Exercise 1

Let m and n be positive integers. Show that:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z},$$

where $\mathbb{Z}$ denotes the integers, and $d = (m,n)$ denotes the greatest common divisor of m and n.

Let $L : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/(m,n)\mathbb{Z}$ be defined as follows. If $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is $\mathbb{Z}$-linear, and if $f(1+m\mathbb{Z}) = k + n\mathbb{Z}$, then

$$L(f) = k + (m,n)\mathbb{Z}.$$

- L does in fact define a map from $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ to $\mathbb{Z}/(m,n)\mathbb{Z}$. For suppose $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is $\mathbb{Z}$-linear. Then since $1 + m\mathbb{Z}$ has order m in $\mathbb{Z}/m\mathbb{Z}$, $f(1+m\mathbb{Z})$ has order dividing m. But since $f(1+m\mathbb{Z})$ is an element of $\mathbb{Z}/n\mathbb{Z}$, $nf(1+m\mathbb{Z}) \equiv 0 \bmod n$ and so $f(1+m\mathbb{Z})$ has order dividing n. Therefore $(m,n)f(1+m\mathbb{Z}) \equiv 0 \bmod n$ and hence we may construe $f(1+m\mathbb{Z})$ as an integer modulo (m,n).

- L is surjective. For given $k + (m,n)\mathbb{Z} \in \mathbb{Z}/(m,n)\mathbb{Z}$, we may define

  $$f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

  by $f(1+m\mathbb{Z}) = k + n\mathbb{Z}$. Since $\mathrm{ord}(k) \mid (m,n)$, this defines a linear map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. And $L(f) = k + (m,n)\mathbb{Z}$.

- L is linear. For if $a, b \in \mathbb{Z}$ and $f, g \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, then

  $$L(af+bg) = (af+bg)(1+m\mathbb{Z}) = af(1+m\mathbb{Z}) + bg(1+m\mathbb{Z}) = aL(f) + bL(g).$$

- L is injective. For any linear map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is determined entirely by $f(1+m\mathbb{Z})$ (since $\mathbb{Z}/m\mathbb{Z}$ is cyclic) and hence any two linear maps $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ differ if and only if they differ at $1 + m\mathbb{Z}$.

Exercise 2

> Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a})\otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. [Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M.]

Let $i : \mathfrak{a} \to A$ be inclusion and $p : A \to A/\mathfrak{a}$ be projection. Then

$$0 \to \mathfrak{a} \xrightarrow{i} A \xrightarrow{p} A/\mathfrak{a} \to 0$$

is exact, and so by the exactness of the tensor product,

$$0 \to \mathfrak{a} \otimes_A M \xrightarrow{f} A \otimes_A M \xrightarrow{g} (A/\mathfrak{a}) \otimes_A M \to 0$$

is exact also.[1][2] Because $A \otimes_A M$ is uniquely isomorphic to M by $a \otimes_A m \mapsto am$, there are maps h, k such that

$$0 \to \mathfrak{a}M \xrightarrow{h} M \xrightarrow{k} (A/\mathfrak{a}) \otimes_A M \to 0$$

is exact. By the exactness of the above sequence, $\ker(k) = \mathfrak{a}M$. Therefore by the first isomorphism theorem,

$$M/\mathfrak{a}M \cong (A/\mathfrak{a}) \otimes_A M.$$

Exercise 3

> Let A be a commutative ring, let I and J be ideals of A, and let M be an A-module. Show that: $(A/I) \otimes_A (A/J) \cong A/(I + J)$.

Here Exercise 2 does the heavy lifting. With $\mathfrak{a} = I$ and $M = A/J$, we obtain

$$\begin{aligned}
(A/I) \otimes_A (A/J) &\cong (A/J)/(I(A/J)) && \text{(Exercise 2)} \\
&= (A/J)/((IA + J)/J) && \text{(rewriting)} \\
&\cong A/(IA + J) && \text{(Proposition 2.1)} \\
&= A/(I + J) && \text{(IA = A)}
\end{aligned}$$

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[1]Here $f = i \otimes_A \mathrm{id}_M$ and $g = p \otimes_A \mathrm{id}_M$.

[2]Here Professor Glaz notes that the first link in this sequence isn't guaranteed.

Exercise 4

Let A be a commutative ring and let $\{M_i\}_{i\in T}$ and N be A-modules. Show that $(\oplus M_i)\otimes N \cong \oplus(M_i \otimes N)$.

