ON THE CHENG-YAU GRADIENT ESTIMATE FOR CARNOT GROUPS AND SUB-RIEMANNIAN MANIFOLDS

FABRICE BAUDOIN[⋆], MARIA GORDINA[†‡], AND PHANUEL MARIANO[†]

Abstract. In this note we show how results in [4,6,11] yield the Cheng-Yau estimate on two classes of sub-Riemannian manifolds: Carnot groups and sub-Riemannian manifolds satisfying a generalized curvature-dimension inequality.

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1. Introduction

Let M be a d-dimensional Riemannian complete non-compact manifold with the Ricci curvature bounded below by $-(d-1)K$. Let u be a positive harmonic function in a Riemannian ball $B(x_0, 2r)$, then we say that u satisfies the Cheng-Yau estimate if

$$\sup_{B(x_0,r)} |\nabla \log u| \leqslant C_d \left(\frac{1}{r} + \sqrt{K}\right), \tag{1.1}$$

where C_d is a global constant depending only on the dimension d. In particular, when $K = 0$ this estimate shows that positive harmonic functions are constant. This estimate was formulated in a more general form in [10, 28], and stated as in (1.1) in [24, Theorem 3.1]. Sharp versions of the Cheng-Yau inequality were given

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in [21, 22]. The stability of (1.1) under certain perturbations of the metric was considered in [29]

The standard curvature arguments are not easily available in the case when M is replaced by a sub-Riemannian manifold. Nevertheless, there has been significant progress in geometric analysis on sub-Riemannian manifolds in [1, 4, 5, 7]. Even in the absence of a Riemannian structure, [7] developed new techniques to prove a number of results which in the Riemannian setting go back to the work of Yau and Li-Yau. The main tool in [7] relied on a generalized curvature-dimension inequality on a class of sub-Riemannian manifolds with transverse symmetries.

Carnot groups also form a large and interesting class of sub-Riemannian manifolds. These are Lie groups whose Lie algebra admits a stratified structure. This stratified structure allows for Hörmander's condition [18, Theorem 1.1] to be satisfied. Hörmander's theorem guarantees that the sub-Laplacian associated with the structure of a Carnot group is hypoelliptic. In particular, this gives us the existence of a smooth heat kernel for this Laplacian. Most Carnot groups do not satisfy a generalized curvature-dimension inequality, so one needs to employ different techniques than in [7].

The Cheng-Yau estimate was proved in [2, Corollary 4.6] for the simplest non-commutative Carnot group, the Heisenberg group, using probabilistic (coupling) techniques. The purpose of this note is to show that this estimate can be proven on two classes of sub-Riemannian manifolds, namely, sub-Riemannian manifolds satisfying a generalized curvature-dimension inequality and Carnot groups, by relying on results from [4, 6, 11]. In particular, this recovers the known fact that global non-negative harmonic functions in these two settings have to be constant (see [8, Theorem 5.8.1] and [5, Theorem 5.1] in more generality).

2. Carnot groups

2.1. **Preliminaries.** We recall that a Carnot group of step N is a simply connected Lie group $\mathbb{G}$ whose Lie algebra can be written as

$$\mathfrak{g} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and $\mathcal{V}_k = 0$ for $k > N$. In particular, Carnot groups are nilpotent.

Let $V_1, \dots, V_d$ be a linear basis for the vector space $\mathcal{V}_1$. The V_is can be viewed as left-invariant vector fields on $\mathbb{G}$. The left-invariant sub-Laplacian on $\mathbb{G}$ is the operator

$$L = \sum_{i=1}^{d} V_i^2. \tag{2.2}$$

Let μ be the bi-invariant Haar measure on $\mathbb{G}$. Since Carnot groups are complete the operator L in (2.2) is essentially self-adjoint on $L^2(\mathbb{G}, \mu)$, with domain being the space of smooth and compactly supported functions $f : \mathbb{G} \to \mathbb{R}$ denoted by $C_c^\infty(\mathbb{G})$. We abuse notation and denote by L the Friedrichs extension of this operator to a unique non-positive self-adjoint operator on $L^2(\mathbb{G}, \mu)$. Then the heat semigroup $(P_t)_{t \geqslant 0}$ on $\mathbb{G}$ can be defined through the spectral theorem. As L is hypoelliptic,

P_t admits a positive smooth fundamental solution to the heat equation called the heat kernel $p_t(g, g')$.

