate.

**Proposition 3.6.3.** *If $X_p$ is the random variable that satisfies (3.6.2), then*

$$\mathbb{E}\left[\log\left(X_p\right)\right] \leq p\log 2. \tag{3.6.7}$$

*Proof.* First, $\log(x+1) \leq \log(2x)$ for $x \geq 1$. Using (3.6.3), we have

$$\begin{aligned} \mathbb{E}\left[\log\left(X_p\right)\right] &= \frac{p}{2}\mathbb{E}\left[\log(X_p+1)\right] \\ &= \frac{p}{2}\mathbb{E}\left[\log(X_p+1)\cdot \mathbf{1}_{(X_p<1)} + \log(X_p+1)\cdot \mathbf{1}_{(X_p>1)}\right] \\ &\leq \frac{p}{2}\left[\log\left(2\right)\mathbb{P}\left(X_p<1\right) + \mathbb{E}\left[\log(2X_p)\cdot \mathbf{1}_{(X_p>1)}\right]\right] \\ &= \frac{p}{2}\left[\log 2 + \mathbb{E}\left[\log(X_p)\cdot \mathbf{1}_{(X_p>1)}\right]\right] \\ &= \frac{p}{2}\log 2 + \frac{1}{2}\mathbb{E}\left[\log\left(X_p\right)\right]. \end{aligned} \tag{3.6.8}$$

Subtracting $\frac{1}{2}\mathbb{E}\left[\log\left(X_p\right)\right]$ from both sides of (3.6.8), we reach the desired result. □

Now we consider the lower bound of the Lyapunov exponent as a function of $p$.

**Proposition 3.6.4.** *If $X_p$ is the random variable that satisfies (3.6.2), then*

$$\frac{p}{3-p}\log 2 \leq \mathbb{E}\left[\log\left(X_p\right)\right]. \tag{3.6.9}$$

*Proof.* Note that $x+1 \geq 2x$ for $x < 1$. Using (3.6.4), we have

$$\begin{aligned} \mathbb{E}\left[\log\left(X_p\right)\right] &= \frac{p}{2}\mathbb{E}\left[\log(X_p+1)\right] \\ &= \frac{p}{2}\mathbb{E}\left[\log(X_p+1)\cdot \mathbf{1}_{(X_p<1)} + \log(X_p+1)\cdot \mathbf{1}_{(X_p>1)}\right] \\ &\geq \frac{p}{2}\left[\mathbb{E}\left[\log(2X_p)\cdot \mathbf{1}_{(X_p<1)}\right] + \log 2\mathbb{P}\left(X_p>1\right)\right] \\ &= \frac{p}{2}\left[\log 2 + \mathbb{E}\left[\log(X_p)\cdot \mathbf{1}_{(X_p<1)}\right]\right] \\ &= \frac{p}{2}\log 2 + \frac{(p-1)}{2}\mathbb{E}\left[\log\left(X_p\right)\right]. \end{aligned} \tag{3.6.10}$$

A simple rearrangement gives us the desired result. □

Using these bounds, we can establish the asymptotics for $\lambda(p)$. Recall that a function $f(x)$ is equal to $\Theta\left(g(x)\right)$ near $x_0$ if there exists constants $C_1, C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ near $x_0$.

**Proposition 3.6.5.** *As $p \to 0$,*

$$\lambda(p) = \Theta(p).$$

*Proof.* We know that $\lambda(p) = \mathbb{E}[\log\left(X_p\right)]$. From (3.6.7), we have the bound

$$|\mathbb{E}[\log\left(X_p\right)]| \leq |p|\log\left(2\right),$$

implying

$$\limsup_{p\to 0}\frac{|\mathbb{E}[\log\left(X_p\right)]|}{|p|} \leq \log\left(2\right),$$

so that $\lambda(p)$ is equivalent to $O(p)$ as $p \to 0$. Now consider the bound from (3.6.9):

$$|\mathbb{E}[\log(X_p)]| \geq \frac{|p|}{3-p} \log(2) .$$

This implies that

$$\limsup_{p \to 0} \frac{|\mathbb{E}[\log(X_p)]|}{|p|} \geq \limsup_{p \to 0} \frac{\log 2}{3-p} = \frac{\log(2)}{3},$$

thus completing the proof. □

In Figure 3.7, we plot $\lambda(p)$ in black and the upper and lower bounds of $\lambda(p)$ (from Propositions 3.6.3 and 3.6.4 respectively) in blue.

