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The Top Lyapunov Exponent of Symplectic Stochastic Differential Equations: Theory and Numerics

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ABSTRACT

We calculate the top Lyapunov exponent for solutions to linear stochastic differential equations with non-commuting drift and diffusion matrices. In particular we consider (1) a class of $\mathbb{R}^d$-valued stochastic differential equations arising in the study of the noisy harmonic oscillator (2) $\mathsf{Sp}(2, \mathbb{R})$-valued stochastic differential equations are considered. Additionally, numerical bounds are provided and simulation techniques are discussed with an example.

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The Top Lyapunov Exponent of Symplectic Stochastic Differential Equations: Theory and Numerics

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Notation

Throughout this thesis we will use the following notation:

- $M(n, \mathbb{R})$ denotes the set of $n \times n$ matrices with real entires
- A^T denotes the transpose of the matrix A
- $A^{i,j}$ denotes the i, j^{th} entry of matrix A
- $\mathrm{Ad}_A[B] = ABA^{-1}$
- W_t is one dimensional standard Brownian motion
- δW_t denotes the Stratonovich differential
- dW_t denotes the Itô differential
- log denotes the base e logarithm
- RDS means random dynamical system
- SDE means stochastic differential equation

Contents

Chapter 1

Introduction

Recently, the Lyapunov exponent of solutions to stochastic differential equations (SDEs) has arisen in many areas. For example, questions related to the propagation of waves through disordered structures, slowing down light by orders of magnitude [MS08], power system engineering [HVK12], medicine [SBvD12], ecosystems, climate change, and finance [Kue11]. In addition, there is an extensive body of mathematical literature devoted to stochastic equations and Lyapunov exponents both in $\mathbb{R}^d$ [BL85, GM89, Arn03, IL99, PW91, IN93, GMV11] and in non-linear spaces such as Lie groups [RM01, Lia04].

Despite its ubiquity and the wealth of the understanding regarding the Lyapunov exponent, it is difficult to calculate [Vis98]. In fact there are very few cases in which the Lyapunov exponent has been explicitly calculated. The goal of this work is to (1) calculate the Lyapunov exponent for vector and matrix valued stochastic differential equations (2) provide some numerical techniques to approximate the Lyapunov exponent of solutions to stochastic differential equations.

1.1 Background

It is well known that the dynamics of a deterministic autonomous linear system in $\mathbb{R}^d$,

$$\frac{d}{dt}x_t = Ax_t, \tag{1.1.1}$$

are completely described by the spectral theory of the matrix A. In particular, if $\{\gamma_i\}_{i=1}^n, \{V_i\}_{i=1}^m$ are the eigenvalues and corresponding eigenspaces of Equation (1.1.1), then for large enough t we have

$$x(t) \approx e^{Re(\gamma_i)t} \iff x_0 \in V_i \backslash V_{i+1}.$$

In 1892, Aleksandr Lyapunov characterized this algebraic concept in terms of dynamical systems. In particular, he studied the possible asymptotic exponential growth rates of Equation (1.1.1) by studying what we now call the **Lyapunov exponent** of the system Equation (1.1.1),

$$\gamma(x_0) = \lim_{t\to\infty} \frac{1}{t} \log \|x_t\|. \tag{1.1.2}$$

In the case of a linear system such as Equation (1.1.1), we have

$$\gamma(x_0) = Re(\gamma_i) \iff x_0 \in V_i \backslash V_{i+1}. \tag{1.1.3}$$

Furthermore, this definition Equation (1.1.2) can be applied to a much more general class of dynamical systems in order to study their stability. In this thesis we are interested in considering the Lyapunov exponent of solutions to stochastic differential equations.

Now let us formally perturb Equation (1.1.1) in the following way,

$$\frac{d}{dt}x_t = Ax_t + b(t, x_t)\xi_t, \tag{1.1.4}$$

where ξ_t is a Markov process. For most choices of the noise processes, ξ_t, Equation (1.1.4) may not have a useful, rigorous interpretation. And even in the case that Equation (1.1.4) is well defined, the analysis of such a system may be very difficult. But if we restrict ourselves to the case when ξ_t is pure white noise, we can interpret Equation (1.1.4) using the theory of Itô stochastic integration. In applications, the decision to interpret Equation (1.1.4) with Itô differential

$$dx_t = Ax_t\,dt + b(t, x_t)\,dW_t, \tag{1.1.5}$$

or with Stratonovich differential

$$\delta x_t = Ax_t\,dt + b(t, x_t)\,\delta W_t \tag{1.1.6}$$

is subtle and not universally agreed upon [MM12]. In both cases the rigorous interpretation is the appropriate corresponding stochastic integral equation. Both the Itô and Stratonovich stochastic integrals arise from the same Riemann sums approach to the construction of stochastic integrals. Itô integral corresponds to choosing left hand samples, and the Stratonovich integral corresponds to choosing middle samples points. In fact we can similarly define the stochastic integral for any choice of sample point $\alpha \in [0, 1]$. But these two choices are the most common because the Itô integral is a martingale and the Stratonovich integral has the standard chain rule. Now in general, it is agreed that the Itô stochastic integral best models processes that are inherently discrete. Whereas the Stratonovich integral is better suited towards the modeling of naturally occurring continuous processes. Without further comment, in this thesis we will use the Stratonovich interpretation of the white noise process.

Now consider the case where $b(x_t, t) = Bx_t$ for some constant matrix $B \in M(n, \mathbb{R})$. Hence our perturbed system is a homogeneous linear stochastic differential equation with multiplicative noise,

$$dx_t = Ax_t\,dt + Bx_t\,\delta W_t. \tag{1.1.7}$$

We will call A the **drift matrix**, B the **diffusion matrix**, and refer to A, B both as **coefficient matrices**. Naively, one might wonder if we can analyze Equation (1.1.7) using spectral analysis methods like for linear ODEs. Somewhat surprisingly, the answer is yes. In fact there is well developed analog of this spectral theory specifically for SDEs (and for random dynamical systems in general). The following famous theorem of Oceledets [Arn03] provides the necessary link.

1.1.1 Osceledets's multiplicative ergodic theorem

For a vector valued stochastic process X_t consider the expression for the Lyapunov exponent,

$$\gamma(X_0, \omega) = \lim_{t\to\infty} \frac{1}{t} \log \|X_t(\omega)\|\,. \tag{1.1.8}$$

This quantity depends on the initial condition, X_0. There is clearly a question of convergence. The multiplicative ergodic theorem of Osceledets (first proven in 1968) allows the theory of Lyapunov exponents to be rigorously applied to stochastic systems (and random dynamical systems in general). Here we state a modern version of this theorem for a very specific case. The multiplicative ergodic theorem is more general and has a much broader scope. For a more general version see Appendix B.

Theorem 1.1.1 (Oceledets, 1968)**.** *Consider the $\mathbb{R}^n$ valued linear stochastic differential equation $X_t = AX_t\,dt + BX_t\,\delta W_t$. Define,*

$$\gamma(X_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|X_t\| .$$

Then with probability one,

1. *$\gamma(\cdot)$ takes at most d many values. Call them $\{\gamma_1, \ldots, \gamma_d\}$ where $d \leq n$.*

2. *There exist nested subspaces $\{V_i\}_{i=1}^n$ of $\mathbb{R}^d$ such that*

$$V_d \subset \cdots \subset V_1 = \mathbb{R}^n$$

and

$$\gamma(X_0) = \gamma_i \iff X_0 \in V_i \backslash V_{i+1}$$

3. *$\gamma(X_0) = \mathbb{E}\gamma(X_0)$*

The largest value taken by the Lyapunov exponent (Equation (1.1.8)) is,

$$\gamma := \gamma_1 = \max_{X_0} \gamma(X_0) \tag{1.1.9}$$

is called the **top Lyapunov exponent** of the system. Understanding the relationship between the stability of Equation (1.1.1) and its perturbation Equation (1.1.6) is a common source of investigations. As γ is determined by the exponential growth rate of the system, it is related to several notions of the stability of the system. A common notion for the stability of solutions to SDEs is due to Khasminskii. We say X_t is *stochastically asymptotically stable* if

$$\lim_{X_0 \to 0} P\left(\lim_{t\to\infty} |X_t| \to 0\right) = 1.$$

It is true that the zero solution is stochastically asymptotically stable if and only if $\gamma < 0$.

1.2 Outline

This thesis is constructed of three main sections. In Chapter 2 an integral expression is calculated for the Lyapunov exponent of a class of $\mathbb{R}^2$-valued stochastic differential equations Equation (2.1.9). The main result is Theorem 2.3.1.

In Chapter 3 the Lyapunov exponent is calculated for a few specific $\mathsf{Sp}(2, \mathbb{R})$-valued stochastic differential equations. The main results are Proposition 3.3.1, Proposition 3.3.2 and Proposition 3.3.3.

In Chapter 4 we provide a technique to numerically approximate the Lyapunov exponent of systems similar to those in Chapter 2 and Chapter 3. The main result is Theorem 4.2.1. In addition, in Section 4.3 a specific example is discussed in detail and numerical simulations are carried out.

Chapter 2

Vector Valued SDEs

2.1 Background

We fix without further mention a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. In this work, a *stochastic process* is a collection of random variables $\{X_t : t \geq t\}$ taking values in either $\mathbb{R}^d$ or $M(n, \mathbb{R})$ the choice of which will be clear in context. A process is said to be adapted if $X_t \in \mathcal{F}_t$ for each $t \geq 0$. A *path* of a stochastic process is a function, $\omega(t)$ defined by

$$\omega(t) = X_t(\omega), \tag{2.1.1}$$

for a fixed $\omega \in \Omega$. In this work we will consider strong solutions to SDEs.

Consider the formal equation for the 1-dimensional harmonic oscillator with a white noise potential studied in [PW91, IL99, GMV11]

$$\varphi''(t) + \epsilon \dot{W}(t)\varphi(t) = -\varphi(t). \tag{2.1.2}$$

In [GMV11, Theorem 2] an integral expression for the top Lyapunov exponent is calculated.

If we define

$$X_t := \begin{pmatrix} \varphi(t) \\ \varphi'(t) \end{pmatrix},$$

then we can rigorously interpret this as the $\mathbb{R}^2$-valued SDE

$$\delta X_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_t \, dt + \epsilon \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X_t \, \delta W_t, \tag{2.1.3}$$

where W_t is a standard $\mathbb{R}$-valued Brownian motion. This Stratonovich stochastic differential equation is equivalent to the Itô stochastic differential equation,

$$dX_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_t \, dt + \epsilon \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X_t \, dW_t. \tag{2.1.4}$$

Notice that the drift and diffusion matrix do not commute. In general, if the drift and diffusion matrix of a linear stochastic differential equation do not commute there is no known technique to calculate or algorithm to approximate the Lyapunov exponent of the resulting system. In this case we will use some algebraic properties of the diffusion matrices to study this system.