Denote by $\nabla = (V_1, \dots, V_d)$ the gradient determined by this basis, and denote by $\|\cdot\|$ the usual Euclidean norm. The *carré du champ* operator of L is defined by

$$\Gamma(f,f) := \frac{1}{2}\left(Lf^2 - 2fLf\right) = \|\nabla f\|^2 = \sum_{i=1}^{d} (V_i f)^2,$$

and it is often thought of as the square of the *length of the gradient* ∇. We let d be the Carnot-Carathéodory distance on $\mathbb{G}$ making $(\mathbb{G}, d)$ a metric space. We refer the reader to [8] for more details and results on Carnot groups.

2.2. **The Cheng-Yau estimate.** We say a function $u : \mathbb{G} \to \mathbb{R}$ is *harmonic* in a domain $D \subset \mathbb{G}$ if $Lu = 0$ on $D \subset \mathbb{G}$.

Theorem 2.1. *If u is any positive harmonic function for L in a ball $B(x, 2r) \subset \mathbb{G}$, then there exists a constant $C > 0$ not dependent on u, r and x such that*

$$\sup_{B(x,r)} \|\nabla \log u\| \leqslant \frac{C}{r}. \tag{2.3}$$

Moreover, if u is a positive harmonic function on $\mathbb{G}$, then u must be equal to a constant.

Proof. In the proof, C will denote a generic positive constant that does not depend on u, r and x whose value might change from line to line. First recall the reverse Poincaré inequality for the heat semigroup obtained for Carnot groups in [4, Proposition 2.5]

$$\|\nabla P_t f(g)\|^2 \leqslant \frac{C}{t}\left(P_t f^2(g) - (P_t f)^2(g)\right) \tag{2.4}$$

for functions $f \in C_c^\infty(\mathbb{G})$. Note that for functions $f \in L^\infty(\mathbb{G})$ we have

$$P_t f^2(g) \leqslant \|f\|^2_{L^\infty(G)}, \tag{2.5}$$

therefore combining (2.4) and (2.5) we obtain

$$\|\nabla P_t f(g)\|^2 \leqslant \frac{C}{t}\|f\|^2_{L^\infty(G)}. \tag{2.6}$$

Taking a square root in (2.6) implies that

$$\|\nabla P_t\|_{\infty\to\infty} \leqslant \frac{C}{\sqrt{t}}. \tag{2.7}$$

Applying [11, Theorem 1.2], in particular that (iii) implies (i), shows that (2.7) implies that there exists a $C > 0$ such that, for every ball $B(x, r)$ and every function u that is harmonic in $B(x, 2r)$ we have

$$\|\nabla u\|_{L^\infty(B(x,r))} \leqslant \frac{C}{r\mu(B(x,2r))}\int_{B(x,2r)} |u|\, d\mu. \tag{2.8}$$

Then we can apply [11, Lemma 2.3] to show that (2.8) implies the Cheng-Yau estimate (2.3).

We rely on results in [11] that require several assumptions that are satisfied for Carnot groups as follows. Their first assumption is that the underlying space is a non-compact doubling Dirichlet metric measure space. The space $\mathbb{G}$ is doubling since by [16, Proposition 11.15] (or more classically by [14,20]) there exists a $C>0$ independent of $x \in \mathbb{G}, r>0$ such that $|B(x,r)| = Cr^{Q}$ where $Q=\sum_{j=1}^{N} j \dim(\mathcal{V}_j)$ is the homogeneous dimension of the $\mathbb{G}$. Here $|E|$ denotes the Lebesgue measure of the set E and recall that the Haar measure μ is Lebesgue measure up to a constant. We also have that $\mathbb{G}$ is Dirichlet space as described in [25, Section 3, pp.233-234]. This is actually true in general for Hörmander's type operators with bounded measurable coefficients on Lie groups having polynomial volume growth in the sense of [23]. Another ingredient in [11] is upper Gaussian bounds on the heat kernel which follow from [27, Theorem VIII2.9]. The space also supports a local scale-invariant L^2-Poincaré inequality by [16, Proposition 11.17].

Finally, if u is a positive harmonic function on all of $\mathbb{G}$, taking $r \to \infty$ in (2.3) gives us that u must be constant.

