INTEGRATION BY PARTS AND QUASI-INVARINANCE
FOR THE HORIZONTAL WIENER MEASURE ON A
FOLIATED COMPACT MANIFOLD

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Abstract. We prove a version of Driver’s integration by parts for-
formula [16] on the horizontal path space of a totally geodesic Riemannian
foliation and prove that the horizontal Wiener measure is quasi-invariant
with respect to the flows generated by suitable tangent processes.

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1. INTRODUCTION

In this paper we study quasi-invariance properties and related integration by parts formulas for the horizontal Wiener measure on a foliated Riemannian manifold equipped with a sub-Riemannian structure. These are most closely related to the well-known results by B. Driver [16] who established such properties for the Wiener measure on a path space over a compact Riemannian manifold. Quasi-invariance in such settings can be viewed as a curved version of the classical Cameron-Martin theorem for the Euclidean space. While the techniques developed for path spaces over Riemannian manifolds are not easily adapted to the sub-Riemannian case we consider, we take advantage of the recent advances in this field. The geometric and stochastic analysis of sub-Riemannian structures on foliated manifolds has attracted a lot of attention in the past few years (see for instance [2,6,19,27–30,43]).

In particular, we make use of the tools such as Weitzenböck formulas for the sub-Laplacian to extend results by J.-M. Bismut, B. Driver et al to foliated Riemannian manifolds. More precisely, the first progress in developing geometric techniques in the sub-Riemannian setting has been made in [4], where a version of Bochner’s formula for the sub-Laplacian was established and generalized curvature-dimension conditions were studied. This Bochner-Weitzenböck formula was then used in [2] to develop a sub-Riemannian stochastic calculus. One of the difficulties in this case is that, a priori, there is no canonical connection on such manifolds such as the Levi-Civita connection in the Riemannian case. However, [2] introduces a one-parameter family of metric connections associated with Bochner’s formula proved in [4] and shows that the derivative of the sub-Riemannian heat semigroup can be expressed in terms of a damped stochastic parallel transport. It should be noted that these connections do not preserve the geometry of the foliation in general. In particular, the corresponding parallel transport does not necessarily transforms a horizontal vector into a horizontal vector, that is, these connections in general are not horizontal. As a consequence, establishing an integration by parts formulas for directional derivatives on the path space of the horizontal Brownian motion, similarly to Driver’s integration by parts formula [16] for the Riemannian Brownian motion, is not straightforward. As a result, the integration by parts formula we prove in the current paper can not be simply deduced from the derivative formula for the corresponding semigroup by applying the standard techniques of covariant stochastic analysis on manifolds as presented for instance in [21, Section 4], in particular [21, Theorems 4.1.1, 4.1.2]. A different approach to proving
quasi-invariance in an infinite-dimensional sub-Riemannian setting has been used in [5].

The history of analysis on path and loop spaces have been developed over several decades, and we will not be able to refer to all the relevant publications, but we mention some which are closer to the subject of this paper. In particular, J.-M. Bismut’s book [8] contains an integration by parts formula on a path space over a compact Riemannian manifold. His methods were based on the Malliavin calculus and Bismut’s motivation was to deal with a hypoelliptic setting as described in [8, Section 5]. A breakthrough has been achieved by B. Driver [16], who established quasi-invariance properties of the Wiener measure over a compact Riemannian manifold, and as a consequence an integration by parts formula. This work has been simplified and extended by E. Hsu [31], and also approached by O. B. Enchev and D. W. Stroock [23]. A review of these techniques can be found in [33]. In [34, 36] the noncompact case has been studied. A different approach to analysis on Riemannian path space can be found in [14], where tangent processes, Markovian connections, structure equations and other elements of what the authors call the renormalized differential geometry on the path space have been introduced.

1.1. Main results and organization of the paper. We now explain in more detail the main results and the organization of the paper. Let \((\mathbb{M}, g)\) be a smooth connected and compact Riemannian manifold equipped with a totally geodesic and bundle-like foliation as described in the beginning of Section 2. On such manifolds, one can define a horizontal Laplacian \(\mathcal{L}\) according to [1, Sections 2.2 and 2.3]), and then the horizontal Brownian motion is defined as the diffusion on \(\mathbb{M}\) with generator \(\frac{1}{2} \mathcal{L}\). The distribution of the horizontal Brownian motion is called the horizontal Wiener measure and its support the horizontal path space. Our main goal is to define a suitable class of tangent processes generating flows such that the horizontal Wiener measure is quasi-invariant under these flows, and to study related integration by parts formulas. As we will see these tangent processes have bounded variation in the horizontal directions but in addition they have a martingale part in the vertical directions. This can be seen even in the simplest case of \(\mathbb{M}\) being the Heisenberg group as we explain in Section 1.2.

In Section 2 we mainly survey known results and introduce the notation and conventions used throughout the paper. Most of this material is based on [6] for the geometric part and [2] for the stochastic part. The most relevant result that will be used later is the Weitzenböck formula given in Theorem 2.3. It asserts that for every \(f \in C_0^\infty(\mathbb{M})\), \(x \in \mathbb{M}\) and every \(\varepsilon > 0\)

\[
dLf(x) = \Box_\varepsilon df(x),
\]

where \(\Box_\varepsilon\) is a one-parameter family of sub-Laplacians on one-forms indexed by the parameter \(\varepsilon > 0\). These sub-Laplacians on one-forms are constructed from a family of metric connections \(\nabla^\varepsilon\) whose adjoint connections \(\hat{\nabla}^\varepsilon\) in the
sense of B. Driver are also metric. Even though Section 2 introduces mostly preliminaries, we present a number of new results there such as Lemma 2.5.

In Section 3, we prove integration by parts formulas for the horizontal Wiener measure with the main result being Theorem 3.12. Suppose $F$ is a cylinder function on the horizontal path space and $v$ is a tangent process in $T_x\mathcal{M}$ as defined in Definition 3.8, then we have

$$E_x(D_vF) = E_x\left(F \int_0^1 \left(v_H'(t) + \frac{1}{2} \langle v_{H,H}^{-1} \mathcal{R}_{H}^{-1}, dB_t \rangle_H \right) dt \right),$$

where $x$ is the starting point of the horizontal Brownian motion, $D_vF$ is the directional derivative of $F$ in the direction of $v$, and $\langle v_{H,H}^{-1} \mathcal{R}_{H}^{-1}, dB_t \rangle_H$ is the stochastic parallel transport for the Bott connection, and $\mathcal{R}_{H}$ is the horizontal Ricci curvature of the Bott connection. The Bott connection is defined in Section 2 and corresponds to the adjoint connection $\hat{\nabla}_\varepsilon$ as $\varepsilon \to \infty$. In the integration by parts formula (1.2), the tangent process $v$ is a $T_x\mathcal{M}$–valued process such that its horizontal part $v_H$ is absolutely continuous and satisfies $E(\int_0^1 \|v_H'(t)\|^2_{T_x\mathcal{M}} dt) < +\infty$ and its vertical part is given by

$$v_V(t) = -\int_0^t \langle v_{H,H}^{-1} T(\|_{0,s} \circ dB_s, v_H(s)) \rangle,$$

where $T$ is the torsion tensor of the Bott connection. Observe that (1.2) looks similar to the integration by parts formulas by J.-M. Bismut and B. Driver. This is not too surprising if one thinks about the special case where the foliation comes from a Riemannian submersion with totally geodesic fibers. We consider this case in Section 3.5.1 and we prove that then Equation (3.12) is actually a horizontal lift of Driver’s formula from the base space of the fibers to $\mathcal{M}$. However, in general foliations do not come from submersions (see for instance [24] for necessary and sufficient conditions) and one therefore needs to develop an intrinsic horizontal stochastic calculus on $\mathcal{M}$ to prove (1.2). Developing such a calculus is one of the main accomplishments of the current paper.

The proof of Theorem 3.12 proceeds in several steps. As in [2], the Weitzenböck formula (1.1) yields a stochastic representation for the derivative of the semigroup of the horizontal Brownian motion in terms of a damped stochastic parallel transport associated to the connection $\nabla_\varepsilon$ (see Lemma 3.13). By using techniques of [3], Lemma 3.13 implies an integration by parts formula for the damped Malliavin derivative as stated in Theorem 3.11. The final step is to prove Theorem 3.12 from Theorem 3.11. The main difficulty is that the connection $\nabla_\varepsilon$ is in general not horizontal. However, it turns out that the adjoint connection $\hat{\nabla}_\varepsilon$ is not only metric but also horizontal. As a consequence, one can use the orthogonal invariance of the horizontal Brownian motion (Lemma 3.18) to filter out the redundant noise which is given by the torsion tensor of $\hat{\nabla}_\varepsilon$. 
It is remarkable that the integration by parts formula for the directional derivatives (3.12) is actually independent of the choice of a particular connection and therefore is independent of \( \varepsilon \) in the one-parameter family of connections used to define the damped Malliavin derivative. While integration by parts formulas for the damped Malliavin derivative may be used to prove gradient bounds for the heat semigroup (as in [2]) and log-Sobolev inequalities on the path space (as in [3]), we prove that the integration by parts formula (1.2) comes from a quasi-invariance property of the horizontal Wiener measure as we show in Section 4.

Quasi-invariance of the horizontal Wiener measure is proven in Theorem 4.12, namely, we show that if \( v \) is a tangent processes on \( T_xM \) whose horizontal part is deterministic, then \( v \) generates a flow in the horizontal path space such that the horizontal Wiener measure is quasi-invariant under this flow. To prove this, it is convenient to work in a bundle over \( M \) which is constructed from horizontal orthonormal frames. This allows us to construct an Itô map between the path space of \( \mathbb{R}^n \) (where \( n \) is the rank of the horizontal bundle) and the horizontal path space of \( M \) (see Corollary 4.6). Following the methods by B. Driver in [16] and E.P. Hsu in [31], one can compute the pullback of the directional derivative \( D_v \) by this Itô map when \( v \) is a tangent process and finally apply Girsanov’s theorem in the form of [16, Lemma 8.2].

**Remark 1.1.** In the current paper, we restrict ourselves to the case of compact manifolds mainly for the sake of conciseness. It is reasonable to conjecture that as in [36], our results may be extended to complete manifolds. Moreover, our first example in Section 1.2 is the path space of the Heisenberg group which is a connected nilpotent (non-compact) Lie group.

### 1.2. Cameron-Martin theorem on \( \mathbb{R}^n \) and its lift to the Heisenberg group.

To conclude the introduction it is instructive to show in the simple case of the Heisenberg group, where the vertical part (1.3) comes from. We start by reminding the classical setting of the Wiener space. Let \( \mathcal{W}(\mathbb{R}^d) \) be the Wiener space of continuous functions \( w : [0, T] \to \mathbb{R}^d \) vanishing at 0, and \( \nu \) be the Wiener measure on \( \mathcal{W}(\mathbb{R}^d) \). In what follows we take \( d = 2n \).

The associated Cameron–Martin space \( H \) is defined

\[
H := H\left(\mathbb{R}^d\right) = \left\{ h \in W : \int_0^T |h'(s)|^2_{\mathbb{R}^d} \, ds < \infty \right\},
\]

wherein \( \int_0^T |h'(s)|^2_{\mathbb{R}^d} \, ds := \infty \) if \( h \) is not absolutely continuous. The Cameron-Martin theorem says that the Wiener measure \( \nu \) on \( \mathcal{W}(\mathbb{R}^d) \) is quasi-invariant with respect to the translations in the directions of the Cameron-Martin subspace \( H \) (see for instance [41, Theorem 2.2, p. 339], [9, Corollary 2.4.3], or the original work in [11,12,38]). More precisely, first we define the map for every \( h \in H \) by
and denote by $\nu^h$ the pushforward of $\nu$ under $p_h$ (that is, $\nu^h(A) = \nu(p_h^{-1}(A))$). By the Cameron-Martin theorem $\nu^h$ is equivalent to $\nu$ and the Radon-Nikodym density is explicitly given by
\[
\frac{d\nu^h}{d\nu} = \exp \left( \int_0^T \langle h'(s), d\omega(s) \rangle - \frac{1}{2} \int_0^T |h'(s)|^2 ds \right),
\]
where $\int_0^T \langle h'(s), d\omega(s) \rangle$ is the Itô stochastic integral.

Now we would like to consider a similar setting on the path space over the Heisenberg group. Recall that the Heisenberg group is the set
\[
\mathbb{H}^{2n+1} = \{(x, y, z), x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}\}
\]
edowed with the group law
\[
(x_1, y_1, z_1) \star (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \langle x_1, y_2 \rangle_{\mathbb{R}^n} - \langle x_2, y_1 \rangle_{\mathbb{R}^n}).
\]
The vector fields
\[
X_i = \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial z}, 1 \leq i \leq n,
\]
\[
Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, 1 \leq i \leq n,
\]
\[
Z = \frac{\partial}{\partial z}
\]
form a basis for the space of left-invariant vector fields on $\mathbb{H}^{2n+1}$. We choose a left-invariant Riemannian metric on $\mathbb{H}^{2n+1}$ in such a way that $\{X_1, ..., X_n, Y_1, ..., Y_n, Z\}$ are orthonormal with respect to this metric. Note that these vector fields satisfy the following commutation relations
\[
[X_i, Y_j] = 2\delta_{ij} Z, \quad [X_i, Z] = [Y_i, Z] = 0, \quad i = 1, ..., n.
\]
Then the map
\[
\pi : \mathbb{H}^{2n+1} \rightarrow \mathbb{R}^n
\]
\[
(x, y, z) \mapsto (x, y)
\]
is a Riemannian submersion with totally geodesic fibers as explained in Section 3.5.1. Let $W(\mathbb{R}^{2n})$ be the Wiener space of continuous functions $[0, T] \rightarrow \mathbb{R}^{2n}$ vanishing at 0. We denote by $(B_t, \beta_t)_{0 \leq t \leq T}$ the coordinate maps on $W(\mathbb{R}^{2n})$ and by $\nu$ the Wiener measure on $W(\mathbb{R}^{2n})$, so that $(B_t, \beta_t)_{0 \leq t \leq T}$ is a $2n$-dimensional Brownian motion under $\nu$. By using the submersion $\pi$, the Brownian motion $(B_t, \beta_t)_{0 \leq t \leq T}$ can be horizontally lifted to the horizontal Brownian motion on $\mathbb{H}^{2n+1}$ which is given explicitly by $X_t = (B_t, \beta_t, \sum_{i=1}^n \int_0^t B_i^t d\beta_i^t$. The distribution of $(X_t)_{0 \leq t \leq T}$ is denoted by $\mu_H$ and referred to as the horizontal Wiener measure on $\mathbb{H}^{2n+1}$. The support of $\mu_H$ is called the
horizontal path space of $H^{2n+1}$ and denoted by $W_{H}^{+}$). The measure
space isomorphism
$$\Pi : (W(R^{2n}), \nu) \rightarrow (W_{H}(H^{2n+1}), \mu_{H})$$
is the horizontal lift map induced by the submersion $\pi$.

