1. Suppose $x^n = 0$ for some $x \in A, n \in \mathbb{Z}^+$. Observe that $(1+x)\sum_{k=0}^{n-1}(-x)^k =$

$$\begin{aligned} &1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} \\ &+x - x^2 + x^3 - \cdots + (-1)^{n-2}x^{n-1} + (-1)^{n-1}x^n \end{aligned}$$

$= 1 \pm x^n = 1 \pm 0 = 1$. So $1 + x$ is a unit in A.

Now if u is any unit in A with $uv = 1$, then $(u+x)v = 1+xv$ and xv is still nilpotent since $(xv)^k = x^k v^k$. Using the above, $\sum_{k=0}^{n-1}(-xv)^k$ is an inverse for $1 + vx$, so $v\sum_{k=0}^{n-1}(-xv)^k$ is an inverse for $u + x$. Therefore the sum of a unit and a nilpotent is a unit.

2. Let $f = \sum_{k=0}^{n} a_k x^k$ be a polynomial in $A[x]$.

i. If a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent, then $(a_1x + \cdots + a_nx^n)$ is nilpotent and by **1.**, $f = a_0 + (a_1x + \cdots + a_nx^n)$ is a unit in $A[x]$.

Conversely, if f is a unit then there is $g = \sum_{k=0}^{m} b_k x^k$ such that $fg = 1$. Since $fg = \sum_{k=0}^{n+m}\left(\sum_{i+j=k} a_i b_j\right)x^k$, we have that the constant term a_0b_0 of fg equals 1 and all the other coefficients $\sum_{i+j=k} a_i b_j$ are 0. This shows that a_0 and b_0 are units (and inverse to each other). We will show that a_n is nilpotent as follows:

Claim: For $0 \le r \le m$, $a_n^{r+1} b_{m-r} = 0$.

Proof: By strong induction on r. Since the leading coefficient a_nb_m of fg is 0, the case $r = 0$ holds. Now assume that for some $1 \le r \le m$ and for all $0 \le s \le r-1$, $a_n^{s+1}b_{m-s} = 0$. The coefficient of x^{n+m-r} in fg is 0 and can be written $\sum_{s=0}^{r} a_{n-r+s}b_{m-s} = 0$. Multiplying on both sides by a_n^r will make the last term on the LHS $a_n^{r+1}b_{m-r}$ and will make the first $r-1$ terms 0 since the s^{th} term is now $a_n^r a_{n-r+s}b_{m-s} = a_{n-r+s}a_n^{r-(s+1)}(a_n^{s+1}b_{m-s}) = 0$ by the induction hypothesis.

Taking $r = m$ we see that $a_n^{m+1}b_0 = 0$. Since b_0 is a unit we can cancel to see that a_n is nilpotent. Noting that $-a_nx^n$ is still nilpotent and using **1**. we see that $f - a_nx^n$ is a unit. We can apply the same argument to the unit $f - a_nx^n = \sum_{k=0}^{n-1} a_kx^k$ to see that the leading coefficient a_{n-1} is nilpotent. Similarly, all the coefficients besides the constant term are nilpotent.

ii. Since nilpotent elements form an ideal, if $a_0, a_1, \ldots, a_n$ are nilpotent, then $\sum_{k=0}^{n} a_kx^k$ is nilpotent. For the other direction, assume $f^r = 0$. Thus for each $0 \le k \le rn$, the coefficient of x^k in f^r is zero. In particular, the leading coefficient $a_n^r = 0$ which says that a_n is nilpotent. If a_n is nilpotent, then $-a_nx^n$ is nilpotent and the sum $f - a_nx^n$ is nilpotent of degree smaller than f. By induction on the degree of f we see that $a_n, a_{n-1}, \ldots, a_1, a_0$ are all nilpotent.

iii. One direction is trivial. Now assume that f is a zero divisor and $g = \sum_{k=0}^{m} b_kx^k$ is of minimal positive degree ($b_m \ne 0$) such that $fg = 0$. Since the leading coefficient a_nb_m of fg is 0, we see that $f(a_ng) = a_n(fg) = 0$. Since the degree of a_ng is strictly less that that of g we have $a_ng = 0$.

Now assume inductively that $a_{n-i}g = 0$ for all $0 \leq i < r$. We will show that $a_{n-r}g = 0$. We can expand $fg = 0$ to see that $\sum_{k=0}^{n} a_k g x^k = 0$. Using the induction assumption, the last $r-1$ terms are 0 and we see that $\sum_{k=0}^{n-r} a_k g x^k = 0$. Thus the leading coefficient $a_{n-r}b_m = 0$ and again $f(a_{n-r}g) = a_{n-r}(fg) = 0$ but $a_{n-r}g$ has degree strictly less that g so $a_{n-r}g = 0$.

Since $a_k g = 0$ for all $0 \leq k \leq n$, in particular we see that the leading coefficients $a_k b_m$ are all 0. This says that b_m kills all the coefficients of f so $b_m f = 0$.

iv. Let $\mathfrak{I}_f = (a_0, \ldots, a_n), \mathfrak{I}_g = (b_0, \ldots, b_m)$ and $\mathfrak{I}_{fg} = (a_0 b_0, \ldots, \sum_{i+j=k} a_i b_j, \ldots, a_n b_m)$ be the ideals generated by the coeffients of f, g and fg respectively. If $\mathfrak{I}_{fg} = (1)$ then $\mathfrak{I}_f = (1) = \mathfrak{I}_g$, since $\mathfrak{I}_{fg} \subset \mathfrak{I}_f$ and $\mathfrak{I}_{fg} \subset \mathfrak{I}_g$.