□

3. Sub-Riemannian manifolds

3.1. **Preliminaries.** In this section we study the setting similar to [5]. We state relevant details here for completeness. Let $(\mathbb{M}, \mu)$ be a measure space, where $\mathbb{M}$ is an n-dimensional C^{∞} connected manifold endowed with a smooth measure μ. Recall (e.g. [26, p.85]) that the measure μ on a smooth manifold $\mathbb{M}$ is called a *smooth measure* if μ is a Radon measure which has a smooth Radon-Nikodym derivative with respect to the Lebesgue measure when viewed in coordinates, that is, for any smooth coordinate chart $\varphi : U \longrightarrow V$, $U \subset M$, $V \subset \mathbb{R}^n$, the pushforward measure $\varphi_*(\mu)$ has a smooth Radon-Nikodym derivative with respect to the Lebesgue measure on V. Let L be a second order diffusion operator on $\mathbb{M}$ which is locally subelliptic (in the sense of [13,19]). We refer to [3, Section 1] for a detailed account on properties of locally subelliptic operators and associated distances that we are going to use in the sequel. In addition, we assume that

$$
\begin{aligned}
& L1 = 0, \\
& \int_{\mathbb{M}} fLg d\mu = \int_{\mathbb{M}} gLf d\mu, \\
& \int_{\mathbb{M}} fLf \leqslant 0
\end{aligned}
$$

for every $f, g \in C_c^{\infty}(\mathbb{M})$, where as before $C_c^{\infty}(\mathbb{M})$ denotes the space of smooth compactly supported functions on $\mathbb{M}$.

The space $\mathbb{M}$ is endowed with a *carré du champ* operator defined by

$$
\Gamma(f,g) := \frac{1}{2}\left(L(fg) - fLg - gLf\right), \ f, g \in C^{\infty}(\mathbb{M}).
$$

We denote $\Gamma(f) = \Gamma(f,f)$. It is not too hard to see that $\Gamma(f) \geqslant 0$ for all $f \in C^{\infty}(\mathbb{M})$. We will also assume the existence of a symmetric, first-order differential bilinear form $\Gamma^Z : C^{\infty}(\mathbb{M}) \times C^{\infty}(\mathbb{M}) \to C^{\infty}(\mathbb{M})$ that satisfies

$$
\begin{aligned}
\Gamma^Z(fg,h) &= f\Gamma^Z(g,h) + g\Gamma^Z(f,h), \\
\Gamma^Z(f) &= \Gamma^Z(f,f) \geqslant 0,
\end{aligned}
$$

for all $f, g, h \in C^{\infty}(\mathbb{M})$. Given the first order bi-linear forms Γ and Γ^Z on $\mathbb{M}$, we can introduce the following second-order differential forms

$$\Gamma_2(f,g) = \frac{1}{2}\left(L\Gamma(f,g) - \Gamma(f, Lg) - \Gamma(g, Lf)\right)$$

and

$$\Gamma_2^Z(f,g) = \frac{1}{2}\left(L\Gamma^Z(f,g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)\right).$$

Similar to Γ, we will use the notation $\Gamma_2(f) := \Gamma_2(f,f), \Gamma_2^Z(f) := \Gamma_2^Z(f,f)$.

We suppose the following assumptions to hold throughout this section.

(I) There exists an increasing sequence $h_k \in C_c^{\infty}(\mathbb{M})$ such that $h_k \uparrow 1$ on $\mathbb{M}$, and

$$\|\Gamma(h_k)\|_{\infty} + \left\|\Gamma^Z(h_k)\right\|_{\infty} \to 0, \text{ as } k \to \infty.$$

(II) For any $f \in C^{\infty}(\mathbb{M})$ one has

$$\Gamma\left(f, \Gamma^Z(f)\right) = \Gamma^Z(f, \Gamma(f)).$$

(III) The *generalized curvature-dimension inequality* $CD(\rho_1, \rho_2, \kappa, d)$ is satisfied with $\rho_1 \geqslant 0$. That is, there exist constants $\rho_1 \geqslant 0, \rho_2 > 0, \kappa \geqslant 0$, and $d \geqslant 2$ such that the following inequality holds

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geqslant \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f),$$

for all $f \in C^{\infty}(\mathbb{M})$ and every $\nu > 0$.

(IV) The heat semigroup generated by L, which will be denoted P_t, is stochastically complete, that is, for $t \geqslant 0, P_t 1 = 1$ and for every $f \in C_c^{\infty}(\mathbb{M})$ and $T \geqslant 0$, one has

$$\sup_{t\in[0,T]} \|\Gamma(P_t f)\|_{\infty} + \left\|\Gamma^Z(P_t f)\right\|_{\infty} < +\infty.$$

(V) Given any two points $x, y \in \mathbb{M}$, there exists a subunit curve (in the sense of [12]), joining them.

