ON THE CHENG-YAU GRADIENT ESTIMATE FOR CARNOT GROUPS AND SUB-RIEMANNIAN MANIFOLDS

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Abstract. In this note we show how results in [4,6,11] yield the Cheng-Yau estimate on two classes of sub-Riemannian manifolds: Carnot groups and sub-Riemannian manifolds satisfying a generalized curvature-dimension inequality.

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1. INTRODUCTION

Let $M$ be a $d$-dimensional Riemannian complete non-compact manifold with the Ricci curvature bounded below by $-(d-1)K$. Let $u$ be a positive harmonic function in a Riemannian ball $B(x_0, 2r)$, then we say that $u$ satisfies the Cheng-Yau estimate if

$$\sup_{B(x_0,r)} |\nabla \log u| \leq C_d \left( \frac{1}{r} + \sqrt{K} \right),$$

where $C_d$ is a global constant depending only on the dimension $d$. In particular, when $K = 0$ this estimate shows that positive harmonic functions are constant.

This estimate was formulated in a more general form in [10,28], and stated as in (1.1) in [24, Theorem 3.1]. Sharp versions of the Cheng-Yau inequality were given

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in [21, 22]. The stability of (1.1) under certain perturbations of the metric was considered in [29].

The standard curvature arguments are not easily available in the case when $M$ is replaced by a sub-Riemannian manifold. Nevertheless, there has been significant progress in geometric analysis on sub-Riemannian manifolds in [1, 4, 5, 7]. Even in the absence of a Riemannian structure, [7] developed new techniques to prove a number of results which in the Riemannian setting go back to the work of Yau and Li-Yau. The main tool in [7] relied on a generalized curvature-dimension inequality on a class of sub-Riemannian manifolds with transverse symmetries.

Carnot groups also form a large and interesting class of sub-Riemannian manifolds. These are Lie groups whose Lie algebra admits a stratified structure. This stratified structure allows for Hörmander’s condition [18, Theorem 1.1] to be satisfied. Hörmander’s theorem guarantees that the sub-Laplacian associated with the structure of a Carnot group is hypoelliptic. In particular, this gives us the existence of a smooth heat kernel for this Laplacian. Most Carnot groups do not satisfy a generalized curvature-dimension inequality, so one needs to employ different techniques than in [7].

The Cheng-Yau estimate was proved in [2, Corollary 4.6] for the simplest non-commutative Carnot group, the Heisenberg group, using probabilistic (coupling) techniques. The purpose of this note is to show that this estimate can be proven on two classes of sub-Riemannian manifolds, namely, sub-Riemannian manifolds satisfying a generalized curvature-dimension inequality and Carnot groups, by relying on results from [4, 6, 11]. In particular, this recovers the known fact that global non-negative harmonic functions in these two settings have to be constant (see [8, Theorem 5.8.1] and [5, Theorem 5.1] in more generality).

2. Carnot Groups

2.1. Preliminaries. We recall that a Carnot group of step $N$ is a simply connected Lie group $G$ whose Lie algebra can be written as

$$
g = V_1 \oplus \cdots \oplus V_N,$$

where

$$[V_i, V_j] = V_{i+j}$$

and $V_k = 0$ for $k > N$. In particular, Carnot groups are nilpotent.

Let $V_1, \ldots, V_d$ be a linear basis for the vector space $V_1$. The $V_i$s can be viewed as left-invariant vector fields on $G$. The left-invariant sub-Laplacian on $G$ is the operator

$$L = \sum_{i=1}^{d} V_i^2.$$  \hfill (2.2)

Let $\mu$ be the bi-invariant Haar measure on $G$. Since Carnot groups are complete the operator $L$ in (2.2) is essentially self-adjoint on $L^2(G, \mu)$, with domain being the space of smooth and compactly supported functions $f : G \to \mathbb{R}$ denoted by $C^\infty_c(G)$. We abuse notation and denote by $L$ the Friedrichs extension of this operator to a unique non-positive self-adjoint operator on $L^2(G, \mu)$. Then the heat semigroup $(P_t)_{t \geq 0}$ on $G$ can be defined through the spectral theorem. As $L$ is hypoelliptic,
$P_t$ admits a positive smooth fundamental solution to the heat equation called the heat kernel $p_t(g, g')$.