Now we use the map $\Pi$ to lift the Cameron-Martin theorem from $W(R^{2n})$ to $W_{H}(H^{2n+1})$. Indeed, for any $h \in H(R^{2n})$ the translation $p_{h}$ induces a map $\tilde{p}_{h}$ so that the following diagram is commutative.

$$W_{H}(H^{2n+1}) \xrightarrow{\tilde{p}_{h}} W_{H}(H^{2n+1})$$

$$W(R^{2n}) \xrightarrow{p_{h}} W(R^{2n})$$

For $h \in H(R^{2n})$ the map $\tilde{p}_{h}$ can be computed explicitly and is given for $w \in W_{H}(H^{2n+1})$ by

$$\tilde{p}_{h}(w)_{t} = w_{t} + \left( h(t), \sum_{i=1}^{n} h^{i}(t)w_{i}^{t+n} - h^{i+n}(t)w_{i}^{t} + \int_{0}^{t} (h^{i}(s) + 2w_{i}^{s})(h^{i+n}(s) - (h^{i+n}(s) + 2w_{i}^{s})dh^{i}(s) \right)$$

Observe that the non-trivial vertical part in $\tilde{p}_{h}(w)_{t}$ is consistent with formula (1.3). Let $\mu_{H}^{h}$ be the pushforward of $\mu_{H}$ under $\tilde{p}_{h}$, then it is immediate that $\mu_{H}^{h}$ is equivalent to $\mu_{H}$ and the density is explicitly given by

$$\frac{d\mu_{H}^{h}}{d\mu_{H}} = \exp \left( \sum_{i=1}^{2n} \int_{0}^{T} h_{i}'(s)dw_{i}^{s} - \frac{1}{2} \int_{0}^{T} |h_{i}'(s)|_{R^{2n}}^{2}ds \right).$$

2. NOTATION AND PRELIMINARIES

Let $M$ be a smooth connected and compact manifold of the dimension $n + m$. We assume that $M$ is equipped with a Riemannian foliation structure, $F$, with a bundle-like metric $g$ and totally geodesic $m$-dimensional leaves. We refer to [1,39,40,44] for details about the geometry of Riemannian foliations.

The subbundle $\mathcal{V}$ formed by the vectors tangent to the leaves is referred to as the set of vertical directions. The subbundle $\mathcal{H}$ which is normal to $\mathcal{V}$ is referred to as the set of horizontal directions. We assume that $\mathcal{H}$ is bracket generating, that is, the Lie algebra of vector fields generated by global $C^\infty$ sections of $\mathcal{H}$ has the full rank at each point in $M$.

**Example 2.1 (Riemannian submersions).** Let $(M, g)$ and $(\mathbb{B}, j)$ be smooth and connected Riemannian manifolds. A smooth surjective map $\pi : (M, g) \rightarrow (\mathbb{B}, j)$ is called a **Riemannian submersion** if its derivative maps $T_{x}\pi : T_{x}M \rightarrow T_{\pi(x)}\mathbb{B}$...
$T_{\pi(x)}\mathcal{B}$ are orthogonal projections, i.e. for every $x \in \mathcal{M}$, the map $T_x\pi(T_x\pi)^* : T_{\pi(x)}\mathcal{B} \to T_{\pi(x)}\mathcal{B}$ is the identity map. The foliation given by the fibers of a Riemannian submersion is then bundle-like (see [1] Section 2.3).

**Example 2.2 (K-contact manifolds).** Another important example of Riemannian foliation is obtained in the context of contact manifolds. Let $(\mathcal{M}, \theta)$ be a $2n+1$-dimensional smooth contact manifold, where $\theta$ is a contact form. On $\mathcal{M}$ there is a unique smooth vector field $Z$, called the Reeb vector field, satisfying

$$\theta(Z) = 1, \quad L_Z(\theta) = 0,$$

where $L_Z$ denotes the Lie derivative with respect to $Z$. On $\mathcal{M}$ there is a foliation, the Reeb foliation, whose leaves are the orbits of the vector field $Z$. As it is well-known (see [12]), that it is always possible to find a Riemannian metric $g$ and a $(1,1)$-tensor field $J$ on $\mathcal{M}$ so that for every vector fields $X,Y$

$$g(X,Z) = \theta(X), \quad J^2(X) = -X + \theta(X)Z, \quad g(X,JY) = (d\theta)(X,Y).$$

The triple $(\mathcal{M}, \theta, g)$ is called a contact Riemannian manifold. We see then that the Reeb foliation is totally geodesic with bundle-like metric if and only if the Reeb vector field $Z$ is a Killing field, that is,

$$L_Z g = 0,$$

as is stated in [10] Proposition 6.4.8. In this case, $(\mathcal{M}, \theta, g)$ is called a K-contact Riemannian manifold. Observe that the horizontal distribution $\mathcal{H}$ is then the kernel of $\theta$ and that $\mathcal{H}$ is bracket generating because $\theta$ is a contact form. We refer to [7][12] for further details on this class of examples.

We will use the following standard notation.

**Notation 2.1.** By $T\mathcal{M}$ we denote the tangent bundle and by $T^*\mathcal{M}$ the cotangent bundle, and by $T_x\mathcal{M}$ ($T^*_x\mathcal{M}$) the tangent (cotangent) space at $x \in \mathcal{M}$. The inner product on $T\mathcal{M}$ induced by the metric $g$ will be denoted by $g(\cdot, \cdot)$, and its restrictions to $\mathcal{H}$ and $\mathcal{V}$ will be denoted by $g_\mathcal{H}(\cdot, \cdot)$ and $g_\mathcal{V}(\cdot, \cdot)$ respectively. As always, for any $x \in \mathcal{M}$ we denote by $g(\cdot, \cdot)_x$ (or $\langle \cdot, \cdot \rangle_x$), $g_\mathcal{H}(\cdot, \cdot)_x$ (or $\langle \cdot, \cdot \rangle_{\mathcal{H}_x}$), $g_\mathcal{V}(\cdot, \cdot)_x$ (or $\langle \cdot, \cdot \rangle_{\mathcal{V}_x}$) the inner product on the fibers $T_x\mathcal{M}$, $\mathcal{H}_x$ and $\mathcal{V}_x$ correspondingly. The space of smooth functions on $\mathcal{M}$ will be denoted by $C^\infty(\mathcal{M})$. The space of smooth and compactly supported functions will be denoted $C^\infty_0(\mathcal{M})$. The space of smooth sections of a vector bundle $\mathcal{E}$ over $\mathcal{M}$ will be denoted $\Gamma^\infty(\mathcal{E})$. The space of smooth and compactly supported sections will be denoted $\Gamma^\infty_0(\mathcal{E})$.

We say that a one-form is horizontal (resp. vertical) if it vanishes on the vertical bundle $\mathcal{V}$ (resp. on the horizontal bundle $\mathcal{H}$). Then the splitting of the tangent space

$$T_x\mathcal{M} = \mathcal{H}_x \oplus \mathcal{V}_x$$

induces a splitting of the cotangent space

$$T^*_x\mathcal{M} = \mathcal{H}^*_x \oplus \mathcal{V}^*_x.$$
The sub-bundle $\mathcal{H}^*$ of the cotangent bundle will be referred to as the co-horizontal bundle. Similarly, $\mathcal{V}^*$ will be referred to as the co-vertical bundle.

2.1. **Bott connection.** On the Riemannian manifold $(\mathbb{M}, g)$ there is the Levi-Civita connection that we denote by $\nabla^R$, but this connection is not adapted to the study of foliations because the horizontal and the vertical bundle may not be parallel with respect to $\nabla^R$. We will rather make use of the *Bott connection* on $\mathbb{M}$ which is defined as follows.

$$\nabla_X Y = \begin{cases} 
\pi_H(\nabla_X^RY), & X, Y \in \Gamma^\infty(\mathcal{H}), \\
\pi_H([X, Y]), & X \in \Gamma^\infty(\mathcal{V}), Y \in \Gamma^\infty(\mathcal{H}), \\
\pi_V([X, Y]), & X \in \Gamma^\infty(\mathcal{H}), Y \in \Gamma^\infty(\mathcal{V}), \\
\pi_V(\nabla_X^RY), & X, Y \in \Gamma^\infty(\mathcal{V}),
\end{cases}$$

where $\pi_H$ (resp. $\pi_V$) is the projection on $\mathcal{H}$ (resp. $\mathcal{V}$). One can check that the Bott connection is metric-compatible, that is, $\nabla g = 0$, though unlike the Levi-Civita connection it is not torsion-free.

Let $T$ be the torsion of the Bott connection $\nabla$. Observe that for $X, Y \in \Gamma^\infty(\mathcal{H})$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$= \pi_H(\nabla_X Y - \nabla_Y X) - [X, Y]$$

$$= \pi_H([X, Y]) - [X, Y]$$

$$= -\pi_V([X, Y]).$$

Similarly one can check that the Bott connection satisfies the following properties that we record here for later use

$$\nabla_X Y \in \Gamma^\infty(\mathcal{H}) \text{ for any } X, Y \in \Gamma^\infty(\mathcal{H}),$$

$$\nabla_X Y \in \Gamma^\infty(\mathcal{V}) \text{ for any } X, Y \in \Gamma^\infty(\mathcal{V}),$$

$$(2.1) \quad T(X, Y) \in \Gamma^\infty(\mathcal{V}) \text{ for any } X, Y \in \Gamma^\infty(\mathcal{H}),$$

$$(2.1) \quad T(U, V) \in \Gamma^\infty(\mathcal{H}) \text{ for any } U, V \in \Gamma^\infty(\mathcal{V}),$$

$$(2.1) \quad T(X, U) = 0 \text{ for any } X \in \Gamma^\infty(\mathcal{H}), U \in \Gamma^\infty(\mathcal{V}),$$

**Example 2.3** (Example 2.2 revisited). Let $(\mathbb{M}, \theta, g)$ be a K-contact Riemannian manifold. The Bott connection coincides with Tanno’s connection that was introduced in [42] and which is the unique connection that satisfies the following.

1. $\nabla \theta = 0$;
2. $\nabla Z = 0$;
3. $\nabla g = 0$;
4. $T(X, Y) = d\theta(X, Y)Z$ for any $X, Y \in \Gamma^\infty(\mathcal{H})$;
5. $T(Z, X) = 0$ for any vector field $X \in \Gamma^\infty(\mathcal{H})$. 
We define the horizontal Ricci curvature $\mathfrak{Ric}_\mathcal{H}$ of the Bott connection as the fiberwise symmetric linear map on one-forms such that for all smooth functions $f, g$ on $\mathbb{M}$

$$\langle \mathfrak{Ric}_\mathcal{H}(df), dg \rangle = \mathfrak{Ric}(\nabla_\mathcal{H} f, \nabla_\mathcal{H} g) = \mathfrak{Ric}_\mathcal{H}(\nabla f, \nabla g),$$

where $\mathfrak{Ric}$ is the Ricci curvature of the Bott connection $\nabla$ and $\mathfrak{Ric}_\mathcal{H}$ its horizontal Ricci curvature (horizontal trace of the full curvature tensor $\mathcal{R}$ of the Bott connection).

### 2.2. Generalized Levi-Civita connections and adjoint connections.

When studying Weitzenböck type identities for the horizontal Laplacian (see Section 2.4), one can not make use of the Bott connection because the adjoint connection of the Bott connection is not a metric connection. We refer to [6, 20, 28, 29] and especially the books [21, 22] for a discussion on Weitzenböck-type identities and adjoint connections. We shall rather make use of a family of connections first introduced in [2] and only keep the Bott connection as a reference connection.