The *real symplectic Lie algebra* of dimension $2n$ is

$$\mathsf{sp}(n,\mathbb{R}) := \{B \in M(2n,\mathbb{R}) : JBJ = B^T\} \tag{2.1.5}$$

where,

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{2.1.6}$$

The following three matrices in $\mathsf{sp}(2,\mathbb{R})$,

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.1.7}$$

form a linear basis for $\mathsf{sp}(2,\mathbb{R})$. Using this basis we can rewrite Equation (2.1.3) as

$$\delta X_t = E_1 X_t\, dt + \frac{\epsilon}{2}(E_3 - E_1) X_t\, \delta W_t. \tag{2.1.8}$$

Now we consider the following generalization of the noisy harmonic oscillator,Equation (2.1.3),

$$\delta X_t = \left(\sum_{i=1}^{3} a_i E_i\right) X_t\, dt + \left(\sum_{i=1}^{3} b_i E_i\right) X_t\, \delta W_t, \tag{2.1.9}$$

where $a_i, b_i \in \mathbb{R}$. The choice of Itô vs. Stratonovich differential does make a difference and this distinction can be quite subtle [MM12]. This generalization serves two purposes. First, it will allow us to consider perturbations of stable ODE systems (i.e., small b_i). Secondly, it will allow us compute the Lyapunov exponent in the case of some non-commuting drift and diffusion matrices. Define the **top Lyapunov exponent** of the system Equation (2.1.9),

$$\gamma(b_1, b_2, b_3, c_1, c_2, c_3) := \max_{X_0 \in \mathbb{R}^2} \lim_{t \to \infty} \frac{1}{t} \log \|X_t\| . \tag{2.1.10}$$

As discussed earlier, the convergence of this quantity is ensured by Osceledets's multiplicative ergodic theorem.

Calculating the Lyapunov exponent of systems with non commuting drift and diffusion matrices is an active area of research. In 1979 the mathematician John Kingman was studying similar ergodic properties of discrete time difference equations. When considering non commuting drift and diffusion matrices he wrote that "pride of place among the unsolved problems of subadditive ergodic theory must go to the calculation of the constant γ". For our current system Equation (2.1.9) we have

$$[aE_1 + bE_2 + cE_3, -dE_1 + dE_3] = \begin{pmatrix} 2(a+c)d & 0 \\ -4bd & -2(a+c)d \end{pmatrix}. \tag{2.1.11}$$

This commutator is the zero matrix in exactly two cases. The first case is when $d = 0$, which means that the diffusion matrix is zero. The second case is when $a = b = c = 0$, which means that the drift matrix is zero. In particular, as long as the drift and diffusion matrices are nonzero they do not commute.

In [GMV11, PW91] an integral expression for

$$\gamma\left(\omega, 0, 0, \frac{-\sigma}{2\omega}, 0, \frac{\sigma}{2\omega}\right) \tag{2.1.12}$$

was calculated. In particular, it was shown in [GMV11] that

$$\gamma\left(\omega, 0, 0, \frac{-\sigma}{2\omega}, 0, \frac{\sigma}{2\omega}\right) = \int_{-\infty}^{\infty} f(z)p(z)\, dz, \tag{2.1.13}$$

where

$$p(z) = Ce^{\Phi(z)} \int_{z}^{\infty} e^{-\Phi(t)}\, dt, \tag{2.1.14}$$

$$\Phi(t) = \frac{2\omega^2}{\sigma^2}(t + t^3), \tag{2.1.15}$$

$$C^{-1} = \int_{-\infty}^{\infty} p(z)\, dz, \tag{2.1.16}$$

$$f(z) = \frac{\sigma^2}{2\omega^2}\frac{1 - z^2}{(1 + z^2)^2}. \tag{2.1.17}$$

In Section 2.3 we calculate an integral expression for $\gamma(a, b, c, -d, 0, d)$ for all $a, b, c, d \in \mathbb{R}$, thus generalizing the above results.

2.2 Preliminaries

In this section we prove a few preliminary results that simplify the proof of the main result of this chapter, Theorem 2.3.1. First we consider the choice of norm in Equation (2.1.10). As stated, the definition of the top Lyapunov exponent seems to depend on the choice of norm. But for finite dimensional vector spaces all norms are equivalent. Hence we have the following result.

Lemma 2.2.1. *In a finite dimensional vector space, the value of the Lyapunov exponents Equation* (2.1.10) *are unaffected by the choice of norm.*

Proof. Consider two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ for the finite dimensional vector space V. Then there exist constants $\alpha, \beta \in \mathbb{R}$ such that $\alpha \|v\|_A \leq \|v\|_B \leq \beta \|v\|_A$ for all $v \in V$. Define

$$\gamma_A = \lim_{t \to \infty} \frac{1}{t} \log \|X_t\|_A \tag{2.2.1}$$

$$\gamma_B = \lim_{t \to \infty} \frac{1}{t} \log \|X_t\|_B \tag{2.2.2}$$

$$\tag{2.2.3}$$

Hence,

$$\gamma_B - \gamma_A = \lim_{t \to \infty} \frac{1}{t} (\log \|X_t\|_B - \log \|X_t\|_A) \tag{2.2.4}$$

$$= \lim_{t \to \infty} \frac{1}{t} \log \left(\frac{\|X_t\|_B}{\|X_t\|_A} \right) \tag{2.2.5}$$

$$\leq \lim_{t \to \infty} \frac{1}{t} \log \left(\frac{\beta \|X_t\|_A}{\|X_t\|_A} \right) \tag{2.2.6}$$

$$= \lim_{t \to \infty} \frac{1}{t} \log(\beta) \tag{2.2.7}$$

$$= 0 \tag{2.2.8}$$

Similarly, $\gamma_B - \gamma_A \geq 0$. Hence, $\gamma_A = \gamma_B$. □

For the system Equation (2.1.9) the expression

$$\lim_{t \to \infty} \frac{1}{t} \log \|X_t\| \tag{2.2.9}$$

can take at most two distinct values which we will call $\gamma_1 > \gamma_2$. This follows from the random splitting of $\mathbb{R}^2$ into nested linear subspaces from Osceledets's multiplicative ergodic theorem. In particular, we have the linear subspaces

$$\{0\} \subseteq V_2(\omega) \subseteq V_1(\omega) \subseteq \mathbb{R}^2$$

such that,

$$\gamma(\omega, x) = \gamma_1(\omega) \iff x \in V_1(\omega) \backslash V_2(\omega) \tag{2.2.10}$$

$$\gamma(\omega, x) = \gamma_2(\omega) \iff x \in V_2(\omega) \backslash \{0\}. \tag{2.2.11}$$

The symplectic structure of the coefficient matrices also provides further constraints on the possible values of the the Lyapunov exponent. In particular, we make use of the following result, the proof of which can be found in [Arn03, Example 3.4.19]

Remark 2.2.2. For the system in Equation (2.1.9), $\gamma_1 = -\gamma_2$.

2.3 Result

In this section we calculate an integral expression for

$$\gamma(a, b, c, -d, 0, d), \tag{2.3.1}$$

the top Lyapunov exponent of the solution to the following $\mathbb{R}^2$-valued stochastic differential equation,

$$\delta X_t = (aE_1 + bE_2 + cE_3)X_t\,dt + (-dE_1 + dE_3)X_t\,\delta W_t, \tag{2.3.2}$$

where $a, b, c, d \in \mathbb{R}$ and W_t is a standard one-dimensional Wiener process.

Theorem 2.3.1. *Assume that $d \neq 0$.*

1. *If any of the following conditions hold:*

   (a) $a + c > 0$,

   (b) $a + c = 0$ *and* $b > 0$,

   (c) $a + c = 0$ *and* $b = 0$ *and* $a - c > 0$,

   then the following formulas for the top Lyapunov exponent hold,

$$\gamma(a, b, c, -d, 0, d) = \left| \int_{-\infty}^{\infty} f(z)p(z)\,dz \right|, \tag{2.3.3}$$

   where

$$p(z) = Ce^{\Phi(z)} \int_z^{\infty} e^{-\Phi(t)}\,dt, \tag{2.3.4}$$

$$\Phi(t) = \frac{1}{4d^2}[(a-c)t + bt^2 + \frac{a+c}{3}t^3], \tag{2.3.5}$$

$$C^{-1} = \int_{-\infty}^{\infty} p(z)\,dz, \tag{2.3.6}$$

$$f(z) = \frac{2b(1-z^2) + 4cz}{1+z^2} + \frac{4d^2(1-z^2)}{(1+z^2)^2}. \tag{2.3.7}$$

2. *If any of the following conditions hold:*

(a) $a + c < 0$,

(b) $a + c = 0$ *and* $b < 0$,

(c) $a + c = 0$ *and* $b < 0$ *and* $a - c < 0$,

then the following formulas for the top Lyapunov exponent hold,

$$\gamma(a, b, c, -d, 0, d) = \left| \int_{-\infty}^{\infty} f(z)p(z)\, dz \right|, \tag{2.3.8}$$

where

$$p(z) = Ce^{\Phi(z)} \int_{-z}^{\infty} e^{\Phi(t)}\, dt, \tag{2.3.9}$$

$$\Phi(t) = \frac{1}{4d^2}[(a - c)t + bt^2 + \frac{a + c}{3}t^3], \tag{2.3.10}$$

$$C^{-1} = \int_{-\infty}^{\infty} p(z)\, dz, \tag{2.3.11}$$

$$f(z) = \frac{2b(1 - z^2) + 4cz}{1 + z^2} + \frac{4d^2(1 - z^2)}{(1 + z^2)^2}. \tag{2.3.12}$$

Proof. If we write $X_t = (x_t, y_t)$, then Equation (2.3.2) can be written in coordinates as

$$\delta x_t = (bx + (a + c)y)dt \tag{2.3.13}$$

$$\delta y_t = ((c - a)x - by)\, dt + 2dx_t\, \delta W_t. \tag{2.3.14}$$

And similarly, can be converted into Itô differential form as

$$dx_t = (bx + (a + c)y)dt \tag{2.3.15}$$

$$dy_t = ((c - a)x - by)\, dt + 2dx_t\, dW_t. \tag{2.3.16}$$

Consider the change of coordinates $z_t := y_t/x_t$. Notice that by Itô's lemma,

$$d\log||X_t|| = \Big[\frac{2b(1-z^2)+4cz}{1+z^2} + \frac{4d^2(1-z^2)}{(1+z^2)^2}\Big]\,dt + \frac{4cz}{1+z^2}\,dW_t. \tag{2.3.17}$$

