LÉVY PROCESSES IN A STEP 3 NILPOTENT LIE GROUP

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Abstract. The infinitesimal generators of Lévy processes in Euclidean space are pseudo-differential operators with symbols given by the Lévy-Khintchine formula. In the absence of a canonical definition of Fourier transform which is sensible for arbitrary Lie groups, a similar characterization of these processes for Lie groups is a subtle matter. We introduce the notion of pseudo-differential operator in a connected, simply connected nilpotent Lie group G using the Weyl functional calculus. We prove that with respect to this definition, the quantized generators of Lévy processes in G are pseudo-differential operators which admit $C_c^\infty(\mathbb{R})$ as a core.

APPROVAL PAGE

Doctor of Philosophy Dissertation

Lévy Processes In a Step 3 Nilpotent Lie Group

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Contents

1. Introduction

In $\mathbb{R}^n$, a Lévy process is defined to be a cádlàg stochastic process with stationary and independent increments (see Appendix B for definitions). Brownian motion and the Poisson process are the most famous examples of these processes. In Euclidean space, these processes are well understood. One of the most useful ways to characterize a stochastic process is to compute its Fourier transform. The Fourier transform of a Lévy process $\rho(t)$ has a particularly nice form, given by the Lévy-Khintchine formula [1]:

$$\mathbb{E}[e^{i\theta\rho(t)}] = \exp(t\varphi(\theta))$$

for

$$\varphi(\theta) = ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + t\int_{\mathbb{R}-\{0\}} (e^{i\theta x} - 1 - i\theta x \mathbf{I}_{|x|<1}\nu(\mathrm{d}x), \tag{1.1}$$

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\mathbf{I}$ is the indicator function and ν is a suitably chosen measure.

Lévy processes are Markov processes, and every Markov process can be described by a corresponding operator called an *infinitesimal generator* [1]. To understand these operators, it is useful to recall some ideas from harmonic analysis. If f is a smooth function on $\mathbb{R}^n$ with some reasonable decay properties, then the Fourier transform $\mathcal{F}(f)(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{k}}\mathrm{d}\mathbf{x}$ converges and from integration by parts one obtains the famous formula

$$\mathcal{F}\left(D^\alpha f\right)(\mathbf{k}) = \mathbf{k}^\alpha \mathcal{F}(f)(\mathbf{k}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is any multi-index,

$$D^\alpha f = \frac{1}{i^{|\alpha|}}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f \qquad \text{and} \qquad \mathbf{k}^\alpha = k_1^{\alpha_1}\cdots k_n^{\alpha_n}.$$

By defining

$$\mathscr{P}_\sigma(f) = \mathcal{F}^{-1}\left[\sigma(\mathbf{k})\mathcal{F}(f)\right]$$

one may recognize that $D^\alpha = \mathscr{P}_\sigma$ where $\sigma(\mathbf{k}) = \mathbf{k}^\alpha$. The function σ is commonly referred to as the *symbol* of the operator D^α. When $\sigma(\mathbf{k})$ is a polynomial, $\mathscr{P}_\sigma$ is a differential operator with constant coefficients; however $\mathscr{P}_\sigma$ is a sensible object for many other choices of $\sigma(\mathbf{k})$. This enables one to define a wide class of so-called *pseudo-differential* operators $\mathscr{P}_\sigma$ (refer to Appendix A for a more detailed exposition). Indeed, given an arbitrary operator $\mathscr{W}$, one may wish to determine if there exists some $\sigma_{\mathscr{W}}$ such that $\mathscr{W} = \mathscr{P}_{\sigma_{\mathscr{W}}}$ with respect to the symbol $\sigma_{\mathscr{W}}$.

If $\rho(t)$ is any Markov process, then one can define the following semigroup of operators:

$$(T_t f)(x) = \mathbb{E}\left(f(\rho(t))|\rho_0 = x\right). \tag{1.2}$$

The *infinitesimal generator* of this semigroup is the operator

$$(L_\rho f)(x) := \lim_{r\to 0} \frac{1}{t}\left(T_t(f)(x) - f(x)\right). \tag{1.3}$$

It is an interesting fact that given appropriate conditions on L, one can determine (up to reasonable equivalency) a process ρ for which $L = L_\rho$ [1]. Indeed, one way to characterize a process is to characterize the infinitesimal generator of its associated semigroup. One has the following classical result, the proof of which appears in Appendix B.

Theorem 1.1. *If $\rho(t)$ is a Lévy process in $\mathbb{R}^n$ then L_ρ is a pseudo-differential operator which is densely defined in $L^2(\mathbb{R})$ and the symbol of this operator is given by the Lévy-Khintchine formula:*

$$(L_p f) = \mathscr{P}_{\varphi(D)} \tag{1.4}$$

where $D = \frac{1}{i^n}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and φ is as in (1.1).

Some of these properties have analogs in more general spaces, for example, in some classes of Lie groups. Lie groups naturally arise in physics; indeed the most well known non-trivial example is the Heisenberg group, which is generated by position and momentum operators from classical quantum mechanics. The matrix groups $SO(n,\mathbb{R})$ and $U(n)$ are other well-known examples.

For a large class of Lie groups one defines the Fourier transform as integration against unitary irreducible representations of G,

$$\mathcal{F}(f(g))(\lambda) = \int_G f(g)\pi_\lambda(g)\mathrm{d}g, \tag{1.5}$$

where π_λ is a unitary irreducible representation of G indexed by λ and $\mathrm{d}g$ is Haar measure [6]. This formula is valid in all locally Abelian and nilpotent groups. For these groups, then, the issue of developing harmonic analysis is intimately connected to the representation theory of the group. For example, if G is locally compact and Abelian, one defines the dual group $\hat{G}$ of G to be the collection of *characters* (i.e. continuous homomorphisms from G into the circle group T) of G. For each $\chi \in \hat{G}$, define

$$\hat{f}(\chi) = \int_G f(g)\chi(g)\mathrm{d}g.$$

Pontryagin duality implies that this map is unitary and invertible.

If G is non-Abelian then this definition is not suitable, however there do exist generalizations of this formula. In the event that G is compact, the Peter-Weyl theorem provides a decomposition of $L^2(G)$ and each irreducible unitary representation of G can be realized as a regular representation on one of these subspaces [6].

Much of the analysis on groups can be reduced to analysis in a corresponding Lie algebra. The tangent space of G at the identity has a natural Lie algebra structure which determines many characteristics of G. For example the Lie algebra of any

n-dimensional locally compact Abelian group is isomorphic to $\mathbb{R}^n$ with the bracket identically 0. If H^n is the Heisenberg group then $\mathfrak{h}$ is the $(2n+1)$-dimensional vector space generated by $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ with the bracket operation $[X_i, Y_j] = \delta_{ij} Z$.

If $\mathfrak{g}$ is a Lie algebra, then one may define $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$. If $\mathfrak{g}_n = \{0\}$ for some n, then $\mathfrak{g}$ is said to be nilpotent. If m is the smallest value satisfying that $\mathfrak{g}_m = \{0\}$ then $\mathfrak{g}$ is said to be *step m nilpotent*. A group G is nilpotent if its corresponding Lie algebra is nilpotent. For example, the Heisenberg group is step 2 nilpotent.

The unitary irreducible representations of H are given, by the Stone-von Neumann Theorem as

$$\pi_\lambda(w, y, z) f(k) = e^{i(\pm\lambda z I + \pm\lambda^{1/2} x \cdot K + \lambda^{1/2} y \cdot D)} f(k)$$

where $f(k) \in L^2(\mathbb{R})$, $Kf(k) := kf(k)$ and $\lambda \in \mathbb{R}^+$. These representations were discovered early in the development of classical quantum mechanics. Using these representations to define the Fourier transform on H, one can analogously form expressions for pseudo-differential operators on H. Indeed, this is given by the classical Weyl functional calculus [11] as

$$a(K, D) = \int \hat{a}(p, q) e^{2\pi i (q \cdot K + p \cdot D)} \mathrm{d}p \mathrm{d}q$$

where $\hat{a}(p, q)$ is the Euclidean Fourier transform of $a(x, y)$. An operator Q on $L^2(\mathbb{R})$ may be called *pseudo-differential* if $Q = \sigma(K, D)$ for some suitably chosen σ.

Lie groups have topological manifold structure, with respect to which multiplication is smooth. In these spaces one can naturally construct a group-valued Lévy process ρ by defining the increment between times s and t to be $\rho(s)^{-1}\rho(t)$ (see Section 5 for a precise definition). Given a Lévy process $\rho(t)$ on H, define the semigroup

$$\mathcal{T}_t^{\pi}(\varphi) = \int_G \pi(g)\varphi \; q_t(\mathrm{d}(g)), \tag{1.6}$$

where $\varphi \in L^2(\mathbb{R})$, $\mathrm{d}g$ is Haar measure and q_t is the law of $\rho(t)$. Let $\mathcal{L}^{\pi}$ denote the infinitesimal generator of this semigroup. The following theorem appears in [2].

Theorem 1.2. *If $\rho(t)$ is a Lévy process in H and π is any irreducible unitary representation of ρ, then the infinitesimal generator $\mathcal{L}^{\pi}$ is a pseudo-differential operator, densely defined in $L^2(\mathbb{R})$. Moreover the symbol of this operator is given by the Lévy-Khintchine formula from $\mathbb{R}^3$:*

$$\sigma_{\rho}(x, y) = \varphi(1, x, y). \tag{1.7}$$

Note that Theorem 1.2 is a statement not about the generator $\mathcal{L}$, but about the *image* of $\mathcal{L}$ through the representation π. The image $\mathcal{L}^{\pi}$ is referred to as the *quantization* of the generator $\mathcal{L}$.

The goal of this thesis is to investigate Lévy processes in a higher step nilpotent group. The first step in extending this analysis is to construct the Fourier transform. In particular, this involves determining a complete set of unitary irreducible representations for these groups.

For general nilpotent groups, these representations are described by Kirillov's method of co-adjoint orbits [4]. Let G be a nilpotent group with Lie algebra $\mathfrak{g}$. If $l \in \mathfrak{g}^*$ is a linear functional on the Lie algebra $\mathfrak{g}$, then a subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ is said to be *subordinate* to l if $l([A, B]) = 0$ for all $A, B \in \mathfrak{m}$. One can construct a unitary irreducible representation π by fixing a specific linear functional l_{π} on $\mathfrak{g}$ and identifying a corresponding subalgebra of maximal dimension, $\mathfrak{m}_l$, subordinate to l_{π}. One has that $M = \exp(\mathfrak{m})$ is a subgroup of G on which π behaves as a character. The restriction of π to $\exp(\mathfrak{m})$ is given by a representation of Euclidean space, and can be uniquely extended by induction [11] to a unitary irreducible representation of G. The space over which the induced representation acts is $L^2(G/M)$.