This family of connections is constructed from the canonical variation of the metric that we recall now. The metric $g$ can be split as

$$(2.2) \quad g = g_\mathcal{H} \oplus g_\mathcal{V}.$$  

Using the splitting of the Riemannian metric $g$ in (2.2) we can introduce the following one-parameter family of Riemannian metrics

$$g_\varepsilon = g_\mathcal{H} \oplus \frac{1}{\varepsilon} g_\mathcal{V}, \quad \varepsilon > 0.$$  

One can check that for every $\varepsilon > 0$, $\nabla g_\varepsilon = 0$. The metric $g_\varepsilon$ then induces a metric on the cotangent bundle which we still denote by $g_\varepsilon$. By using similar notation and conventions as before we have

$$\|\eta\|_\varepsilon^2 = \|\eta\|_\mathcal{H}^2 + \varepsilon \|\eta\|_\mathcal{V}^2, \quad \text{for every } \eta \in T^*_x \mathbb{M}.$$  

For each $Z \in \Gamma(\mathcal{V})$ there is a unique skew-symmetric endomorphism $J_Z : \mathcal{H}_x \rightarrow \mathcal{H}_x$, $x \in \mathbb{M}$ such that for all horizontal vector fields $X, Y \in \mathcal{H}_x$

$$(2.3) \quad g_\mathcal{H}(J_Z(X), Y)_x = g_\mathcal{V}(Z, T(X, Y))_x,$$

where $T$ is the torsion tensor of $\nabla$. We then extend $J_Z$ to be 0 on $\mathcal{V}_x$. Also, to ensure (2.3) holds also for $Z \in \Gamma(\mathcal{H})$, taking into account (2.1) we set $J_Z \equiv 0$.

We now introduce the following family of connections (see [2])

$$\nabla_\varepsilon^X Y = \nabla X Y - T(X, Y) + \frac{1}{\varepsilon} J_Y X, \quad X, Y \in \Gamma(\mathcal{M}).$$

It is easy to check that $\nabla_\varepsilon^X g_\varepsilon = 0$. The torsion of $\nabla_\varepsilon$ is equal to

$$T^\varepsilon(X, Y) = -T(X, Y) + \frac{1}{\varepsilon} J_Y X - \frac{1}{\varepsilon} J_X Y, \quad X, Y \in \Gamma(\mathcal{M}).$$
The adjoint connection of $\nabla^\varepsilon$ (in the sense of B. Driver [16], see [21, Section 1.3] for a discussion about adjoint connections) is then given by
\[ \hat{\nabla}^\varepsilon_X Y := \nabla_X Y - T^\varepsilon(X,Y) = \nabla_X Y + \frac{1}{\varepsilon} J_X Y, \]
thus $\hat{\nabla}^\varepsilon$ is also a metric connection. Moreover, it preserves the horizontal and vertical bundles. For later use, we record that the torsion of $\hat{\nabla}^\varepsilon$ is
\[ \hat{T}^\varepsilon(X,Y) = -T^\varepsilon(X,Y) = T(X,Y) - \frac{1}{\varepsilon} J_Y X + \frac{1}{\varepsilon} J_X Y. \]
The Riemannian curvature tensor of $\hat{\nabla}^\varepsilon$ also can be computed easily in terms of the Riemannian curvature tensor $R$ of the Bott connection and it is given by the following lemma.

**Lemma 2.2.** For $X, Y, Z \in \Gamma^\infty(M)$,
\[ \hat{R}^\varepsilon(X,Y)Z = R(X,Y)Z + \frac{1}{\varepsilon} J_{T(X,Y)} Z + \frac{1}{\varepsilon^2} (J_X J_Y - J_Y J_X) Z + \]
\[ \frac{1}{\varepsilon} (\nabla_X J_Y) Z - \frac{1}{\varepsilon} (\nabla_Y J_X) Z, \]
where $R$ is the curvature tensor of the Bott connection.

**Proof.**
\[ \hat{R}^\varepsilon(X,Y)Z = \hat{\nabla}^\varepsilon_X \hat{\nabla}^\varepsilon_Y Z - \hat{\nabla}^\varepsilon_Y \hat{\nabla}^\varepsilon_X Z - \hat{\nabla}^\varepsilon_{[X,Y]} Z \]
\[ = (\nabla_X \nabla_Y + \frac{1}{\varepsilon} (\nabla_X J_Y) + \frac{1}{\varepsilon} J_X \nabla_Y + \frac{1}{\varepsilon} J_Y \nabla_X + \frac{1}{\varepsilon} J_{\nabla_X Y} + \frac{1}{\varepsilon^2} J_X J_Y) Z \]
\[ - (\nabla_Y \nabla_X + \frac{1}{\varepsilon} (\nabla_Y J_X) + \frac{1}{\varepsilon} J_Y \nabla_X + \frac{1}{\varepsilon} J_X \nabla_Y + \frac{1}{\varepsilon} J_X J_Y + \frac{1}{\varepsilon^2} J_Y J_X) Z \]
\[ - \nabla_{[X,Y]} Z - \frac{1}{\varepsilon} J_{[X,Y]} Z \]
\[ = R(X,Y)Z + \frac{1}{\varepsilon^2} (J_X J_Y - J_Y J_X) Z + \]
\[ \frac{1}{\varepsilon} (\nabla_X J_Y) Z - \frac{1}{\varepsilon} (\nabla_Y J_X) Z + \frac{1}{\varepsilon} J_{T(X,Y)} Z. \]
\[ \Box \]

2.3. **Horizontal Brownian motion and stochastic parallel transport.**
We define the horizontal gradient $\nabla^H f$ of a function $f$ as the projection of the Riemannian gradient of $f$ on the horizontal bundle $H$. Similarly, we define the vertical gradient $\nabla^V f$ of a function $f$ as the projection of the Riemannian gradient of $f$ on the vertical bundle $V$.

Consider the pre-Dirichlet form
\[ \mathcal{E}_H(f,h) = \int_M g_H(\nabla^H f, \nabla^H h) \, d\text{Vol}, \quad f, h \in C^\infty_0(M), \]
where $d\text{Vol}$ is the Riemannian volume measure. Then there exists a unique diffusion operator $L$ on $\mathcal{M}$ such that for all $f, h \in C_0^\infty(\mathcal{M})$

$$\mathcal{E}_H(f, h) = -\int_\mathcal{M} fLh \, d\text{Vol} = -\int_\mathcal{M} hLf \, d\text{Vol}.$$ 

The operator $L$ is called the horizontal Laplacian of the foliation. If $X_1, \ldots, X_n$ is a local orthonormal frame of horizontal vector fields, then we can write $L$ in this frame

$$L = \sum_{i=1}^n X_i^2 - \nabla_{X_i}X_i.$$ 

Observe that the subbundle $\mathcal{H}$ satisfies Hörmander’s (bracket generating) condition, therefore by Hörmander’s theorem the operator $L$ is locally subelliptic (for comments on this terminology introduced by Fefferman-Phong we refer to [26], see for instance the survey paper [37] or [18, p. 944]).

By [1, Proposition 5.1] the completeness of the Riemannian metric $g$ implies that $L$ is essentially self-adjoint on $C_0^\infty(\mathcal{M})$ and thus that $\mathcal{E}_H$ is closable. Then we can define the semigroup $P_t = e^{tL}$ by using the spectral theorem. The diffusion process $\{X_t\}_{t \geq 0}$ corresponding to the semigroup $\{P_t\}_{t \geq 0}$ will be called the horizontal Brownian motion on the Riemannian foliation $$(\mathcal{F}, g).$$ Since $\mathcal{M}$ is assumed to be compact, $1 \in \text{dom}(\mathcal{E}_H)$ and thus $P_11 = 1$. This implies that $\{X_t\}_{t \geq 0}$ is a non-explosive diffusion. At the same time the horizontal Brownian motion $\{X_t\}_{t \geq 0}$ is a semimartingale on $\mathcal{M}$ which can be constructed from a globally defined stochastic differential equation on a bundle over $\mathcal{M}$ (see [19, Theorem 3.8] or Corollary 4.6).

2.4. Weitzenböck formulas. A key ingredient in studying the horizontal Brownian motion $\{X_t\}_{t \geq 0}$ is the Weitzenböck formula that has been proven in [26]. We recall here this formula. If $Z_1, \ldots, Z_m$ is a local vertical frame, then the $(1, 1)$ tensor

$$J^2 := \sum_{\ell=1}^m J_{Z_\ell}J_{Z_\ell}$$

does not depend on the choice of the frame and may be defined globally.

**Example 2.4** (Example 2.2 revisited). If $\mathcal{M}$ is a K-contact manifold equipped with the Reeb foliation, then, by taking $Z$ to be the Reeb vector field, one gets $J^2 = J_Z^2 = -\text{Id}_{\mathcal{H}}$.

The horizontal divergence of the torsion $T$ is the $(1, 1)$ tensor which in a local horizontal frame $X_1, \ldots, X_n$ is defined by

$$\delta_H T(X) := -\sum_{j=1}^n (\nabla_{X_j}T)(X_j, X).$$

**Example 2.5** (Example 2.2 revisited). If $\mathcal{M}$ is a K-contact manifold equipped with the Reeb foliation, then $\delta_H T$ can be expressed as the horizontal trace
of the Tanno tensor (see [7, 42]). Therefore if \((M, \theta)\) is a CR manifold, then 
\[ \delta_H T = 0 \]

By using the duality between the tangent and cotangent bundles with respect to the metric \(g\), we can identify the \((1,1)\) tensors \(J^2\) and \(\delta_H T\) with linear maps on the cotangent bundle \(T^*M\).

Namely, let \(\sharp : T^*M \to TM\) be the standard musical (raising an index) isomorphism which is defined as the unique vector \(\omega^\sharp\) such that for any \(x \in M\)

\[ g\left( \omega^\sharp, X \right)_x = \omega(X) \quad \text{for all } X \in T_xM, \]

while in local coordinates the isomorphism \(\sharp\) can be written as follows

\[ \omega = \sum_{i=1}^{n+m} \omega_i dx^i \mapsto \omega^\sharp = \sum_{j=1}^{n+m} \omega^\sharp_j \partial_j = \sum_{j=1}^{n+m} \sum_{i=1}^{n+m} g_{ij} \omega_i \partial_j. \]

The inverse of this isomorphism is the (lowering an index) isomorphism \(\flat : TM \to T^*M\) defined by

\[ X^\flat = g\left( X, \cdot \right)_x, X \in T_xM \]

and in local coordinates

\[ X = \sum_{i=1}^{n+m} X^i \partial_i \mapsto X^\flat = \sum_{i=1}^{n+m} X_i dx^i = \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} g_{ij} X^j dx^i. \]

If \(\eta\) is a one-form, we define the horizontal gradient in a local adapted frame of \(\eta\) as the \((0,2)\) tensor

\[ \nabla_H \eta = \sum_{i=1}^n \nabla_{X_i} \eta \otimes \theta_i, \]

where \(\theta_i, i = 1, \ldots, n\) is the dual of \(X_i\).

Finally, for \(\varepsilon > 0\), we consider the following operator which is defined on one-forms by

\[ \Box_\varepsilon := \sum_{i=1}^n (\nabla_{X_i} - T^\varepsilon_{X_i})^2 - (\nabla \nabla_{X_i} X_i - \nabla T^\varepsilon_{X_i} X_i) - \frac{1}{\varepsilon} J^2 + \frac{1}{\varepsilon} \delta_H T - \mathfrak{Ric}_H, \]

where \(T^\varepsilon\) is the \((1,1)\) tensor defined by

\[ T^\varepsilon_X Y = -T(X, Y) + \frac{1}{\varepsilon} J_Y X, \quad X, Y \in \Gamma^\infty(M). \]

Similarly as before, we will use the notation

\[ T^\varepsilon_H \eta := \sum_{i=1}^n T^\varepsilon_{X_i} \eta \otimes \theta_i. \]
The expression (2.4) does not depend on the choice of the local horizontal frame and thus $\Box_\varepsilon$ may be globally defined. Formally, we have

\begin{equation}
\Box_\varepsilon = - (\nabla_H - \mathcal{F}_H^\varepsilon)(\nabla_H - \mathcal{F}_H^\varepsilon) - \frac{1}{\varepsilon} J^2 + \frac{1}{\varepsilon} \delta_H T - 9\text{ic}_H,
\end{equation}

where the adjoint is understood with respect to the $L^2(M, g_\varepsilon, \mu)$ inner product on sections, i.e. $\int_M \langle \cdot, \cdot \rangle_\varepsilon d\mu$ (see [1, Lemma 5.3] for more detail). The main result of [6] is the following.

**Theorem 2.3** (Lemma 3.3, Theorem 3.1 in [6]). Let $f \in C_0^\infty(M)$, $x \in M$ and $\varepsilon > 0$, then

\begin{equation}
dL f(x) = \Box_\varepsilon df(x).
\end{equation}

**Remark 2.4.** From [6, Lemma 3.4], we see that for $\varepsilon_1, \varepsilon_2 > 0$, the operator $\Box_{\varepsilon_1} - \Box_{\varepsilon_2}$ vanishes on exact one-forms. It is therefore no surprise that the left hand side of (2.6) does not depend of $\varepsilon$.

To conclude this section we remark, and this is not a coincidence (see Section 4), that the potential term in the Weitzenböck identity can be identified with the horizontal Ricci curvature of the adjoint connection $\hat{\nabla}^\varepsilon$.

**Lemma 2.5.** The horizontal Ricci curvature of the adjoint connection $\hat{\nabla}^\varepsilon$ is given by

$$\hat{\text{Ric}}^\varepsilon_H = \text{Ric}_H - \frac{1}{\varepsilon} \delta_H^* T + \frac{1}{\varepsilon} J^2,$$

where $\delta_H^* T$ denotes the adjoint of $\delta_H T$ with respect to the metric $g$.

