4.2 SOLUTIONS

3. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix},$$

the general solution is $x_1 = 2x_3 - 4x_4$, $x_2 = -3x_3 + 2x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

11. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

14. An element $\mathbf{w}$ on W may be written as

$$\mathbf{w} = s \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

where a and b are any real numbers. So $W = \text{Col}\, A$ where $A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \\ 5 & -1 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

17. The matrix A is a 4×2 matrix. Thus

(a) Nul A is a subspace of $\mathbb{R}^2$, and

(b) Col A is a subspace of $\mathbb{R}^4$.

21. Either column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = (2/3)x_2$, with x_2 free. Letting x_2 be a nonzero value (say $x_2 = 3$) gives the nonzero vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

which is in Nul A.

23. Consider the system with augmented matrix $[A \quad \mathbf{w}]$. Since

$$[A \quad \mathbf{w}] \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col A. Also, since

$$A\mathbf{w} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul A.

25. a. True. See the definition before Example 1.

b. False. See Theorem 2.

c. True. See the remark just before Example 4.

d. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* $\mathbf{b}$. See #7 in the table on page 204.

e. True. See Figure 2.

f. True. See the remark after Theorem 3.

3. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the matrix has only two pivot positions. Thus its columns do not form a basis for $\mathbb{R}^3$; the set of vectors is linearly independent and does not span $\mathbb{R}^3$.

4. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 2 & 2 & -8 \\ -1 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the matrix has three pivot positions. Thus its columns form a basis for $\mathbb{R}^3$.

9. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $x_1 = 2x_3$, $x_2 = -x_3$, $x_4 = 0$, with x_3 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

11. Let $A = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x = 3y - 2z$ with y and z free. So

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

13. Since B is a row echelon form of A, we see that the first and second columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix},\begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables: $x_1=-6x_3-5x_4$, $x_2=(-5/2)x_3-(3/2)x_4$, with x_3 and x_4 free. So

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}\right\}.$$

15. This problem is equivalent to finding a basis for Col A, where $A=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5\end{bmatrix}$. Since the reduced echelon form of A is

$$\begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\-2&2&-8&10&-6\\3&3&0&3&9\end{bmatrix}\sim\begin{bmatrix}1&0&2&0&0\\0&1&-2&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$

we see that the first, second, fourth and fifth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{\begin{bmatrix}1\\0\\-2\\3\end{bmatrix},\begin{bmatrix}0\\1\\2\\3\end{bmatrix},\begin{bmatrix}2\\-1\\10\\3\end{bmatrix},\begin{bmatrix}3\\-1\\-6\\9\end{bmatrix}\right\}.$$

23. Let $A=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix}$. Then A is square and its columns span $\mathbb{R}^4$ since $\mathbb{R}^4$ $=\text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$. So its columns are linearly independent by the Invertible Matrix Theorem, and $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

24. Let $A=\begin{bmatrix}\mathbf{v}_1 & \dots & \mathbf{v}_n\end{bmatrix}$. Then A is square and its columns are linearly independent, so its columns span $\mathbb{R}^n$ by the Invertible Matrix Theorem. Thus $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.