Let $B : (\oplus M_i)\times N \to \oplus(M_i\otimes N)$ be given by

$$(\{m_i\}_{i\in T}, n) \mapsto \{m_i \otimes n\}_{i\in T}.$$

Then B is bilinear:

$$\begin{aligned} B(\{am_i\}_{i\in T}, n) &= \{(am_i)\otimes n\}_{i\in T} \\ &= a\cdot\{m_i\otimes n\}_{i\in T}\{m_i\otimes(an)\}_{i\in T} = B(\{m_i\}_{i\in T}, an) \end{aligned}$$

and the additivity properties follow from those of the direct sum. Therefore by the universal property of the tensor product there is a unique linear map

$$L : (\oplus M_i)\otimes N \to \oplus(M_i\otimes N)$$

satisfying $L(x\otimes y) = B(x,y)$ on elementary tensors. Similarly, there are unique linear maps (one for each $i \in T$)

$$K_i : M_i \otimes N \to (\oplus M_i)\otimes N$$

satisfying

$$K_i(m_i\otimes n) = (0,\ldots,\underset{\substack{\uparrow \\ i^{\text{th}}\text{ place}}}{m_i},\ldots,0)\otimes n$$

Let $K : \oplus(M_i\otimes N) \to (\oplus M_i)\otimes N$ be given by

$$K\left(\{m_i\otimes n\}_{i\in T}\right) = \sum_{i\in T} K_i(m_i\otimes n).$$

Then K is linear because K_i is for each $i\in T$, and L and K are inverses.

Exercise 5

Let A be a commutative ring. Do Exercise 2.4 from the book, and conclude that any free A-module is flat.

Exercise 2.4

Let $M_i\,(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Optional addition to this exercise (only if you learned about projective modules): One definition of a projective module A is: P is a projective A-module iff P is a direct summand of a free A-module. Conclude that projective modules are flat.

We will use the notation 1 to denote the identity on M and 1_i to denote the identity on M_i.

Suppose M is flat. Then if N', N are A-modules, and $f : N' \to N$ is injective, $f \otimes 1 : N' \otimes M \to N \otimes M$ is injective also (Proposition 2.19). By Exercise 4 and Proposition 2.14, it follows that there are isomorphisms h, k such that $h : N' \otimes M \to \oplus_{i\in I}(N' \otimes M_i)$ and $k : N \otimes M \to \oplus_{i\in I}(N \otimes M_i)$ and

$$k \circ f \circ h^{-1} : \oplus_{i\in I}(N' \otimes M_i) \to \oplus_{i\in I}(N \otimes M_i)$$

is injective. Call the function above g, so that

$$g : \oplus_{i\in I}(N' \otimes M_i) \to \oplus_{i\in I}(N \otimes M_i)$$

is injective, and for $i \in I$ let g_i denote the i^{th} component of g. Then since g is injective, g_i is injective for each $i \in I$.[3] But g_i is just

$$f \otimes 1_i : N' \otimes M_i \to N \otimes M_i.$$

So by Proposition 2.19, M_i is flat (for each $i \in I$, since the choice of index was arbitrary).

Conversely, suppose that M_i is flat for each $i \in I$. Then if N', N are A-modules and $f : N' \to N$ is injective,

$$f \otimes 1_i : N' \otimes M_i \to N \otimes M_i$$

[3] If $j \in I$ is such that $x \neq y \in N' \otimes M_i$ but $g_j(x) = g_j(y)$, then $g(\hat{x}) = g(\hat{y})$, where $\hat{x}$ is the tuple with x in the j^{th} place and 0 elsewhere, and similarly for $\hat{y}$.

is injective for each i. Hence the direct sum of these maps,

$$\oplus_{i\in I}(f\otimes 1_i):\oplus_{i\in I}(N'\otimes M_i)\to\oplus_{i\in I}(N\otimes M_i),$$

is injective. But as before, the map displayed above is—up to composition with isomorphisms—the same as

$$f\otimes 1:N'\otimes(\oplus_{i\in I}M_i)\to N\otimes(\oplus_{i\in I}M_i),$$

i.e.

$$f\otimes 1:N'\otimes M\to N\otimes M$$

is injective. Hence by Proposition 2.19, M is flat.[4]

Exercise 6

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. [Use Exercise 4.]