**Proof.** Let $X, Y \in \Gamma^\infty(TM)$ and $X_1, \cdots, X_n$ be a local horizontal orthonormal frame. By the definition of the horizontal Ricci curvature and Lemma 2.2 we have

$$\text{Ric}_H^\varepsilon(X, Y)$$

$$= \sum_{i=1}^n g_H(\hat{\text{Ric}}_H^\varepsilon(X_i, X)Y, X_i)$$

$$= \sum_{i=1}^n g_H(R(X_i, X)Y, X_i) + \sum_{i=1}^n g_H \left( \frac{1}{\varepsilon} J_T(X_i, X)Y, X_i \right)$$

$$+ \sum_{i=1}^n g_H \left( \frac{1}{\varepsilon} (\nabla X J)_X Y - \frac{1}{\varepsilon} (\nabla X J)_X Y, X_i \right).$$

For the first term, we have

$$\sum_{i=1}^n g_H(R(X_i, X)Y, X_i) = \text{Ric}_H(X, Y).$$
For the second term, we easily see that
\[
\sum_{i=1}^{n} g_{\mathcal{H}} (J_{T(X_i, X)} Y, X_i) = - \sum_{i=1}^{n} g_{V} (T(X, X_i), T(Y, X_i))
\]
\[= g_{\mathcal{H}} (J^{2} X, Y).\]

For the third term, we first observe that \(g_{\mathcal{H}}((\nabla X J)_{X} Y, X_i) = 0\). Then, we have
\[
\sum_{i=1}^{n} g_{\mathcal{H}} ((\nabla X J)_{X} Y, X_i) = - \sum_{i=1}^{n} g_{V} ((\nabla X J)_{X} Y, X_i)
\]
\[= - \sum_{i=1}^{n} g_{V} ((\nabla X T)(X_i, Y), X)
\]
\[= g_{V} (\delta_{\mathcal{H}} T(Y), X).\]

\[\square\]

3. Integration by Parts Formula on the Horizontal Path Space

Let \(x \in \mathbb{M}\) and \(\{X_t\}_{t \geq 0}\) be a horizontal Brownian motion on \(\mathbb{M}\) started at \(x\), defined on a probability space \((\Omega, \mu)\). As usual we will denote \(\mu_x = \mu(\cdot | X_0 = x)\). We fix \(\varepsilon > 0\) throughout the section. Our goal in this section is to prove integration by parts formulas on the path space of the horizontal Brownian motion. Some of the integration by parts formulas for the damped Malliavin derivative have been already announced in a less general and slightly different setting in [3]. The integration by part formulas for the intrinsic Malliavin derivative are new. We point out a significant difference of our techniques from what have been used in [1–3]. Namely, we shall mostly make use of the adjoint connection \(\hat{\nabla}^\varepsilon\) instead of the Bott connection. Three important properties of the connection \(\hat{\nabla}^\varepsilon\) are as follows.

**Metric connection:** \(\hat{\nabla}^\varepsilon g^\varepsilon = 0;\)

**Horizontal connection:** if \(X \in \Gamma^\infty (\mathcal{H})\) and \(Y \in \Gamma^\infty (\mathbb{M})\) then \(\hat{\nabla}^\varepsilon Y X \in \Gamma^\infty (\mathcal{H});\)

**Skew-symmetry:** the torsion tensor \(\hat{T}^\varepsilon\) of \(\hat{\nabla}^\varepsilon\) satisfies Driver’s total skew-symmetry condition ([16, p. 272]) as follows. For \(X, Y, Z \in \Gamma^\infty (\mathbb{M})\)
\[
\langle \hat{T}^\varepsilon (X, Y), Z \rangle_\varepsilon = - \langle \hat{T}^\varepsilon (X, Z), Y \rangle_\varepsilon.
\]

The latter can be seen from the formula
\[
\hat{T}^\varepsilon (X, Y) = T(X, Y) - \frac{1}{\varepsilon} J_{Y} X + \frac{1}{\varepsilon} J_{X} Y
\]
and the definition of \(J\).

We need to use a stochastic parallel transport on forms. Let \(\hat{\nabla}\) be a general connection on \(\mathbb{M}\), and \(\{X_t\}_{t \geq 0}\) be a semimartingale on \(\mathbb{M}\). We denote by \(\parallel_{0,t} : T_{X_0} \mathbb{M} \to T_{X_t} \mathbb{M}\) the stochastic parallel transport of vector
fields along the paths of \{X_t\}_{t \geq 0} (see \cite[p. 50]{35}). Then by duality we can define the stochastic parallel transport on one-forms as follows. We have \( \overline{\gamma}_{0,t} : T^*_X \to T^*_x \) such that for \( \alpha \in T^*_x \)

\[
\langle \overline{\gamma}_{0,t} \alpha, v \rangle = \langle \alpha, \overline{\gamma}_{0,t} v \rangle, \quad v \in T_x.
\]

(3.1)

In particular, the stochastic parallel transport for the adjoint connection \( \hat{\nabla}^\epsilon \) along the paths of the horizontal Brownian motion \( \{X_t\}_{t \geq 0} \) will be denoted by \( \hat{\Theta}^\epsilon_t \). Since the adjoint connection \( \hat{\nabla}^\epsilon \) is horizontal, the map \( \hat{\Theta}^\epsilon_t : T_x \to T_x \) is an isometry that preserves the horizontal bundle, that is, if \( u \in H_x \), then \( \hat{\Theta}^\epsilon_t u \in H_x \). We see then that the anti-development of \( \{X_t\}_{t \geq 0} \) defined as

\[
B_t := \int_0^t (\hat{\Theta}^\epsilon_s)^{-1} \circ dX_s,
\]

is a Brownian motion in the horizontal space \( H_x \). The usual completion of the natural filtration of \( \{B_t\}_{t \geq 0} \) will be denoted by \( \{\mathcal{F}_t\}_{t \geq 0} \), and this is the filtration we will be using in what follows.

\textbf{Remark 3.1.} Observe that on one-forms the process \( \hat{\Theta}^\epsilon_t : T^*_X \to T^*_x \) is a solution to the following covariant Stratonovich stochastic differential equation

\[
d[\hat{\Theta}^\epsilon_t \alpha(X_t)] = \hat{\Theta}^\epsilon_t \hat{\nabla}^\epsilon_{odX_t} \alpha(X_t),
\]

where \( \alpha \) is any smooth one-form. Since \( \hat{\nabla}^\epsilon_{odX_t} = \nabla_{odX_t} + \frac{1}{\epsilon} J_{odX_t} = \nabla_{odX_t} \), we deduce that \( \hat{\Theta}^\epsilon_t \) is actually independent of \( \epsilon \) and is therefore also the stochastic parallel transport for the Bott connection. As a consequence, the Brownian motion \( (B_t)_{t \geq 0} \) and its filtration are also independent of the particular choice of \( \epsilon \).

We define a \textit{damped parallel transport} \( \tau^\epsilon_t : T^*_X \to T^*_x \) by the formula

\[
\tau^\epsilon_t = \mathcal{M}^\epsilon_t \hat{\Theta}^\epsilon_t,
\]

(3.2)

where the process \( \Theta^\epsilon_t : T^*_X \to T^*_x \) is the stochastic parallel transport of one-forms with respect to the connection \( \nabla^\epsilon = \nabla - \mathcal{F}^\epsilon \) along the paths of \( \{X_t\}_{t \geq 0} \). The multiplicative functional \( \mathcal{M}^\epsilon_t : T^*_x \to T^*_x \), \( t \geq 0 \), is defined as the solution to the following ordinary differential equation

\[
\frac{d\mathcal{M}^\epsilon_t}{dt} = -\frac{1}{2} \mathcal{M}^\epsilon_t \hat{\Theta}^\epsilon_t \left( \frac{1}{\epsilon} J^2 - \frac{1}{\epsilon} \delta_H T + 2 \text{Ric}_H \right) (\hat{\Theta}^\epsilon_t)^{-1},
\]

\[
\mathcal{M}^\epsilon_0 = \text{I}d.
\]

(3.3)
Observe that the process $\tau_t^\varepsilon : T^*_X M \to T^*_x M$ is a solution of the following covariant Stratonovich stochastic differential equation

$$d[\tau_t^\varepsilon \alpha(X_t)] = \tau_t^\varepsilon \left( \nabla_{\od X_t} - \tau_t^\varepsilon \od X_t - \frac{1}{2} \left( \varepsilon J^2 - \frac{1}{\varepsilon} \delta_H T + \mathcal{R}ic_H \right) dt \right) \alpha(X_t),$$

$\tau_0 = \text{Id},$

where $\alpha$ is any smooth one-form.

Also observe that $M^\varepsilon_t$ is invertible and that its inverse is the solution of the following ordinary differential equation

$$d(M_t^\varepsilon)^{-1} = \frac{1}{2} \Theta_t^\varepsilon \left( \varepsilon J^2 - \frac{1}{\varepsilon} \delta_H T + \mathcal{R}ic_H \right) (\Theta_t^\varepsilon)^{-1} (M_t^\varepsilon)^{-1}.$$

In particular, it implies that $\tau_t^\varepsilon$ is invertible.

### 3.1. Malliavin and directional derivatives.

The horizontal Wiener measure on $C_x([0, T], M)$ is defined as the distribution of the horizontal Brownian motion. The support of this measure will be called the horizontal path space and will be denoted by $W^H(M)$. The coordinate process on $C_x([0, T], M)$ will be denoted by $(w_t)_{0 \leq t \leq T}$.

**Definition 3.2.** A function $F : W^H(M) \to \mathbb{R}$ is called a $C^k$-cylinder function if there exists a partition

$$\pi := \{0 = t_0 < t_1 < t_2 < \cdots < t_n \leq T\}$$

of the interval $[0, T]$ and $f \in C^k(M^n)$ such that

$$F(w) = f(w_{t_1}, \ldots, w_{t_n}) \text{ for all } w \in W^H(M).$$

The function $F$ is called a smooth cylinder function on $W^H(M)$ of $M$, if there exists a partition $\pi$ and $f \in C^\infty(M^n)$ such that (3.6) holds.

We denote by $\mathcal{F}C^k(W^H(M))$ the space of $C^k$-cylinder functions, and by $\mathcal{F}C^\infty(W^H(M))$ the space of $C^\infty$-cylinder functions.

**Remark 3.3.** The representation (3.6) of a cylinder function is not unique. However, let $F \in \mathcal{F}C^\infty(W^H(M))$ and $n \geq 0$ be the minimal $n$ such that there exists a partition

$$\bar{\pi} := \{0 = \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n \leq T\}$$

of the interval $[0, T]$ and $\tilde{f} \in C^k(M^n)$ such that

$$F(w) = \tilde{f}(w_{\tilde{t}_1}, \ldots, w_{\tilde{t}_n}) \text{ for all } w \in W^H(M).$$

In that case, if

$$\tilde{\pi} = \{0 = \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n \leq T\}$$

is another partition of the interval $[0, T]$ and $\tilde{f} \in C^k(M^n)$ is such that

$$F(w) = \tilde{f}(w_{\tilde{t}_1}, \ldots, w_{\tilde{t}_n}) \text{ for all } w \in W^H(M),$$
then $\pi = \tilde{\pi}$ and $f = \tilde{f}$. Indeed, since

$$f(w_{t_1}, ..., w_{t_n}) = \tilde{f}(w_{\tilde{t}_1}, ..., w_{\tilde{t}_n})$$

we first deduce that $t_1 = \tilde{t}_1$. Otherwise $d_1 f = 0$ or $d_1 \tilde{f} = 0$, where $d_1$ denotes the differential with respect to the first component. This contradicts the fact that $n$ is minimal. Similarly, $t_2 = \tilde{t}_2$ and more generally $t_k = \tilde{t}_k$.

The representation (3.3) will be referred to as the minimal representation of $F$.

We now turn to the definition of directional derivative on the horizontal path space.

**Definition 3.4.** Let $F = f(w_{t_1}, ..., w_{t_n}) \in FC^\infty(W_H(M))$. For an $\mathcal{F}$-adapted and $T_2M$-valued semimartingale $(v(t))_{0 \leq t \leq T}$ such that $v(0) = 0$, we define the directional derivative

$$D_v F = \sum_{i=1}^{n} \left\langle d_i f(X_{t_1}, \cdots, X_{t_n}), \tilde{\Theta}_i^\varepsilon v(t_i) \right\rangle$$

**Definition 3.5.** For $F = f(w_{t_1}, ..., w_{t_n}) \in FC^\infty(W_H(M))$ we define the damped Malliavin derivative by

$$\tilde{D}_v F := \sum_{i=1}^{n} 1_{[0,t_i]}(t) (\tau_i^\varepsilon)^{-1} \tau_i^\varepsilon d_i f(X_{t_1}, \cdots, X_{t_n}), \quad 0 \leq t \leq T.$$

Observe that from this definition $\tilde{D}_v F \in T^*_X M$.

**Remark 3.6.** Note that the directional derivative $D$ is independent of $\varepsilon$, but the damped Malliavin derivative depends on $\varepsilon$. In addition, both the directional derivatives and damped Malliavin derivatives are independent of the representation of $F$. Indeed, let $F = f(w_{t_1}, ..., w_{t_n})$ be the minimal representation of $F$. If $\tilde{f}(w_{\tilde{t}_1}, ..., w_{\tilde{t}_n})$ is another representation of $F$, then for every $1 \leq j \leq N$, we have either that there exists $i$ such that $t_i = \tilde{t}_j$ in which case $d_i f = d_j \tilde{f}$, or for all $i$, $t_i \neq \tilde{t}_j$ in which case $d_j \tilde{f} = 0$.

Before we can formulate the main result, we need to define an analog of the Cameron-Martin subspace.

**Definition 3.7.** An $\mathcal{F}_t$-adapted absolutely continuous $\mathcal{H}_x$-valued process $(\gamma(t))_{0 \leq t \leq T}$ such that $\gamma(0) = 0$ and $\mathbb{E}_x \left( \int_0^T \|\gamma'(t)\|^2_H dt \right) < \infty$ will be called a horizontal Cameron-Martin process. The space of horizontal Cameron-Martin processes will be denoted by $\mathcal{CM}_H(M, \Omega)$.