Conversely, suppose that $\mathfrak{I}_f = (1) = \mathfrak{I}_g$ and $\mathfrak{I}_{fg} \neq (1)$. Since $\mathfrak{I}_{fg}$ is a proper ideal, it is contained in a maximal ideal $\mathfrak{m}$. Now $A/\mathfrak{m}$ is a field, so $(A/\mathfrak{m})[x] \cong A[x]/\mathfrak{m}[x]$ is an integral domain and $(f + \mathfrak{m}[x])(g + \mathfrak{m}[x]) = (fg + \mathfrak{m}[x]) = (0 + \mathfrak{m}[x])$ since the coefficients of fg are in $\mathfrak{I}_{fg} \subset \mathfrak{m}$. Hence $(f + \mathfrak{m}[x])$ or $(g + \mathfrak{m}[x])$ is zero in $(A/\mathfrak{m})[x]$ and either $\mathfrak{I}_f$ or $\mathfrak{I}_g$ is properly contained in $A = (1)$, a contradiction.

6. Since the nilradical $\mathfrak{N} = \bigcap_{\mathfrak{p} prime} \mathfrak{p}$ and the Jacobson radical $\mathfrak{J} = \bigcap_{\mathfrak{m} maximal} \mathfrak{m}$ and maximal ideals are always prime, it is always the case that $\mathfrak{N} \subset \mathfrak{J}$. If $\mathfrak{N} \neq \mathfrak{J}$, then $\mathfrak{J}$ is an ideal not properly contained in $\mathfrak{N}$ and by assumption we have that there is a nonzero idempotent $e \in \mathfrak{J}$. From Proposition 1.9 we know that for any element $j \in \mathfrak{J}$, $1 - rj$ is a unit for all $r \in A$. So $(1-e)$ is a unit, but we have that $e(1-e) = e - e^2 = e - e = 0$. Cancelling implies that $e = 0$, a contradiction.

7. Let $\mathfrak{p}$ be a prime ideal in A. We will show that the integral domain $A/\mathfrak{p}$ is a field. Let $(x + \mathfrak{p}) \neq (0 + \mathfrak{p})$ in $A/\mathfrak{p}$. By assumption we have that $x^n = x$ for some $n > 1$ i.e. $x^n - x = x(x^{n-1} - 1) = 0$. Looking mod $\mathfrak{p}$, we see that $(x + \mathfrak{p})((x^{n-1} - 1) + \mathfrak{p}) = (0 + \mathfrak{p})$. Since $A/\mathfrak{p}$ is a domain and $(x + \mathfrak{p}) \neq (0 + \mathfrak{p})$, we have that $((x^{n-1} - 1) + \mathfrak{p}) = (0 + \mathfrak{p}) \iff (x^{n-1} + \mathfrak{p}) = (1 + \mathfrak{p}) \iff (x + \mathfrak{p})(x^{n-2} + \mathfrak{p}) = (1 + \mathfrak{p})$. Since $x \notin \mathfrak{p}$ was arbitrary, we see that every nonzero element of $A/\mathfrak{p}$ is invertible. So $A/\mathfrak{p}$ is a field, and $\mathfrak{p}$ is maximal.

11. Suppose A is a Boolean ring.

i. Fix $x \in A$ and observe that since A is Boolean, $(x + x)^2 = (x + x) \iff x^2 + x^2 + x^2 + x^2 = x + x \iff x + x + x + x = x + x \iff x + x = 0 \iff 2x = 0$.

ii. Since A satisfies the conditions of **7**. (with $n = 2$ for every x) we see that every prime ideal in A is maximal. Let $\mathfrak{p}$ be a prime (hence maximal) ideal in A and consider the field $A/\mathfrak{p}$. If $(x + \mathfrak{p}) \neq (0 + \mathfrak{p})$, then since $((x^2 - x) + \mathfrak{p}) = (0 + \mathfrak{p}) \iff (x + \mathfrak{p})((x - 1) + \mathfrak{p}) = (0 + \mathfrak{p})$. Since $A/\mathfrak{p}$ is a domain and $(x + \mathfrak{p}) \neq (0 + \mathfrak{p})$, we have that $((x-1) + \mathfrak{p}) = (0 + \mathfrak{p}) \iff (x + \mathfrak{p}) = (1 + \mathfrak{p})$. So every nonidentity element in $A/\mathfrak{p}$ is $(1 + \mathfrak{p})$. And $A/\mathfrak{p} = \{(0 + \mathfrak{p}), (1 + \mathfrak{p})\}$ is a field with two elements.

iii. By induction, it suffices to show that $(a, b) = (d)$ for any $a, b \in A$. Let $d = a + b - ab$. Clearly, $(d) \subset (a, b)$ but observe also that $a = ad$ and $b = bd$. Thus $a, b \in (d)$ and $(d) \supset (a, b)$.

12. Let A be a local ring with maximal ideal $\mathfrak{m}$. Hence the Jacobson radical of A is $\mathfrak{m}$. Suppose there is a nonzero idempotent $e \in A$. If $e \in \mathfrak{m}$, then $1 - e$ is a unit and $e(1-e) = 0 \implies e = 0$ a contradiction. If $e \notin \mathfrak{m}$, then in the field $A/\mathfrak{m}$, since $((e-e^2)+\mathfrak{m}) = (0+\mathfrak{m}) \iff (e+\mathfrak{m})((1-e)+\mathfrak{m}) = (0+\mathfrak{m}) \iff ((1-e)+\mathfrak{m}) = (0+\mathfrak{m}) \iff (1-e) \in \mathfrak{m}$. Again, since $(1-e)$ is in the Jacobson radical $\mathfrak{m}$, $1-(1-e) = e$ is a unit in A. So $e(1-e) = 0 \implies (1-e) = 0 \implies e = 1$. So in a local ring, the only nonzero idempotent is 1.