Now,

$$\gamma(a,b,c,-d,0,d) = \lim_{t\to\infty}\frac{1}{t}\log||X_t|| \tag{2.3.18}$$

$$= \lim_{t\to\infty}\frac{1}{t}\int_0^t \Big[\frac{2b(1-z_s^2)+4cz_s}{1+z_s^2} + \frac{4d^2(1-z_s^2)}{(1+z_s^2)^2}\Big]\,ds \tag{2.3.19}$$

$$+ \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{4cz_s}{1+z_s^2}\,dW_s \tag{2.3.20}$$

$$= \mathbb{E}\Big[\lim_{t\to\infty}\frac{1}{t}\int_0^t \Big[\frac{2b(1-z_s^2)+4cz_s}{1+z_s^2} + \frac{4d^2(1-z_s^2)}{(1+z_s^2)^2}\Big]\,ds\Big] \tag{2.3.21}$$

$$+ \mathbb{E}\Big[\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{4cz_s}{1+z_s^2}\,dW_s\Big], \tag{2.3.22}$$

where the last equality follows from Oceledets's multiplicative ergodic theorem. First, consider the second expression in Equation (2.3.22). Notice that the integrand in bounded. In particular,

$$\frac{4cz_s}{1+z_s^2} \le 2c. \tag{2.3.23}$$

Hence we can apply the dominated convergence theorem and use the fact that the Wiener integral is a martingale,

$$\mathbb{E}\Big[\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{4z_s}{1+z_s^2}\,dW_s\Big] = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \frac{4z_s}{1+z_s^2}\,dW_s \tag{2.3.24}$$

$$= 0. \tag{2.3.25}$$

Now consider the first expression in Equation (2.3.22). Since the metric DS induced on Wiener space is ergodic [Arn03] we have,

$$\mathbb{E}\Big[\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big[\frac{4z_s}{1+z_s^2}+\frac{4(1-z_s^2)}{(1+z_s^2)^2}\Big]\,ds\Big]=\int_{\mathbb{R}}\Big[\frac{4z}{1+z^2}+\frac{4(1-z^2)}{(1+z^2)^2}\Big]p(z)\,ds \tag{2.3.26}$$

where p is the invariant density of the process z_t. We calculate p directly by inspecting the appropriate Fokker-Planck equation for z_t.

Let $\mathcal{L}$ denote the generator of z_t. The density for z_t is the solution to following partial differential equation,

$$\mathcal{L}^* = \partial_t, \tag{2.3.27}$$

and the invariant density for z_t is the solution to

$$\mathcal{L}^* = 0. \tag{2.3.28}$$

By Itô's lemma we have

$$dz_t = [(a-c)+(2b+d^2)z+(a+c)z^2]\,dt+[d+(a+c)z^2]\,dW_t. \tag{2.3.29}$$

And so the formula for the adjoint of its generator is

$$\mathcal{L}^*[p(z)] = [-2b-2(a+c)]p(z)-[a-c+2bz+(a+c)z^2]p'(z)+4d^2p''(z). \tag{2.3.30}$$

One can check that Equation (2.3.4) and Equation (2.3.9) both satisfy this Fokker-

Planck equation under their respective conditions. Furthermore, they are indeed probability densities. The result follows.

□

Chapter 3

Matrix Valued SDEs

As an extension of the linear model discussed in Chapter 2, we also consider matrix valued solutions to the stochastic differential equation

$$\delta X_t = AX_t\,dt + BX_t\,\delta W_t. \tag{3.0.1}$$

Due to properties of the matrix exponential we will show that X_t is $\mathsf{Sp}(2,\mathbb{R})$-valued with probability 1. We will then use a structural decomposition of $\mathsf{Sp}(2,\mathbb{R})$ known as the Iwasawa decomposition in order to simplify our analysis of the process X_t.

3.1 Background

Let X_t be a $M(n, \mathbb{R})$ valued, adapted stochastic process and write

$$X_t = \begin{pmatrix} x_t^{1,1} & \dots & x_t^{1,n} \\ \vdots & \ddots & \vdots \\ x_t^{n,1} & \dots & x_t^{n,n} \end{pmatrix}.$$

Each coordinate entry, $x_t^{i,j}$, is an $\mathbb{R}$-valued stochastic process and is adapted to the same filtration. We will refer to the $x_t^{i,j}$ as the **coordinate processes of** X_t. We will say that X_t has a certain property (continuity, martingale property, etc.) if and only if each $x_t^{i,j}$ has the property. We will adhere to the following formal coordinatewise differential notion,

$$\delta X_t = \delta \begin{pmatrix} x_t^{1,1} & \dots & x_t^{1,n} \\ \vdots & \ddots & \vdots \\ x_t^{n,1} & \dots & x_t^{n,n} \end{pmatrix} = \begin{pmatrix} \delta x_t^{1,1} & \dots & \delta x_t^{1,n} \\ \vdots & \ddots & \vdots \\ \delta x_t^{n,1} & \dots & \delta x_t^{n,n} \end{pmatrix}.$$

In addition we will use the following one dimensional notation convention,

$$(\delta x_t) y_t := y_t \, \delta x_t \tag{3.1.1}$$

whenever x_t and y_t are one-dimensional processes for which the right hand side is well defined. Now consider the following matrix valued SDE with $X_t \in M(n, \mathbb{R})$,

$$\delta X_t = a(X_t, t) \, dt + b(X_t, t) \, \delta W_t. \tag{3.1.2}$$

By inspecting the matrix entries we can interpret this as a system of n^2, one-dimensional SDEs,

$$\delta[X_t]^{i,j} = [a(X_t,t)]^{i,j}\,dt + [b(X_t,t)]^{i,j}\,\delta W_t. \tag{3.1.3}$$

Which is shorthand for the rigorous integral equation

$$x_t^{i,j} - x_0^{i,j} = \int_0^t [a(X_s,s)]^{i,j}\,ds + \int_0^t [b(X_s,s)]^{i,j}\,\delta W_s.$$

We have the following matrix valued product rule.

Proposition 3.1.1. *If the entries of the matrix valued process X_t, Y_t are continuous semimartingales, we have the product formula*

$$\delta(X_tY_t) = (\delta X_t)Y_t + X_t(\delta Y_t). \tag{3.1.4}$$

Proof. For each coordinate process of the product X_tY_t we can directly calculate that,

$$\begin{aligned}
\delta[X_tY_t]^{ij} &= \delta\sum_{k=1}^n (X_t^{i,k}Y_t^{k,j}) && (3.1.5)\\
&= \sum_{k=1}^n \delta(X_t^{i,k}Y_t^{k,j}) && (3.1.6)\\
&= \sum_{k=1}^n X_t^{i,k}\delta Y_t^{k,j} + Y_t^{k,j}\delta X_t^{ik} && (3.1.7)\\
&= [X_t\delta Y_t]^{i,j} + [(Y_t^T(\delta X_t)^T)^T]^{i,j} && (3.1.8)\\
&= [X_t\delta Y_t]^{i,j} + [(\delta X_t)Y_t]^{i,j}. && (3.1.9)
\end{aligned}$$

Where Equation (3.1.7) follows from the one-dimensional Stratonovich product rule

[Yor01]. □

If the coordinate functions $[a(X_t,t)]^{i,j}, [b(X_t,t)]^{i,j}$ in the stochastic differential equation Equation (3.1.2) are well behaved, the existence and uniqueness of the coordinate solutions are guaranteed by standard one-dimensional existence and uniqueness theorems [Yor01]. Hence the existence and uniqueness of the solution X_t is guaranteed. For example if we take $A, B \in M(n, \mathbb{R})$ and define

$$a(X_t, t) = AX_t, \tag{3.1.10}$$

$$b(X_t, t) = BX_t, \tag{3.1.11}$$

we recover the time homogeneous linear stochastic differential equation with constant coefficient matrices Equation (3.0.1),

$$\delta X_t = AX_t\,dt + BX_t\,\delta W_t. \tag{3.1.12}$$

In the case that A and B commute it is well known [KP92] that the fundamental matrix,

$$\Phi_t = \exp(At + BW_t)X_0, \tag{3.1.13}$$

is the strong solution to Equation (3.1.12). In particular this can be interpreted coordinatewise,

$$\delta[\Phi_t]^{i,j} = [A\Phi_t]^{i,j}\,dt + [B\Phi_t]^{i,j}\,\delta W_t. \tag{3.1.14}$$

But in general, for non-commuting drift and diffusion matrices the situation is more complicated and a closed form solution may not be possible to derive. Writing a formula for the solution to Equation (3.1.12) would require using the Trotter product formula or the Baker-Campbell-Hausdorff formula, neither of which are discussed here. For example see [FJ12, Section 7.3].

Another closely related example would be

$$a(X_t, t) = X_t A, \tag{3.1.15}$$

$$b(X_t, t) = X_t B. \tag{3.1.16}$$

This generates the stochastic differential equation

$$\delta X_t = X_t A \, dt + X_t B \, \delta W_t. \tag{3.1.17}$$

Now consider a matrix Lie group, G, and its Lie algebra, g. For definitions and discussion see Appendix A. Due to some well known properties of exponential map,

$$\exp : \mathsf{g} \to \mathsf{G}, \tag{3.1.18}$$

matrix Lie groups and matrix Lie algebras provide a natural framework to study matrix valued linear SDE. In particular, if the diffusion matrices are elements of g, the solution X_t will be G-valued. The following proposition clarifies this relationship.

Proposition 3.1.2. *Assume that $A, B \in \mathfrak{sp}(2,\mathbb{R})$. Consider the matrix valued SDEs*

$$\delta X_t = AX_t\,dt + BX_t\,\delta W_t, \tag{3.1.19}$$

$$\delta Y_t = Y_t A\,dt + Y_t B\,\delta W_t. \tag{3.1.20}$$

If $X_0 \in Sp(2,\mathbb{R})$ then $X_t \in Sp(2,\mathbb{R})$ with probability one for all $t \geq 0$. Similarly, if $Y_0 \in Sp(2,\mathbb{R})$ then $Y_t \in Sp(2,\mathbb{R})$ with probability one for all $t \geq 0$.

Proof. We'll prove the result for X_t. By the product rule,

$$\begin{aligned}
\delta(X_t^T J X_t) &= (\delta X_t^T) J X_t + X_t^T J(\delta X_t) && (3.1.21)\\
&= (AX_t\,dt + BX_t\,\delta W_t)^T J X_t + X_t^T J(AX_t\,dt + BX_t\,\delta W_t) && (3.1.22)\\
&= X_t^T[(A\,dt + B\,\delta W_t)^T J + J(A\,dt + B\,\delta W_t)]X_t. && (3.1.23)
\end{aligned}$$

And since $A, B \in \mathsf{sp}(2,\mathbb{R})$, we know that $At + BW_t \in \mathsf{sp}(2,\mathbb{R})$. Hence $\delta(X_t^T J X_t) = 0$. Which means in particular,

$$X_t^T J X_t - X_0^T J X_0 = 0.$$

Therefore since $X_0 \in \mathsf{Sp}(n,\mathbb{R})$ we have $X_t^T J X_t = J$ for all $t \geq 0$. $\square$

3.2 Iwasawa decomposition of $\mathsf{Sp}(2, \mathbb{R})$

Recall that the coefficient matrices discussed in Chapter 2.