**Definition 3.8.** An $\mathcal{F}_t$-adapted $T_2M$-valued continuous semimartingale $(v(t))_{0 \leq t \leq T}$ such that $v(0) = 0$ and $\mathbb{E}_x \left( \int_0^T \|v(t)\|^2 dt \right) < \infty$ will be called a tangent process if the process

$$v(t) + \int_0^t (\tilde{\Theta}_s^\varepsilon)^{-1} T(\tilde{\Theta}_s^\varepsilon \circ dB_s, \tilde{\Theta}_s^\varepsilon v(s))$$
is a horizontal Cameron-Martin process. The space of tangent processes will be denoted by $T_xW_H(M, \Omega)$.

**Remark 3.9.** By Remark 3.1 the stochastic parallel transport $\hat{\Theta}_s^\varepsilon$ is independent of $\varepsilon$, therefore the notion of a tangent process is itself independent of $\varepsilon$ as well.

**Remark 3.10.** As the torsion $T$ is a vertical tensor, then an $\mathcal{F}_t$-adapted and $T_xM$-valued continuous semimartingale $\{v(t)\}_{0 \leq t \leq T}$ such that

$$
E_x \left( \int_0^T \|v(t)\|^2 dt \right) < \infty, \quad v(0) = 0
$$

is in $TW_H(M)$ if and only if

1. The horizontal part $v_H \in \mathcal{CM}_H(M, \Omega)$;
2. The vertical part $v_V$ is given by

$$
v_V(t) = -\int_0^t (\hat{\Theta}_s^\varepsilon)^{-1}T(\hat{\Theta}_s^\varepsilon \circ dB_s, \hat{\Theta}_s^\varepsilon v_H(s)).
$$

The main results of this section are the following two theorems.

**Theorem 3.11 (Integration by parts for the damped Malliavin derivative).** Suppose $F \in F\mathcal{C}^\infty(W_H(M))$ and $\gamma \in \mathcal{CM}_H(M, \Omega)$, then

$$
E_x \left( \int_0^T \langle \tilde{D}_s^\varepsilon F, \hat{\Theta}_s^\varepsilon \gamma'(s) \rangle ds \right) = E_x \left( F \int_0^T \langle \gamma'(s), dB_s \rangle_H \right).
$$

**Theorem 3.12 (Integration by parts for the directional derivatives).** Suppose $F \in F\mathcal{C}^\infty(W_H(M))$ and $v \in T_xW_H(M, \Omega)$, then

$$
E_x (D_v F) = E_x \left( F \int_0^T \left\langle v_H'(t) + \frac{1}{2} (\hat{\Theta}_t^\varepsilon)^{-1}\mathbf{Ric}_H \hat{\Theta}_t^\varepsilon v_H(t), dB_t \right\rangle_H \right).
$$

Even though these two integration by parts formulas seem similar, they are quite different in nature. The damped derivative is used to derive gradient bounds and functional inequalities on the path space (see [2][3]). The directional derivative, however, is more related to quasi-invariance properties (see Section 4), and the expression

$$
\int_0^T \left\langle v_H'(t) + \frac{1}{2} (\hat{\Theta}_t^\varepsilon)^{-1}\mathbf{Ric}_H \hat{\Theta}_t^\varepsilon v_H(t), dB_t \right\rangle_H
$$

can be viewed as a horizontal divergence on the path space.

The remainder of the section is devoted to the proof of Theorem 3.11 and Theorem 3.12. We shall adapt the methods from Markovian stochastic calculus developed by Fang-Malliavin [25] and E. Hsu [32] in the Riemannian case to our framework.
3.2. Gradient formula. In this preliminary section we recall the gradient formula for $P_t$. In the case $\delta_H T = 0$, the operator $\Box_\varepsilon$ is essentially self-adjoint on the space of one-forms, and the gradient representation was first proved in [2].

Lemma 3.13 (Theorem 4.6 and Corollary 4.7 in [2], Theorem 2.7 in [30]). Let $T > 0$. For $f \in C^\infty_0(\mathbb{M})$, the process

$$N_s = \tau^\varepsilon_s (dP_{T-s}f)(X_s), \quad 0 \leq s \leq T,$$

is a martingale, where $dP_{T-s}f$ denotes here the exterior derivative of the function $P_{T-s}f$. As a consequence, for every $t \geq 0$,

$$dP_t f(x) = \mathbb{E}_x(\tau^\varepsilon_t df(X_t)).$$

Proof. From Itô’s formula and the definition of $\tau^\varepsilon$, we have

$$dN_s = \tau^\varepsilon_s \left( \nabla_{\partial_d X_s} - \Sigma^\varepsilon_{\partial_d X_s} - \frac{1}{2} \left( \frac{1}{\varepsilon} J^2 - \frac{1}{\varepsilon} \delta_H T + \mathfrak{R}ic_H \right) \right) ds (dP_{T-s}f)(X_s) + \tau^\varepsilon_s \frac{d}{ds} (dP_{T-s}f)(X_s) ds.

We now see that

$$\frac{d}{ds} (dP_{T-s}f) = -\frac{1}{2} dP_{T-s}Lf = -\frac{1}{2} dLP_{T-s}f = -\frac{1}{2} \Box_\varepsilon dP_{T-s}f,$$

where we used Theorem 2.3. Observe that the bounded variation part of

$$\tau^\varepsilon_s \left( \nabla_{\partial_d X_s} - \Sigma^\varepsilon_{\partial_d X_s} - \frac{1}{2} \left( \frac{1}{\varepsilon} J^2 - \frac{1}{\varepsilon} \delta_H T + \mathfrak{R}ic_H \right) \right) ds (dP_{T-s}f)(X_s)$$

is given by $\frac{1}{2} \tau^\varepsilon_s \Box_\varepsilon dP_{T-s}f(X_s) ds$ which cancels out with the expression $\tau^\varepsilon_s \frac{d}{ds} (dP_{T-s}f)(X_s) ds$. The martingale property follows from the bound as in Lemma 4.3 in [2] or [28,29, Theorem 2.7].

3.3. Integration by parts formula for the damped Malliavin derivative. We prove Theorem 3.11 in this section. Some of the key arguments may be found in [2,3], however since our framework is more general here (for example, we do not assume the Yang-Mills condition $\delta_H T = 0$) and we now use the adjoint connection instead of the Bott connection, for the sake of self-containment, we give a complete proof.

Lemma 3.14. For $f \in C^\infty_0(\mathbb{M})$, and $\gamma \in \mathcal{CM}_H(\mathbb{M}, \Omega)$

$$\mathbb{E}_x \left( f(X_T) \int_0^T \langle \gamma'(s), dB_s \rangle_H \right) =$$

$$\mathbb{E}_x \left( \langle \tau^\varepsilon_T f(X_T), \int_0^T (\tau^\varepsilon_s)^{-1} \tilde{\Theta}_s^\varepsilon \gamma'(s) ds \rangle \right).$$
Proof. Consider again the martingale process $N_s$ defined by (3.9). We have then for $f \in C_0^\infty(\mathcal{M})$

$$\mathbb{E}_x \left( f(X_t) \int_0^t \langle \gamma'(s), dB_s \rangle_{\mathcal{H}} \right) = \mathbb{E}_x \left( f(X_t) \int_0^t \langle \hat{\Theta}_s^\varepsilon \gamma'(s), \hat{\Theta}_s^\varepsilon dB_s \rangle_{\mathcal{H}} \right)$$

$$= \mathbb{E}_x \left( (f(X_t) - \mathbb{E}_x (f(X_t))) \int_0^t \langle \hat{\Theta}_s^\varepsilon \gamma'(s), \hat{\Theta}_s^\varepsilon dB_s \rangle_{\mathcal{H}} \right)$$

$$= \mathbb{E}_x \left( \int_0^t \langle dP_{t-s} f(X_s), \hat{\Theta}_s^\varepsilon dB_s \rangle \int_0^t \langle \hat{\Theta}_s^\varepsilon \gamma'(s), \hat{\Theta}_s^\varepsilon dB_s \rangle_{\mathcal{H}} \right)$$

$$= \mathbb{E}_x \left( \int_0^t \langle \hat{\Theta}_s^\varepsilon \gamma'(s) \rangle ds \right)$$

where we integrated by parts in the last equality.

Remark 3.15. A similar proof as above actually yields that for $f \in C_0^\infty(\mathcal{M})$, $\gamma \in \mathcal{CM}_H(\mathcal{M}, \Omega)$ and $t \leq T$,

$$\mathbb{E}_x \left( f(X_T) \int_t^T \langle \gamma'(s), dB_s \rangle_{\mathcal{H}} \mid \mathcal{F}_t \right) =$$

$$\mathbb{E}_x \left( \langle \hat{\Theta}_s^\varepsilon \gamma'(s) \rangle ds \mid \mathcal{F}_t \right)$$

Lemma 3.14 shows that integration by parts formula (3.8) holds for cylinder functions of the type $F = f(X_t)$. We now turn to the proof of Theorem 3.11 by using induction on $n$ in a representation of a cylinder function $F$. To run the induction argument we need the following fact.

Proposition 3.16. Let $F = f(X_{t_1}, \ldots, X_{t_n}) \in \mathcal{FC}^\infty(\mathcal{W}_H(\mathcal{M}))$. We have

$$d\mathbb{E}_x(F) = \mathbb{E}_x \left( \sum_{i=1}^n \tau_{i}^\varepsilon d_i f(X_{t_1}, \ldots, X_{t_n}) \right)$$

Proof. We will proceed by induction. Consider a cylinder function $F = f(X_{t_1}, \ldots, X_{t_n})$. For $n = 1$ the statement follows from Lemma 3.13 which implies that

$$d\mathbb{E}_x(f(X_{t_1})) = dP_{t_1} f(x) = \mathbb{E}_x(\tau_{t_1}^\varepsilon df(X_{t_1}))$$

Now we assume that the claim holds for any cylinder function $F = f(X_{t_1}, \ldots, X_{t_k})$ for any $k \leq n - 1$. By the Markov property we have

$$\mathbb{E}_x(F) = \mathbb{E}_x(\mathbb{E}(F \mid \mathcal{F}_{t_1})) = \mathbb{E}_x(g(X_{t_1})),$$
where $g(y) = \mathbb{E}_y(f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1}))$. Therefore
\[
d\mathbb{E}_x(F) = \mathbb{E}(\tau_{t_1}^\varepsilon dg(X_{t_1})).
\]
By using the induction hypothesis, we obtain
\[
dg(y) = \mathbb{E}_y(d_1 f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1}) + \\
\mathbb{E}_y \left( \sum_{i=2}^n \tau_{t_i-t_1}^\varepsilon d_i f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1}) \right) \\
= \mathbb{E}_y \left( \sum_{i=1}^n \tau_{t_i-t_1}^\varepsilon d_i f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1}) \right).
\]
By the multiplicative property of $\tau^\varepsilon$ and the Markov property of $X$ we have
\[
\mathbb{E}_{X_{t_1}} \left( \tau_{t_i-t_1}^\varepsilon d_i f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1}) \right) = \\
(\tau_{t_1}^\varepsilon)^{-1} \mathbb{E} \left( \tau_{t_i}^\varepsilon d_i f(X_{t_1}, \cdots, X_{t_n}) \mid \mathcal{F}_{t_1} \right).
\]
Therefore we conclude
\[
d\mathbb{E}_x(F) = \mathbb{E}_x \left( \sum_{i=1}^n \tau_{t_i}^\varepsilon d_i f(X_{t_1}, \cdots, X_{t_n}) \right).
\]

\[\Box\]

**Remark 3.17.** As expected, the expression
\[
\mathbb{E}_x \left( \sum_{i=1}^n \tau_{t_i}^\varepsilon d_i f(X_{t_1}, \cdots, X_{t_n}) \right)
\]
is independent of the choice of the representation of the cylinder function $F$ (see Remark 3.6).