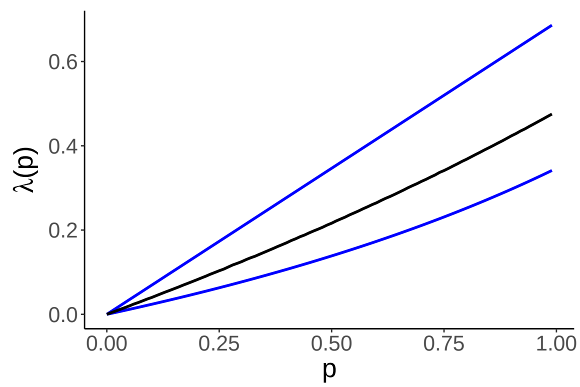



Figure 3.7: $k = 0.01$, $n = 1\,000\,000$

We simulate, usng our simulation method from Section 3.3.2 , $\text{Var}(L_p)$ with $n = 1000$ and $m = 1\,000\,000$. We plot the resulting points in Figure 3.8.

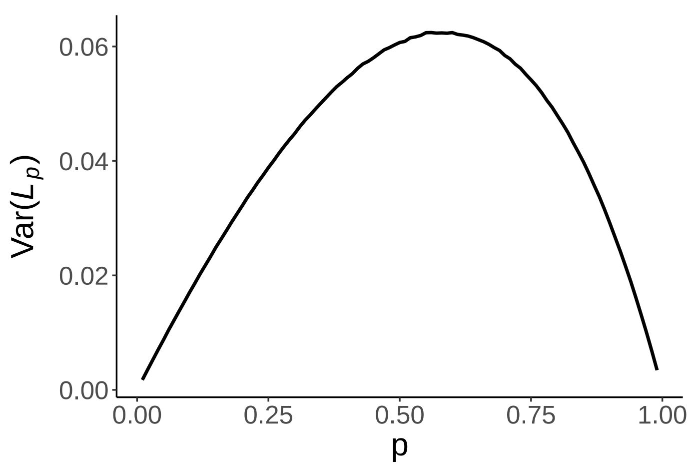



Figure 3.8

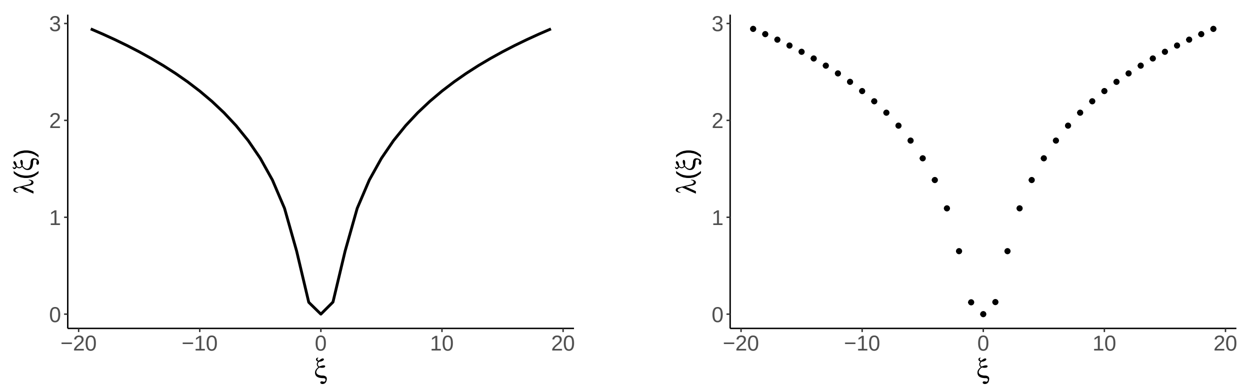
With $p = 0$ and $p = 1$, the variance is zero. The distribution of variances is moderately skewed to the left. The maximum variance is achieved at $p = 0.56$.