Denote by $\nabla = (V_1, \ldots, V_d)$ the gradient determined by this basis, and denote by $\| \cdot \|$ the usual Euclidean norm. The carré du champ operator of $L$ is defined by

$$\Gamma(f, f) := \frac{1}{2} (Lf^2 - 2fL_0) = \|\nabla f\|^2 = \sum_{i=1}^d (V_if)^2,$$

and it is often thought of as the square of the length of the gradient $\nabla$. We let $d$ be the Carnot-Carathéodory distance on $\mathbb{G}$ making $(\mathbb{G}, d)$ a metric space. We refer the reader to [8] for more details and results on Carnot groups.

2.2. The Cheng-Yau estimate. We say a function $u : \mathbb{G} \to \mathbb{R}$ is harmonic in a domain $D \subset \mathbb{G}$ if $Lu = 0$ on $D \subset \mathbb{G}$.

**Theorem 2.1.** If $u$ is any positive harmonic function for $L$ in a ball $B(x, 2r) \subset \mathbb{G}$, then there exists a constant $C > 0$ not dependent on $u, r$ and $x$ such that

$$\sup_{B(x, r)} \|\nabla \log u\| \leq \frac{C}{r}.$$ 

Moreover, if $u$ is a positive harmonic function on $\mathbb{G}$, then $u$ must be equal to a constant.

**Proof.** In the proof, $C$ will denote a generic positive constant that does not depend on $u, r$ and $x$. First recall the reverse Poincaré inequality for the heat semigroup obtained for Carnot groups in [4, Proposition 2.5]

$$(2.4) \quad \|\nabla P_t f(g)\|^2 \leq \frac{C}{t} \left( P_t f^2(g) - (P_t f)^2(g) \right)$$

for functions $f \in C^\infty_c(\mathbb{G})$. Note that for functions $f \in L^\infty(\mathbb{G})$ we have

$$(2.5) \quad P_t f^2(g) \leq \|f\|_{L^\infty(\mathbb{G})}^2,$$

therefore combining (2.4) and (2.5) we obtain

$$(2.6) \quad \|\nabla P_t f(g)\|^2 \leq \frac{C}{t} \|f\|_{L^\infty(\mathbb{G})}^2.$$ 

Taking a square root in (2.6) implies that

$$(2.7) \quad \|\nabla P_t\|_{\infty \to \infty} \leq \frac{C}{\sqrt{t}}.$$ 

Applying [11] Theorem 1.2, in particular that $(iii)$ implies $(i)$, shows that (2.7) implies that there exists a $C > 0$ such that, for every ball $B(x, 2r)$ and every function $u$ that is harmonic in $B(x, 2r)$ we have

$$(2.8) \quad \|\nabla u\|_{L^\infty(B(x,r))} \leq \frac{C}{r\mu(B(x,2r))} \int_{B(x,2r)} |u| \, d\mu.$$ 

Then we can apply [11] Lemma 2.3 to show that (2.8) implies the Cheng-Yau estimate (2.3).

We rely on results in [11] that require several assumptions that are satisfied for Carnot groups as follows. Their first assumption is that the underlying space is...
a non-compact doubling Dirichlet metric measure space. The space $G$ is doubling since by [16, Proposition 11.15] (or more classically by [14,20]) there exists a $C > 0$ independent of $x \in G, r > 0$ such that $|B(x,r)| = Cr^Q$ where $Q = \sum_{j=1}^{N} j \dim (V_j)$ is the homogeneous dimension of the $G$. Here $|E|$ denotes the Lebesgue measure of the set $E$ and recall that the Haar measure $\mu$ is Lebesgue measure up to a constant. We also have that $G$ is Dirichlet space as described in [25, Section 3, pp.233-234]. This is actually true in general for Hörmander’s type operators with bounded measurable coefficients on Lie groups having polynomial volume growth in the sense of [23]. Another ingredient in [11] is upper Gaussian bounds on the heat kernel which follow from [27, Theorem VIII2.9]. The space also supports a local scale-invariant $L^2$-Poincaré inequality by [16, Proposition 11.17].