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.2.1}$$

form a basis for the Lie algebra $\mathsf{sp}(2, \mathbb{R})$. As a direct result of Proposition 3.1.2 we know that if X_t is the solution to a linear stochastic differential equation with coefficient matrices from $\mathsf{sp}(2, \mathbb{R})$ then X_t will be $\mathsf{Sp}(2, \mathbb{R})$ valued with probability one. More specifically,

$$X_t \in \exp(\mathsf{sp}(2, \mathbb{R})) \subset \mathsf{Sp}(2, \mathbb{R}) \tag{3.2.2}$$

for all $t \geq 0$. To simplify our analysis of X_t we will utilize an algebraic decomposition of $\mathsf{Sp}(2, \mathbb{R})$ known as the Iwasawa decomposition. First we notice that in this low-dimensional case of 2×2 matrices

$$\mathsf{Sp}(2, \mathbb{R}) \cong \mathsf{Sl}(2, \mathbb{R}), \tag{3.2.3}$$

a fact that can be verified directly from the definition of the two spaces.

Consider the following Iwasawa decomposition (Theorem A.0.6) of $\mathsf{Sp}(2, \mathbb{R})$,

$$\mathsf{Sp}(2, \mathbb{R}) = \mathsf{KAN},$$

where

$$\mathsf{K} = \left\{ \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix} : k \in \mathbb{R} \right\}, \tag{3.2.4}$$

$$\mathsf{A} = \left\{ \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} : a \in \mathbb{R} \right\}, \tag{3.2.5}$$

$$\mathsf{N}^+ = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \geq 0 \right\}. \tag{3.2.6}$$

Specifically this means that for all $X \in \mathsf{Sp}(2,\mathbb{R})$ there exist $K \in \mathsf{K}, A \in \mathsf{A}, N \in \mathsf{N}^+$ such that $X = KAN$. So if we take an arbitrary $X \in \mathsf{Sp}(2,\mathbb{R})$ there exists $k, a \in \mathbb{R}$ and $n \geq 0$ such that

$$X = \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix} \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \tag{3.2.7}$$

$$= \begin{pmatrix} e^a \cos k & e^a n_t \cos k + e^{-a} \sin k \\ -e^a \sin k & e^{-a} \cos k - e^a n \sin k \end{pmatrix}. \tag{3.2.8}$$

Now, since we are dealing with finite dimensional matrices, our choice of norm will not affect the numerical value of the Lyapunov exponent (see Lemma 2.2.1). Hence it will suffice to use the **Hilbert Schmidt norm** of a matrix X,

$$\|X\|_{HS} := \mathrm{trace}(X^* X). \tag{3.2.9}$$

In these Iwasawa coordinates a direct calculation shows that the Hilbert-Schmidt norm of a matrix $X \in \mathsf{Sp}(2,\mathbb{R})$ is given by,

$$\|X\|_{HS} = e^{2a} + e^{-2a}(1+n^2). \tag{3.2.10}$$

3.2.1 Iwasawa coordinate processes

Consider X_t, the solution to Equation (3.1.12). Now we discuss how the Iwasawa decomposition of $\mathsf{Sp}(2,\mathbb{R})$ induces one-dimensional Iwasawa coordinate processes from X_t. For each $t \geq 0$, $X_t \in \mathsf{Sp}(2,\mathbb{R})$ has an Iwasawa decomposition $X_t = K_t A_t N_t$ where

$$\mathsf{K}_t = \left\{ \begin{pmatrix} \cos k_t & \sin k_t \\ -\sin k_t & \cos k_t \end{pmatrix} : k_t \in \mathbb{R} \right\}, \tag{3.2.11}$$

$$\mathsf{A}_t = \left\{ \begin{pmatrix} e^{a_t} & 0 \\ 0 & e^{-a_t} \end{pmatrix} : a_t \in \mathbb{R} \right\}, \tag{3.2.12}$$

$$\mathsf{N}_t^+ = \left\{ \begin{pmatrix} 1 & n_t \\ 0 & 1 \end{pmatrix} : n_t \geq 0 \right\}. \tag{3.2.13}$$

Hence we have the following three parameter decomposition of X_t,

$$X_t = \begin{pmatrix} e^{a_t}\cos k_t & e^{a_t} n_t \cos k_t + e^{-a_t}\sin k_t \\ -e^{a_t}\sin k_t & e^{-a_t}\cos k_t - e^{a_t} n_t \sin k_t \end{pmatrix}.$$

So we can study the matrix valued process X_t by studying the one-dimensional coordinate processes k_t, a_t, n_t. In particular the SDE that the matrix valued process, X_t, satisfies will induce three stochastic differential equations satisfied by k_t, a_t, n_t respectively.

3.2.2 Previous Results

In [RM01] the Lyapunov exponent is calculated for certain matrix valued linear stochastic differential equations. A much broader class of Lie groups is considered. In fact the class of semi-simple matrix Lie groups are considered. The stochastic differential equations considered is

$$\delta X_t = \sum_{i=1}^{n} A^i X_t \, \delta W_t \tag{3.2.14}$$

where $\{A^1, \ldots, A^n\}$ is an orthonormal basis of the symmetric space $\mathfrak{a} \oplus \mathfrak{n}$. In this work we will consider some stochastic differential equations that do not fit this model. In particular we will consider an example in which the diffusion matrices do not form a full basis for $\mathfrak{a} \oplus \mathfrak{n}$. In addition we will consider a case which has a nonzero drift coefficient matrix.

3.3 Results

Here we calculate the top Lyapunov exponent for a linear system with non-symmetric diffusion matrix E_1.

Proposition 3.3.1. *Consider the matrix valued stochastic differential equation*

$$\delta X_t = X_t (cE_1) \, \delta W_t \tag{3.3.1}$$

where W_t is a 1-dim Brownian motion and $c \in \mathbb{R}$. Then

$$\lim_{t \to \infty} \frac{1}{t} \log \|X_t\| = 0.$$

Proof. If we write $X_t = N_t A_t K_t$ we can explicitly calculate the Iwasawa coordinate processes (n_t, a_t, k_t). Due to Proposition 3.1.2, $X_t \in \mathsf{Sp}(2, \mathbb{R})$ with probability one. In particular, X_t is invertible with probability one and the following identity holds,

$$X_t^{-1} \delta X_t = cE_1 \, \delta W_t. \tag{3.3.2}$$

Furthermore as W_t is one dimensional,

$$Ad_{K_t}[X_t^{-1} \delta X_t] = Ad_{K_t}[cE_1] \, \delta W_t. \tag{3.3.3}$$

Now consider the LHS of Equation (3.3.3). By Itô's Lemma,

$$\begin{aligned}
Ad_{K_t}[X_t^{-1} \delta X_t] &= Ad_{K_t}[K_t^{-1} A_t^{-1} N_t^{-1} \delta(N_t A_t K_t)] && (3.3.4)\\
&= Ad_{K_t}[K_t^{-1} A_t^{-1} N_t^{-1}[(\delta N_t) A_t K_t + N_t(\delta A_t) K_t + N_t A_t(\delta K_t)]] && (3.3.5)\\
&= A_t^{-1} N_t^{-1} (\delta N_t) A_t + A_t^{-1} \delta A_t + (\delta K_t) K_t^{-1} && (3.3.6)\\
&= \begin{pmatrix} \delta a_t & \delta k_t \\ e^{2a_t} \delta n_t - \delta k_t & 0 \end{pmatrix}. && (3.3.7)
\end{aligned}$$

Which we can calculate explicitly from,

$$A_t^{-1} N_t^{-1} (\delta N_t) A_t = \begin{pmatrix} 0 & 0 \\ e^{2a_t} \delta n_t & 0 \end{pmatrix}, \tag{3.3.8}$$

$$A_t^{-1} \delta A_t = \begin{pmatrix} \delta a_t & 0 \\ 0 & -\delta a_t \end{pmatrix}, \tag{3.3.9}$$

$$(\delta K_t)K_t^{-1} = \begin{pmatrix} 0 & \delta k_t \\ -\delta k_t & 0 \end{pmatrix}. \tag{3.3.10}$$

Now let's consider the RHS of Equation (3.3.3). We have the following commutation relations,

$$\mathrm{Ad}_{K_t} E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.3.11}$$

$$\mathrm{Ad}_{K_t} E_2 = \begin{pmatrix} \cos(2k_t) & -\sin(2k_t) \\ -\sin(2k_t) & -\cos(2k_t) \end{pmatrix}, \tag{3.3.12}$$

$$\mathrm{Ad}_{K_t} E_3 = \begin{pmatrix} \sin(2k_t) & \cos(2k_t) \\ \cos(2k_t) & -\sin(2k_t) \end{pmatrix}. \tag{3.3.13}$$

In particular, the subspace generated by E_1 is Ad_K invariant. Hence the RHS of Equation (3.3.3) is

$$\mathrm{Ad}_{K_t}[cE_1]\delta W_t = cE_1 \delta W_t. \tag{3.3.14}$$

Now, by the equality of Equation (3.3.7) and Equation (3.3.14) we have,

$$\begin{pmatrix} \delta a_t & \delta k_t \\ e^{2a_t}\delta n_t - \delta k_t & 0 \end{pmatrix} = c \begin{pmatrix} 0 & \delta W_t \\ -\delta W_t & 0 \end{pmatrix}. \tag{3.3.15}$$

From this we can determine the coordinate stochastic differential equations,

$$\delta n_t = 0, \tag{3.3.16}$$

$$\delta a_t = 0, \tag{3.3.17}$$

$$\delta k_t = c\,\delta W_t. \tag{3.3.18}$$

Which yield the coordinate processes,

$$n_t = n_0, \tag{3.3.19}$$

$$a_t = a_0, \tag{3.3.20}$$

$$k_t = cW_t + k_0. \tag{3.3.21}$$

Finally,

$$\gamma(X) = \lim_{t\to\infty} \frac{1}{t} \log \|X_t\| \tag{3.3.22}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \sqrt{e^{2a_t} + e^{-2a_t}(1 + n_t^2))} \tag{3.3.23}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \sqrt{e^{2a_0} + e^{-2a_0}(1 + n_0^2))} \tag{3.3.24}$$

$$= 0 \tag{3.3.25}$$

□

Now we calculate the top Lyapunov exponent for a system with non-symmetric diffusion coefficient matrix E_1 that also has a non-zero drift coefficient matrix.