Every unitary irreducible representation of a nilpotent group can be realized as an representation induced in this manner. The exponential map is a global diffeomorphism in nilpotent groups, and so if $g \in G$ then $g = \exp(\gamma)$ for some $\gamma \in \mathfrak{g}$. G acts on $\mathfrak{g}^*$ by $g(l)(X) = l([\gamma, X])$ for each $l \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. This is called the *co-adjoint* action of G on $\mathfrak{g}^*$. If l_1 and l_2 are linear functionals corresponding to two representations π_1 and π_2 then $\pi_1 \cong \pi_2$ if and only if l_1 and l_2 lie in the same co-adjoint orbit. Thus the co-adjoint orbits of $\mathfrak{g}^*$ parameterize unitary irreducible representations of G (hence the λ from (1.5) in fact indexes the co-adjoint orbit corresponding to π_λ).

In nilpotent groups, Haar measure is the push-forward of Lebesgue measure on the Lie algebra [4], and so every function in $L^2(G/M)$ can be pulled back to a function in $L^2(\mathfrak{g}/\mathfrak{m})$. Thus every representation of G can be realized as a representation acting in $L^2(\mathbb{R}^m)$. Therefore, for each reasonable function $f(g)$, the Fourier transform of f (given by (1.5)) operates on $L^2(\mathbb{R}^m)$.

There is a general Weyl functional calculus for nilpotent groups [3, 11]. Given a unitary irreducible representation π of G there exists a set of multiplication and differentiation operators $A_1, \ldots, A_k$ in $L^2(\mathbb{R}^n)$ such that for each suitably chosen function σ on G, the Weyl calculus allows for a sensible definition of $\sigma(A_1, \ldots, A_k)$ as an operator on $L^2(\mathbb{R}^n)$. Indeed an operator Q on $L^2(\mathbb{R})$ may be called *pseudo-differential* if $Q = \sigma(A_1, \ldots, A_k)$ for some suitably chosen σ. Given a Lévy process $\rho(t)$ on G one defines the semigroup $\mathcal{T}_t^\pi$ and its infinitesimal generator $\mathcal{L}^\pi$ as in the Heisenberg group case.

These techniques are quite general and it appears that they can be applied to arbitrary nilpotent groups. In section 3 of this volume we utilize Kirillov's method of coadjoint orbits to determine a complete set of representations of the simplest step 3 nilpotent Lie group. These representations are then used in section 4 to formulate the Weyl functional calculus for this group. In section we prove the following main result

Theorem 1.3. *Let G be a Lie group with step 3 nilpotent Lie algebra $\mathfrak{g}$ generated by W, X, Y and Z with the following commutation relations:*

$$[W, X] = Y \qquad \textit{and} \qquad [W, Y] = Z,$$

and let $\rho(t)$ be a Lévy process in G, and let π be an irreducible unitary representation of G. The quantized generator $\mathcal{L}^\pi$ is a pseudo-differential operator which is densely defined in $L^2(\mathbb{R})$.

2. A Step 3 Nilpotent Lie Group

Let G denote $\mathbb{R}^4$ with the multiplication law

$$\{w_1, x_1, y_1, z_1\} * \{w_2, x_2, y_2, z_2\} = \Big\{w_1 + w_2,\ x_1 + x_2,\ y_1 + y_2 + w_1 x_2,\ z_1 + z_2 + w_1\left(y_2 + \frac{w_1 x_2}{2}\right)\Big\}. \tag{2.1}$$

Proposition 2.1. *(Characterization of G)*

With respect to the multiplication law in (2.1) G is a Lie group with identity $\{0,0,0,0\}$ *and inversion given by*

$$\{w, x, y, z\}^{-1} \;=\; \Big\{-w,\ -x,\ -y + wx,\ -z + w\left(y - \frac{wx}{2}\right)\Big\}.$$

Proof. Clearly G is closed under $*$. We verify associativity:

$$(\{w_1, x_1, y_1, z_1\} * \{w_2, x_2, y_2, z_2\}) * \{w_3, x_3, y_3, z_3\} =$$

$$\begin{aligned}
&= \Big\{w_1 + w_2,\ x_1 + x_2,\ y_1 + y_2 + w_1 x_2,\ z_1 + z_2 + w_1\left(y_2 + \frac{w_1 x_2}{2}\right)\Big\} * \{w_3, x_3, y_3, z_3\} \\
&= \Big\{(w_1 + w_2) + w_3,\ (x_1 + x_2) + x_3,\ (y_1 + y_2 + w_1 x_2) + y_3 + (w_1 + w_2)\, x_3, \\
&\qquad \left(z_1 + z_2 + w_1\left(y_2 + \frac{w_1 x_2}{2}\right)\right) + z_3 + (w_1 + w_2)\left(y_3 + \frac{(w_1 + w_2) x_3}{2}\right)\Big\} \\
&= \Big\{w_1 + (w_2 + w_3),\ x_1 + (x_2 + x_3),\ y_1 + (y_2 + y_3 + w_2 x_3) + w_1(x_2 + x_3), \\
&\qquad z_1 + \left(z_2 + z_3 + w_2\left(y_3 + \frac{w_2 x_3}{2}\right)\right) + w_1\left((y_2 + y_3 + w_2 x_3) + \frac{w_1(x_2 + x_3)}{2}\right)\Big\} \\
&= \{w_1, x_1, y_1, z_1\} * (\{w_2, x_2, y_2, z_2\} * \{w_3, x_3, y_3, z_3\}).
\end{aligned}$$

We have that

$$\{w, x, y, z\} * \{0,0,0,0\} = \{w, z, y, z\} = \{0,0,0,0\} * \{w, x, y, z\}$$

and

$$\{w,x,y,z\} * \left\{-w,\ -x\ -y+wx,\ -z+w\left(y-\frac{wx}{2}\right)\right\} =$$

$$\begin{aligned} &= \Big\{w+(-w),\ x+(-x),\ y+(-y+wx)+w(-x), \\ &\quad z+\left(-z+w\left(y-\frac{wx}{2}\right)\right)+w\left((-y+wx)+\frac{w(-x)}{2}\right)\Big\} \\ &= \{0,0,0,0\}. \end{aligned}$$

Therefore G is a group. We have that G is a manifold with a global chart $\Phi : \mathbb{R}^4 \to G$ given by $\Phi(w,x,y,z) = \{w,x,y,z\}$. We equip G with the standard topology on $\mathbb{R}^4$, realized as the topology induced by Φ. Define $\mu : G \times G \to G$ by $\mu(g_1,g_2) = g_2^{-1} * g_1$ for all $g_1, g_2 \in G$. The product space is equipped with product topology, and

$$\begin{aligned} &\mu\left(\{w_1,x_1,y_1,z_1\},\{w_2,x_2,y_2,z_2\}\right) \\ &= \left\{-w_2,-x_2,-y_2+w_2x_2,-z_2+w_2\left(y_2-\frac{w_2x_2}{2}\right)\right\} * \{w_1,x_1,y_1,z_1\} \\ &= \left\{w_1-w_2,\ x_1-x_2,\ y_1-y_2+w_2x_2,\ z_1-z_2+w_2\left(y_2-\frac{w_2x_2}{2}\right)-w_2\left(y_1-\frac{w_2x_1}{2}\right)\right\}. \end{aligned}$$

Thus μ is a polynomial map from $\mathbb{R}^8$ into $\mathbb{R}^4$, and as such, is smooth. Therefore G is a Lie group. $\square$

Because both the underlying manifold of G and $\mathfrak{g}$ are $\mathbb{R}^4$, we adopt the convention of $\{w,x,y,z\}$ when referring to a point in G and (w,x,y,z) when referring to a point in $\mathfrak{g}$.

We have the following characterization of the Lie algebra of G.

Proposition 2.2.

(1) *The collection of left-invariant vector fields of G is spanned by the following set:*

$$W = \frac{\partial}{\partial w} \qquad X = \frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + \frac{w^2}{2}\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad Z = \frac{\partial}{\partial z}.$$

Moreover, the nonzero commutators of these vector fields are given by

$$[W, X] = Y \qquad [W, Y] = Z.$$

(2) *The linear span of $\{W, X, Y, Z\}$ is a vector space $\mathfrak{g}$ which is a Lie algebra with respect to the bracket operation*

$$[W, X] = Y \qquad [W, Y] = Z.$$

(3) *The Lie algebra $\mathfrak{g}$ is step 3 nilpotent.*

(4) *The exponential map $\exp : \mathfrak{g} \to G$ is given by*

$$\exp(w, x, y, z) = \left\{w,\ x,\ y + \frac{wx}{2},\ z + \frac{wy}{2} + \frac{w^2 x}{6}\right\}. \tag{2.2}$$

Proof. Let $f \in C^1(G)$. We have

$$\begin{aligned}
&\lim_{t\to 0} \frac{f(\{w, x, y, z\} * \{t, 0, 0, 0\}) - f(\{w, x, y, z\})}{t} \\
&= \lim_{t\to 0} \frac{f(\{w + t, x, y, z\}) - f(\{w, x, y, z\})}{t} \\
&= \frac{\partial f}{\partial w}
\end{aligned}$$

$$
\begin{aligned}
&\lim_{t\to 0}\frac{f(\{w,x,y,z\}*\{0,t,0,0\})-f(\{w,x,y,z\})}{t}\\
&=\lim_{t\to 0}\frac{f(\{w,x+t,y+wt,z+\frac{w^2t}{2}\})-f(\{w,x,y,z\})}{t}\\
&=\frac{\partial f}{\partial x}+w\frac{\partial f}{\partial y}+\frac{w^2}{2}\frac{\partial f}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\to 0}\frac{f(\{w,x,y,z\}*\{0,0,t,0\})-f(\{w,x,y,z\})}{t}\\
&=\lim_{t\to 0}\frac{f(\{w,x,y+t,z+wt\})-f(\{w,x,y,z\})}{t}\\
&=\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\to 0}\frac{f(\{w,x,y,z\}*\{0,0,0,t\})-f(\{w,x,y,z\})}{t}\\
&=\lim_{t\to 0}\frac{f(\{w,x,y,z+t\})-f(\{w,x,y,z\})}{t}\\
&=\frac{\partial f}{\partial z}
\end{aligned}
$$

We have that

$$
\begin{aligned}
[W,X]&=\frac{\partial}{\partial w}\left(\frac{\partial}{\partial x}+w\frac{\partial}{\partial y}+\frac{w^2}{2}\frac{\partial}{\partial z}\right)-\left(\frac{\partial}{\partial x}+w\frac{\partial}{\partial y}+\frac{w^2}{2}\frac{\partial}{\partial z}\right)\frac{\partial}{\partial w}\\
&=\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\\
&=Y
\end{aligned}
$$

and also that

$$\begin{aligned}[W, Y] &= \frac{\partial}{\partial w}\left(\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) - \left(\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\frac{\partial}{\partial w} \\ &= \frac{\partial}{\partial z} \\ &= Z.\end{aligned}$$

Thus (1) is proven, and (2) is clear. To prove (3), write

$$\begin{aligned}&[(w_1, x_1, y_1, z_1), [(w_2, x_2, y_2, z_2), [(w_3, x_3, y_3, z_3), (w_4, x_4, y_4, z_4)]]] \\ &\qquad = [(w_1, x_1, y_1, z_1), [(w_2, x_2, y_2, z_2), (0, 0, w_3x_4 - w_4x_3, w_3y_4 - w_4y_3)]] \\ &\qquad = [(w_1, x_1, y_1, z_1), (0, 0, 0, w_2(w_3x_4 - w_4x_3))] \\ &\qquad = 0.\end{aligned}$$

Thus (3) is proven. To verify (4) we write the following

$$\exp(w, 0, 0, 0) = \{w, 0, 0, 0\}$$

$$\exp(0, x, 0, 0) = \{0, x, 0, 0\}$$

$$\exp(0, 0, y, 0) = \{0, 0, y, 0\}$$

$$\exp(0, 0, 0, z) = \{0, 0, 0, z\}$$

then the formula (2.2) follows from a standard application of the Baker-Campbell-Hausdorff-Dynkin formula (refer to Appendix D for details). □

Any Lie group naturally acts on its Lie algebra via the adjoint representation. We have the following proposition.