(VI) The metric space $(\mathbb{M}, d)$ is complete with respect to the intrinsic distance defined by

$$d(x,y) := \sup\left\{|f(x) - f(y)| : f \in C^{\infty}(\mathbb{M}), \|\Gamma(f)\|_{\infty} \leqslant 1\right\}, \tag{3.9}$$

for all $x, y \in \mathbb{M}$ and where we define $\|g\|_{\infty} = \operatorname{ess\,sup}_{\mathbb{M}} |g|$.

Note that by [9, Lemma 5.29] and [5, Equation (2.4)] we know that Assumption (V) implies that d is indeed a metric on $\mathbb{M}$.

As a consequence of Assumption (VI), the operator L is essentially self-adjoint on $C_c^{\infty}(\mathbb{M})$ (e.g. [3, Proposition 1.20, Proposition 1.21]). Thus, the pre-Dirichlet form $\mathcal{E}(f,g)$ defined on $C_c^{\infty}(\mathbb{M})$ by

$$\mathcal{E}(f,g) = \int_{\mathbb{M}} \Gamma(f,g)d\mu,$$

has a unique closure as a Dirichlet form, and the generator of this Dirichlet form is the Friedrichs extension of L. We define the Sobolev space $W^{1,2}(\mathbb{M})$ to be the domain $\mathcal{D}$ of $\mathcal{E}$ with the norm on $W^{1,2}(\mathbb{M})$ given by

$$\|f\|_{W^{1,2}(\mathbb{M})} = \sqrt{\|f\|_2^2 + \mathcal{E}(f,f)}.$$

It is a consequence of Assumption (III) that the metric measure space $(\mathbb{M}, d, \mu)$ satisfies the volume doubling property and supports a scale invariant L^2-Poincaré inequality on metric balls (see [5]). In particular, by [17, Chapter 8] locally Lipschitz continuous functions form a dense subclass in $W^{1,2}(\mathbb{M})$.

For an open set $U \subset \mathbb{M}$, one can define the local Sobolev space $W^{1,2}_{\mathrm{loc}}(U)$ to be the space of all functions f such that for any compact set $K \subset U$ there exists $F \in \mathcal{D}$ satisfying $f = F$ a.e. on K. For each $p \geqslant 2$, we define $W^{1,p}(U)$ to be space of functions $f \in W^{1,2}_{\mathrm{loc}}(U)$ satisfying $f, \sqrt{\Gamma(f)} \in L^p(U)$.

3.2. **Examples.** We remark that so far the approach has been purely analytical as we have not mentioned any geometric structure of these sub-Riemannian manifolds. In fact $\mathbb{M}$ and L are rather general, even though we have sub-Riemannian manifolds in mind for $\mathbb{M}$.

3.2.1. *Sum of squares operators.* We start by recalling a natural setting where assumption (V) is satisfied. Let us consider L that are *sums of squares* operators in the form of

$$L = \sum_{i=1}^{m} X_i^2 + X_0, \tag{3.10}$$

where the X_i are C^∞ vector fields. We refer the reader to [15] for more details on operators of the form given in (3.10) in the context of sub-Riemannian manifolds. Consider the following assumption.

Assumption 3.1. *(Hörmander's condition) We will say that L satisfies* Hörmander's (bracket generating) condition *if the vector fields $\{X_1, \dots, X_m\}$ with their Lie brackets span the tangent space $T_x\mathbb{M}$ at every point $x \in \mathbb{M}$.*

Hörmander's condition guarantees analytic and topological properties such as hypoellipticity of L and topological properties of $\mathbb{M}$. The Chow-Rashevski theorem says that Hörmander's condition is sufficient to ensure that any two points in $\mathbb{M}$ can be connected by a finite length sub-unit curve. Thus, operators L of the form (3.10) that satisfy Hörmander's condition automatically satisfy assumption (V).

3.2.2. *Other examples.* We note that the assumptions (I)-(VI) are satisfied for a large class of sub-Riemannian manifolds. Such a class includes all Sasakian manifolds whose horizontal Webster-Tanaka-Ricci curvature is bounded below, a wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is bounded below, and Carnot groups of step 2. We remark that in general we do not know if $CD(\rho_1, \rho_2, \kappa, d)$ is satisfied for Carnot groups of an arbitrary step. This shows the need to treat Carnot groups separately in Theorem 2.1. We refer the reader to [7] for a comprehensive treatment on sub-Riemannian manifolds satisfying assumptions (I)-(VI).