Proposition 3.3.2. *Consider the matrix valued stochastic differential equation,*

$$\delta X_t = X_t(cE_1)\,dt + X_t(dE_1)\,\delta W_t$$

where W_t *is a 1-dim Brownian motion. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log \|X_t\| = 0.$$

Proof. Write $X_t = N_t A_t K_t$. Using the same method as above we can calculate the coordinate processes,

$$n_t = n_0, \tag{3.3.26}$$

$$a_t = a_0, \tag{3.3.27}$$

$$k_t = ct + dW_t + k_0. \tag{3.3.28}$$

And again since a_t and n_t are constant, the claim is proven. □

Everything in this section can be done with multiplication on the other side. We state the result below for completeness. The proofs are nearly identical to those of Proposition 3.3.1 and Proposition 3.3.2.

Proposition 3.3.3. *Consider the matrix valued stochastic differential equation,*

$$\delta X_t = (cE_1)X_t\,dt + (dE_1)X_t\,\delta W_t$$

where W_t *is a 1-dim Brownian motion and* $c, d \in \mathbb{R}$*. Then*

$$\lim_{t\to\infty} \frac{1}{t} \log \|X_t\| = 0.$$

Chapter 4

Numerical Results

Osceledets's multiplicative ergodic theorem provides an analogue of the law of large numbers by stating that under appropriate conditions on X_t,

$$\frac{1}{t} \log \|X_t\| \tag{4.0.1}$$

converges a.s. to a constant. Furthermore, analytically calculating or algorithmically approximating this constant is difficult. In fact, there is a field of research being undertaken regarding the theoretical computability of the top Lyapunov exponent for various systems. For example, in [TB97] it is shown that algorithmically approximating the Lyapunov exponent of related discrete time systems is NP hard. And in [TB97, Vis98, Vis01], it is shown that for any given degree of accuracy no algorithm exists that can approximate the Lyapunov exponent of related discrete time systems.

The exponential decay/growth of the solution X_t leads to practical computation issues. Simulated trajectories of X_t rapidly approach either 0 or ∞. Consider the case when $X_T < \delta$ for some small $0 < \delta < 1$. There are two possible sources of computational error. First, if δ is smaller than the working precision of the computational tool being used, the numerical value of X_T will be 0. Secondly, $\log X_T$ can be so close to $-\infty$ that its magnitude is larger than the largest number available to the computational tool used. Due to these facts, naive computations can be very misleading.

Now if the convergence rate of Equation (4.0.1) were fast enough, we would be able to effectively approximate γ by only simulating X_t for relatively small t, stopping the simulation before incorrect values of 0 or ∞ were achieved. To that end we define the **finite time Lyapunov exponent** of a stochastic process X_t to be

$$\gamma_t := \frac{1}{t} \log \|X_t\| . \tag{4.0.2}$$

Due to the difficulties outlined above we will numerically inspect γ_t. Now, Osceledets's theorem ensures that the limit

$$\lim_{t \to \infty} \gamma$$

is constant a.s. But no such guarantee is made for the finite time Lyapunov exponent. So the distribution of λ_t is sought after. We will approximate $\mathbb{E}\gamma_t$.

In this work we will bypass these difficulties in the following way. We will write γ_t as the solution to a stochastic differential equation. This process will not have exponential growth/decay properties and so we can calculate its Euler approximation,

$\tilde{\gamma}_t$ (see Section 4.1 for definitions). In short, notice that

$$|\mathbb{E}\gamma_t - \hat{g}_t| \leq |\mathbb{E}\gamma_t - \mathbb{E}\tilde{\gamma}_t| + |\mathbb{E}\tilde{\gamma}_t - \hat{g}_t|. \tag{4.0.3}$$

Theorem 4.2.1 ensures that $\mathbb{E}\gamma_t$ can be made arbitrarily close to $\mathbb{E}\tilde{\gamma}_t$ by taking the time discretization interval as small as necessary. We statistically estimate $\mathbb{E}\tilde{\gamma}_t$ and probabilistically bound the second summand using the method of confidence intervals.

4.1 Background

4.1.1 Euler Approximation to SDE

Consider the SDE

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t. \tag{4.1.1}$$

In order to define the Euler approximation to the solution X_t on the interval $[t_0, T]$ we will discretize the time interval time and make some related definitions.

Definition 4.1.1. For a time discretization $t_0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ define

$$\Delta_n = \tau_{n+1} - \tau_n \tag{4.1.2}$$

$$\Delta W_n = W_{\tau_n+1} - W_{\tau_n} \tag{4.1.3}$$

For example, it is common to consider equidistant time intervals in which case the above simplifies to

$$\Delta_n = (T - t_0)/N =: \Delta \tag{4.1.4}$$

$$\tau_n = t_0 + n\Delta \tag{4.1.5}$$

$$\Delta W_n \sim \mathcal{N}(0, \Delta) \tag{4.1.6}$$

Definition 4.1.2 (Euler Approximation)**.** Assume that X_t satisfies

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t. \tag{4.1.7}$$

Then the **Euler approximation to** X_t is the discrete time stochastic process, denoted $\tilde{X}_n$, defined iteratively, $\{\tilde{X}_n : n = 0, 1, \dots, N\}$ as follows,

$$\tilde{X}_0 := X_0, \tag{4.1.8}$$

$$\tilde{X}_{i+1} := \tilde{X}_i + a(\tilde{X}_i)\Delta_i + b(\tilde{X}_i)\Delta W_i, \tag{4.1.9}$$

for $i = 0, 1, \dots, N - 1$. It is useful to linearly interpolate and consider $\tilde{X}_t$ as a continuous time process with piecewise linear paths.

In practice Bernoulli random variables are another useful means of approximating the Brownian increments ΔW_n. In this work we will use the following definition for ΔW_n,

$$\Delta W_n = \begin{cases} \sqrt{\Delta} & \text{w.p. } 1/2 \\ -\sqrt{\Delta} & \text{w.p. } 1/2 \end{cases} \tag{4.1.10}$$

We remark that $\tilde{X}_t$ is a stochastic process in its own right, and in particular Osceledets's multiplicative ergodic theorem applies. But unfortunately,

$$\lim_{t\to\infty} \frac{1}{t} \log \|X_t\| \neq \lim_{t\to\infty} \frac{1}{t} \log \left\|\tilde{X}_t\right\|. \tag{4.1.11}$$

The relationship between these two quantities was studied in [Tal99].

4.1.2 Confidence Intervals

In general, analysis of the Euler approximation to an SDE is just as difficult as that of the original SDE. But the iterative nature of its definition allows for simple simulation and estimation. For example, one way to estimate $\mathbb{E}\tilde{X}_t$ is the method of confidence intervals. We will simulate M many batches, each consisting of N simulated trajectories of $\tilde{X}_t$. Label $X_t^{j,k}$ the k^{th} simulated trajectory from the j^{th} batch for $k \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, M\}$. And for each batch, define the **batch sample average**

$$\hat{\epsilon}_j = \frac{1}{N} \sum_{k=1}^{N} \tilde{X}_t^{j,k}$$

for $j \in \{1, 2, \ldots, M\}$. Now define the sample mean and sample variance for the random variable $\hat{\epsilon}_j$,

$$\hat{\epsilon} = \frac{1}{M} \sum_{j=1}^{M} \hat{\epsilon}_j, \tag{4.1.12}$$

$$\hat{\sigma}^2 = \frac{1}{M-1} \sum_{j=1}^{M} (\hat{\epsilon}_j - \hat{\epsilon})^2, \tag{4.1.13}$$

which are both unbiased estimators. Then the scaled random variable

$$T := \frac{\hat{\epsilon} - \mathbb{E}\tilde{X}_t}{\sqrt{\hat{\sigma}^2/M}} \tag{4.1.14}$$

has the Student t-distribution with mean zero, variance $(M-1)/(M-3)$ and $M-1$ degrees of freedom. Hence,

$$\mathbb{P}(|\hat{\epsilon} - \mathbb{E}\tilde{X}_t| < a) = \mathbb{P}(|T| < a\sqrt{M/\hat{\sigma}^2})$$

Now define $t_{1-\alpha,M-1}$ to be the quantity that satisfies,

$$\mathbb{P}(|T| < t_{1-\alpha,M-1}) = 1 - \alpha.$$

And we can conclude that

$$\mathbb{P}\Big(|\hat{\epsilon} - \mathbb{E}\tilde{X}_t| < t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2}{M}}\Big) = 1 - \alpha,$$

or

$$\mathbb{P}\Big[\mathbb{E}\tilde{X}_t \in \Big(\hat{\epsilon} - t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2}{M}}, \hat{\epsilon} + t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2}{M}}\Big)\Big] = 1 - \alpha. \tag{4.1.15}$$

It should be noted here that the quantity of interest, $\mathbb{E}\tilde{X}_t$, is deterministic. It is the *interval* that is random. This is because the quantities $\hat{\sigma}^2$ and $\hat{\epsilon}$ are random variables. So the statement in Equation (4.1.15) can be interpreted as follows: Every time that $\hat{\sigma}^2$ and $\hat{\epsilon}$ are simulated, the interval Equation (4.1.15) is constructed. The deterministic quantity $\mathbb{E}X_t$ lies within that interval with probability $1 - \alpha$.

4.2 Results

Now we will consider the one dimensional noisy harmonic oscillator from Chapter 2. Specifically, we consider the solution to

$$dX_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_t \, dt + \epsilon \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X_t \, dW_t. \tag{4.2.1}$$

The following result is the main content of the chapter. We will prove this result with Lemma 4.2.2 and Lemma 4.2.3 in this section. Notation conventions and discussion thereof are contained in Section 4.1.

Theorem 4.2.1. *Let γ_t be the finite time Lyapunov exponent for Equation* (4.2.1) *and let $\tilde{\gamma}_t$ be the Euler approximation to γ_t with step size Δ. Then,*

$$\mathbb{P}\Big\{\mathbb{E}\gamma_t \in \Big(\hat{g}_t \pm \Big[t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2}{M}} + \frac{4\epsilon^2\sqrt{\Delta}\sqrt{(1+\epsilon^2)^2+\epsilon^2}}{t}\int_0^t e^{16\epsilon^2(s^2+s)}\,ds\Big]\Big)\Big\} = 1-\alpha. \tag{4.2.2}$$

Where,

$$\hat{g}_t^j = \frac{1}{N}\sum_{k=1}^{N}\tilde{\gamma}_t^{j,k} \tag{4.2.3}$$

$$\hat{g}_t = \frac{1}{MN}\sum_{j=1}^{M}\hat{g}_t^j, \tag{4.2.4}$$

$$\hat{\sigma}^2 = \frac{1}{M-1}\sum_{j=1}^{M}(\hat{g}_t^j - \hat{g}_t)^2, \tag{4.2.5}$$

Lemma 4.2.2. *For the system Equation* (4.2.1) *the finite time Lyapunov exponent* *satisfies,*

$$\mathbb{E}\gamma_t = \frac{\epsilon^2}{8} + \frac{\epsilon^2}{4t}\int_0^t \mathbb{E}\cos 2\theta_s\, ds + \frac{\epsilon^2}{8t}\int_0^t \mathbb{E}\cos 4\theta_s\, ds \tag{4.2.6}$$

Proof. Consider the change to polar coordinates in Equation (4.2.1). Specifically,

$$X_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} =: \begin{pmatrix} r_t \sin\theta_t \\ y_t \cos\theta_t \end{pmatrix}. \tag{4.2.7}$$

Then by Itô's Lemma we can calculate

$$\delta\theta_t = -dt + \epsilon\cos^2\theta_t\,\delta W_t \tag{4.2.8}$$

which we can convert to Stratonovich differential form as

$$d\theta_t = -(1 + \epsilon^2\cos^3\theta_t\sin\theta_t)\,dt + \epsilon\cos^2\theta_t\,dW_t. \tag{4.2.9}$$

Then we can apply Itô's lemma directly to $\gamma_t = \frac{1}{t}\log\|X_t\|$ which yields the desired result. □

Notice that the constant term, $\frac{\epsilon^2}{8}$, in Lemma 4.2.2 agrees with the asymptotic results from [GMV11]. Now consider the following bound. The proof techniques are standard and inspired by [KP92].