**Proof of Theorem 3.11.** We use induction on $n$ in a representation of the cylinder function $F$. More precisely, we would like to show that for any $F = f(X_{t_1}, \cdots, X_{t_n}) \in \mathcal{F}C^\infty(W_H(M))$ and $t \leq t_1$ we have
\[
(3.11) \quad \mathbb{E}_x \left( F \int_t^{t_n} \langle \gamma'(s), dB_s \rangle \mid \mathcal{F}_t \right) = \\
\mathbb{E}_x \left( \sum_{i=1}^n \langle d_i f(X_{t_1}, \cdots, X_{t_n}), \tau_{t_i}^\varepsilon \rangle \int_t^{t_i} (\tau_{s_i}^\varepsilon)^{-1} \gamma'(s) ds \mid \mathcal{F}_t \right).
\]
The case $n = 1$ is Lemma 3.14 and Remark 3.15. Assume that (3.11) holds for any cylinder function $F$ represented by a partition of size $n - 1$ for
\[ n \geq 2. \text{ Let } F = f(X_{t_1}, \cdots, X_{t_n}) \in \mathcal{F}C^\infty(W_H(\mathbb{M})). \text{ We have for } t \leq t_1, \]
\[
\mathbb{E}_x \left( F \int_t^T \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) \\
= \mathbb{E}_x \left( F \int_t^{t_1} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) + \mathbb{E}_x \left( F \int_{t_1}^T \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) \\
= \mathbb{E}_x \left( \mathbb{E}_x(F | \mathcal{F}_{t_1}) \int_t^{t_1} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) + \mathbb{E}_x \left( \mathbb{E}_x(F | \mathcal{F}_{t_1}) \int_{t_1}^{t_n} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) \\
= \mathbb{E}_x \left( \mathbb{E}_x(F | \mathcal{F}_{t_1}) \int_t^{t_1} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) + \mathbb{E}_x \left( \mathbb{E}_x(F | \mathcal{F}_{t_1}) \int_{t_1}^{t_n} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right).
\]

By the Markov property we have
\[
\mathbb{E}_x(F | \mathcal{F}_{t_1}) = g(X_{t_1}),
\]
where \( g(y) = \mathbb{E}_y(f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1})) \). Thus by Lemma 3.14 and Remark 3.15
\[
\mathbb{E}_x \left( \mathbb{E}_x(F | \mathcal{F}_{t_1}) \int_t^{t_1} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) = \mathbb{E}_x \left( g(X_{t_1}) \int_t^{t_1} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) \\
= \mathbb{E}_x \left( dg(X_{t_1}), (\tau_{t_1}^s)^* \int_t^{t_1} (\tau_{t_1}^s)^* \tilde{\Theta}_s \gamma'(s) ds \right) | \mathcal{F}_t \right).
\]

Now according to Proposition 3.16
\[
dg(y) = \mathbb{E}_y \left( \sum_{i=1}^n \tau_{t_i-t_1}^e d_i f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1}) \right).
\]

Using the fact that
\[
\mathbb{E}_{X_{t_1}} (\tau_{t_i-t_1}^e d_i f(y, X_{t_2-t_1}, \cdots, X_{t_n-t_1})) = (\tau_{t_i}^s)^{-1} \mathbb{E}_x (\tau_{t_i}^e d_i f(X_{t_1}, \cdots, X_{t_n}) | \mathcal{F}_{t_i}),
\]
we conclude
\[
\mathbb{E}_x \left( \mathbb{E}_x(F | \mathcal{F}_{t_1}) \int_t^{t_1} \langle \gamma'(s), dB_s \rangle_H | \mathcal{F}_t \right) \\
= \mathbb{E}_x \left( \sum_{i=1}^n d_i f(X_{t_1}, \cdots, X_{t_n}), (\tau_{t_i}^{e_i})^* \int_t^{t_1} (\tau_{t_i}^s)^* \tilde{\Theta}_s \gamma'(s) ds \right) | \mathcal{F}_t \right).
\]
Using the induction hypothesis that (3.11) holds for \(n-1\) we see that
\[
\mathbb{E}_x \left( F \int_{t_1}^{t_n} \langle \gamma'(s), dB_s \rangle \right| \mathcal{F}_{t_1} ) = \\
\mathbb{E}_x \left( \sum_{i=1}^{n} (d_i f(X_{t_1}, \ldots, X_{t_n}), \tau_{t_i}^{\varepsilon,s} \int_{t_1}^{t_i} (\tau_{s}^{\varepsilon,s})^{-1} \gamma'(s) ds ) | \mathcal{F}_{t_1} \right).
\]

\[\square\]

3.4. Integration by parts formula for the directional derivatives. In this section we prove Theorem 3.12. One of the main ingredients B. Driver used in [16] in the Riemannian case was the orthogonal invariance of the Brownian motion to filter out redundant noise. As a complement to Lemma 3.14, we first prove the following result.

Lemma 3.18. Let \((\mathcal{O}_s)_{s \geq 0}\) be a continuous and \(\mathcal{F}\)-adapted process taking values in the space of skew-symmetric endomorphisms of \(\mathcal{H}_x\) such that 
\[
\mathbb{E} \left( \int_0^T \| \mathcal{O}_s \|^2 ds \right) < +\infty,
\]
where \(\| \mathcal{O}_s \|^2 = \text{Tr}(\mathcal{O}_s^* \mathcal{O}_s)\). For \(f \in C_0^\infty(M)\), we have
\[
\mathbb{E}_x \left( \left\langle \tau_{t}^{\varepsilon,s} df(X_T), \int_0^T (\tau_{s}^{\varepsilon,s})^{-1} \hat{\Theta}_s^{\varepsilon} \left( \mathcal{O}_s dB_s - \frac{1}{2} T_{\mathcal{O}_s} ds \right) \right\rangle \right) = 0,
\]
where \(T_{\mathcal{O}_s}^{\varepsilon} = \sum_{i=1}^{n} (\hat{\Theta}_s^{\varepsilon})^{-1} T_s^{\varepsilon}(e_i, \hat{\Theta}_s^{\varepsilon} \mathcal{O}_s (\hat{\Theta}_s^{\varepsilon})^{-1} e_i).\)

Proof. Recall that we considered the following martingale in (3.9)
\[
N_s = \tau_{s}^{\varepsilon} (dP_{T-s} f)(X_s), \quad 0 \leq s \leq T.
\]

We have then
\[
\mathbb{E}_x \left( \left\langle \tau_{t}^{\varepsilon,s} df(X_T), \int_0^T (\tau_{s}^{\varepsilon,s})^{-1} \hat{\Theta}_s^{\varepsilon} \mathcal{O}_s dB_s \right\rangle \right) = \\
\mathbb{E}_x \left( \left\langle N_T, \int_0^T (\tau_{s}^{\varepsilon,s})^{-1} \hat{\Theta}_s^{\varepsilon} \mathcal{O}_s dB_s \right\rangle \right).
\]

From the proof of Lemma 3.13 we have
\[
dN_s = \tau_{s}^{\varepsilon} \left( \nabla_{\partial dX_s} - \mathcal{T}_s^{\varepsilon} \partial dX_s - \frac{1}{2} \left( - J^2 - \frac{1}{\varepsilon} \delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}} \right) \right) (dP_{T-s} f)(X_s)
\]
\[+ \tau_{s}^{\varepsilon} \frac{d}{ds} (dP_{T-s} f)(X_s) ds \]
\[= \tau_{s}^{\varepsilon} \left( \nabla_{\hat{\Theta}_s^{\varepsilon} dB_s} - \mathcal{T}_s^{\varepsilon} \hat{\Theta}_s^{\varepsilon} dB_s \right) (dP_{T-s} f)(X_s) = \tau_{s}^{\varepsilon} \nabla_{\hat{\Theta}_s^{\varepsilon} dB_s} dP_{T-s} f(X_s),\]
where, as before, \(\nabla^{\varepsilon}\) denotes the connection \(\nabla - \mathcal{T}^{\varepsilon}\). Let us denote by \(\text{Hess}^{\varepsilon}\) the Hessian for the connection \(\nabla^{\varepsilon}\). One has therefore
\[ \mathbb{E}_x \left( \left\langle N_T, \int_0^T (\tau_s^e)^{-1} \tilde{\Theta}_s \mathcal{O}_s dB_s \right\rangle \right) = \]
\[ \mathbb{E}_x \left( \int_0^T \text{Hess}^e P_{T-s} f(\tilde{\Theta}_s dB_s, \tilde{\Theta}_s \mathcal{O}_s dB_s)(X_s) \right) \]

Due to the skew symmetry of \(\mathcal{O}\) and the fact that for \(h \in C_0^\infty(\mathbb{M}), X,Y \in \Gamma^\infty(\mathbb{M}),\)
\[ \text{Hess}^e h(X,Y) - \text{Hess}^e h(Y,X) = T^e(X,Y)h, \]
we deduce
\[ \mathbb{E}_x \left( \left\langle N_T, \int_0^T (\tau_s^e)^{-1} \tilde{\Theta}_s \mathcal{O}_s dB_s \right\rangle \right) = \]
\[ \frac{1}{2} \mathbb{E}_x \left( \int_0^T \left\langle dP_{T-s} f, \tilde{\Theta}_s T_{\tilde{\Theta}_s} \right\rangle \, ds \right) = \]
\[ \frac{1}{2} \mathbb{E}_x \left( \int_0^T \left\langle N_s, (\tau_s^e)^{-1} \tilde{\Theta}_s T_{\tilde{\Theta}_s} \right\rangle \, ds \right). \]
Integrating by parts the right hand side yields the conclusion. \(\square\)

We are now in position to prove the integration by parts formula for cylinder functions of the type \(F = f(X_t).\)

**Lemma 3.19.** Let \(v \in T_x W_\mathcal{H}(\mathbb{M}, \Omega).\) For \(f \in C_0^\infty(\mathbb{M}),\)
\[ \mathbb{E}_x \left( \left\langle df(X_T), \tilde{\Theta}_T^e v(T) \right\rangle \right) = \]
\[ \mathbb{E}_x \left( f(X_T) \int_0^T \left\langle v'_H(t) + \frac{1}{2} (\tilde{\Theta}_t^e)^{-1} \text{Ric}_H \tilde{\Theta}_t^e v_H(t), dB_t \right\rangle_{\mathcal{H}} \right). \]

**Proof.** Let \(v \in T_x W_\mathcal{H}(\mathbb{M}, \Omega).\) We define
\[ h(t) = v(t) + \int_0^t (\tilde{\Theta}_s^e)^{-1} T(\tilde{\Theta}_s^e \circ dB_s, \tilde{\Theta}_s^e v(s)). \]
By definition of \(T_x W_\mathcal{H}(\mathbb{M}, \Omega),\) we have \(h = v_H \in C \mathcal{M}_H(\mathbb{M}, \Omega).\) From the formula
\[ \tilde{T}^e(X,Y) = T(X,Y) - \frac{1}{\varepsilon} J_Y X + \frac{1}{\varepsilon} J_X Y, \]
we have
\[ \tilde{T}^e(\circ dX_t, \tilde{\Theta}_t^e v(t)) = T(\tilde{\Theta}_t^e \circ dB_t, \tilde{\Theta}_t^e v(t)) - \frac{1}{\varepsilon} J_{\tilde{\Theta}_t^e v(t)}(\tilde{\Theta}_t^e \circ dB_t). \]
We get therefore
\[ dv(t) + (\widehat{\Theta}_t^\varepsilon)^{-1} \left( \widehat{F}^\varepsilon \circ dX_t, \cdot \right) + \frac{1}{2} \left( \frac{1}{\varepsilon} J^2 - \frac{1}{\varepsilon} \delta_H^* T + \mathfrak{Ric}_H \right) dt \right) \widehat{\Theta}^\varepsilon v(t) \]
\[ = dh(t) - \frac{1}{\varepsilon} (\widehat{\Theta}_t^\varepsilon)^{-1} J_{\widehat{\Theta}^\varepsilon v(t)} (\widehat{\Theta}_t^\varepsilon \circ dB_t) + \frac{1}{2} (\widehat{\Theta}_t^\varepsilon)^{-1} \left( \frac{1}{\varepsilon} J^2 - \frac{1}{\varepsilon} \delta_H^* T + \mathfrak{Ric}_H \right) \widehat{\Theta}_t^\varepsilon h(t) dt \]
\[ = dh(t) - \frac{1}{\varepsilon} (\widehat{\Theta}_t^\varepsilon)^{-1} J_{\widehat{\Theta}^\varepsilon v(t)} \widehat{\Theta}_t^\varepsilon dB_t + \frac{1}{2} (\widehat{\Theta}_t^\varepsilon)^{-1} (\mathfrak{Ric}_H) \widehat{\Theta}_t^\varepsilon h(t) dt. \]

It is then a consequence of Itô’s formula that
\[ v(t) = (\widehat{\Theta}_t^\varepsilon)^{-1} \tau_{\varepsilon, s} \int_0^t \int_0^t \left( \frac{1}{2} (\widehat{\Theta}_s^\varepsilon)^{-1} (\mathfrak{Ric}_H) \widehat{\Theta}_s^\varepsilon h(t) dt \right) \]
\[ = \mathcal{O}_t dB_t - \frac{1}{2} T_{\varepsilon, t} \]
with
\[ \mathcal{O}_t = -\frac{1}{\varepsilon} (\widehat{\Theta}_t^\varepsilon)^{-1} J_{\widehat{\Theta}^\varepsilon v(t)} \widehat{\Theta}_t^\varepsilon. \]

Since \( \mathcal{O}_t \) is a skew-symmetric horizontal endomorphism, one can conclude from Lemmas 3.14 and 3.18 that
\[ \mathbb{E}_x \left( \left< df(X_t), \widehat{\Theta}_t^\varepsilon v(t) \right> \right) \]
\[ = \mathbb{E}_x \left( f(X_t) \int_0^T \left< v_{\mathcal{H}}^\varepsilon(t) + \frac{1}{2} (\widehat{\Theta}_t^\varepsilon)^{-1} \mathfrak{Ric}_H \widehat{\Theta}_t^\varepsilon v_{\mathcal{H}}(t), dB_t \right> \right), \]
because \( h(t) = v_{\mathcal{H}}(t) \).

Now Theorem 3.12 can be proven using induction on \( n \) in the presentation of a cylinder function \( F \). The case \( n = 1 \) is Lemma 3.19 and showing the induction step is similar to how Theorem 3.11 has been proven, so for the sake of conciseness of the paper, we omit the details. As a direct corollary of Theorem 3.12, we obtain the following.

**Corollary 3.20.** Let \( F, G \in \mathcal{F}C^\infty (W_\mathcal{H}(\mathbb{M})) \) and \( v \in T_s W_\mathcal{H}(\mathbb{M}) \). We have
\[ \mathbb{E}_x (FD_v G) = \mathbb{E}_x (GD_v F), \]
where
\[ D_v^* = -D_v + \int_0^T \left< v_{\mathcal{H}}^\varepsilon(t) + \frac{1}{2} (\widehat{\Theta}_t^\varepsilon)^{-1} \mathfrak{Ric}_H \widehat{\Theta}_t^\varepsilon v_{\mathcal{H}}(t), dB_t \right> \mathcal{H}. \]
**Proof.** By Theorem 3.12 we have

\[ E_x(D_v(FG)) = E_x \left( FG \int_0^T \left< v'_H(t) + \frac{1}{2}(\tilde{\Theta}_t^\epsilon)^{-1} \mathfrak{R}ic_H \tilde{\Theta}_t^\epsilon v_H(t), dB_t \right>_{\mathcal{H}} \right). \]

Since \( D_v(FG) = FD_v(G) + GD_v(F) \), the conclusion follows immediately. \( \square \)

### 3.5. Examples.

#### 3.5.1. Riemannian submersions.

In this section, we check that the integration by parts formula we obtained for the directional derivatives is consistent with and generalizes the formulas known in the Riemannian case. Let us assume here that the foliation on \( M \) comes from a totally geodesic submersion \( \pi : (M,g) \to (B,j) \) (see example 2.1). Since the submersion has totally geodesic fibers, \( \pi \) is harmonic and the projected process:

\[ X^B_t = \pi (X_t) \]

is a Brownian motion on \( B \) started at \( \pi (x) \). Observe that from the definition of submersion, the derivative map \( T_x \pi \) is an isometry from \( \mathcal{H} \) to \( T_x B \).