Proposition 2.3. *The adjoint representation of G on $\mathfrak{g}$ is given by*

$$Ad(\{w,x,y,z\})(a,b,c,d) = \left(a,\ b,\ c+(wb-ax),\ d+(wc-ay)+\frac{w^2b}{2}\right).$$

Proof.

$$\begin{aligned}
&\mathrm{Ad}(\{w,x,y,z\})(a,b,c,d)\\
&:= \frac{\mathrm{d}}{\mathrm{d}t}\left(\{w,x,y,z\}*\exp(t(a,b,c,d))*\{w,x,y,z\}^{-1}\right)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\{w,x,y,z\}*\left\{ta,\ tb,\ tc+\frac{t^2ab}{2},\ td+\frac{t^2ac}{2}+\frac{t^3a^2b}{6}\right\}\\
&\quad *\left\{-w,\ -x,\ -y+wx,\ -z+w\left(y-\frac{wx}{2}\right)\right\}\Bigg)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\{w,x,y,z\}*\{ta,\ tb,\ tc,\ td\}*\left\{-w,\ -x,\ -y+wx,\ -z+w\left(y-\frac{wx}{2}\right)\right\}\right)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\left\{w+ta,\ x+tb,\ y+t(c+wb),\ z+td+w\left(tc+\frac{twb}{2}\right)\right\}\\
&\quad\quad *\left\{-w,\ -x,\ -y+wx,\ -z+w\left(y-\frac{wx}{2}\right)\right\}\Bigg)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\Bigg\{ta,\ tb,\ t(c+wb)+wx+(w+ta)(-x),\\
&td+w\left(tc+\frac{twb}{2}\right)+w\left(y-\frac{wx}{2}\right)+(w+ta)\left((-y+wx)+\frac{(w+ta)(-x)}{2}\right)\Bigg\}\Bigg)\Big|_{t=0}\\
&= \left(a,\ b,\ c+(wb-ax),\ d+(wc-ay)+\frac{w^2b}{2}\right)
\end{aligned}$$

∎

Let $\mathfrak{g}^*$ denote the linear dual of $\mathfrak{g}$. The adjoint action induces the *co-adjoint action of G on $\mathfrak{g}^*$*, defined for each $l \in \mathfrak{g}^*$ as

$$\mathrm{Ad}^*(\{w,x,y,z\})\left(l(a,b,c,d)\right) = l(\mathrm{Ad}(\{w,x,y,z\}^{-1})(a,b,c,d)).$$

In the following sections we will make use of unitary irreducible representations of G. Classifications of these representations for nilpotent Lie groups follows from Kirillov's "method of co-adjoint orbits," as stated in Theorem 3.1. To this end we must identify the co-adjoint orbits of G in $\mathfrak{g}^*$. If

$$l(a,b,c,d) = \alpha a + \beta b + \gamma c + \delta d$$

for $(a,b,c,d) \in \mathfrak{g}$ then we will adopt the convention of writing $l = [\alpha, \beta, \gamma, \delta]$. Utilizing these coordinates, we have the following proposition.

Proposition 2.4. *The co-adjoint Representation of G on $\mathfrak{g}^*$ is given by*

$$Ad^*(\{w,x,y,z\})[\alpha,\beta,\gamma,\delta] = \left[\alpha + x\gamma + (y-wx)\delta,\ \beta - w\gamma + \frac{w^2\delta}{2},\ \gamma - w\delta,\ \delta\right].$$

Proof.

$$\begin{aligned}
\mathrm{Ad}^*(\{w,x,y,z\})[\alpha,\beta,\gamma,\delta](a,b,c,d) &= [\alpha,\ \beta,\ \gamma,\ \delta](\mathrm{Ad}(\{w,x,y,z\}^{-1})(a,b,c,d)) \\
&= [\alpha,\ \beta,\ \gamma,\ \delta]\left(\mathrm{Ad}\left(\left\{-w,-x,-y+wx,-z+w\left(y-\frac{wx}{2}\right)\right\}\right)(a,b,c,d)\right) \\
&= [\alpha,\ \beta,\ \gamma,\ \delta]\left(a,\ b,\ c+(ax-bw),\ d+(ay-cw)-awx+\frac{w^2b}{2}\right) \\
&= \alpha a + \beta b + \gamma\,(c+(ax-bw)) + \delta\left(d+(ay-cw)-awx+\frac{w^2b}{2}\right) \\
&= (\alpha + \gamma x + \delta(y-wx))\,a + \left(\beta - \gamma w + \frac{\delta w^2}{2}\right) b + (\gamma - \delta w)\,c + \delta d
\end{aligned}$$

$\square$

3. Representation Theory of G

To form a complete set of unitary irreducible representations of G, we make use of the fact that G is nilpotent. The following result, due to Kirillov is presented in Section 2.2 of [4].

Theorem 3.1. *(Kirillov) Let K be any locally compact, simply connected connected nilpotent Lie group with Lie algebra $\mathfrak{k}$.*

(1) *If $l \in \mathfrak{k}^*$ then there exists a subalgebra $\mathfrak{m}_l$ of $\mathfrak{k}$ of maximal dimension such that $l([m_1, m_2]) = 0$ for all $m_1, m_2 \in \mathfrak{m}_l$.*
(2) *$M_l = \exp(\mathfrak{m}_l)$ is a closed subgroup of K and $\rho_l(\exp(m)) = e^{2\pi i l(m)}$ is one dimensional representation of M_l.*
(3) *$Ind^K_{M_l,\rho_l}$ is a unitary irreducible representation of K.*
(4) *If π is any unitary irreducible representation of K, then there exists $l \in \mathfrak{k}^*$ such that π is unitarily equivalent to $Ind^K_{M_l,\rho_l}$. (Refer to Appendix E for elementary theory of induced representations).*
(5) *Two irreducible representations $\pi_1 = Ind^K_{M_{l_1},\rho_{l_1}}$ and $\pi_2 = Ind^K_{M_{l_2},\rho_{l_2}}$ are unitarily equivalent if and only if l_1 and l_2 are elements of the same coadjoint orbit of K in $\mathfrak{k}^*$.*

If l and $\mathfrak{m}_l$ are as in Theorem 3.1, then the subalgebra $\mathfrak{m}_l$ is said to be a *maximal subordinate algebra* for l.

Theorem 3.1 implies that the set of unitary irreducible representations of G is indexed by the set of co-adjoint orbits of G in $\mathfrak{g}^*$. The coadjoint action described in Proposition 2.4 allows for an explicit parametrization of these orbits. This parametrization can be used to give an explicit expression of unitary dual of G, as presented in the following proposition. This calculation can be found in [4, 9], but we include it here for completeness.

Proposition 3.2. *If π is a unitary irreducible representation of G then π is unitarily equivalent to a representation of one of the following types:*

I. *π is a unitary character of G given by*

$$\pi(\{w,x,y,z\})(z) = e^{2\pi i(\alpha w+\beta x)}z$$

for some $\alpha,\beta \in \mathbb{R}$ and any $z \in \mathbb{C}$.

II. *π is a representation on $L^2(\mathbb{R})$ given by*

$$\pi(\{w,x,y,z\})f(k) = e^{2\pi i\gamma\left(y+\frac{kx}{2}\right)}f(k+w)$$

for some $\gamma \in \mathbb{R}$.

III. *π is a representation on $L^2(\mathbb{R})$ given by*

$$\pi(\{w,x,y,z\})f(k) = e^{2\pi i\left(\beta x+\delta\left(z+k\left(y+\frac{kx}{2}\right)\right)\right)}f(k+w)$$

where $\delta \in \mathbb{R}^\times$, $\beta \in \mathbb{R}$.

Proof. If $[\alpha,\beta,\gamma,\delta] \in \mathfrak{g}^*$ and $\{w,x,y,z\} \in G$ then $\pi_{[\alpha,\beta,\gamma,\delta]}(\{w,x,y,z\})$ can be computed by considering some individual cases.

Type I: $(\delta=\gamma=0)$. In this case $\mathrm{Ad}^*(w,x,y,z)[\alpha,\beta,0,0] = [\alpha,\beta,0,0]$ for all w,x,y,z. These are 1 point orbits determined by α and β. The maximal subordinate algebra corresponding to any such orbit is the entire Lie algebra $\mathfrak{g}$, since $[A,B] \in \mathrm{Span}\,\{Y,Z\}$ for each $A,B \in \mathfrak{g}$. Therefore $M_l = G$ and $G/M_l \cong 0$. For any point $\{w,x,y,z\} \in G$, we write

$$\{w,x,y,z\} = \exp\left(w,\ x,\ y-\frac{wx}{2},\ z-\frac{x}{2}\left(y-\frac{wx}{2}+\frac{w^2}{6}\right)\right)$$

and $\pi_{[\alpha,\beta,0,0]}$ is the one dimensional representation of G given in $\mathbb{C}$ as

$$\begin{aligned}\pi_{[\alpha,\beta,0,0]}\{w,x,y,z\}z &= e^{2\pi i[\alpha,\beta,0,0]\left(w,x,y-\frac{wx}{2},z-\frac{x}{2}\left(y-\frac{wx}{2}+\frac{w^2}{6}\right)\right)}z\\ &= e^{2\pi i(\alpha w+\beta x)}z,\end{aligned}$$

for each $z\in\mathbb{C}$.