## 3.7 $\xi \cdot$ Rademacher Parameter Model

Our final parameter model involves matrices where the first entry is a parameter $\xi \in \mathbb{R}$ multiplied by a Rademacher random variable. A Rademacher random variable takes on the value 1 with probability $\frac{1}{2}$ and $-1$ with probability $\frac{1}{2}$. We consider a measure $\mu_\xi$ on $\mathrm{GL}(2, \mathbb{R})$ given by

$$Y_i = \begin{pmatrix} \xi\epsilon_i & 1 \\ 1 & 0 \end{pmatrix}, \ \epsilon_i \sim \text{Rademacher}, \ \xi \in \mathbb{R}. \tag{3.7.1}$$

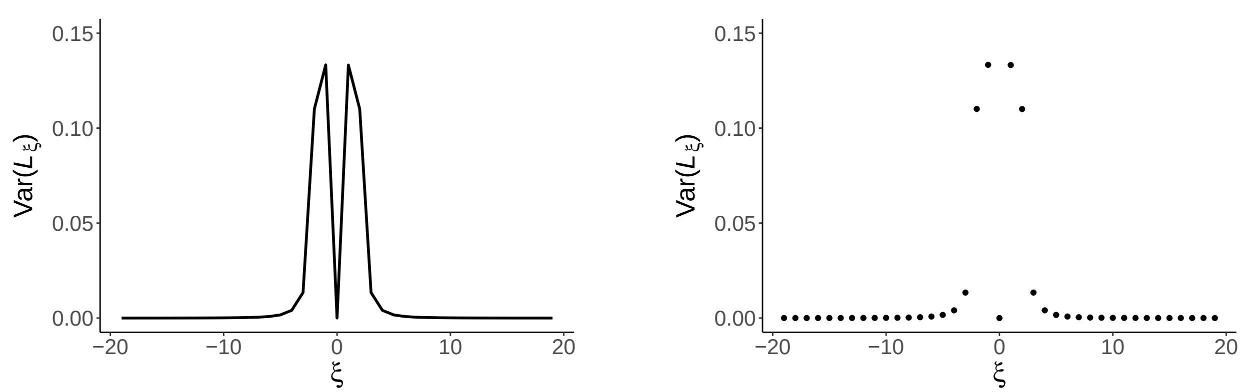
We do not have an explicit formula for $\lambda(\xi)$, so we approximate it using our standard method from Section 3.3.1. Figure 3.9 displays the resulting $\lambda(\xi)$ plot for $\xi \in [-20, 20]$ with intervals of length of 1.




(a) (b)

Figure 3.9

We use these $\lambda(\xi)$ to simulate $\mathrm{Var}(L_\xi)$, shown in Figure 3.10. The shape of the variance curve in Figure 3.10a is difficult to interpret, so we examine the individual points in Figure 3.10b.




(a) (b)

Figure 3.10

The variance is 0 when $\xi = 0$, but sharply increases at $\xi = 1$ (and $\xi = -1$) before decreasing in the same manner between $\xi = 1$ and $\xi = 3$ (and between $\xi = -1$ and $\xi = -3$).

To better understand the behavior of the variance between $-3$ and 3, we look at finer intervals ($k = 0.05$) between $\xi = 0$ and $\xi = 3$. The sixty variance points that we obtain in Figure 3.11 clearly indicate that the region between $\xi = 1.5$ and $\xi = 1.8$ experiences the most volatility.

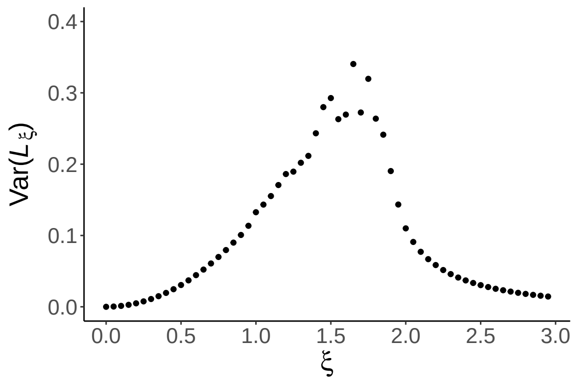



Figure 3.11

Thus, we examine this region between $\xi = 1.5$ and $\xi = 1.8$ in greater detail. Figure 3.12 shows the variance of 31 sample points in this region. There is no clearly discernable pattern, although it is worth mentioning that the Golden Ratio lies in this region. For future research, it would be interesting to investigate this phenomenon further.