The stochastic parallel transport \( \tilde{\Theta}_t^\epsilon \) projects down to the stochastic parallel transport for the Levi-Civita connection along the paths of \( (X^B_t)_{0 \leq t \leq T} \). More precisely,

\[ //_{0,t} = T_{X^B_t} \pi \circ \tilde{\Theta}_t^\epsilon \circ (T_x \pi)^{-1}, \]

where \( //_{0,t} : T_{\pi (x)} B \to T_{X^B_t} B \) is the stochastic parallel transport for the Levi-Civita connection along the paths of \( (X^B_t)_{0 \leq t \leq T} \). Consider now a Cameron-Martin process \( (h(t))_{0 \leq t \leq T} \) in \( T_{\pi (x)} B \) and a cylinder function \( F = f(X^B_{t_1}, \cdots, X^B_{t_n}) \) on \( B \). The function \( F = f(\pi(X_{t_1}), \cdots, \pi(X_{t_n})) \) is then in \( FC^{\infty}(W_H(M)) \). Using Theorem 3.12 one gets

\[ E_x(D_v F) = E_x \left( F \int_0^T \left< v'_H(t) + \frac{1}{2}(\tilde{\Theta}_t^\epsilon)^{-1} \mathfrak{R}ic_H \tilde{\Theta}_t^\epsilon v_H(t), dB_t \right>_{\mathcal{H}} \right), \]

where \( v_H \) is the horizontal lift of \( h \), that is, \( v_H = (T_x \pi)^{-1} h \). By definition, we have

\[ D_v F = \sum_{i=1}^n \left< d_i f(X^B_{t_1}, \cdots, X^B_{t_n}), (T_{X^B_{t_i}} \pi) \circ \tilde{\Theta}_t^\epsilon v_{t_i} \right> \]

\[ = \sum_{i=1}^n \left< d_i f(X^B_{t_1}, \cdots, X^B_{t_n}), //_{0,t_i} h(t_i) \right> \]

It is easy to check that \( \mathfrak{R}ic_H \) is the horizontal lift of the Ricci curvature \( \mathfrak{R}ic^B \) of \( B \). Therefore, the integration by parts formula for the directional
derivative $D_vF$ can be rewritten as follows.
\[
\mathbb{E}_x \left( \sum_{i=1}^n \left\langle d_i f(X^B_{t_i}, \cdots, X^B_{t_n}), /_0, t_i h(t_i) \right\rangle \right) = \mathbb{E}_x \left( F \int_0^T \left\langle h'(t) + \frac{1}{2} /_0, t R_{0,t} /_0, t h(t), dB_t^B \right\rangle T_{e(x)}^B \right),
\]
where $B^B$ is the Brownian motion on $T_{e(x)}^B$ given by $B^B = T_e \pi(B)$. This is exactly Driver’s integration by parts formula [16] for the Riemannian Brownian motion $X^B$.

3.5.2. $K$-contact manifolds. In this section, we assume that the Riemannian foliation on $\mathbb{M}$ is the Reeb foliation of a $K$-contact structure. The Reeb vector field on $\mathbb{M}$ will be denoted by $R$ and the complex structure by $J$. The torsion of the Bott connection is then
\[
T(X, Y) = \langle JX, Y \rangle_H R.
\]
Therefore with the previous notation, one has
\[
J_Z X = \langle Z, R \rangle JX.
\]
and the vertical part of a tangent process is given by
\[
v_V(t) = -\int_0^t (\hat{\Theta}_e)^{-1} T(\hat{\Theta}_e \circ dB_s, \hat{\Theta}_e v_H(s))
= \int_0^t ((\hat{\Theta}_e)^{-1} R \langle J \hat{\Theta}_e v_H(s), \hat{\Theta}_e \circ dB_s \rangle_H.
\]

4. Quasi-invariance of the horizontal Wiener measure

To prove quasi-invariance we adapt to our framework a method due to B. Driver [16] and later simplified by E. Hsu [31]. The crucial step is to use a construction similar to the one by Eells-Elworthy-Malliavin relying on the orthonormal frame bundle.

In the section we fix $0 < \varepsilon \leq +\infty$. For $\varepsilon = +\infty$, it will be understood that the adjoint connection $\hat{\nabla}^e = \nabla + \frac{1}{2} J$ is equal to the Bott connection. Interestingly, we will show that the flows we construct are independent of $\varepsilon$.

4.1. Normal frames. For computations in local coordinates, it will be convenient to work in the normal frame setting introduced in [6].

Lemma 4.1. Let $x_0 \in \mathbb{M}$. Around $x_0$, there exist a local orthonormal horizontal frame $\{X_1, \cdots, X_n\}$ and a local orthonormal vertical frame $\{Z_1, \cdots, Z_m\}$ such that the following structure relations hold
\[
[X_i, X_j] = \sum_{k=1}^n \omega_{ij}^k X_k + \sum_{k=1}^m \gamma_{ij}^k Z_k
\]
\[ [X_i, Z_k] = \sum_{j=1}^{m} \beta_{ik}^j Z_j, \]

where \( \omega_{ij}, \gamma_{ij}, \beta_{ik}^j \) are smooth functions such that:

\[ \beta_{ik}^j = -\beta_{ij}^k. \]

Moreover, at \( x_0 \) we have

\[ \omega_{ij}^k = 0, \beta_{ij}^k = 0. \]

We record the fact that in this frame the Christoffel symbols of the Bott connection \( \nabla \) are given by

\[
\begin{align*}
\nabla_{X_i} X_j &= \frac{1}{2} \sum_{k=1}^{n} \left( \omega_{ij}^k + \omega_{ki}^j + \omega_{kj}^i \right) X_k \\
\nabla_{Z_i} X_j &= 0 \\
\nabla_{X_i} Z_j &= \sum_{k=1}^{m} \beta_{ij}^k Z_k 
\end{align*}
\]

Thus, the Christoffel symbols of the adjoint connection \( \hat{\nabla}^\varepsilon = \nabla + \frac{1}{\varepsilon} J \) are given by

\[
\begin{align*}
\hat{\nabla}^\varepsilon_{X_i} X_j &= \frac{1}{2} \sum_{k=1}^{n} \left( \omega_{ij}^k + \omega_{ki}^j + \omega_{kj}^i \right) X_k \\
\hat{\nabla}^\varepsilon_{Z_i} X_j &= \frac{1}{\varepsilon} J_{Z_j} X_i = -\frac{1}{\varepsilon} \sum_{k=1}^{n} \gamma_{ik}^j X_k \\
\hat{\nabla}^\varepsilon_{X_i} Z_j &= \sum_{k=1}^{m} \beta_{ij}^k Z_k 
\end{align*}
\]

4.2. Construction of the horizontal Brownian motion from the orthonormal frame bundle. An orthonormal map at \( x \in M \) is an isometry \( u : (\mathbb{R}^{n+m}, \langle \cdot, \cdot \rangle) \rightarrow (T_x M, g) \). The orthonormal map bundle will be denoted by \( \mathcal{O}(M) \). The adjoint connection \( \hat{\nabla}^\varepsilon \) allows to lift vector fields on \( M \) to vector fields on \( \mathcal{O}(M) \). Let \( e_1, \ldots, e_n, f_1, \ldots, f_m \) be the canonical basis of \( \mathbb{R}^{n+m} \).

Notation 4.2. We denote by \( A_i \) the vector field on \( \mathcal{O}(M) \) such that \( A_i(x, u) \) is the lift of \( u(e_i) \), and we denote by \( V_i \) the vector field on \( \mathcal{O}(M) \) such that \( V_i(x, u) \) is the lift of \( u(f_i) \).

To take into account the foliation structure on \( M \), we shall be interested in a special sub-bundle of \( \mathcal{O}(M) \), the horizontal frame bundle. An isometry \( u : (\mathbb{R}^{n+m}, \langle \cdot, \cdot \rangle) \rightarrow (T_x M, g) \) will be called horizontal if

\[
\begin{align*}
&u(\mathbb{R}^n \times \{0\}) \subset H_x; \\
u(\{0\} \times \mathbb{R}^m) \subset V_x.
\end{align*}
\]

The horizontal frame bundle is then defined as the set of \( (x, u) \in \mathcal{O}(M) \) such that \( u \) is horizontal. It will be denoted \( \mathcal{O}_H(M) \). For notational convenience, when needed, we identify \( \mathbb{R}^n \) with \( \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+m} \) and \( \mathbb{R}^m \) with \( \{0\} \times \mathbb{R}^m \subset \mathbb{R}^{n+m} \).

We can write the vector fields \( A_i \)'s locally in terms of the normal frames described in Section 4.1. For \( x_0 \in M \) we let \( \{X_1, \ldots, X_n, Z_1, \ldots, Z_m\} \) be a normal frame around \( x_0 \).
If $u : \mathbb{R}^{n+m} \to T_x M$ is a horizontal isometry, we can find an orthogonal matrix $e^j_i$ such that $u(e_i) = \sum_{j=1}^n e^j_i X_j$ and $u(f_i) = \sum_{j=1}^m f^j_i Z_j$. Let $\tilde{X}_j$ be the vector field on $\mathcal{O}_H(M)$ defined by

$$\tilde{X}_j f(x,u) = \lim_{t \to 0} \frac{f(e^j_i X_i)(x,u) - f(x,u)}{t}.$$ 

**Lemma 4.3.** Let $x_0 \in M$ and $(x,u) \in \mathcal{O}_H(M)$

$$A_i(x,u) = \sum_{j=1}^n e^j_i \tilde{X}_j - \sum_{j,k,l,r=1}^n e^j_i e^l_r \langle \nabla_{\tilde{X}_j} X_l, X_k \rangle \frac{\partial}{\partial e^k_r} - \sum_{j=1}^n \sum_{k,l,r=1}^m e^j_i f^l_r \langle \nabla_{\tilde{X}_j} Z_l, Z_k \rangle \frac{\partial}{\partial f^k_r}.$$ 

In particular, at $x_0$ we have

$$A_i(x_0,u) = \sum_{j=1}^n e^j_i \tilde{X}_j.$$ 

**Proof.** Let $u : \mathbb{R}^{n+m} \to T_x M$ be a horizontal isometry and $x(t)$ be a smooth curve in $M$ such that $x(0) = x$ and $x'(0) = u(e_i)$. We denote by $x^*(t) = (x(t), u(t))$ the lift to $\mathcal{O}(M)$ of $x(t)$ and by $x'_1(t), \ldots, x'_n(t)$ the components of $x'(t)$ in the horizontal frame $X_1, \ldots, X_n$. Since the adjoint connection preserves the horizontal and vertical bundles, the curve $x^*(t)$ takes its values in $\mathcal{O}_H(M)$. By definition of $A_i$, one has

$$A_i = \sum_{j=1}^n x'_j(0) \tilde{X}_j + \sum_{k,l=1}^n u'_{kl}(0) \frac{\partial}{\partial e^k_l} + \sum_{k,l=1}^m v'_{kl}(0) \frac{\partial}{\partial f^k_l},$$

where $u_{kl}(t) = \langle u(t)(e_k), X_l \rangle$ and $v_{kl}(t) = \langle u(t)(f_k), Z_l \rangle$. Since $u(t)(e_k)$ and $u(t)(f_k)$ are parallel along $x(t)$, one has

$$\nabla^\varepsilon_{x'(t)} u(t)(e_k) = 0, \quad \nabla^\varepsilon_{x'(t)} u(t)(f_k) = 0.$$ 

At $t = 0$, this yields the expected result. 

**Remark 4.4.** Since the vector fields $X_j$ are horizontal, the covariant derivatives $\nabla_{\tilde{X}_j} X_l$ and $\nabla_{\tilde{X}_j} Z_l$ are independent of $\varepsilon$. As a consequence, the vector fields $A_i$ are themselves independent of a particular choice of $\varepsilon$.

In particular, Lemma 4.3 implies the following statement.

**Proposition 4.5.** Let $\pi : \mathcal{O}(M) \to M$ be the bundle projection map. For a smooth $f : M \to \mathbb{R}$, and $(x,u) \in \mathcal{O}_H(M)$,

$$\left( \sum_{i=1}^n A_i^2 \right) (f \circ \pi)(x,u) = Lf \circ \pi(x,u).$$
Proof. It is enough to prove this identity at $x_0$. Using the fact that at $x_0$ we have $\langle \hat{\nabla}_{X_j} X_l, X_k \rangle = \langle \hat{\nabla}_{X_j} Z_l, Z_k \rangle = 0$, we see that
\[ \sum_{i=1}^{n} A_i^2 = \sum_{j=1}^{n} \tilde{X}_j^2. \]
The conclusion follows. \qed

As a straightforward corollary, we obtain the corresponding statement about Brownian motion.

**Corollary 4.6.** Let $(B_t)_{t \geq 0}$ be an $n$-dimensional Brownian motion and let $(U_t)_{t \geq 0}$ be a solution to the stochastic differential equation
\[ dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dB_i^t, \quad U_0 \in \mathcal{O}_H(M), \]
then $X_t = \pi(U_t)$ is a horizontal Brownian motion on $M$, that is, a Markov process with generator $\frac{1}{2}L$.