Type II: $(\delta=0,\ \gamma\neq 0)$. In this case $\mathrm{Ad}^*(w,x,y,z)[\alpha,\beta,\gamma,0]=[\alpha+x\gamma,\beta-w\gamma,\gamma,0]$, and so

$$\mathrm{Ad}^*(G)[\alpha,\beta,\gamma,\delta]=\{[p,q,\gamma,0]:p,q\in\mathbb{R}\}.$$

These are 2-dimensional orbits parametrized by γ. For any such orbit, the unitary irreducible representations induced by elements of the orbit are all unitarily equivalent and so it suffices to choose a convenient representative. There is a one-to-one correspondence between the set

$$R_2=\{[0,0,\gamma,0]:\gamma\in\mathbb{R}^{\times}\}$$

and the collection of orbits of this type. Since $\gamma\neq 0$,

$$l_\gamma([W,X])=\gamma\neq 0$$

and so $\mathfrak{g}$ is not subordinate to $[0,0,\gamma,0]$. The three dimensional subalgebra $\mathfrak{m}=\mathrm{Span}\{X,Y,Z\}$ is Abelian and is therefore maximal subordinate to any element of $\mathfrak{g}^*$. The subgroup

$$M=\exp(\mathfrak{m})=\{\{w,x,y,0\}:w,x,y\in\mathbb{R}\}$$

and $G/M\cong\mathbb{R}$. As indicated in [11], $\pi_{[0,0,\gamma,0]}$ acts on

$$\begin{aligned}\mathscr{H}_\gamma=\Big\{f:G\to\mathbb{C}\,\Big|\,&f\in L^2(G/M)\text{ and}\\ &f(\exp(q)g)=e^{2\pi i l_\gamma(q)}f(g)\quad\text{for each } q\in\mathfrak{m}\text{ and } g\in G\ \Big\}.\end{aligned}$$

We have that Haar measure μ on G is given by $\mu(\exp(E)) = \Lambda(E)$ where Λ is Lebesgue measure on $\mathfrak{g}$, and so $\mathscr{H}_\pi := L^2(G/M, \mu) \cong L^2(\mathbb{R}, \Lambda)$. We have that

$$\begin{aligned}
&\pi_{[0,0,\gamma,0]}(\{w,x,y,z\})f(k)\\
&\qquad = f(\{k,0,0,0\} * \{w,x,y,z\})\\
&\qquad = f\left(\left\{k+w,\ x,\ y+kx,\ z+k\left(y+\frac{kx}{2}\right)\right\}\right)\\
&\qquad = f\left(\left\{0,\ x,\ y+\frac{kx}{2},\ z+k\left(y+\frac{kx}{2}\right)\right\} * \{k+w,0,0,0\}\right)\\
&\qquad = e^{2\pi i\gamma\left(y+\frac{kx}{2}\right)} f(k+w).
\end{aligned}$$

Type III: $(\delta \neq 0)$. We have that

$$\begin{aligned}
&\mathrm{Ad}^*(\{w,x,y,z\})[\alpha,\beta,\gamma,\delta]\\
&\qquad = \left[\alpha + x\gamma + (y-wx)\delta,\ \beta - w\gamma + \frac{w^2\delta}{2},\ \gamma - w\delta,\ \delta\right].
\end{aligned}$$

Defining $q = \gamma - w\delta$ we have that $w = \frac{\gamma - q}{\delta}$ and so

$$\begin{aligned}
&\mathrm{Ad}^*(\{w,x,y,z\})[\alpha,\beta,\gamma,\delta]\\
&\qquad = \left[\alpha + x\gamma + (y-wx)\delta,\ \left(\beta - \frac{\gamma^2}{2\delta}\right) + \frac{q^2}{2\delta},\ q,\ \delta\right].
\end{aligned}$$

Hence

$$\mathrm{Ad}^*(G)[\alpha,\beta,\gamma,\delta] = \left\{\left[p,\ \left(\beta - \frac{\gamma^2}{2\delta}\right) + \frac{q^2}{2\delta},\ q,\ \delta\right] : p,q \in \mathbb{R}\right\}.$$

These orbits are 2-dimensional parabolic cylinders parametrized by δ and the quantity $\beta - \frac{\gamma^2}{2\delta}$. As in the previous case we have that

$$R_3 = \{[0,\beta,0,\delta] : \delta \in \mathbb{R}^\times, \beta \in \mathbb{R}\}$$

is a collection of orbit representatives and $M = \mathrm{Span}\{X, Y, Z\}$ is a maximal subordinate subalgebra for each representative. Therefore, $\mathscr{H}_{\beta,\delta} = L^2(\mathbb{R})$ and

$$\begin{aligned}
&\pi_{[0,\beta,0,\delta]}(\{w, x, y, z\})f(k) \\
&\qquad = f\left(\{k, 0, 0, 0\} * \{w, x, y, z\}\right) \\
&\qquad = f\left(\left\{0,\ x,\ y + \frac{kx}{2},\ z + k\left(y + \frac{kx}{2}\right)\right\} * \{k + w, 0, 0, 0\}\right) \\
&\qquad = e^{2\pi i\left(\beta x + \delta\left(z + k\left(y + \frac{kx}{2}\right)\right)\right)} f(k + w).
\end{aligned}$$

$\square$

4. The Weyl Functional Calculus for G

In Euclidean space, there is a well-developed theory of pseudo-differential operators and the corresponding symbolic calculus (see, for example [12]). The classical Weyl functional calculus provides an analogous construction for the simplest step-2 nilpotent case. A functional calculus for general connected and simply connected nilpotent groups has been developed in [3]. We will characterize this functional calculus for G, and begin by stating the general construction for arbitrary nilpotent groups.

Definition 4.1. As above, let K be an n dimensional locally compact nilpotent Lie group with corresponding Lie algebra $\mathfrak{k}$.

(1) Let $\xi_0 \in \mathfrak{k}^*$ with corresponding co-adjoint orbit $\mathscr{O}$. The *isotropy group of K at ξ_0* is $K_{\xi_0} := \{k \in K | \mathrm{Ad}^*(k)\xi_0 = \xi_0\}$.

(2) K_{ξ_0} is a Lie group with corresponding *isotropy Lie algebra*

$$\mathfrak{k}_{\xi_0} = \{X \in \mathfrak{k} | \xi_0 \circ \mathrm{ad}(\mathfrak{k})X = 0\}.$$

(3) Fix a sequence of ideals in $\mathfrak{k}$,

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_n = \mathfrak{k}$$

such that $\dim(\mathfrak{k}_j/\mathfrak{k}_{j-1}) = 1$ and $[\mathfrak{k}, \mathfrak{k}_j] \subset \mathfrak{k}_{j-1}$ for $j = 1, \ldots, n$. Pick any $X_j \in \mathfrak{k}_j \setminus \mathfrak{k}_{j-1}$ for $j = 1, \ldots, n$ so that the set $\{X_1, \ldots, X_n\}$ is a *Jordan-Hölder basis* in $\mathfrak{k}$.

(4) Consider the set of jump indices of the coadjoint orbit $\mathscr{O}$ with respect to the Jordan-Hölder basis,

$$\begin{aligned} J_{\xi_0} &= \{j \in \{1, \ldots, n\} | \mathfrak{k}_j \not\subseteq \mathfrak{k}_{j-1} + \mathfrak{k}_{\xi_0}\} \\ &= \{j \in \{1, \ldots, n\} | X_j \not\subseteq \mathfrak{k}_{j-1} + \mathfrak{k}_{\xi_0}\} \end{aligned}$$

and then define the corresponding *predual of the coadjoint orbit* $\mathscr{O}$,

$$\mathfrak{k}_e := \operatorname{Span}\{X_j | j \in J_{\xi_0}\}.$$

(5) The *Fourier transform* $\mathscr{S}(\mathscr{O}) \to \mathscr{S}(\mathfrak{g}_e)$ is given by the formula:

$$P \in \mathfrak{g}_e \qquad\qquad \hat{a}(P) = \int_{\mathscr{O}} e^{-i\langle \xi, P\rangle} a(\xi)\mathrm{d}\xi.$$

where $\mathrm{d}\xi$ is Liouville measure on $\mathscr{O}$.

(6) The *Weyl calculus* $\mathrm{Op}^{\pi}(\cdot)$ for the unitary representation π is defined for every $a \in \mathscr{S}(\mathscr{O})$ by

$$\mathrm{Op}^{\pi}(a) = \int_{\mathfrak{k}_e} \hat{a}(V)\pi(\exp_K V)\mathrm{d}V,$$

where $\hat{a}(V)$ is the *Fourier transform* of $a \in \mathscr{S}(\mathscr{O})$. The operator $\mathrm{Op}^{\pi}(a)$ is called the *pseudo-differential operator* with symbol a.

The following result appears in [3].

Theorem 4.2. *The Weyl calculus has the following properties:*

(1) *For every symbol $a \in \mathscr{S}(\mathscr{O})$ we have $Op^{\pi}(a) \in \mathscr{B}(\mathscr{H})_{\infty}$ (the space of smooth operators for the representation π) and the mapping*

$$\mathscr{S}(\mathscr{O}) \to \mathscr{B}(\mathscr{H})_{\infty} \qquad\qquad a \mapsto Op^{\pi}(a)$$

is a linear topological isomorphism.

(2) *For every $T \in \mathscr{B}(\mathscr{H})_{\infty}$ we have $T = Op^{\pi}(a)$ where $a \in \mathscr{S}(\mathscr{O})$ satisfies the condition $\hat{a}(V) = \mathrm{Tr}(\pi(\exp_K V)^{-1}A)$ for every $V \in \mathfrak{k}_e$.*

If π is a representation of the nilpotent group G, then π can be classified as in Proposition (3.2). If π is of type 1 or type 2 then $\mathrm{Op}^{\pi}(\cdot)$ is understood [11]. From above results one can compute the Weyl functional calculus for type 3 representations of G.

Proposition 4.3. *If π is the Type III irreducible unitary representation of G corresponding to the orbit $\mathscr{O}$ and $a \in \mathscr{S}(\mathscr{O})$ then the Fourier transform of a is given by*

$$\hat{a}\left(yY + wW\right) = \int_{\mathbb{R}^2} e^{-i(qy+pw)} a\left(q,p\right) \mathrm{d}q\mathrm{d}p$$

and the pseudo-differential operator $\mathrm{Op}^\pi(a)$ is given for each $f \in L^2(\mathbb{R}^2)$ by

$$\mathrm{Op}^\pi(a)f(k) = \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} e^{-i(qy+pw)} a(q,p)\mathrm{d}q\mathrm{d}p\right] e^{2\pi i(\delta ky + \frac{1}{2}\delta yw)} f(k+w)\mathrm{d}y\mathrm{d}w.$$

Proof. The basis $\{W, X, Y, Z\}$ is a Jordan-Hölder basis for G, and the predual of the co-adjoint orbit $\mathscr{O}$ is given by $\mathfrak{g}_e = \{W, Y\}$. The chart

$$\mathscr{O} \to \mathbb{R} \qquad\qquad pW^* + \left[\beta - \frac{q^2}{2\delta}\right] X^* + qY^* + \delta Z^* \mapsto (p,q)$$

is a map which brings Liouville measure on $\mathscr{O}$ to Lebesgue measure on $\mathbb{R}$. Direct substitution implies that the Fourier transform is given by

$$\hat{a}\left(yY + wW\right) = \int_{\mathbb{R}^2} e^{-i(qy+pw)} a\left(q,p\right) \mathrm{d}q\mathrm{d}p.$$

For $\pi(\{w,x,y,z\})f(k) = e^{2\pi i(\beta x + \delta(z + k(y + \frac{kx}{2})))} f(k+w)$ and $(w,0,y,0) \in \mathfrak{g}_e$ we have that

$$\pi(\exp(w,0,y,0))f(k) = \pi(\{w,0,y,0\})f(k) = e^{2\pi i(\delta(ky + \frac{k^2x}{2}))} f(k+w),$$

and direct substitution yields the result.