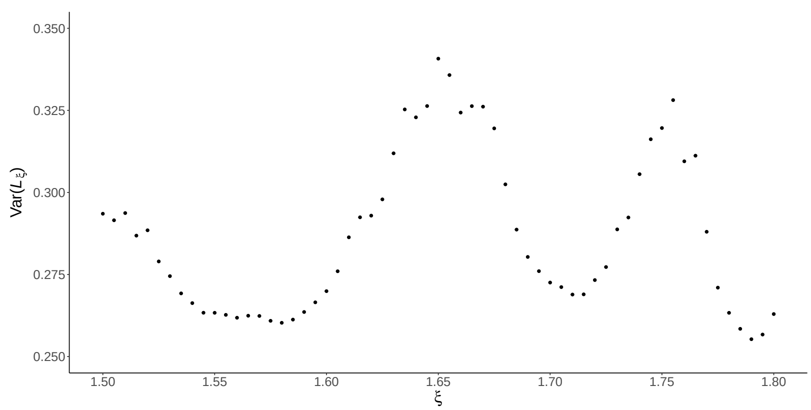



Figure 3.12

# References


[1] Patrick Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, second edition, 1986.

[2] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[3] Philippe Bougerol and Jean Lacroix. *Products of Random Matrices with Applications to Schrödinger Operators*, volume 8 of *Progress in Probability and Statistics*. Birkhäuser.

[4] Philippe Chassaing, Gérard Letac, and Marianne Mora. Brocot sequences and random walks on $sl_2(r)$. In Herbert Heyer, editor, *Probability Measures on Groups VII*, pages 36–48, Berlin, Heidelberg, 1984. Springer Berlin Heidelberg.

[5] R.M. Dudley. *Real Analysis and Probability*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002.

[6] Hillel Furstenberg and Harry Kesten. Products of random matrices. 31(2):457–469.

[7] Hillel Furstenberg and Yuri Kifer. Random matrix products and measures on projective spaces. *Israel J. Math.*, 46(1-2):12–32, 1983.

[8] Alok Goswami. Random continued fractions: a Markov chain approach. *Econom. Theory*, 23(1):85–105, 2004. Symposium on Dynamical Systems Subject to Random Shock.

[9] R. Lima and M. Rahibe. Exact lyapunov exponent for infinite products of random matrices. *Journal of Physics A: Mathematical and General*, 27:3427–3437.

[10] Jens Marklof, Yves Tourigny, and Lech Wolowski. Explicit invariant measures for products of random matrices. *Trans. Amer. Math. Soc.*, 360:3391–3427, 2008.

[11] Robert C. Merton. Theory of rational option pricing. *Bell J. Econom. and Management Sci.*, 4:141–183, 1973.

[12] Emile Le Page. Theoremes limites pour les produits de matrices aleatoires. 928:258–303.

[13] Valentin V. Petrov. *Limit theorems of probability theory*, volume 4 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York, 1995. Sequences of independent random variables, Oxford Science Publications.

[14] Mark Pollicott. Maximal lyapunov exponents for random matrix products. *Inventiones mathematicae*, 181(1):209–226, Jul 2010.

[15] Sheldon M. Ross. *An elementary introduction to mathematical finance*. Cambridge University Press, Cambridge, third edition, 2011.

[16] A. Sengupta. *Pricing Derivatives*. McGraw-Hill, 2005.

[17] S.R.S. Varadhan. *Probability Theory*. Courant Lecture Notes. Courant Institute of Mathematical Sciences, 2001.

[18] Divakar Viswanath. Random Fibonacci sequences and the number $1.13198824\ldots$. *Math. Comp.*, 69(231):1131–1155, 2000.