Finally, if $u$ is a positive harmonic function on all of $G$, taking $r \to \infty$ in (2.3) gives us that $u$ must be constant.

\[ \square \]

3. Sub-Riemannian manifolds

3.1. Preliminaries. In this section we study the setting similar to [5]. We state relevant details here for completeness. Let $(\mathcal{M}, \mu)$ be a measure space, where $\mathcal{M}$ is an $n$-dimensional $C^\infty$ connected manifold endowed with a smooth measure $\mu$. Recall (e.g. [26, p.85]) that the measure $\mu$ on a smooth manifold $\mathcal{M}$ is called a smooth measure if $\mu$ is a Radon measure which has a smooth Radon-Nikodym derivative with respect to the Lebesgue measure when viewed in coordinates, that is, for any smooth coordinate chart $\varphi : U \rightarrow V, U \subset M, V \subset \mathbb{R}^n$, the pushforward measure $\varphi_*(\mu)$ has a smooth Radon-Nikodym derivative with respect to the Lebesgue measure on $V$. Let $L$ be a second order diffusion operator on $\mathcal{M}$ which is locally subelliptic (in the sense of [13,19]). We refer to [3, Section 1] for a detailed account on properties of locally subelliptic operators and associated distances that we are going to use in the sequel. In addition, we assume that

\[ L1 = 0, \]
\[ \int_{\mathcal{M}} f Lg d\mu = \int_{\mathcal{M}} g Lf d\mu, \]
\[ \int_{\mathcal{M}} f Lf \leq 0 \]

for every $f, g \in C_c^\infty (\mathcal{M})$, where as before $C_c^\infty (\mathcal{M})$ denotes the space of smooth compactly supported functions on $\mathcal{M}$.

The space $\mathcal{M}$ is endowed with a carré du champ operator defined by

\[ \Gamma(f,g) := \frac{1}{2} (L(fg) - f Lg - g Lf), \quad f, g \in C^\infty (\mathcal{M}). \]

We denote $\Gamma(f) = \Gamma(f,f)$. It is not too hard to see that $\Gamma(f) \geq 0$ for all $f \in C^\infty (\mathcal{M})$. We will also assume the existence of a symmetric, first-order differential bilinear form $\Gamma^Z : C^\infty (\mathcal{M}) \times C^\infty (\mathcal{M}) \rightarrow C^\infty (\mathcal{M})$ that satisfies

\[ \Gamma^Z (fg, h) = f \Gamma^Z (g, h) + g \Gamma^Z (f, h), \]
\[ \Gamma^Z (f) = \Gamma^Z (f, f) \geq 0, \]
for all \(f, g, h \in C^\infty(\mathbb{M})\). Given the first order bi-linear forms \(\Gamma\) and \(\Gamma^Z\) on \(\mathbb{M}\), we can introduce the following second-order differential forms

\[
\Gamma_2(f, g) = \frac{1}{2} (L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf))
\]

and

\[
\Gamma^Z_2(f, g) = \frac{1}{2} (L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)).
\]

Similar to \(\Gamma\), we will use the notation \(\Gamma_2(f) := \Gamma_2(f, f)\), \(\Gamma^Z_2(f) := \Gamma^Z_2(f, f)\).

We suppose the following assumptions to hold throughout this section.

(I) There exists an increasing sequence \(h_k \in C^\infty_c(\mathbb{M})\) such that \(h_k \uparrow 1\) on \(\mathbb{M}\), and

\[
\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to 0, \text{ as } k \to \infty.
\]

(II) For any \(f \in C^\infty(\mathbb{M})\) one has

\[
\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, f).
\]

(III) The generalized curvature-dimension inequality CD\((\rho_1, \rho_2, \kappa, d)\) is satisfied with \(\rho_1 \geq 0\). That is, there exist constants \(\rho_1 \geq 0, \rho_2 > 0, \kappa \geq 0,\) and \(d \geq 2\) such that the following inequality holds

\[
\Gamma_2(f) + \nu \Gamma^Z_2(f) \geq \frac{1}{d} (Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f) + \rho_2 \Gamma^Z(f),
\]

for all \(f \in C^\infty(\mathbb{M})\) and every \(\nu > 0\).