$\square$

5. Lévy Processes in G

The introductory material of this section follows from [2]. Suppose that K is an arbitrary (not necessarily nilpotent) Lie group with Lie algebra $\mathfrak{k}$. A Lévy process in K is a K-valued stochastic process $\rho = (\rho(t), t \geq 0)$ which satisfies the following:

(1) ρ has stationary and independent left increments, where the increment between s and t with $s \leq t$ is $\rho(s)^{-1}\rho(t)$.
(2) $\rho(0) = e$ a.s.
(3) ρ is stochastically continuous, i.e.

$$\lim_{s \to t} P(\rho(s)^{-1}\rho(t) \in A) = 0$$

for all $A \in \mathcal{B}(K)$ such that $e \notin \overline{A}$.

Let $C_0(K)$ be the Banach space (with respect to the supremum norm) of functions on K which vanish at infinity. Just as in the Euclidean case, one obtains a Feller semigroup on $C_0(K)$ by the prescription

$$T(t)f(\tau) = \mathbf{E}(f(\tau\rho(t))),$$

for each $t \geq 0$, $\tau \in K$, $f \in C_0(K)$ and its infinitesimal generator will be denoted as $\mathcal{L}$.

We fix a basis $\{Z_1, \ldots, Z_n\}$ for $\mathfrak{k}$ and define a dense subspace $C_2(K)$ of $C_0(K)$ as follows:

$$C_2(K) =$$

$$\{f \in C_0(K); Z_i^L(f) \in C_0(K) \text{ and } Z_i^L Z_j^L(f) \in C_0(K) \text{ for all } 1 \leq i, j \leq n\},$$

where Z^L denotes the left invariant vector field associated to $Z \in \mathfrak{k}$.

In [8], Hunt proved that there exist local coordinate functions $y_i \in C_2(K)$, $1 \le i \le n$ so that each

$$y_i(e) = 0 \text{ and } Z_i^L y_j(e) = \delta_{ij},$$

and a map $h \in \mathrm{Dom}(\mathcal{L})$ which is such that:

(1) $h > 0$ on $K - \{e\}$.
(2) There exists a compact neighborhood of the identity U such that for all $\tau \in U$,

$$h(\tau) = \sum_{i=1}^{n} y_i(\tau)^2.$$

Any such function is called a *Hunt function* in K. A positive measure ν defined on $\mathcal{B}(Q - \{e\})$ is called a *Lévy measure* whenever

$$\int_{Q-\{e\}} h(\sigma)\nu(\mathrm{d}\sigma) < \infty.$$

Theorem 5.1 (Hunt). *Let ρ be a Lévy process in K with infinitesimal generator $\mathcal{L}$ then,*

(1) $C_2(K) \subset \mathrm{Dom}(\mathcal{L})$.
(2) *For each $\tau \in K$, $f \in C_2(K)$*

$$\mathcal{L}(\tau) = \sum_{i=1}^{n} b_i Z_i^L f(\tau) + \sum_{i,j=1}^{n} c_{ij} Z_i^L Z_j^L f(\tau)$$

$$+ \int_{K-\{e\}} (f(\tau\sigma) - f(\tau) - \sum_{i=1}^{n} y_i(\sigma) Z_i^L f(\tau))\nu(\mathrm{d}\sigma), \tag{5.1}$$

where $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $c = (c_{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $K - \{e\}$.

Furthermore, any linear operator with a representation as in 5.1 is the restriction to $C_2(K)$ of a unique weakly continuous, convolution semigroup of probability measures in K.

Let $\mathscr{H}$ be a complex, separable Hilbert space and $U(\mathscr{H})$ be the group of unitary operators in $\mathscr{H}$. Let $\pi : K \to U(\mathscr{H})$ be a strongly continuous unitary representation of K in $\mathscr{H}$ and let $C^\infty(\pi) = \{\psi \in \mathscr{H}; k \to \pi(k)\psi \text{ is } C^\infty\}$ be the dense linear space of *smooth vectors for* π in $\mathscr{H}$. Define a strongly continuous contraction semigroup $\mathcal{T}_t$ of linear operators on $\mathscr{H}$ by

$$\mathcal{T}_t\psi = \mathbf{E}(\pi(\rho(t))\psi)$$

for each $\psi \in \mathscr{H}$. Let $\mathcal{L}^\pi$ denote the infinitesimal generator of this semigroup; i.e.

$$\mathcal{L}^\pi\psi = \lim_{t\to 0} \frac{\mathcal{T}_t\psi - \psi}{t}. \tag{5.2}$$

It follows from the work in [2] that $C^\infty(\pi) \subseteq \mathrm{Dom}(\mathcal{L}^\pi)$ and for $f \in C^\infty(\pi)$ we have

$$\begin{aligned} \mathcal{L}^\pi f = &\sum_{i=1}^n b_i \mathrm{d}\pi(Z_i) f + \sum_{i,j=1}^n c_{ij} \mathrm{d}\pi(Z_i)\mathrm{d}\pi(Z_j) f + \\ &+ \int_{K-\{e\}} \left(\pi(\sigma) - I - \sum_{i=1}^n y_i(\sigma)\mathrm{d}\pi(Z_i) \right) f \nu(\mathrm{d}\sigma). \end{aligned} \tag{5.3}$$

We now investigate $\mathcal{L}^\pi$ where $K = G$. Since G is nilpotent, the Haar measure $\mathrm{d}\sigma$ is related to Lebesgue measure on $\mathfrak{g}$ via the exponential map. Therefore it will be convenient to adopt exponential coordinates in G. To this end we impose the identification of (w, x, y, z) with $\exp(w, x, y, z)$. Fix real numbers β and $\delta \neq 0$. Let $\pi = \pi_{\delta,\beta}$ be a representation of the third type. Define

$$Kf(k) = kf(k) \qquad \text{and} \qquad Df(k) = \frac{1}{i}\frac{\mathrm{d}f}{\mathrm{d}k}.$$

We have that

$$\pi(w, x, y, z) f(k) = e^{2\pi i((\beta x + \delta(z + \frac{xy}{2} + \frac{w^2 x}{6}))I + (y + \frac{wx}{2})K + \frac{x}{2}K^2 + 2\pi i w D} f(k) \tag{5.4}$$

and

$$\begin{aligned} \mathrm{d}\pi(W) &= 2\pi i D \\ \mathrm{d}\pi(X) &= 2\pi i \beta I + \pi i K^2 \\ \mathrm{d}\pi(Y) &= 2\pi i \delta K \\ \mathrm{d}\pi(Z) &= 2\pi i \delta I. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{L}_1^\pi &= \sum_{i=1}^n b_i \mathrm{d}\pi(Z_i) \\ \mathcal{L}_2^\pi &= \sum_{i,j=1}^n c_{ij} \mathrm{d}\pi(Z_i)\mathrm{d}\pi(Z_j) \\ \mathcal{L}_3^\pi &= \int_{G-\{e\}} \left(\pi(\sigma) - I - \sum_{i=1}^n y_i(\sigma)\mathrm{d}\pi(Z_i) \right) f\nu(\mathrm{d}\sigma). \end{aligned}$$

We have that the drift part

$$\mathcal{L}_1^\pi = b_1(2\pi i \delta I) + b_2(2\pi i \delta K) + b_3(2\pi i \beta I + \pi i K^2) + b_4(2\pi i D). \tag{5.5}$$

With respect to the Weyl functional calculus expressed in Proposition 4.3, $\mathcal{L}_1^\pi$ is a pseudo-differential operator with symbol given by

$$\mathcal{S}_1^\pi = 2\pi i \delta b_1 + 2\pi i \delta b_2 t + b_3(2\pi i \beta + \pi i t^2) + 2\pi i b_4 \frac{\partial}{\partial t}.$$

The Brownian part can be expressed

$$
\begin{aligned}
\mathcal{L}_2^\pi = \; & c_{11}(-4\pi^2\delta^2 I) + c_{22}(-4\pi^2\delta^2 K^2) \\
& + c_{33}(-4\pi^2\beta^2 I - 4\pi^2\beta K^2 - \pi^2 K^4) \\
& + c_{44}(-4\pi^2 D^2) + 2c_{12}(-2\pi^2\delta^2 K) + 2c_{13}(-4\pi^2\delta\beta I - 2\pi^2\delta K^2) \\
& + 2c_{14}(-4\pi^2\delta D) + 2c_{23}(-4\pi^2\delta\beta K - 2\pi^2\delta K^3) + c_{24}(-4\pi^2\delta K D) \\
& + c_{34}(-4\pi^2\beta D - 2\pi^2 K^2 D) + c_{42}(-4\pi^2\delta(KD + I)) \\
& + c_{43}(-4\pi^2\beta D - 2\pi^2(2K + K^2 D)),
\end{aligned}
\tag{5.6}
$$

which is a pseudo-differential operator with symbol

$$
\begin{aligned}
\mathcal{S}_2^\pi = \; & -4\pi^2\delta^2 c_{11} - 4\pi^2\delta^2 c_{22} t^2 + c_{33}(-4\pi^2\beta^2 - 4\pi^2\beta t^2 - \pi^2 t^4) \\
& + c_{44}\left(-4\pi^2\frac{\partial^2}{\partial t^2}\right) - 4\pi^2\delta^2 c_{12} t + 2c_{13}(-4\pi^2\delta\beta - 2\pi^2\delta t^2) \\
& - 8\pi^2\delta c_{14}\frac{\partial}{\partial t} + 2c_{23}(-4\pi^2\delta\beta t - 2\pi^2\delta t^3) - 8\pi^2\delta c_{24} t\frac{\partial}{\partial t} \\
& + c_{34}\left(-4\pi^2\beta\frac{\partial}{\partial t} - 2\pi^2 t^2\frac{\partial}{\partial t}\right) - 4\pi^2\delta c_{42}\left(t\frac{\partial}{\partial t} + 1\right) \\
& + c_{43}\left(-4\pi^2\beta\frac{\partial}{\partial t} - 2\pi^2\left(2t + t^2\frac{\partial}{\partial t}\right)\right).
\end{aligned}
$$