Lemma 4.2.3. *Let $\tilde{\theta}_t$ be the Euler approximation to θ_t with step size Δ. Then for all $a \in \mathbb{R}$,*

$$\mathbb{E}|\cos(a\tilde{\theta}_t) - \cos(a\theta_t)| \leq a\sqrt{\Delta}\Big[4e^{16\epsilon^2(t^2+t)}\sqrt{[(1+\epsilon^2)^2 + \epsilon^2]}\Big] \tag{4.2.10}$$

Proof. Define $n_t = \max\{0, 1, \ldots, N : \tau_n \leq t\}$. By the definition of the Euler approximation,

$$\sup_{0\leq s\leq t} \mathbb{E}(|\tilde{\theta}_s - \theta_s|^2) \tag{4.2.11}$$

$$= \sup_{0\leq s\leq t} \mathbb{E}(|\theta_{n_s} + \sum_{n=0}^{n_s-1} a(\theta_n^\delta)\Delta_n - a(\theta_n^\delta)\Delta_n + b(\theta_n^\delta)\Delta W_n - b(Y_n^\delta)\Delta W_n \tag{4.2.12}$$

$$- \int_0^s a(\theta_r)\, dr - \int_0^s a(\theta_r)\, dW_r|^2) \tag{4.2.13}$$

$$= \sup_{0\leq s\leq t} \mathbb{E}(|\sum_{n=0}^{n_s-1} a(\theta_n^\delta)\Delta_n + b(\theta_n^\delta)\Delta W_n \int_0^s a(\theta_r)\, dr - \int_0^s a(\theta_r)\, dW_r|^2) \tag{4.2.14}$$

$$= \mathbb{E}(|\int_0^{\tau_{n_s}} a(\tilde{\theta}_{n_r}) - a(\theta_r)\, dr + \int_0^{\tau_{n_s}} b(\tilde{\theta}_{n_r}) - b(\theta_r)\, dW_r \tag{4.2.15}$$

$$- \int_{\tau_{n_s}}^s a(\theta_r)\, dr - \int_{\tau_{n_s}}^s a(\theta_r)\, dW_r|^2) \tag{4.2.16}$$

$$\leq \mathbb{E}\Big(\Big[|\int_0^{\tau_{n_s}} a(\tilde{\theta}_{n_r}) - a(\theta_r)\, dr| + |\int_0^{\tau_{n_s}} b(\tilde{\theta}_{n_r}) - b(\theta_r)\, dW_r| \tag{4.2.17}$$

$$+ |\int_{\tau_{n_s}}^s a(\theta_r)\, dr| + |\int_{\tau_{n_s}}^s a(\theta_r)\, dW_r|\Big]^2\Big) \tag{4.2.18}$$

$$\leq 2^4\mathbb{E}\Big(|\int_0^{\tau_{n_s}} a(\tilde{\theta}_{n_r}) - a(\theta_r)\, dr|^2 + |\int_0^{\tau_{n_s}} b(\tilde{\theta}_{n_r}) - b(\theta_r)\, dW_r|^2 \tag{4.2.19}$$

$$+ |\int_{\tau_{n_s}}^s a(\theta_r)\, dr|^2 + |\int_{\tau_{n_s}}^s a(\theta_r)\, dW_r|^2\Big) \tag{4.2.20}$$

$$\leq 2^4\Big(\mathbb{E}\int_0^{\tau_{n_s}} |a(\tilde{\theta}_{n_r}) - a(\theta_r)\, dr|^2 + \mathbb{E}|\int_0^{\tau_{n_s}} b(\tilde{\theta}_{n_r}) - b(\theta_r)\, dW_r|^2 \tag{4.2.21}$$

$$+ \mathbb{E}\int_{\tau_{n_s}}^s |a(\theta_r)\, dr|^2 + \mathbb{E}|\int_{\tau_{n_s}}^s b(\theta_r)\, dW_r|^2\Big) \tag{4.2.22}$$

$$\leq 2^4\Big(\epsilon^2 T\mathbb{E}\int_0^{\tau_{n_s}} |\tilde{\theta}_{n_r} - \theta_r|^2\, dr + \epsilon^2\mathbb{E}\int_0^{\tau_{n_s}} |\tilde{\theta}_{n_r} - \theta_r|^2\, dr \tag{4.2.23}$$

$$+ \mathbb{E}\Delta\int_{\tau_{n_s}}^s |a(\theta_r)|^2\, dr + \mathbb{E}\int_{\tau_{n_s}}^s |b(\theta_r)|^2\, dW_r\Big) \tag{4.2.24}$$

$$\leq 2^4\Big(\epsilon^2(T+1)\int_0^{\tau_{n_s}} \mathbb{E}|\tilde{\theta}_{n_r} - \theta_r|^2\, dr + (1+\epsilon^2)^2\Delta + \epsilon^2\delta\Big), \tag{4.2.25}$$

where we've made use of the inequality.

$$(x_1 + \cdots + x_n)^2 \leq 2^n (x_1 + \cdots x_n). \tag{4.2.26}$$

So we can apply the Gronwall inequality (Theorem C.0.13) with

$$L = 16\epsilon^2 (T + 1), \tag{4.2.27}$$

$$\alpha_r = \mathbb{E}|\tilde{\theta}_{n_r} - \theta_r|^2, \tag{4.2.28}$$

$$\beta_r = 16\Delta[(1 + \epsilon^2)^2 + \epsilon^2]. \tag{4.2.29}$$

In particular this means,

$$\sup_{0 \leq s \leq t} \mathbb{E}|\tilde{\theta}_{n_r} - \theta_r|^2 \leq e^{LT}\beta.$$

Hence,

$$\mathbb{E}|\tilde{\theta}_{n_t} - \theta_t| \leq \sqrt{\mathbb{E}|\tilde{\theta}_{n_t} - \theta_t|^2} \tag{4.2.30}$$

$$\leq \sqrt{\sup_{0 \leq s \leq t} \mathbb{E}|\tilde{\theta}_{n_s} - \theta_s|^2} \tag{4.2.31}$$

$$\leq \sqrt{e^{LT}\beta} \tag{4.2.32}$$

$$= \sqrt{\Delta}\Big[4e^{16\epsilon^2(T^2+T)}\sqrt{[(1 + \epsilon^2)^2 + \epsilon^2]}\Big]. \tag{4.2.33}$$

Finally using the Lipschitz bound, $|\cos(ax) - \cos(ay)| \leq a|x - y|$, yields

$$|\mathbb{E}\cos a\theta_t - \mathbb{E}\cos a\tilde{\theta}| = |\mathbb{E}(\cos a\theta_t - \cos a\tilde{\theta})| \tag{4.2.34}$$

$$\leq a|\mathbb{E}(\tilde{\theta}_t - \theta_t)| \tag{4.2.35}$$

$$\leq a\mathbb{E}|\tilde{\theta}_t - \theta_t| \tag{4.2.36}$$

$$\leq a\sqrt{\delta}\Big[4e^{16\epsilon^2(T^2+T)}\sqrt{[(1+\epsilon^2)^2+\epsilon^2]}\Big] \tag{4.2.37}$$

□

Now we can verify a bound between the Lyapunov exponent of the exact system and that of its Euler approximation.

Proposition 4.2.4. *Let* γ_t *be the finite time Lyapunov exponent for the system Equation* (4.2.1)*. Let* $\tilde{\gamma}_t$ *be the Euler approximation to* γ_t *with step size* Δ*. Then we have the following bound,*

$$|\mathbb{E}\gamma_t - \mathbb{E}\tilde{\gamma}_t| \leq \frac{4\epsilon^2\sqrt{\Delta}\sqrt{(1+\epsilon^2)^2+\epsilon^2}}{t}\int_0^t e^{16\epsilon^2(s^2+s)}\,ds \tag{4.2.38}$$

Proof. From Lemma 4.2.3 we know that

$$|\mathbb{E}\gamma_t - \mathbb{E}\tilde{\gamma}_t| \leq \frac{\epsilon^2}{4t}\int_0^t |\mathbb{E}\cos 2\theta_s - \mathbb{E}\cos 2\tilde{\theta}_s|\,ds + \frac{\epsilon^2}{8t}\int_0^t |\mathbb{E}\cos 4\theta_s - \mathbb{E}\cos 4\tilde{\theta}_s|\,ds \tag{4.2.39}$$

$$= \frac{\epsilon^2}{4t}\int_0^t 2|\mathbb{E}\theta_s - \mathbb{E}\tilde{\theta}_s|\,ds + \frac{\epsilon^2}{8t}\int_0^t 4|\mathbb{E}\theta_s - \mathbb{E}\tilde{\theta}_s|\,ds \tag{4.2.40}$$

$$= \frac{\epsilon^2}{t}\int_0^t |\mathbb{E}\theta_s - \mathbb{E}\tilde{\theta}_s|\,ds \tag{4.2.41}$$

And by Lemma 4.2.3 we have the upper bound

$$\frac{\epsilon^2}{t}\int_0^t |\mathbb{E}\theta_s - \mathbb{E}\tilde{\theta}_s|\,ds \leq \frac{4\epsilon^2\sqrt{\Delta}\sqrt{(1+\epsilon^2)^2+\epsilon^2}}{t}\int_0^t e^{16\epsilon^2(s^2+s)}\,ds, \tag{4.2.42}$$

which completes the proof. □

Now we combine the deterministic bound from Proposition 4.2.4 and the probabilistic bound for the confidence intervals for $\tilde{\gamma}_t$ to complete the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. By the triangle inequality we have

$$|\mathbb{E}\gamma_t - \hat{g}_t| \leq |\mathbb{E}\gamma_t - \mathbb{E}\tilde{\gamma}_t| + |\mathbb{E}\tilde{\gamma}_t - \hat{g}_t| \tag{4.2.43}$$