(IV) The heat semigroup generated by \(L\), which will be denoted \(P_t\), is stochastically complete, that is, for \(t \geq 0, P_t1 = 1\) and for every \(f \in C^\infty_c(\mathbb{M})\) and \(T \geq 0\), one has

\[
\sup_{t \in [0, T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty.
\]

(V) Given any two points \(x, y \in \mathbb{M}\), there exists a subunit curve (in the sense of [12]), joining them.

(VI) The metric space \((\mathbb{M}, d)\) is complete with respect to the intrinsic distance defined by

\[
d(x, y) := \sup \{|f(x) - f(y)| : f \in C^\infty(\mathbb{M}), \|\Gamma(f)\|_\infty \leq 1\},
\]

for all \(x, y \in \mathbb{M}\) and where we define \(\|g\|_\infty = \text{ess sup}_{\mathbb{M}} |g|\).

As a consequence of Assumption (VI) the operator \(L\) is essentially self-adjoint on \(C^\infty_c(\mathbb{M})\) (e.g., [3, Proposition 1.20, Proposition 1.21]). Thus, the pre-Dirichlet form \(\mathcal{E}(f, g)\) defined on \(C^\infty_c(\mathbb{M})\) by

\[
\mathcal{E}(f, g) = \int_{\mathbb{M}} \Gamma(f, g)d\mu,
\]

has a unique closure as a Dirichlet form, and the generator of this Dirichlet form is the Friedrichs extension of \(L\). We define the Sobolev space \(W^{1, 2}(\mathbb{M})\) to be the domain \(\mathcal{D}\) of \(\mathcal{E}\) with the norm on \(W^{1, 2}(\mathbb{M})\) given by

\[
\|f\|_{W^{1, 2}(\mathbb{M})} = \sqrt{\|f\|_2^2 + \mathcal{E}(f, f)}.
\]

It is a consequence of Assumption (III) that the metric measure space \((\mathbb{M}, d, \mu)\) satisfies the volume doubling property and supports a scale invariant \(L^2\)-Poincaré
inequality on metric balls (see [5]). In particular, by [17, Chapter 8] locally Lipschitz continuous functions form a dense subclass in $W^{1,2}(\mathbb{M})$.

For an open set $U \subset \mathbb{M}$, one can define the local Sobolev space $W^{1,2}_{\text{loc}}(U)$ to be the space of all functions $f$ such that for any compact set $K \subset U$ there exists $F \in \mathcal{D}$ satisfying $f = F$ a.e. on $K$. For each $p \geq 2$, we define $W^{1,p}(U)$ to be space of functions $f \in W^{1,2}_{\text{loc}}(U)$ satisfying $f, \sqrt{\Gamma(f)} \in L^p(U)$.

3.2. Examples. We remark that so far the approach has been purely analytical as we have not mentioned any geometric structure of these sub-Riemannian manifolds. In fact $\mathbb{M}$ and $L$ are rather general, even though we have sub-Riemannian manifolds in mind for $\mathbb{M}$.

3.2.1. Sum of squares operators. We start by recalling a natural setting where assumption (V) is satisfied. Let us consider $L$ that are sums of squares operators in the form of

$$L = \sum_{i=1}^{m} X_i^2 + X_0,$$

where the $X_i$ are $C^\infty$ vector fields. We refer the reader to [15] for more details on operators of the form given in (3.10) in the context of sub-Riemannian manifolds. Consider the following assumption.

**Assumption 3.1.** (Hörmander’s condition) We will say that $L$ satisfies Hörmander’s (bracket generating) condition if the vector fields $\{X_1, \ldots, X_m\}$ with their Lie brackets span the tangent space $T_x \mathbb{M}$ at every point $x \in \mathbb{M}$.

Hörmander’s condition guarantees analytic and topological properties such as hypoellipticity of $L$ and topological properties of $\mathbb{M}$. The Chow-Rashevski theorem says that Hörmander’s condition is sufficient to ensure that any two points in $\mathbb{M}$ can be connected by a finite length sub-unit curve. Thus, operators $L$ of the form (3.10) that satisfy Hörmander’s condition automatically satisfy assumption (V).