Before expressing the jump part $\mathcal{L}_3^\pi$, observe that (5.4) can be rewritten as

$$\pi(w,x,y,z)f(k) = \exp(i\Phi(w,x,y,z))f(k)$$

where

$$\Phi(w,x,y,z) = 2\pi\left[\delta I\left(z+\frac{1}{2}w^2x+\frac{1}{2}wy\right)+\delta K\left(y+\frac{1}{2}wx\right)\right.$$
$$\left.+\left(\beta I+\frac{1}{2}K^2\right)(x)+wD\right]$$

is essentially self-adjoint. This form suggests the following choices for local coordinate functions:

$$\begin{aligned}
y_1(w,x,y,z) &= w\chi_B(w,x,y,z) \\
y_2(w,x,y,z) &= x\chi_B(w,x,y,z) \\
y_3(w,x,y,z) &= \left(y+\frac{1}{2}wx\right)\chi_B(w,x,y,z) \\
y_4(w,x,y,z) &= \left(z+\frac{1}{2}w^2x+\frac{1}{2}wy\right)\chi_B(w,x,y,z),
\end{aligned}$$

where $y_i(w,x,y,z) = y_i(\exp(w,x,y,z))$, $B = \exp(B(0,1))$. With respect to these local coordinate functions we have that

$$\mathcal{L}_3^\pi =$$
$$\int_{\mathbb{R}^4-\{0\}} \left(\pi(w,x,y,z) - I - i\Phi(w,x,y,z)\chi_B(w,x,y,z)\right)\nu(\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}w).$$

We have that $\mathcal{L}_3^\pi$ is a pseudo-differential operator with symbol

$$\mathcal{S}_3^\pi =$$
$$\int_{\mathbb{R}^4-\{0\}} \left(\tau(w,x,y,z) - I \quad - i\Theta(w,x,y,z)\chi_B(w,x,y,z)\right)\nu(\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}w).$$

where

$$\tau(w,x,y,z) = \exp(i\Theta(w,x,y,z))$$

for

$$\Theta(w,x,y,z) = 2\pi \left[\delta\left(z + \frac{1}{2}w^2 x + \frac{1}{2}wy\right) + \delta t\left(y + \frac{1}{2}wx\right) + \left(\beta + \frac{1}{2}t^2\right)(x) + w\frac{\partial}{\partial t}\right]$$

and π is as in (5.4). We are now ready to state the main theorem of this paper.

Theorem 5.2. *The operator $\mathcal{L}^\pi$ is a pseudo-differential operator. Moreover, $C_c^\infty(\mathbb{R})$ is a core for $\mathcal{L}^\pi$.*

Proof. We have that

$$\mathcal{L}^\pi = \mathcal{L}_1^\pi + \mathcal{L}_2^\pi + \mathcal{L}_3^\pi,$$

and consequently we have shown that $\mathcal{L}^\pi$ is pseudo-differential with symbol

$$\mathcal{S}^\pi = \mathcal{S}_1^\pi + \mathcal{S}_2^\pi + \mathcal{S}_3^\pi.$$

We write $\mathcal{L}_3^\pi = \mathcal{L}_{3,1}^\pi + \mathcal{L}_{3,2}^\pi$ with

$$\mathcal{L}_{3,1}^\pi = \int_{B^c} (\pi(w,x,y,z) - I)\, \nu(\mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}w)$$
$$\mathcal{L}_{3,2}^\pi = \int_{B-\{0\}} \Big(\pi(w,x,y,z) - I \quad - i\Phi(w,x,y,z)\Big)\, \nu(\mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}w).$$

For each $f \in C_c^\infty(\mathbb{R})$, we have that

$$\begin{aligned}\|\mathcal{L}_{3,1}^\pi f\| \leq &\int_{B^c} \|(\pi(w,x,y,z) - I)\, f\|\, \nu(\mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}w) \\ \leq &2\nu(B^c)\|f\|.\end{aligned}$$

Let $P(w,x,y,z)$ denote the projection-valued measure associated to the spectral decomposition of the self adjoint operator Φ. By the spectral theorem and Taylor's theorem, and referring again to (5.4) we see that

$$\begin{aligned}
&\|(\pi(w,x,y,z) - I - i\Phi(w,x,y,z))f\|^2 \\
&\qquad = \int_{\mathbb{R}^4} \left|e^{i\lambda} - 1 - i\lambda\right|^2 \|P(w,x,y,z)(\mathrm{d}\lambda)f\|^2 \\
&\qquad \leq \frac{1}{4}\int_{\mathbb{R}^4} |\lambda|^4 \|P(w,x,y,z)(\mathrm{d}\lambda)f\|^2 \\
&\qquad = \frac{1}{4}\left\| 2\pi \left[\delta I\left(z + \frac{1}{2}w^2x + \frac{1}{2}wy\right) + \delta K\left(y + \frac{1}{2}wx\right)\right.\right. \\
&\qquad\qquad \left.\left. + \left(\beta I + \frac{1}{2}K^2\right)(x) + wD\right]^2 f\right\|^2 \\
&\qquad \leq \pi^2 \left\| \left[y_1(w,x,y,z)D + y_2(w,x,y,z)\left(\beta I + \frac{1}{2}K^2\right)\right.\right. \\
&\qquad\qquad \left.\left. + y_3(w,x,y,z)\delta K + y_4(w,x,y,z)\delta I\right]^2 f\right\|^2 \\
&\qquad \leq 16\pi^2 C_f^2 \cdot h^2(w,x,y,z).
\end{aligned}$$

The last inequality follows from Young's inequality. The Hunt function h corresponds to the local coordinate functions $\{y_i\}_{i=1}^4$ and

$$\begin{aligned}
C_f =& ((\beta+\delta)^2 + \delta)\|f\| + (2\delta(\beta+\delta)+1)\|Kf\| + (\delta^2 + \beta + \delta)\|K^2 f\| \\
&+ \delta\|K^3 f\| + \frac{1}{4}\|K^4 f\| + 2(\beta+\delta)\|Df\| + 2\delta\|KDf\| + 2\|K^2 Df\| \\
&+ \|D^2 f\|.
\end{aligned}$$

Therefore we have that

$$\|\mathcal{L}_{3,2}^{\pi} f\| \leq 4\pi C_f \int_B h(w,x,y,z)\nu(\mathrm{d}w,\mathrm{d}x,\mathrm{d}y,\mathrm{d}z),$$

and the latter integral is finite by the defining characteristic of ν. Applying these bounds for $\mathcal{L}_{3,1}^{\pi}$ and $\mathcal{L}_{3,2}^{\pi}$ and the expressions (5.5) and (5.6) there exist non-negative constants $\omega(f)_{ij}$ such that

$$\|\mathcal{L}^{\pi} f\| \leq \sum_{i=1}^{4}\sum_{j=1}^{2} \omega(f)_{ij} \|K^i D^j f\|. \tag{5.7}$$

Let $f \in \mathrm{Dom}(\mathcal{L}^{\pi})$, then we can find $(f_n, n \in \mathbb{N})$ in $C_c^{\infty}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Applying (5.7) to the sequence $f_n - f_m$, we deduce by integration by parts and the Schwarz inequality that $\lim_{m,n\to\infty} \|\mathcal{L}^{\pi}(f_n - f_m)\| = 0$. Hence the sequence $(\mathcal{L}^{\pi} f_n, n \in \mathbb{N})$ is Cauchy and so convergent to some $g \in L^2(\mathbb{R})$. The operator $\mathcal{L}^{\pi}$ is closed, hence $g = \mathcal{L}^{\pi} f$ and the result is established.

□

Appendix A. Fourier Transforms and Pseudo-differential Operators in $\mathbb{R}^n$.

For a rigorous treatment of the material in this appendix, the reader may refer to [7]. We include some results relevant to the subject of this thesis, with proofs omitted.

Definition A.1. If $f : \mathbb{R}^n \to \mathbb{C}$ is any integrable function, then the *Fourier transform* $(\mathcal{F}f)(\xi)$ of f is defined by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\mathrm{d}x$$

for each $\xi \in \mathbb{R}^n$.

Proposition A.2. *If $f, g \in L^1(\mathbb{R}^n)$ then*

(1) *For any complex numbers a and b,*

$$(\mathcal{F}(af + bg))(\xi) = a(\mathcal{F}f)(\xi) + b(\mathcal{F}g)(\xi).$$

(2) *For any $x_0 \in \mathbb{R}^n$, if $h(x) = f(x - x_0)$ then*

$$(\mathcal{F}h)(\xi) = e^{2\pi i x_0 \xi}(\mathcal{F}f)(\xi).$$

(3) *For any $\xi_0 \in \mathbb{R}^n$, if $h(x) = e^{2\pi i \xi_0} f(x)$ then*

$$(\mathcal{F}h)(\xi) = (\mathcal{F}f)(\xi - \xi_0).$$

(4) *For any $a \in \mathbb{R}^\times$, if $h(x) = f(ax)$ then*

$$(\mathcal{F}h) = \frac{1}{|a|}(\mathcal{F}f)\left(\frac{\xi}{a}\right).$$

(5) *If* $h(x) = (f * g)(\xi) := \int_{\mathbb{R}^n} f(\xi - \tau)g(\tau)\mathrm{d}\tau$ *then*

$$(\mathcal{F}h)(\xi) = (\mathcal{F}f)(\xi) \cdot (\mathcal{F}g)(\xi).$$

In general, the Fourier transform of an integrable function is not Lebesgue integrable (for example, if $f(x) = I_{[-a,a]}$ then $(\mathcal{F}f)(\xi) = \frac{2\sin(a\xi)}{\xi}$). We will now identify a suitable subclass of integrable functions with integrable Fourier transforms. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be multi-indices ($\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}$ for all i), and let

$$x^\alpha f(x) := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} f(x)$$
$$D^\beta f(x) := \frac{\partial^{\beta_1}}{x_1^{\beta_1}} \frac{\partial^{\beta_2}}{x_2^{\beta_2}} \cdots \frac{\partial^{\beta_n}}{x_n^{\beta_n}} f(x)$$

for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ $f \in C^\infty(\mathbb{R}^n)$. Define the following family of norms indexed by α and β:

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

Definition A.3. The set of functions

$$\mathscr{S}(\mathbb{R}^n) = \left\{ f :\in C^\infty(\mathbb{R}^n) \,\middle|\, \|f\|_{\alpha,\beta} < \infty \text{ for all multi-indices } \alpha \text{ and } \beta \right\}.$$

is called the *Schwartz class* of functions on $\mathbb{R}^n$. We have that $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for each $1 \leq p$, and so in particular we may equip $\mathscr{S}(\mathbb{R}^n)$ with the standard norm of $L^2(\mathbb{R}^n)$.

Proposition A.4. *Suppose* $f \in \mathscr{S}(\mathbb{R}^n, x)$.

(1) $(\mathcal{F}f)(\xi) \in \mathscr{S}(\mathbb{R}^n, \xi)$.

(2) *(Inversion) The transformation* $\mathcal{F}^{-1} : L^1(\mathbb{R}^n, \xi) \to L^\infty(\mathbb{R}^n, x)$ *defined by*

$$(\mathcal{F}^{-1}g)(x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi i\xi\cdot x}\mathrm{d}\xi$$

satisfies that $(F^{-1}g)(x) \in \mathscr{S}(\mathbb{R}^n, x)$ *whenever* $g \in \mathscr{S}(\mathbb{R}^n, \xi)$ *and moreover*

$$(\mathcal{F}^{-1}(\mathcal{F}f))(x) = f(x).$$

(3) *(Plancherel Theorem)* $\mathcal{F} : \mathscr{S}(\mathbb{R}^n, x) \to \mathscr{S}(\mathbb{R}^n, \xi)$ *is a isometric linear isomorphism with respect to the standard norm on* $L^2(\mathbb{R}^n)$.