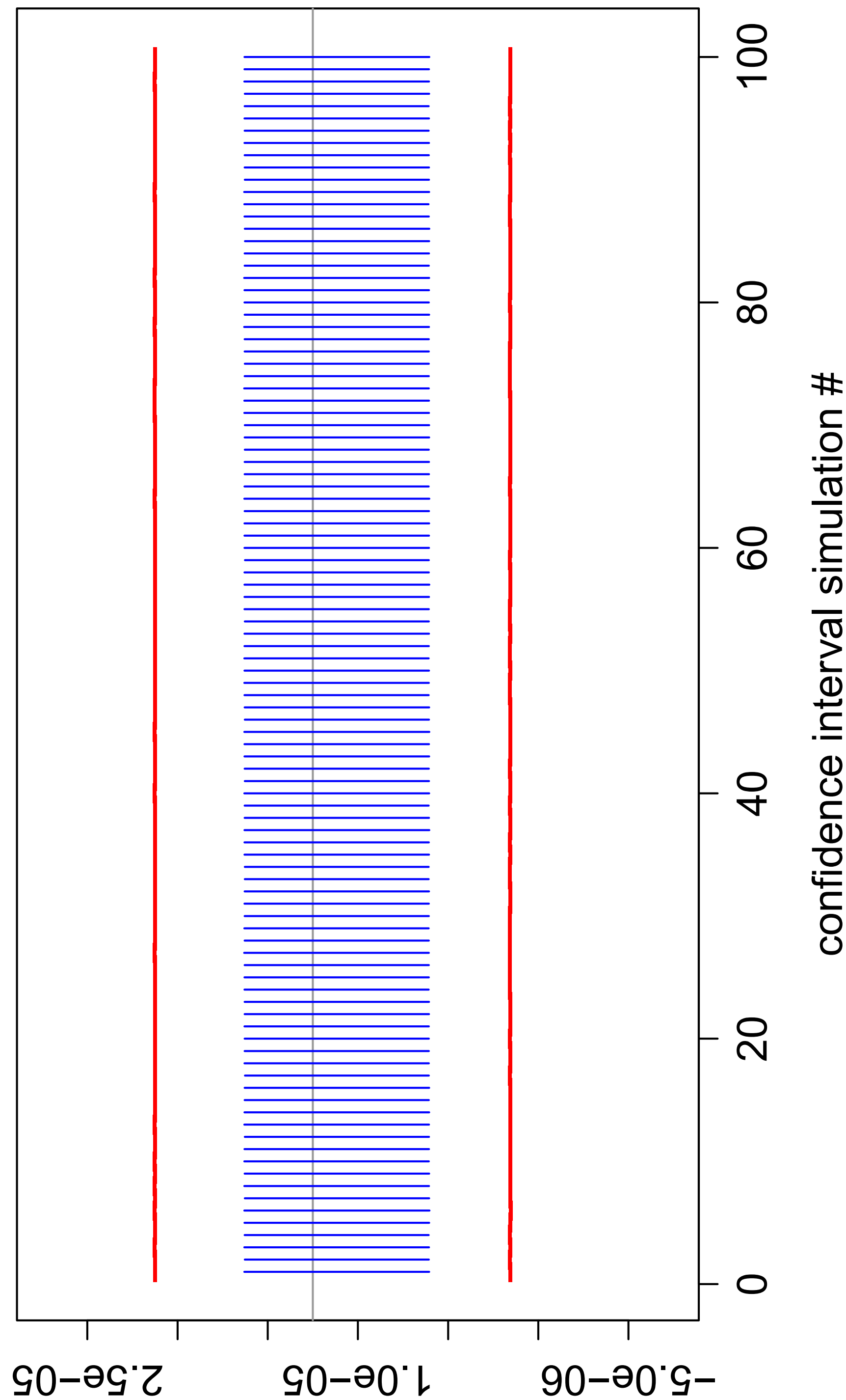
The upper bound for the first summand is provided by Proposition 4.2.4. And by the method of confidence intervals discussed in Section 4.1, we know that

$$\mathbb{P}\left\{|\mathbb{E}\tilde{\gamma}_t - \hat{g}| \leq t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2}{M}}\right\} = 1 - \alpha.$$

The result follows. □

4.3 Example

In this section we implement numerical simulations to exemplify the bounds in Theorem 4.2.1. Specifically we will consider Equation (4.2.1) with $\epsilon = 10^{-2}$. Now to get numerical bounds for $\mathbb{E}\tilde{\gamma}_{25}$ we will simulate the confidence interval in Theorem 4.2.1. Each time a simulation is run, the confidence interval will have potentially different numerical values. We will construct and simulate the confidence interval for $\mathbb{E}\tilde{\gamma}_{25}$ a total of 100 times. For each confidence interval simulation we ran a total of $M = 100$ batches each consisting of $N = 5000$ trajectories per batch, all with $\Delta = 10^{-3}$. We take $\alpha = .02$. Hence, our T-value is $t_{1-\alpha,M-1} = 3.174$.



confidence interval simulation #
0
20
40
60
80
100
−5.0e−06
1.0e−05
2.5e−05

Each red vertical lines denote the endpoints of the interval for one of the simulated intervals from Theorem 4.2.1,

$$\left(\hat{g}_{25} \pm \left[t_{1-\alpha,M-1}\sqrt{\frac{\hat{\sigma}^2}{M}} + \frac{4\epsilon^2\sqrt{\Delta}\sqrt{(1+\epsilon^2)^2+\epsilon^2}}{t}\int_0^t e^{16\epsilon^2(s^2+s)}\,ds\right]\right) \tag{4.3.1}$$

Each blue line is a single realization of the $1-\alpha\% = 99.98\%$ confidence interval,

$$\left(\hat{g}_{25} - 3.174\sqrt{\frac{\hat{\sigma}^2}{1000}}, \hat{g}_{25} + 3.174\sqrt{\frac{\hat{\sigma}^2}{1000}}\right).$$

Each green dotted line represents the deterministic portion of the interval from Theorem 4.2.1 that is contributed by the Euler approximation. In our current example that is,

$$\frac{4\epsilon^2\sqrt{\Delta}\sqrt{(1+\epsilon^2)^2+\epsilon^2}}{t}\int_0^t e^{16\epsilon^2(s^2+s)}\,ds \tag{4.3.2}$$

$$= \frac{4(10^{-2})^2\sqrt{10^{-3}}\sqrt{(1+(10^{-2})^2)^2+(10^{-2})^2}}{t}\int_0^{25} e^{16(10^{-2})^2(s^2+s)}\,ds \tag{4.3.3}$$

$$\approx 4.736\times 10^{-6}. \tag{4.3.4}$$

The grey, horizontal line is the the asymptotic value for γ calculated in [GMV11],

$$\gamma \approx \frac{\epsilon^2}{8} = \frac{(10^{-2})^2}{8}.$$

4.3.1 MATLAB® Code

The following MATLAB® code was used to produce each of the confidence interval simulations in Section 4.3.

```
%%%%%%%%%%%%%%%%%%%%
%coefficients
%%%%%%%%%%%%%%%%%%%%
e=.01; %size of noise

%%%%%%%%%%%%%%%%%%%%
%simulation variables
%%%%%%%%%%%%%%%%%%%%
dt=.001; %step size
n=1/dt; %steps per unit time
T=25; %total time length
M=100; %number of batches
N=5000; %number of simulations per batch
time = 0:dt:T-dt; %time vector
x = zeros(M,N); %preallocate space for theta
y = zeros(M,N); %preallocate space for gamma

%%%%%%%%%%%%%%
%preallocate space for simulations
%%%%%%%%%%%%%%
x(:,:) = 0; %initial value
y(:,:) = 0; %initial value

%%%%%%%%%%%%%%%%
%Simulation
%%%%%%%%%%%%%%%
for k=1:M  %k counts the batch number

    for j=1:N  %j counts the simulation #
```

```
            dW = sqrt(dt)*randn(1,n*T-1); %discrete approx to of ...
                brownian increments

            for i=1:T*n-1 %i counts the time step in the simulation
                X=x(k,j);
                Y=y(k,j);

                x(k,j)  = X + (1+2*e^2*sin(X)^3*cos(X))*dt - ...
                    (e*sin(X)^2)*dW(i);
                y(k,j)  = Y - (.5*e^2*sin(X)^2*(sin(X)^2-cos(X)^2))*dt ...
                    + (e*sin(X)*cos(X))*dW(i);
            end
        end
    end

%%%%%%%%%%%%%%%%%%%
%Statistics
%%%%%%%%%%%%%%%%%%%
L=y(:,:)/T;
E= mean(mean(L));
sigSq=1/(M-1)*sum((mean(L,2)-E).^2);
```

Appendix A

Lie Goups

For a rigorous treatment of the material in this appendix, the reader is reffered to [Kna02,Hal03]. We include some results relevant to this thesis, with proofs omitted.

Definition A.0.1. A **Lie group** is a smooth manifold, G, with group structure such that the following maps ,

1. $g \mapsto hg$

2. $g \mapsto g^{-1}$

are C^∞ for all $g, h \in \mathsf{G}$.

Definition A.0.2. A **matrix Lie group** is a closed subset of $\mathsf{Gl}(n, \mathbb{R})$.

Matrix Lie groups are also referred to as closed linear groups [Kna02]. Some examples include

$$\mathsf{Gl}(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}) : \det X \neq 0\}, \tag{A.0.1}$$

$$\mathsf{Sl}(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) : \det X = 1\}, \tag{A.0.2}$$

$$\mathsf{Sp}(2n,\mathbb{R}) = \{X \in M(2n,\mathbb{R}) : X^T J X = J\}, \tag{A.0.3}$$

where

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{A.0.4}$$

The proof of the following proposition can be found in [Kna02, Theorem 0.15].

Proposition A.0.3. *Every matrix Lie group is a Lie group.*

The converse of this statement is false. There are certainly Lie groups that are not matrix Lie groups. Two such example are $\mathbb{R} \times \mathbb{R} \times S^1$ and the universal cover of $\mathsf{Sl}(2,\mathbb{R})$, $\widetilde{\mathsf{Sl}(2,R)}$. The **Lie algebra of G**, denoted by $\mathfrak{g}$, is the tangent space of G at the identity. Specifically,

$$\mathfrak{g} = T_{I_n}\mathsf{G} = \{\gamma'(0) : \gamma : \mathbb{R} \to \mathsf{G} \text{ is a smooth curve with } \gamma(0) = I_n\}. \tag{A.0.5}$$

This is an $\mathbb{R}-$linear space. Some examples include

$$\mathsf{gl}(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) : \det X \neq 0\}, \tag{A.0.6}$$

$$\mathsf{sl}(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) : \det X = 1\}, \tag{A.0.7}$$

$$\mathsf{sp}(2n,\mathbb{R}) = \{X \in M(2n,\mathbb{R}) : X^T J X = J\}. \tag{A.0.8}$$

In fact it carries the algebraic structure of a **Lie Algebra**. In particular, there is a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{A.0.9}$$

satisfying:

1. Anti-symmetry: For all $X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$,
2. Jacobi identity: For all $X, Y, Z \in \mathfrak{g}$, $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

For matrix Lie groups the Lie bracket is the multiplication commutator,

$$[X, Y] = XY - YX. \tag{A.0.10}$$

Now, for each $X \in \mathfrak{g}$ there is a small neighborhood about 0 such that there is a unique integral curve, γ_X, satisfying:

1. $\gamma'_X(0) = X$,
2. $\gamma_X(t + s) = \gamma_X(t)\gamma_X(s)$.