We claim that

$$A[x]\cong\bigoplus_{i\in\mathbb{N}}A.$$

Then by Exercise 2.4 (our Exercise 5), $A[x]$ is flat if and only if A is. But A is a flat A-algebra (because $A\otimes_A B\cong B$ for any A-algebra B). To see that

$$A[x]\cong\bigoplus_{i\in\mathbb{N}}A,$$

observe that the map

$$a_0+a_1x+\cdots+a_nx^n\mapsto(a_0,a_1,\ldots,a_n,0,0,0,\ldots)$$

is linear and has inverse

$$\{a_i\}_{i\in\mathbb{N}}\mapsto a_0+a_1x+\cdots+a_nx^n,$$

where a_n is the nonzero element of greatest index in the sequence $\{a_i\}_{i\in\mathbb{N}}$.

[4] Professor Glaz adds: If F is free, $F\cong A^I=\oplus_I A$; since A is A-flat, F is flat.

Exercise 7

Let G and H be $\mathbb{Z}$-modules (abelian groups). Determine the structure of $G \otimes_{\mathbb{Z}} H$ in each of the following cases:

(i) G and H are infinite cyclic

(ii) G and H are finite cyclic

(iii) G is finite cyclic and H is infinite cyclic

(iv) G and H are finitely generated

(v) G and H are free

(i) If G and H are infinite cyclic, then $G \cong H \cong \mathbb{Z}$. So

$$G \otimes_{\mathbb{Z}} H \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}.$$

Thus $G \otimes_{\mathbb{Z}} H$ is an infinite cyclic group with generator $1 \otimes_{\mathbb{Z}} 1$, hence isomorphic to $\mathbb{Z}$; i.e. $\boxed{G \otimes_{\mathbb{Z}} H \text{ is infinite cyclic.}}$

(ii) If G and H are finite cyclic, then there are $m, n \in \mathbb{Z}$ such that $G \cong \mathbb{Z}/m\mathbb{Z}$ and $H \cong \mathbb{Z}/n\mathbb{Z}$. So

$$G \otimes_{\mathbb{Z}} H \cong (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}).$$

By Exercise 3, this means that

$$G \otimes_{\mathbb{Z}} H \cong \mathbb{Z}/(m\mathbb{Z} + n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z},$$

i.e. $\boxed{G \otimes_{\mathbb{Z}} H \text{ is finite cyclic.}}$ To check that $m\mathbb{Z}+n\mathbb{Z} = (m,n)\mathbb{Z}$, we observe that since $\mathbb{Z}$ is a PID, there is some $k \in \mathbb{Z}$ such that $m\mathbb{Z} + n\mathbb{Z} = k\mathbb{Z}$. Furthermore, since $m\mathbb{Z} \subset m\mathbb{Z} + n\mathbb{Z}$ and $n\mathbb{Z} \subset m\mathbb{Z} + n\mathbb{Z}$, $k \mid m$ and $k \mid n$. Thus $k \mid (m,n)$, meaning that $(m,n)\mathbb{Z} \subset k\mathbb{Z}$ (alternatively, this follows immediately from Bezout's identity). But suppose $m = k(m,n)$ and $n = l(m,n)$. Then for any $a, b \in \mathbb{Z}$,

$$am + bn = ak(m,n) + bl(m,n) = (ak + bl)(m,n) \in (m,n)\mathbb{Z},$$

so $k\mathbb{Z} \subset (m,n)\mathbb{Z}$. This completes the proof.

(iii) If G is finite cyclic and H is infinite cyclic, then there is $m \in \mathbb{Z}$ such that $G \cong \mathbb{Z}/m\mathbb{Z}$ and $H \cong \mathbb{Z}$. So

$$G \otimes_{\mathbb{Z}} H \cong (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z},$$

i.e. $\boxed{G \otimes_{\mathbb{Z}} H \text{ is finite cyclic.}}$

(iv) If G and H are finitely generated, then there are finite subsets $\{g_1, \ldots, g_m\} \subset G$ generating G and $\{h_1, \ldots, h_n\} \subset H$ generating H. For convenience we take each generating set to already contain the inverse of each of its elements. Take any two elements $g = \sum_i g_i \in G$ and $h = \sum_j h_j \in H$ (where the sums are finite and each g_i and h_j is among the finite set of generators; note that we don't strictly need coefficients from $\mathbb{Z}$ since the generating sets are closed under inversion). Then

$$\begin{aligned} g \otimes_{\mathbb{Z}} h &= \left(\sum_i g_i\right) \otimes_{\mathbb{Z}} \left(\sum_j h_j\right) \\ &= \sum_i \left(g_i \otimes_{\mathbb{Z}} \left(\sum_j h_j\right)\right) \\ &= \sum_i \sum_j (g_i \otimes_{\mathbb{Z}} h_j) \\ &= \sum_{i,j} (g_i \otimes_{\mathbb{Z}} h_j). \end{aligned}$$