If $f \in \mathscr{S}(\mathbb{R}^n)$ then it can be easily shown by repeated integration by parts that

$$\mathcal{F}(D^\alpha f) = i^{|\beta|}\xi^\beta\mathcal{F}(f). \tag{A.1}$$

By taking the inverse Fourier transform of both sides of (A.1) we have that

$$D^\alpha f = \mathcal{F}^{-1}\left(i^{|\beta|}\xi^\beta\mathcal{F}(f)\right). \tag{A.2}$$

The function $\sigma_D = |i|^{|\beta|}\xi^\beta$ is said to be the *symbol* of the differential operator D. In principle, any function $\sigma(x, \xi)$ for which $\sigma(x, \xi)(\mathcal{F}f)(\xi) \in \mathscr{S}(\mathbb{R}^n, \xi)$ (for all x) corresponds to an operator $\mathscr{P}_\sigma : \mathscr{S}(\mathbb{R}^n, x) \to \mathscr{S}(\mathbb{R}^n, x)$ defined by

$$(\mathscr{P}_\sigma f)(x) := \mathcal{F}^{-1}(\sigma(x, \xi)(\mathcal{F}f)(\xi))(x) \tag{A.3}$$

Definition A.5. The operator $\mathscr{P}_\sigma$ is said to be a *pseudo-differential operator* with *symbol* $\sigma(x, \xi)$.

By restricting $\sigma(x, \xi)$ to functions of specified class, one can often make statements about the domain of $\mathscr{P}_\sigma$. Indeed, one commonly chosen class is the Hormänder class $\mathcal{H}$ defined by

$$\mathcal{H} = \bigcap_{\alpha,\beta}\mathcal{H}_{\alpha,\beta}$$

where α and β are multi-indices and

$$\mathcal{H}_{\alpha,\beta} = \left\{ \sigma(x,\xi) \;\middle|\; \left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \sigma(x,\xi) \right| \leq C_{\alpha,\beta}(1+|\xi|)^m \text{ for some } m \in \mathbb{R} \right\}.$$

We have the following result.

Proposition A.6. *If $\sigma(x,\xi) \in \mathcal{H}$ then $\mathscr{P}_\sigma : \mathscr{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.*

Refer to [12] for a comprehensive development of the theory of psuedo-differential operators in $\mathbb{R}^n$.

Appendix B. Lévy Processes in $\mathbb{R}^n$.

This appendix contains material from [1] and [5].

Definition B.1. A stochastic process $\rho(t)$ his said to be a Lévy process if

(1) $\rho(0) = 0$ almost surely.
(2) (Independent Increments) For any $0 \leq t_1, < t_2 < \cdots < t_n < \infty$ the random variables $\rho(t_2) - \rho(t_1)$, $\rho(t_3) - \rho(t_2)$,..., $\rho(t_n) - \rho(t_{n-1})$ are independent.
(3) (Stationary Increments) For any $s < t$, the random variable $\rho(t) - \rho(s)$ is equal in distribution to the random variable $\rho(t - s)$.
(4) ρ_t is almost surely right continuous with left limits.

Example B.1. The process $B(t)$ is said to be a Brownian motion if

(1) $B(0) = 0$ almost surely.
(2) $B(t)$ is almost surely continuous.
(3) $B(t)$ has independent increments as in B.1.2, and moreover for each s, $B(s)$ is a random variable with distribution $\mathcal{N}(0, s)$.

Brownian motion is perhaps the most well-known example of a Lévy process.

Example B.2. The process $P(t)$ is said to be a homogeneous Poisson process if

(1) $P(0) = 0$ almost surely.
(2) For each $0 \leq s \leq t < \infty$

$$\mathbf{P}(P(t) - P(s)) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!} \quad \text{with} \quad k = 0, 1, \ldots$$

The parameter λ is a non-negative real number, called the *intensity* of the process P. The Poisson process is another well-known example of a Lévy process.

One way to characterize a stochastic process is to compute its *characteristic function*. The characteristic function of a Lévy process $\rho(t)$ in $\mathbb{R}^n$ can be expressed by the Lévy-Khintchine formula

$$\mathbf{E}(e^{iu\cdot\rho(t)}) = e^{t\varphi(u)}$$

where for all $u \in \mathbb{R}^n$

$$\varphi(u) = i(m\cdot u) - \frac{1}{2}u\cdot Au + \int_{\mathbb{R}^n-\{0\}} \left(e^{iu\cdot y} - 1 - i\frac{u\cdot y}{1+|y|^2}\right)\nu(\mathrm{d}y), \tag{B.1}$$

with $m \in \mathbb{R}^n$, A an $n\times n$ non-negative symmetric matrix and ν is a measure on $\mathbb{R}^n - \{0\}$ satisfying

$$\int_{\mathbb{R}^n-\{0\}} (|y|^2 \wedge 1)\nu(\mathrm{d}y) < \infty.$$

Such a measure is called a *Lévy measure*.

Let $C_0(\mathbb{R}^n)$ denote the Banach space of continuous functions on $\mathbb{R}^n$ which vanish at infinity. We obtain a semigroup $(T(t), t\geq 0)$ on $C_0(\mathbb{R}^n)$ by defining

$$(T(t)f)(x) = \mathbf{E}(f(x+\rho(t))).$$

Define the *infinitesimal generator* of $\rho(t)$ to be the operator

$$\mathcal{A}_\rho f = \lim_{t\to 0}\frac{(T(t)-I)f}{t}$$

taking the domain of $\mathcal{A}_\rho$ to be the collection of f for which the above limit exists.

The following proposition appears in [2].

Proposition B.2. *$\mathcal{A}$ is a pseudo-differential operator of the form*

$$\mathcal{A} = \varphi(D).$$

Proof. We have the following Fourier inversion formula:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iu\cdot x} (\mathcal{F}f)(u) \mathrm{d}u.$$

Applying the definition

$$\begin{aligned}(T(t)f)(x) &= \mathbf{E}(f(x+\rho(t))) \\ &= \mathbf{E}\left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iu\cdot(x+\rho(t))} (\mathcal{F}f)(u) \mathrm{d}u\right).\end{aligned}$$

Applying Fubini and subsequently the Lévy-Khintchine formula we obtain

$$\begin{aligned}&= \int_{\mathbb{R}^n} e^{iu\cdot(x)} \mathbf{E}(e^{iu\cdot\rho(t)}) (\mathcal{F}f)(u) \mathrm{d}u \\ &= \int_{\mathbb{R}^n} e^{iu\cdot(x)} e^{t\varphi(u)} (\mathcal{F}f)(u) \mathrm{d}u.\end{aligned}$$

We have that by direct substitution the dominated convergence

$$\begin{aligned}\mathcal{A}_\rho f &= \lim_{t\to 0} \frac{(T(t)-I)f}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \left(\int_{\mathbb{R}^n} e^{iu\cdot(x)} e^{t\varphi(u)} (\mathcal{F}f)(u) \mathrm{d}u - \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iu\cdot x} (\mathcal{F}f)(u) \mathrm{d}u\right) \\ &= \int_{\mathbb{R}^n} e^{iu\cdot(x)} \left(\lim_{t\to 0} \frac{1}{t} \left(e^{t\varphi(u)} - 1\right)\right) (\mathcal{F}f)(u) \mathrm{d}u \\ &= \int_{\mathbb{R}^n} e^{iu\cdot(x)} \varphi(u) (\mathcal{F}f)(u) \mathrm{d}u \\ &= \varphi(D)f.\end{aligned}$$

□

Appendix C. Manifolds, Lie Groups and Lie Algebras.

Lie groups are manifolds that have group structure with a multiplication operation that is in some sense smooth. We begin this section with general information about manifolds and then proceed to elementary theory of Lie groups and Lie algebras. The material below, including all omitted proofs, can be found in greater detail in [10, Ch.1-4].

Definition C.1. Suppose that K is a topological space. K is a *topological manifold of dimension n* (or simply an *n-manifold*) if the following properties are satisfied:

- K is *Hausdorff*. For each points $k_1, k_2 \in K$ there exist open subsets U_1 and U_2 such that $k_1 \in U_1$, $k_2 \in U_2$ and $U_1 \cap U_2 = \varnothing$.
- K is *second-countable*. There exists a countable base for the topology of K.
- K is locally homeomorphic to $\mathbb{R}^n$. For every point $k \in M$ there exists an open neighborhood U_k of k and a homeomorphism $\varphi_k : U_k \to \mathbb{R}^n$ (called a local coordinate chart).

Suppose that $I \subseteq K$. If $\{U_k : k \in I\}$ covers K then the collection $\mathcal{A} = \{(U_k, \varphi_k) : k \in I\}$ is called an *atlas*. Suppose (U, φ_U) and (V, φ_V) are elements of $\mathcal{A}$ with $U \cap V \neq \varnothing$. The map $\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$ is called the *transition map* from U to V. If $\varphi_V \circ \varphi_U^{-1}$ is smooth for each U, V with $(U, \varphi_U), (V, \varphi_V) \in \mathcal{A}$ and $U \cap V \neq \varnothing$ then $\mathcal{A}$ is said to be a *smooth atlas*. The manifold K is said to be a *smooth manifold* if K has a smooth atlas.

Suppose k is a point on a smooth manifold K, that $k \in U_k \in \mathcal{K}$ and φ_k is the coordinate chart corresponding to U_k. Let $C^m(k)$ denote the set of complex valued functions f on K satisfying that $f \circ \varphi_m^{-1}$ has m continuous derivatives at $\varphi_k(k) \in \mathbb{R}^n$.

Any such function is said to be m times differentiable at the point k. The set

$$C^m(K) = \bigcap_{k \in K} C^m(k)$$

is the collection of functions of K which are smooth up to mth order, and

$$C^\infty(K) = \bigcap_{m \in \mathbb{N}} C^m(K)$$

is the collection of *smooth functions* on K.

Let K_1 and K_2 be smooth manifolds with respect to smooth atlases $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively and let $\Psi : K_1 \to K_2$. Let $(U, \varphi_U) \in \mathcal{K}_2$. For each $(V, \varphi_V) \in \mathcal{K}_1$ with $V \cap \Psi^{-1}(U) \neq \varnothing$, let $\Psi_{V,U} : \varphi_V(V \cap \Psi^{-1}(U)) \to \varphi_U(U \cap \Psi(V))$ defined by $\Psi_{V,U} = \varphi_U \circ \Psi \circ \varphi_V^{-1}$. Ψ is said to be *smooth* if $\Psi_{V,U}$ is smooth for each $(U, \varphi_U) \in \mathcal{K}_2$ and $(V, \varphi_V) \in \mathcal{K}_1$. If Ψ is a smooth bijective map from K_1 to K_2 with a smooth inverse, then Ψ is said to be a *diffeomorphism* and K_1 is said to be *diffeomorphic* to K_2 and one writes $K_1 \cong K_2$.