3.2.2. Other examples. We note that the assumptions (I)-(VI) are satisfied for a large class of sub-Riemannian manifolds. Such a class includes all Sasakian manifolds whose horizontal Webster-Tanaka-Ricci curvature is bounded below, a wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is bounded below, and Carnot groups of step 2. We remark that in general we do not know if $CD(\rho_1, \rho_2, \kappa, d)$ is satisfied for Carnot groups of an arbitrary step. This shows the need to treat Carnot groups separately in Theorem 2.1. We refer the reader to [7] for a comprehensive treatment on sub-Riemannian manifolds satisfying assumptions (I)-(VI).

3.3. The Cheng-Yau estimate. Recall that we say a function $u : \mathbb{M} \to \mathbb{R}$ is harmonic in a domain $D \subset \mathbb{M}$ if $Lu = 0$ on $D \subset \mathbb{M}$. We can now state the main result of this section.

**Theorem 3.2.** Suppose assumptions (I)-(VI) hold for $\mathbb{M}$ and $L$. If $u$ is any positive harmonic function for $L$ in a ball $B(x, 2r) \subset \mathbb{M}$, then there exists a constant $C > 0$ not dependent on $u, r$ and $x$ such that

$$\sup_{B(x, r)} \sqrt{\Gamma(u)} \leq \frac{C}{r}. \quad (3.11)$$
Moreover, if $u$ is any positive harmonic function on $\mathbb{M}$, then $u$ must be equal to a constant.

**Proof.** In the proof, $C, C_2$ will denote generic positive constants that do not depend on $u, r$ and $x_0$, whose values might change from line to line. First we check that the assumptions of the results in [11, Theorem 1.2, Lemma 2.3] are satisfied. The results in [11] require the assumption that the underlying space is a non-compact doubling Dirichlet metric measure space. Doubling is shown in (1.11) of [5, Theorem 1.5]. Further, we need $\mathbb{M}$ to satisfy upper Gaussian bounds on the heat kernel, which is given in [5, Theorem 4.1]. Finally we need $\mathbb{M}$ to support a local $L^2$-Poincaré inequality of the form

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_B| \, d\mu \leq C_2 r \left( \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \Gamma(f) \, d\mu \right)^{\frac{1}{2}},$$

for some $C_2 > 0$, for every ball $B(x,r)$ and each $f \in W^{1,2}(B(x,r))$. Here we used the notation $f_B = \mu(B)^{-1} \int_B f \, d\mu$. To see this, by (1.12) in [5, Theorem 1.5] we have that there exists constant $C_2 > 0$, depending only on $\rho_1, \rho_2, \kappa, d$, for which one has for every $x \in \mathbb{M}$ and every $r > 0$,

$$\int_{B(x,r)} |f - f_B|^2 \leq C_2 r^2 \int_{B(x,r)} \Gamma(f) \, d\mu,$$

for every $f \in C^1(B(x,r))$. Using Cauchy-Schwarz inequality, followed by (3.13) we have that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_B| \, d\mu \leq \frac{1}{\mu(B(x,r))} \left( \int_{B(x,r)} |f - f_B|^2 \, d\mu \right)^{\frac{1}{2}} \sqrt{\mu(B(x,r))}$$

$$\leq \frac{1}{\sqrt{\mu(B(x,r))}} \left( C_2 r^2 \int_{B(x,r)} \Gamma(f) \, d\mu \right)^{\frac{1}{2}}$$

$$= \sqrt{C_2 r \left( \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \Gamma(f) \, d\mu \right)^{\frac{1}{2}}}.$$

This shows (3.12) holds for all $f \in C^1(B(x,r))$. By a density argument we can show that (3.12) holds for all $f \in W^{1,2}(B(x,r))$, as needed.

To finish off the proof, we note that Corollary 3.5 in [6] shows that

$$\left\| \sqrt{\Gamma(P_t f)} \right\|_{\infty \to \infty} \leq C \frac{1}{\sqrt{t}},$$

where $C = n \sqrt{\frac{2\kappa + \rho_2}{4\rho_2}}$. Using the estimate (3.14), the rest of the proof becomes similar to the proof of Theorem 2.1. \hfill $\Box$

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