3.3. **The Cheng-Yau estimate.** Recall that we say a function $u : \mathbb{M} \to \mathbb{R}$ is *harmonic* in a domain $D \subset \mathbb{M}$ if $Lu = 0$ on $D \subset \mathbb{M}$. We can now state the main result of this section.

Theorem 3.2. *Suppose assumptions (I)-(VI) hold for $\mathbb{M}$ and L. If u is any positive harmonic function for L in a ball $B(x, 2r) \subset \mathbb{M}$, then there exists a constant $C > 0$ not dependent on u, r and x such that*

$$\sup_{B(x,r)} \sqrt{\Gamma(\log u)} \leqslant \frac{C}{r}. \tag{3.11}$$

Moreover, if u is any positive harmonic function on $\mathbb{M}$, then u must be equal to a constant.

Proof. In the proof, C, C_2 will denote generic positive constants that do not depend on u, r and x_0, whose values might change from line to line. First we check that the assumptions of the results in [11, Theorem 1.2, Lemma 2.3] are satisfied. The results in [11] require the assumption that the underlying space is a non-compact doubling Dirichlet metric measure space. Doubling is shown in (1.11) of [5, Theorem 1.5]. Further, we need $\mathbb{M}$ to satisfy upper Gaussian bounds on the heat kernel, which is given in [5, Theorem 4.1]. Finally we need $\mathbb{M}$ to support a local L^2-Poincaré inequality of the form

$$\frac{1}{\mu\left(B(x,r)\right)}\int_{B(x,r)}|f-f_B|\,d\mu \leqslant C_2 r\left(\frac{1}{\mu\left(B(x,r)\right)}\int_{B(x,r)}\Gamma\left(f\right)d\mu\right)^{\frac{1}{2}}, \tag{3.12}$$

for some $C_2>0$, for every ball $B(x,r)$ and each $f\in W^{1,2}\left(B(x,r)\right)$. Here we used the notation $f_B=\mu\left(B\right)^{-1}\int_B f d\mu$. To see this, by (1.12) in [5, Theorem 1.5] we have that there exists constant $C_2>0$, depending only on ρ_1,ρ_2,κ,d, for which one has for every $x\in\mathbb{M}$ and every $r>0$,

$$\int_{B(x,r)}|f-f_B|^2 \leqslant C_2 r^2\int_{B(x,r)}\Gamma(f)d\mu, \tag{3.13}$$

for every $f\in C^1\left(\overline{B}(x,r)\right)$. Using Cauchy-Schwarz inequality, followed by (3.13) we have that

$$\begin{aligned}
\frac{1}{\mu\left(B(x,r)\right)}\int_{B(x,r)}|f-f_B|\,d\mu &\leqslant \frac{1}{\mu\left(B(x,r)\right)}\left(\int_{B(x,r)}|f-f_B|^2\,d\mu\right)^{\frac{1}{2}}\sqrt{\mu\left(B(x,r)\right)}\\
&\leqslant \frac{1}{\sqrt{\mu\left(B(x,r)\right)}}\left(C_2 r^2\int_{B(x,r)}\Gamma(f)d\mu\right)^{\frac{1}{2}}\\
&= \sqrt{C_2}\,r\left(\frac{1}{\mu\left(B(x,r)\right)}\int_{B(x,r)}\Gamma(f)d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

This shows (3.12) holds for all $f\in C^1\left(\overline{B}(x,r)\right)$. By a density argument we can show that (3.12) holds for all $f\in W^{1,2}\left(B(x,r)\right)$, as needed.

To finish off the proof, we note that Corollary 3.5 in [6] shows that

$$\left\|\sqrt{\Gamma\left(P_t f\right)}\right\|_{\infty\to\infty}\leqslant\frac{C}{\sqrt{t}}, \tag{3.14}$$

where $C=n\sqrt{\frac{(2\kappa+\rho_2)}{2\rho_2}}$. Using the estimate (3.14), the rest of the proof becomes similar to the proof of Theorem 2.1. □

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Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
Email address: fabrice.baudoin@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
Email address: maria.gordina@uconn.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
Email address: pmariano@purdue.edu