A linear map $X : C^\infty(K) \to \mathbb{R}$ satisfying

$$X(fg) = f(k)X(g) + g(k)X(f) \qquad f, g \in C^\infty(K)$$

is said to be a *derivation at* k. Let T_kK be the set of all such X. From [10, Ch. 3] we have that T_kK is a real linear space of dimension n. The set T_kK is the *tangent space of* K *at* k.

Suppose that $\Psi : K_1 \to K_2$ is a diffeomorphism, and let $k \in K_1$, $X \in T_kK_1$. If $f \in C^\infty(K_2)$ then $f \circ \Psi \in C^\infty(K_1)$ and $X[f \circ \Psi] \in C^\infty(K_1)$.

Proposition C.2. *For each $X \in T_k K_1$ and $f \in C^\infty(K_2)$ define $\Psi_* X f = X[f \circ \Psi] \circ \Psi^{-1} \in C^\infty(K_1)$. Then $\Psi_* X \in T_{\Psi(k)} K_2$ and the map $\Psi_* : T_k K_1 \to T_{\Psi(k)} K_2$ is a vector space isomorphism.*

The map Ψ_* is called the *pushforward* of Ψ.

The disjoint union

$$TK := \coprod_{k \in K} T_k K$$

is the collection of pairs (k, X) with $k \in K$ and $X \in T_k K$. If K is a smooth n-manifold then there exists a natural topology on TK and a smooth atlas with respect to which TK is a smooth $2n$-manifold, called the *tangent bundle* of K. There is a natural map $A : TK \to K$ defined by $A[(k, X)] = k$. For each $k \in K$ define $B_k : A^{-1}(k) \to T_k K$ by $B_k[(k, X)] = X$.

A map (resp. smooth map) $X : K \to TK$ is called a *vector field (resp. smooth vector field)* if $X(k) \in (k, T_k K)$. For each vector field X and $f \in C^\infty(K)$, we define the function Xf at each point k by

$$Xf(k) = B_k(X(k))f(k).$$

From [10, Lemma 4.6] one can deduce the following proposition:

Proposition C.3. *If X is a smooth vector field and $f \in C^\infty(K)$ then $Xf \in C^\infty(K)$. Moreover, if Y is another smooth vector field then, for each $a, b \in \mathbb{R}$, $aX + bY$ is a smooth vector field defined by $(aX + Y)f(k) = a(Xf)(k) + b(Yf)(k)$ for each $k \in K$.*

This proposition allows for the following definition:

Definition C.4. If X, Y are smooth vector fields then the *Lie bracket* of X and Y, denoted by $[X, Y]$ is a smooth vector field defined for each $f \in C^\infty(K)$ as

$$[X, Y]f = X(Yf) - Y(Xf).$$

The properties of the Lie bracket are given in the next proposition, the proof of which is given as [10, Lemma 4.15].

Proposition C.5. *The Lie bracket satisfies each of the following properties for all vector fields X, Y and Z.*

(1) *Bilinearity: For each $a, b \in \mathbb{R}$*

$$\begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z] \qquad \textit{and} \\ [X, aY + bZ] &= a[X, Y] + b[X, Z]. \end{aligned}$$

(2) *Anti-symmetry:*

$$[X, Y] = -[Y, X].$$

(3) *Jacobi Identity:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We now define the notions of Lie group and Lie algebra.

Definition C.6. Suppose that G is a smooth manifold with group structure. G is a *Lie group* if the maps

- $\mu_g : G \to G$ given by $\mu_g(h) = gh$, and
- $i : G \to G$ given by $i(h) = h^{-1}$

are smooth for all $g \in G$.

It is a straightforward matter to prove the following elementary result:

Lemma C.7. *Let G be a smooth manifold with group structure. G is a Lie group if and only if*

$$\nu_g : G \to G \text{ given by } \nu_g(h) = gh^{-1}$$

is smooth for all $g \in G$.

A vector field X on G is said to be *left-invariant* if $(Xf)(e) = (X[f \circ \mu_g])(g)$ for each $g \in G$.

Proposition C.8. *If X and Y are smooth left-invariant vector fields on G then for each $a, b \in \mathbb{R}$ $aX + bY$ and $[X, Y]$ are a smooth left-invariant vector fields on G.*

Let $\mathfrak{g}$ denote the set of smooth left-invariant vector fields on G. Proposition C.8 implies that $\mathfrak{g}$ is closed under linear combinations and the Lie bracket. In this respect, $\mathfrak{g}$ is an algebra with linear combination given in the obvious way and multiplication given by $X \cdot Y := [X, Y]$. The algebra $\mathfrak{g}$ is said to be the *Lie algebra* corresponding to the Lie group G.

Proposition C.9. $\mathfrak{g} \cong T_eG$.

Appendix D. Nilpotent Lie Groups

The material in this section can be found in greater detail in [4] and [9].

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{g}_0 = \mathfrak{g}$, and for each $k \geq 1$ let $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$. One has the following *descending central series*:

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_k \supseteq \cdots .$$

If there exists some n such that $\mathfrak{g}_n = \{0\}$, and if $\mathfrak{g}_k \neq \{0\}$ for any $k < n$ then $\mathfrak{g}$ is said to be *step n nilpotent*. This is equivalent to the condition that the descending central series has finite length:

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \mathfrak{g}_{n-1} \supsetneq \mathfrak{g}_n = \{0\}.$$

A Lie group is said to be *nilpotent* if its corresponding Lie algebra is nilpotent.

If G is any connected (not necessarily nilpotent) Lie group with exponential map $\exp_G : \mathfrak{g} \to G$ and define

$$X * Y = \log(\exp X \cdot \exp Y), \qquad X, Y. \in \mathfrak{g}$$

This function is analytic and well-defined near $X = Y = 0$ and does not depend on the choice of locally isomorphic connected Lie group associated to the Lie algebra $\mathfrak{g}$. This product is given by a universal power series involving only commutators:

$$X * Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_i + q_i \in \mathbb{N} \\ 1 \leq i \leq n}} \left(\frac{\sum_{i=1}^{n} (p_i + q_i)^{-1}}{p_1! q_1! \cdots p_n! q_n!} \right. \\ \left. \times (\mathrm{ad} X)^{p_1} (\mathrm{ad} Y)^{q_1} \cdots (\mathrm{ad} X)^{p_n} (\mathrm{ad} Y)^{q_n} Y \right). \tag{D.1}$$

This expression is known as the Baker-Campbell-Hausdorff-Dynkin formula (BCHD). The low-order non-zero terms in (D.1) are well known:

$$\begin{aligned} X * Y =& X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] \\ & - \frac{1}{48}[Y, [X, [X, Y]]] - \frac{1}{48}[X, [Y, [X, Y]]] \\ & + \text{(commutators with five or more terms)}. \end{aligned}$$

The following theorem appears as Theorem 1.2.1 in [4].

Theorem D.1. *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$*

(a) $\exp : \mathfrak{g} \to G$ *is an analytic diffeomorphism.*

(b) *The Baker-Campbell-Hausdorff-Dynkin formula holds for all $X, Y \in \mathfrak{g}$.*

Indeed, if G is step m nilpotent then $(D.1)$ reduces to a finite sum. If G is finite dimensional with ordered basis $\{Z_1, \dots, Z_n\}$ then

$$(a_1 Z_1 + \cdots + a_n Z_n) * (b_1 Z_1 + \cdots + b_n Z_n) = c_1 Z_1 + \cdots + c_n Z_n$$

then for each $1 \leq j \leq n$ we have

$$c_j = p_j(a_1, \dots, a_n, b_1, \dots, b_n)$$

where p_j is a polynomial of degree no more than $m + 1$.

Appendix E. Induced Representations

Important in the calculations presented within this thesis are the *unitary irreducible representations* of nilpotent groups. When calculating these representations one makes use of *induced representations*. We present a short synopsis of this theory. The results below can be found in [11].

Let H be a closed subgroup of G, and let

$$M = G/H.$$

Suppose that M has a G-invariant measure $\mathrm{d}\mu$. Let π_H be a unitary representation of H on a Hilbert space $\mathcal{H}_H$. Let $\mathcal{H}_G$ denote the collection of measureable functions f on G with values in $\mathcal{H}_H$, satisfying that

$$f(gh) = \pi_H(h)[f(g)], \qquad h \in H,$$

and

$$\int_M \|f([g])\|^2_{\mathcal{H}_H} \mathrm{d}\mu([g]) < \infty,$$

where $[g]$ is the element of M corresponding to $g \in G$. Define the following inner product on $\mathcal{H}_G$.

$$(f_1, f_2)_{\mathcal{H}_G} = \int_M (f_1(x), f_2(x))_{\mathcal{H}_H} \mathrm{d}\mu(x). \tag{E.1}$$

Let Ind^G_{H,π_H} denote the representation of G on $\mathcal{H}_G$ defined by

$$\mathrm{Ind}^G_{H,\pi_H} f(x) = f(g^{-1}x), \qquad g, x \in G,\ f \in \mathcal{H}_G.$$

Proposition E.1. *With respect to the inner product in (E.1) the space $\mathcal{H}_G$ is a Hilbert space in which Ind^G_{H,π_H} is a unitary representation of G.*

References

1. David Applebaum, *Lévy processes and stochastic calculus*, second ed., Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, 2009. MR 2512800 (2010m:60002)
2. David Applebaum and Serge Cohen, *Lévy processes, pseudo-differential operators and Dirichlet forms in the Heisenberg group*, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), no. 2, 149–177. MR 2126741 (2006b:60097)
3. Ingrid Beltiţă and Daniel Beltiţă, *Magnetic pseudo-differential Weyl calculus on nilpotent Lie groups*, Ann. Global Anal. Geom. **36** (2009), no. 3, 293–322. MR 2544305 (2011b:22010)
4. Lawrence J. Corwin and Frederick P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Mathematics, vol. 18, Cambridge University Press, Cambridge, 1990, Basic theory and examples. MR 1070979 (92b:22007)
5. R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002, Revised reprint of the 1989 original. MR 1932358 (2003h:60001)
6. Gerald B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397028 (98c:43001)
7. ———, *Introduction to partial differential equations*, second ed., Princeton University Press, Princeton, NJ, 1995. MR 1357411 (96h:35001)
8. G. A. Hunt, *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264–293. MR 0079232 (18,54a)
9. A. A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Mathematics, vol. 64, The American Mathematical Society, Providence, 2004, Basic theory and examples.
10. John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003. MR 1930091 (2003k:58001)
11. Michael E. Taylor, *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs, vol. 22, American Mathematical Society, Providence, RI, 1986. MR 852988 (88a:22021)
12. ———, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics, vol. 100, Birkhäuser Boston Inc., Boston, MA, 1991. MR 1121019 (92j:35193)