### 4.3. Quasi-invariance of the horizontal Wiener measure.

Let $(B_t)_{t \geq 0}$ be an $n$-dimensional Brownian motion and let $(U_t)_{t \geq 0}$ be the solution to the stochastic differential equation
\[ dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dB_i^t + \sum_{i=1}^{m} V_i(U_t) \circ d\beta_i^t, \quad U_0 \in \mathcal{O}_H(M). \]

By Corollary 4.6, $X_t = \pi(U_t)$ is a horizontal Brownian motion on $M$ which is started at $x = \pi(U_0)$. The map $\phi_H : B \mapsto X$ will be referred to as the horizontal development map. Our goal in this section is to prove quasi-invariance of the law of $X$ with respect to a suitable family of flows. Our argument follows relatively closely the one by B. Driver [16] and then E. Hsu [31] (see also [13, 14, 23]). The only difference is that, in [16, 31], the authors consider the Itô map associated to the stochastic differential equation
\[ dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dB_i^t + \sum_{i=1}^{m} V_i(U_t) \circ d\beta_i^t, \quad U_0 \in \mathcal{O}(M), \]
where $(B_t, \beta_t)$ is a $\mathbb{R}^{n+m}$-valued Brownian motion. All of the formulas and results in [16, 31] can essentially be used by changing the driving path $(B_t, \beta_t)$ into the driving path $(B_t, 0)$. The only difference is that we need to consider adapted vector fields on the horizontal path space whose pullbacks by the horizontal development map are horizontal processes.

We now introduce some notation and formulate the precise framework in which we establish quasi-invariance. We will mainly follow the presentation in [15, 17].
Definition 4.8. We define the horizontal Itô map \( M \) and the horizontal Brownian motion on the Stratonovitch stochastic differential equation \( A \) denoted respectively by \( \omega_t \) and \( \omega_t^H \). The law of the horizontal Brownian motion will be called the horizontal Wiener (or Cameron-Martin) path. The space of tangent processes will be denoted by \( \mathcal{B}_t \). We decompose
\[
\omega_t = \omega_t^H + \omega_t^V,
\]
where \( \omega_t^H \in \mathbb{R}^n \times \{0\} \simeq \mathbb{R}^n \) and \( \omega_t^V \in \{0\} \times \mathbb{R}^m \simeq \mathbb{R}^m \). The process \( (\omega_t^H)_{0 \leq t \leq T} \) will be called the horizontal Brownian motion in \( \mathbb{R}^{n+m} \) and the law of the horizontal Brownian motion will be called the horizontal Wiener measure on \( \mathbb{R}^{n+m} \) and will be denoted by \( \mu_H \).

Definition 4.9. We define the horizontal Cameron-Martin space denoted by \( \mathcal{CM}_H(\mathbb{R}^{n+m}) \) as the space of absolutely continuous \( \mathbb{R}^n \)-valued (deterministic) functions \( (\gamma(t))_{0 \leq t \leq T} \) such that \( \gamma(0) = 0 \) and
\[
\int_0^T |\gamma'(t)|^2_{\mathbb{R}^n} \, dt < \infty.
\]
If \( h \in \mathbb{R}^{n+m} \) is a vector, we denote by \( Ah = \sum_{i=1}^n h_i A_i \) and \( Vh = \sum_{i=1}^m h_{i+n} V_i \), where \( A_i \) and \( V_i \) are defined in Notation 4.2. \( Ah \) and \( Vh \) are therefore vector fields on \( \mathcal{O}(\mathbb{M}) \) whose values at some \( u \in \mathcal{O}(\mathbb{M}) \) will be denoted respectively by \( A_u h \) and \( V_u h \). We consider the solution \( (U_t)_{0 \leq t \leq T} \) to the Stratonovitch stochastic differential equation
\[
dU_t = A_{U_t} \circ d\omega_t^H, \quad U_0 \in \mathcal{O}_H(\mathbb{M}),
\]
and the horizontal Brownian motion on \( \mathbb{M} \) given by \( X_t = \pi(U_t) \). The horizontal Itô map \( I_H \) is defined as the map \( I_H : \omega^H \rightarrow U \) and the horizontal development map \( \phi_H \) is defined as the map \( \phi_H : C_0(\mathbb{R}^n) \rightarrow W_H(\mathbb{M}), \omega^H \rightarrow X \). We refer to Definition 2.5 in [15] and the associated comments for a discussion of the Itô and development map in the classical Riemannian setting.

In view of Definition 3.8 we define a set of tangent processes which is suitable in the frame bundle setting. As we will see, intuitively, tangent processes are the adapted vector fields on \( W_H(\mathbb{M}) \) (in the sense of [15] Definition 4.1)) whose pullbacks by the horizontal development map \( \phi_H \) are horizontal processes.

Definition 4.9. A \( \mathcal{B} \)-adapted \( \mathbb{R}^{n+m} \)-valued continuous semimartingale \( (v(t))_{0 \leq t \leq T} \) such that \( v(0) = 0 \) and
\[
\mathbb{E} \left( \int_0^T |v(t)|^2_{\mathbb{R}^{n+m}} \, dt \right) < \infty
\]
will be called a tangent process if the process
\[
v(t) + \int_0^t T_{U_s} (A \circ d\omega_s^H, Av(s))
\]
is a horizontal Cameron-Martin path. The space of tangent processes will be denoted by \( TW_H(\mathbb{M}) \).
Remark 4.10. In Definition 4.9 we denote by $T$ the torsion of the Bott connection. Observe that since $T$ is a vertical tensor, a $B$-adapted and $\mathbb{R}^{n+m}$-valued continuous semimartingale $(v(t))_{0 \leq t \leq T}$ such that $v(0) = 0$ and $\mathbb{E} \left( \int_0^T |v(t)|^2_{\mathbb{R}^{n+m}} dt \right) < \infty$ is in $TW_H(M)$ if and only if

1. The horizontal part $v_H$ is in $CM_H(\mathbb{R}^{n+m})$;
2. The vertical part $v_V$ is given by

$$v_V(t) = - \int_0^t T_{U_s}(A \circ d\omega^H_s, Av_H(s)).$$

If $v \in TW_H(M)$ is a tangent process, we denote

$$p_v(t) = v(t) + \int_0^t \tilde{T}_{U_s}^\varepsilon(A \circ d\omega^H_s, Av(s) + Vv(s)) + \int_0^t \left( \int_0^s \tilde{\Omega}_{U_s}^\varepsilon(A \circ d\omega^H_s, Av(\tau) + Vv(\tau)) \right) \circ d\omega^H_s,$$

where $\tilde{\Omega}^\varepsilon$ is the curvature form of $\tilde{\nabla}^\varepsilon$. We observe that $p_v(t)$ can be seen as the pullback by the horizontal development map $\phi_H$ of the adapted vector field $v$ (see Section 3.1 in [15]).

Since $\tilde{\nabla}^\varepsilon$ is a horizontal metric connection, the stochastic integrand $\int_0^s \tilde{\Omega}_{U_s}^\varepsilon(A \circ d\omega^H_s, Av(\tau) + Vv(\tau))$ takes values in the space of skew-symmetric endomorphisms of $\mathbb{R}^n$. Also, we have

$$\int_0^t \tilde{T}_{U_s}^\varepsilon(A \circ d\omega^H_s, Av(s) + Vv(s)) =$$

$$\int_0^t T_{U_s}(A \circ d\omega^H_s, Av(s)) - \frac{1}{\varepsilon} \int_0^t J_{Vv(s)}(A \circ d\omega^H_s)_{U_s}.$$

As a consequence, $p_v(t)$ is actually a horizontal process, that is, it is $\mathbb{R}^n$-valued. Moreover, by Lemma 2.5 the tensor $\frac{1}{\varepsilon} J^2 - \frac{1}{\varepsilon} \delta^H_T + 2\text{Ric}_H$ is equal to the horizontal Ricci curvature of the adjoint connection. As a consequence, we can rewrite $p_v(t)$ by using Itô’s integral, and we obtain

$$p_v(t) =$$

$$v_H(t) + \frac{1}{2} \int_0^t \left( \frac{1}{\varepsilon} J^2 - \frac{1}{\varepsilon} \delta^H_T + 2\text{Ric}_H \right)_{U_s}(Av(s) + Vv(s))ds$$

$$- \frac{1}{\varepsilon} \int_0^t J_{Vv(s)}(A \circ d\omega^H_s)_{U_s} + \int_0^t \left( \int_0^s \tilde{\Omega}_{U_s}^\varepsilon(A \circ d\omega^H_s, Av(\tau) + Vv(\tau)) \right) d\omega^H_s.$$
We can further simplify this expression as follows.

\[ J_{Vv}(s) = \sum_{i=1}^{n} J_{Vv}(A) \circ d\omega^H_{s} = \sum_{i=1}^{n} J_{Vv}(A_i) \circ d\omega^i_{s} + \frac{1}{2} \sum_{i=1}^{n} A_i J_{Vv}(A_i) \circ d\omega^i_{s} \]

\[ - \frac{1}{2} \sum_{i=1}^{n} J_{T(A_i, AvH(s))}(A_i) \circ d\omega^i_{s} = J_{Vv}(A \circ d\omega^H_{s}) - \frac{1}{2}(\delta^H_T)(Vv(s)) + \frac{1}{2} J^2(AvH(s))ds. \]

As a result, we see that

\[ p_v(t) = \int_0^t q_v(s) d\omega^H_{s} + \int_0^t r_v(s) ds, \]

where \( q_v \) is an \( \mathfrak{so}(n) \)-valued adapted process and \( r_v \) is an \( \mathbb{R}^n \)-valued adapted process such that \( \int_0^t |r_v(s)|^2_{\mathbb{R}^n} ds < +\infty \) a.e.

The process \( p_v \) is therefore an adapted vector field on \( W(\mathbb{R}^n) \) in the sense of \( [15, \text{Definition 3.2}] \). In particular, one deduces the following analogue of \( [17, \text{Theorem 7.28}] \) (see \( [15] \) for the details).

**Theorem 4.11** (Differential of the horizontal development map). Let \( v \in TW_H(\mathbb{M}) \) be a tangent process such that \( |v'|_{H} \leq K \), where \( K \) is a non-random constant. Let

\[ b^v = \int_0^s e^{tq_v(u)} d\omega^H_u + t \int_0^s r_v(u) du, \]

where \( q_v \) and \( r_v \) are defined as in (4.1). Then, there exists a version of \( \phi_H(b^v) \) which is continuous in \((s,t)\) differentiable in \( t \) and such that

\[ \frac{d}{dt}|_{t=0} \phi_H(b^v) = \tilde{\Theta}^v, \]

where, as in Sections 2 and 3, \( \tilde{\Theta}^v \) denotes the parallel transport along the paths of the horizontal Brownian motion \( X \) for the connection \( \tilde{\nabla}^v \).
The following theorem is an analogue of in [17, Theorem 7.29] and can be proved similarly to [16, 31] (the main ingredients are Girsanov theorem in the form of [16, Lemma 8.2] and the previous theorem). We recall that, under $\mu$, $\phi_H(\omega_H)$ is a horizontal Brownian motion on $M$ started at $x$ with distribution denoted $\mu_X$ (the horizontal Wiener measure on $M$).

**Theorem 4.12** (Quasi-invariance of the horizontal Wiener measure). Let $h \in \mathcal{CM}_H(\mathbb{R}^{n+m})$. Consider the $\mu$-a.e. defined process given by

$$v_t(\omega) = h(t) - \int_0^t T_{\mathcal{X_H}(\omega_H)}(A \circ d\omega_H^s, Ah(s)).$$

Consider the $\mu$-a.e. defined process $\eta^v(\omega)$, given by parallel transport of $v(\omega)$ along $\phi_H(\omega_H)$ for the connection $\nabla^\varepsilon$. Then, $\eta^v \circ \phi_H^{-1}$ generates a flow on the horizontal path space $W_H(M)$ and this flow leaves the measure $\mu_X$ quasi-invariant.

**Remark 4.13.** Observe that $\eta^v \circ \phi_H^{-1}$ is indeed $\mu_X$-a.e. well defined since $\mu$-a.e. $\eta^v(\omega) = \eta^v(\omega_H)$.

In the case where the foliation on $M$ comes from a totally geodesic submerion $(M, g) \to (B, j)$ (see example 2.1 and section 3.5.1) then, as expected, theorem 4.12 projects down to [17, Theorem 7.29].

In the Heisenberg group, the situation is described in section 1.2. In the general case, with the notations of theorem 4.12 and denoting $\Pi$ the horizontal lift, one has a commutative diagram

$$\begin{array}{c}
\mathcal{W}_H(M) \xrightarrow{e^{\eta^v \circ \phi_H^{-1}}} \mathcal{W}_H(M) \\
\Pi \downarrow \quad \Pi \downarrow \\
\mathcal{W}(B) \xrightarrow{e^{\zeta}} \mathcal{W}(B)
\end{array}$$

where $e^{\eta^v \circ \phi_H^{-1}}$ is the flow generated by $\eta^v \circ \phi_H^{-1}$. The induced flow $e^{\kappa}$ on the path space $\mathcal{W}(B)$ leaves then the Wiener measure on $\mathcal{W}(B)$ quasi-invariant. Since the torsion $T$ is vertical and the connection $\nabla^\varepsilon$ projects down to the Levi-Civita connection on $B$, the generator $\zeta$ of the flow $e^{\kappa}$ is the $\mu_B$-a.e. defined process given by parallel transport (for the Levi-Civita connection on $B$) of the Cameron-Martin path $h$ along the Brownian motion on $B$.

**References**