Thus each elementary tensor may be written as a finite sum of elementary tensors whose components are generators. Since the elementary tensors generate $G \otimes_{\mathbb{Z}} H$, it follows that $\boxed{G \otimes_{\mathbb{Z}} H \text{ is finitely generated.}}$[5]

[5]Professor Glaz observes that this can be made more explicit by writing

$$\begin{aligned} G &\cong \left(\bigoplus_{i=1}^{r_1} \mathbb{Z}\right) \oplus \left(\bigoplus_{i=1}^{k_1} \mathbb{Z}/m_i\mathbb{Z}\right) \\ H &\cong \left(\bigoplus_{i=1}^{r_2} \mathbb{Z}\right) \oplus \left(\bigoplus_{i=1}^{k_3} \mathbb{Z}/n_i\mathbb{Z}\right) \end{aligned}$$

and now using Exercise 4 to get

$$G \otimes_{\mathbb{Z}} H \cong \bigoplus_{i=1}^{r_1} \left(\left(\bigoplus_{i=1}^{r_2} \mathbb{Z}\right) \oplus \left(\bigoplus_{i=1}^{k_1} \mathbb{Z}/n_i\mathbb{Z}\right)\right) \oplus \bigoplus_{i=1}^{k} \left(\left(\bigoplus_{i=1}^{r_2} \mathbb{Z}/m_i\mathbb{Z}\right) \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Z}/(m_i, n_i)\mathbb{Z}\right)\right).$$

(v) If G and H are free, then there are index sets I, I' such that $G = \oplus_{i\in I}\mathbb{Z}$ and $H = \oplus_{i\in I'}\mathbb{Z}$. So (using Exercise 4 twice)

$$\begin{aligned} G \otimes_{\mathbb{Z}} H &= \left(\bigoplus_{i\in I}\mathbb{Z}\right) \otimes_{\mathbb{Z}} \left(\bigoplus_{i\in I'}\mathbb{Z}\right) \\ &\cong \bigoplus_{i\in I}\left(\mathbb{Z} \otimes_{\mathbb{Z}} \left(\bigoplus_{i\in I'}\mathbb{Z}\right)\right) \\ &\cong \bigoplus_{i\in I}\bigoplus_{i\in I'}(\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}) \\ &= \bigoplus_{i\in I\times I'}(\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}) \\ &\cong \bigoplus_{i\in I\times I'}\mathbb{Z}, \end{aligned}$$

so $\boxed{G \otimes_{\mathbb{Z}} H \text{ is free.}}$

Exercise 8

Use Exercise 7(ii) to do Exercise 1 on page 31. Also, find an alternative proof for Exercise 1.

Exercise 1

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

► In Exercise 7(ii), we showed that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z}.$$

Since $(m,n) = 1$, this means that $(\mathbb{Z}/m\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to (therefore equal to) 0.

- Alternatively, consider the exact sequence

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

where f is the map $a \mapsto ma$ and g is reduction modulo m. Then by Proposition 2.18, the following sequence (with the appropriate arrows) is exact:

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{f \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{g \otimes 1} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Since $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$, we have another exact sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{a \mapsto ma} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Since $(m, n) = 1$, $a \mapsto ma$ is onto and hence the kernel of the second map is all of $\mathbb{Z}/n\mathbb{Z}$, i.e. the map

$$\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

in the above sequence is the zero map. But since the sequence is exact, this map is surjective, which is impossible unless $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$.

- As another alternative, let $a \otimes b$ denote $(a + m\mathbb{Z}) \otimes_{\mathbb{Z}} (b + n\mathbb{Z}) \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Then

$$\begin{aligned} \underbrace{a \otimes b + \cdots + a \otimes b}_{m \text{ times}} &= (\underbrace{a + \cdots + a}_{m \text{ times}}) \otimes b \\ &= 0 \\ &= a \otimes (\underbrace{b + \cdots + b}_{n \text{ times}}) = \underbrace{a \otimes b + \cdots + a \otimes b}_{n \text{ times}}, \end{aligned}$$

so $a \otimes b$ has order dividing m and dividing n. Since $(m, n) = 1$ by hypothesis, this means that $a \otimes b = 0$. But this holds for arbitrary $a \otimes b \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, so $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.