Hence, we can make the following definition.

Definition A.0.4. The **exponential map**, $\exp : \mathfrak{g} \to \mathsf{G}$, is defined to be

$$\exp(tX) = \gamma_X(t). \tag{A.0.11}$$

For matrix Lie algebras the exponential map is simply the matrix exponential,

$$\exp(X) = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}. \tag{A.0.12}$$

which converges for al $X \in M(n, \mathbb{R})$. The exponential map is local diffeomorphism in some small neighborhood of the identity. But in general the map exp is not a global diffeomorphism. For example consider

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathsf{Sl}(2, \mathbb{R}).$$

It can be shown that there is no $X \in \mathsf{sl}(2, \mathbb{R})$ such that $\exp X = A$ (see [Hal03]).

A.0.2 Semi-Simple Lie groups

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A subspace $\mathsf{h} \subset \mathsf{g}$ is called an **ideal** if $[\mathsf{h}, \mathsf{g}] \subset \mathsf{h}$. Now define $\mathsf{g}^0 = \mathsf{g}$ and $\mathsf{g}^{k+1} = [\mathsf{g}^k, \mathsf{g}^k]$ for $k \in \{0, 1, \dots\}$. We say that g is solvable if $\mathsf{g}^n = 0$ for some n.

Definition A.0.5. A finite-dimensional Lie algebra $\mathfrak{g}$ is called **semisimple** if it has no nonzero solvable ideals. A Lie group is said to be **semisimple** if its Lie algebra g is semisimple.

Semisimple Lie algebras and Lie groups admit a structural decomposition known as the **Iwasawa decomposition** the details of which are contained in the following theorem [Kna02].

Theorem A.0.6. *For a semisimple matrix Lie algebra* g *there exists a direct sum decomposition,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \tag{A.0.13}$$

where

$$k := \{X \in g : X^T = -X\}, \tag{A.0.14}$$

$$a := \{\text{maximal abelian subspace of } g \ominus k\}, \tag{A.0.15}$$

$$n^+ := g \ominus (k \oplus a). \tag{A.0.16}$$

For example, let's consider $\mathsf{sl}(2,\mathbb{R})$. We have $\mathsf{sl}(2,\mathbb{R}) = \mathsf{k} \oplus \mathsf{a} \oplus \mathsf{n}^+$ where

$$\mathsf{k} = Span\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \tag{A.0.17}$$

$$\mathsf{a} = Span\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{A.0.18}$$

$$\mathsf{n}^+ = Span\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}. \tag{A.0.19}$$

This Lie algebra structure theorem also provides a decomposition for semisimple Lie groups. Let K, A and N^+ be the analytic subgroups of G with Lie algebras k, a and n^+.

Theorem A.0.7. *For semisimple Lie groups G, the map*

$$(k, a, n) \mapsto g = kan$$

is a diffeomorphism between $K \times A \times N^+$ and G.

Appendix B

Random Dynamical Systems

For a rigorous treatment of the material in this appendix, the reader is reffered to [Arn03]. We include some results relevant to this thesis, with proofs omitted.

Definition B.0.8. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a **Dynamical System (DS)** over Ω is a mapping

$$\theta : T \times \Omega \to \Omega$$

satisfying the following conditions:

1. $\theta_0 = \mathrm{id}_\Omega$,

2. $\theta_{t+s} = \theta_t \circ \theta_s$.

It can be helpful to interpret T as the time index set. For the purpose of this thesis T will be either the set of non-negative integers or non-negative real numbers. The choice will be clear from context.

Definition B.0.9. A dynamical system is called **ergodic** if the set of θ-invariant sets,

$$\mathcal{I} = \{A \subset X : \theta_t A = A\},$$

is trivial.

Now we describe Brownian motion as a dynamical system. The shift operator, $(\theta_t f)(s) := f(s+t) - f(t)$ is an ergodic DS over classical Weiner space,

$$X = \{f \in C([0,T];\mathbb{R}^n) \mid f(0) = 0\}. \tag{B.0.1}$$

Denote the law of standard one dimensional Brownian motion by $\mathbb{P}$. This DS preserves Wiener measure, $\theta_t\mathbb{P} = \mathbb{P}$. Furthermore, all shift invariant sets have measure 1 or 0. And since Brownian motion has stationary, independent increments, the Kolmogorov 0-1 law implies that the tail sigma-algebra, T^∞, is trivial. And since $\mathcal{I} \subset T^\infty$, Brownian motion is an ergodic dynamical system.

Definition B.0.10. Given a DS, $(\Omega, \mathcal{F}, \theta)$, a **Random Dynamical System (RDS)** acting on the measure space $(X, \mathcal{B})$ is a mapping

$$\varPhi : T \times \Omega \times X \to X$$

satisfying:

1. $\varPhi_0(\omega, \cdot) = \mathrm{id}_X$,
2. $\varPhi_{t+s}(\omega, \cdot) = \varPhi_t(\theta_s\omega, \cdot) \circ \varPhi_s(\omega, \cdot)$.

Condition 2 above is called the cocycle condition.

Example B.0.1. *Take $A, B \in M(n, \mathbb{R})$ such that $[A, B] = 0$. The $\mathbb{R}^d$-valued linear SDE,*

$$\delta X_t = AX_t\, dt + BX_t\, \delta B_t$$

has the fundamental matrix solution,

$$\Phi_t = e^{At+BW_t} X_0,$$

which generates a RDS over the ergodic DS generated by Brownian motion. We check the second condition:

$$\Phi_{t+s}(\omega) = e^{At+BW_{t+s}}(\omega) \tag{B.0.2}$$

$$= e^{At+BW_t}(\theta_s\omega)e^{As+BW_s}(\omega) \tag{B.0.3}$$

$$= \Phi_t(\theta_s\omega) \circ \Phi_s(\omega). \tag{B.0.4}$$

In 1968 Oseledets proved a multiplicative ergodic theorem which allowed the theory of Lyapunov exponents to be useful in the study of stochastic systems (and random dynamical systems in general). The following modern form of Oseledets's multiplicative ergodic theorem appears in [Arn03].

Theorem B.0.11 (Oseledets '68)**.** *If Φ is a linear RDS over a measure preserving DS θ, satisfying $\sup_{0\le t\le 1} \log^+ \|\Phi_t(\omega)\| < \infty$ then there exists a set of full measure $\tilde{\Omega} \in \mathcal{F}$ such that for each $\omega \in \tilde{\Omega}$*

1. *The limit $\Psi(\omega) := \lim_{t\to\infty}(\Phi_t(\omega)^*\Phi_t(\omega))^{1/2t}$ exists*

2. *The eigenvalues of Ψ are $e^{\gamma_{p(\omega)}(\omega)} < \cdots < e^{\gamma_1(\omega)}$ with eigenspaces $U_{n(\omega)}(\omega), \ldots, U_1(\omega)$ and multiplicites d_i. Write $V_i(\omega) := U_p(\omega) \oplus \cdots \oplus U_i(\omega)$.*

3. For each $x \in \mathbb{R}^d \backslash \{0\}$ *the Lyapunov exponent*

$$\gamma(\omega, x) := \lim_{t \to \infty} \log \|\Phi(t, \omega)x\|$$

exists and,

$$\gamma(\omega, x) = \gamma_i(\omega) \iff x \in V_i(\omega) \backslash V_{i+1}(\omega) \tag{B.0.5}$$

4. $\gamma_i(\omega), V_i(\omega), p(\omega)$ *are invariant under* θ*.*

Due to the last statement, if the underlying metric dynamical system, θ, is ergodic there are a important consequences for the subject of this thesis.

Appendix C

Stratonovich SDEs

For a rigorous treatment of the material in this appendix, the reader is reffered to [RY94, Øks10, Pro05, KP92]. We include some results relevant to this thesis, with proofs omitted. Throughout this section W_t is a standard, one-dimensional Brownian motion.

Definition C.0.12 (Itô and Stratonovich Wiener Integral)**.** For continuous, adapted, bounded one-dimensional processes $f(t,\omega)$ we have

$$\int_0^t f(t,\omega)\, dW_t(\omega) := \lim_{|\Delta|\to 0} \sum_{j\in\Delta} f(t_j,\omega)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)) \tag{C.0.1}$$

$$\int_0^t f(t,\omega)\, \delta W_t(\omega) := \lim_{|\Delta|\to 0} \sum_{j\in\Delta} f\Big(\frac{t_j + t_{j+1}}{2}, \omega\Big)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)) \tag{C.0.2}$$

Itô's lemma provides a simple relationship between the Itô and Stratonovich integrals. It can be shown that if X and Y are continuous semimartingales then,

$$\int_0^t X_s \, \delta Y_s = \int_0^t X_s \, dY_s + \frac{1}{2}\langle X, Y\rangle_t. \tag{C.0.3}$$

We can use (C.0.3) to convert SDEs between Stratonovich and Itô differential. Take X_t to be a solution to the $\mathbb{R}^d$-valued SDE

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t \tag{C.0.4}$$

where a, b are well behaved and W_t is a one dimensional Brownian motion. The Stratonovich SDE that corresponds to this Itô SDE is

$$\delta X_t = \tilde{a}(X_t, t)\, dt + b(X_t, t)\, \delta W_t \tag{C.0.5}$$

where the Stratonovich drift vector is defined by

$$\tilde{a}^i(X_t) = a^i(X_t) - \frac{1}{2}\sum_{j=1}^{d} b^j(X_t)\frac{\partial b^i}{\partial x_j}(X_t). \tag{C.0.6}$$

We include the following well known inequality [KP92] as a reference,

Theorem C.0.13. *[The Gronwall Inequality] Let $\alpha, \beta : [t_0, T] \to \mathbb{R}$ be integrable with*

$$0 \le \alpha_t \le \beta_t + L\int_{t_0}^t \alpha_s \, ds$$

for $t \in [t_0, T]$ where $L > 0$. The

$$\alpha_t \le \beta_t + L\int_{t_0}^t e^{L(t-s)}\beta_s \, ds$$

for $t \in [t_0, T]$.

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